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INVESTIGATION OF COVID-19 MATHEMATICAL MODEL UNDER FRACTIONAL ORDER DERIVATIVE

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Abstract. The given article is devoted to presentation of some results regarding existence and uniqueness of solution to a fractional order model that addressing the effect of immigration on the transmission dynamics of a population model. Further, in view of this investigation the effect of immigration have been checked on transmission of recent pandemic known as Corona virus COVID-19. The concerned results have been established by using fixed point theory approach. After investigation qualitative analysis of the considered model, by applying Laplace transform along with decomposition method, we have calculated some series type results for the concerned model. The unknown quantities of each equation have been decomposed into small quantities to calculate each small quantity very easily for the series solution by adding first few terms of the said quantities. Approximate results of some testing data with different cases are given to illustrate the results.

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1. Introduction

Recently the pandemic known as COVID-19 largely affected about two third of the world population by some aspect. The origin and time of this disease is the Wuhan city of China and December 2019 respectively. Up to now about forty hundred thousand people have got the infection of this pandemic and nearly fourteen hundred thousand of the people have been died [44]. The interesting point is that much more of the human population have been recovered from the said disease. Every country and their government are trying to make different polices for controlling and reducing this outbreak. One of the main cause of this disease is the immigration and interactions of infected human to the territory or area of uninfected population. For this precautions and cares habits should be followed by each country on the globe. Since the beginning of the said pandemic up to now various steps like banned on air traffic, immigration, non-quarrantina and social gathering in different areas, haven been taken.

Keywords and phrases: Existence and uniqueness, Laplace transform, approximate results.

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So for different countries of the globe have announced lock down in various activities or reduce insufficient traveling and crowded bossiness activities [17]. Peoples are afraid of this because such like pandemic in the past have become the cause of death for more than hundred thousand of humans. For each and every terrible outbreak different scientists and researchers trying to discover their vaccine from the very beginning and may be applied in the future. Understanding the behavior of the pandemic need statistical data and some concepts of mathematical formulation. Such concept of mathematical modeling was applied for the first time in 1927. By applying the statistical data on different diseases will lead us to the construction of mathematical formula. The said formula will be than used for modeling different real world phenomenon. So for, various infection have been molded by single mathematical formula as can be study in [5, 7, 13, 26].

Different scholars have been analyzed the recent pandemic cased by corona virus by some methods of analysis as can be seen in [6, 9, 18, 22, 42]. While some others formulated it by mathematical models which predicts the future information about the said disease. The past, present and future information of the said pandemic can helps the policy makers to construct some well strategies for controlling it. Because of importance of modeling the aforesaid disease have been discussed in various research articles [15]. Constructing a mathematical models involve some parameters which are based on some assumption like immigration is one of the most important cause of spreading the disease will be include as a term in constructing models for it. As COVID-19 can spread easily in social gathering, therefore some scholars have analyzed the dynamics of such type of disease for transmission due to immigration as given follows

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x(t)a - y(t)x(t)b + x(t)e, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = (-d - e + c)y(t) + x(t)by(t), \\ x(0) = x_0, \quad y(0) = y_0. \end{cases} \tag{1.1}$$

Such type of mathematical models that we used in this paper is inspired from the classic Lotka-Volterra model [14, 16] for analyzing predator-prey dynamics. The classic model has been suitably modified to build the healthy-infected individual population dynamics model. The healthy individual population at any time t is given by $x(t)$. The infected individual population is represented by $y(t)$ at any time t. The infection rate is given by $b = (1 - \textit{protection rate})$. The immigration rate of healthy ones is given by a. Immigration rate of infected population is given by c. The death rate is d and the treatment rate is given by e. This model is just an indication to see what happen in a community, if immigration of individuals does not control in it.

Keeping in mind the aforementioned points, we have to study the modified model given in (1.1) by involving birth rate β and natural death rate δ for non-integer order of derivative with $0 < \alpha \leq 1$ is given by

$$\begin{cases} \frac{{}^{c}d^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \beta + x(t)a - y(t)bx(t) + (e - \delta)x(t), \\ \frac{{}^{c}d^{\alpha}y(t)}{\mathrm{d}t^{\alpha}} = y(t)bx(t) + (c - d - e - \delta)y(t), \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{1.2}$$

where y_0 and x_0 are the starting values for healthy and infected individuals density in percent. Further the total population $X = x + y$ holds. Also $a, b, c, d, e, \delta, \beta > 0$. We investigating the equation (1.2) in fractional order sense because it gives much better result than integer order. Due to this the modern calculus has gained more attention as compare to classical calculus. Fractional calculus in which fractional global operators can model physical and biological problems relating to real world phenomena with very well degree of freedom. Sa for a large numbers of books and research articles have been touched by the scholars of fractional calculus. Various phenomena like qualitative and numerical analysis of non-integer order differential and integral equations have been presented in different books and monographs [8, 14, 21, 23, 28, 30, 31].

Modern calculus generalized the classical calculus of integer order differentiation and integration to rational or complex numbers, describing the situation between two integers as in [29, 32, 33, 40]. So for, many real world problems have been modeled by integer order differential equation, like population model,logistic equations, HIV, SEIR, HIV, Cancer model, Predator-prey model, etc. Further the scientists converted these equations to arbitrary order of differential equations which give much more real solution [4, 11, 19, 25, 34–37, 41, 45]. They have analyzed these equations for exitence and uniqueness by applying some of the properties and theorems of fixed point theory which is given in [1, 2, 10, 24]. Some scholars apply Banach contraction theorem, topological degree theory and Leray-Shaudar theorem [43]. The FDEs have also interesting for obtaining analytical as well as numerical solution for different scholars and researchers. To find exact solution, it is very difficult task, therefore, most of the scholars investigating FDEs for optimize and approximate solution applying the pre-existing techniques. For numerical analysis they applied Taylor's series method, Modified Euler techniques, Adams Bash-Forth techniques, predictor-corrector method and different integral transforms along with wavelets methods as in [3, 12, 14, 19, 27, 38, 39]. Now we will investigate our considered problem for qualitative and some numerical analysis using the concepts of fixed point theory and Laplace transform along with some decomposition techniques. We will also investigate the effect of immigration on the pandemic transmission in the society using different fractional or arbitrary orders for future policy.

2. Basics of fractional calculus

In this section we will present some basic work from modern calculus [14, 21, 45].

Definition 2.1. Let we have any operator say $x(t)$, then we may define the arbitrary order integration w.r.t t as

$$I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \mathrm{d}\zeta,\ \theta > 0,$$

such that integral on right side converges.

Definition 2.2. For any mapping $x(t)$, one may define the non-integer order derivative in Caputo sense w.r.t t as

$$\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} \frac{d^n}{\mathrm{d}\zeta^n}[x(\zeta)]\mathrm{d}\zeta,\ \alpha > 0,$$

with right side is point wise continuous on $\mathbf{R}_+$ and $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then we have

$$\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\zeta}[x(\zeta)]\mathrm{d}\zeta.$$

Lemma 2.3. *[14] The solution of*

$$\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} = z(t), \quad 0 < \alpha \leq 1$$

is

$$x(t) = c_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \mathrm{d}\zeta.$$

Lemma 2.4. *[14] The Laplace transform of* $\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha}$ *for* $0 < \alpha \leq 1$ *is provided by*

$$\mathscr{L}\left[\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha}\right] = s^\alpha \mathscr{L}[x(t)] - s^{\alpha-1} x(0)].$$

Lemma 2.5. *"The boundedness and feasibility of the solution region for the given model is*

$$\mathbb{S} = \left\{(x,y) \in \mathbf{R}_+^2 :\ 0 \leq X(t) \leq \frac{\beta}{\delta-(a+c+e)}\right\},\ \textit{where}\ \ \delta > a+e+c\text{''}.$$

Proof. For the proof we have to add both the equation of (1.2), as

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= (e+a)x(t) + \beta - \delta(y(t)+x(t)) + y(t)c - (e+d)y(t) \\ &\leq \beta + (e+a)X(t) - \delta X(t) + X(t)c \\ &= \beta - (\delta - (e+a+c))X(t) \end{aligned}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} + (\delta-(e+a+c))X(t) \leq \beta. \tag{2.1}$$

Taking integration of (2.1), as follows

$$X(t) \leq \frac{\beta}{\delta-(a+e+c)} + C\exp(-(\delta-(a+e+c))t), \tag{2.2}$$

where C is constant of integration. From (2.2), we see that as $t \to \infty$, then

$$X(t) \leq \frac{\beta}{\delta-(a+e+c)}.$$

So proved. □

3. Qualitative analysis

In this section we will check our considered model of fractional order for qualitative properties. These types of properties can be easily handled by fixed point theory. We will apply some basic theorems like Banach contraction and Schauder theorems to receives our required result as

$$\begin{aligned} \frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} &= \Phi_1(x(t),y(t),t), \\ \frac{{}^c d^\alpha y(t)}{\mathrm{d}t^\alpha} &= \Phi_2(x(t),y(t),t), \\ x(0) &= x_0,\ y(0) = y_0,\ \alpha \in (0,1]. \end{aligned} \tag{3.1}$$

Upon integration for $0 < \alpha \leq 1$ to equation (3.1), we get the given system as:

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi_1(s,x(s),y(s))\mathrm{d}s, \\ y(t) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi_2(s,x(s),y(s))\mathrm{d}s. \end{aligned} \tag{3.2}$$

Now we will take $\infty > T > t > 0$ and define Banach space as $E_1 = C([0,T] \times R^2+, R_+)$, then $E = E_1 \times E_2$ will also be the Banach space having the norm $\|(x,y)\| = \max_{t\in[0,T]}|x(t)| + \max_{t\in[0,T]}|y(t)|$. Expressing the system (3.2) as

$$\mathbf{Z}(t) = \mathbf{Z}_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \phi(\zeta, \mathbf{Z}(\zeta)) \mathrm{d}\zeta, \tag{3.3}$$

where

$$\mathbf{Z}(t) = \begin{cases} y(t) \\ x(t), \end{cases} \qquad \mathbf{Z}_0(t) = \begin{cases} y_0(t) \\ x_0(t), \end{cases} \qquad \phi(t, \mathbf{Z}(t)) = \begin{cases} \Phi_1(x(t), y(t), t) \\ \Phi_2(x(t), y(t), t). \end{cases} \tag{3.4}$$

To derive the proof for qualitative analysis, we take the conditions of growth on non-linear vector operator $\phi : [0,T] \times R_+^2 \to R_+$ as:

(A1) $\exists$ a constants $L_\phi > 0$; $\forall$ $\mathbf{Z}(t),\ \bar{\mathbf{Z}(t)} \in R \times R$;

$$|\phi(t, \mathbf{Z}(t)) - \phi(t, \bar{\mathbf{Z}(t)})| \leq L_\phi |\mathbf{Z}(t) - \bar{\mathbf{Z}(t)}|.$$

(A2) $\exists$ a constants $C_\phi > 0$ & $M_\phi > 0$;

$$|\phi(t, \mathbf{Z}(t))| \leq C_\phi |\mathbf{Z}| + M_\Phi.$$

Theorem 3.1. *"Under the continuity of ϕ together with assumption (A2), system (3.1) has at least one solution".*

Proof. With the help of "Schauder fixed point theorem", we will derive our result. Let us take a bounded subset B of E as

$$B = \{\mathbf{Z} \in E : \ \|\mathbf{Z}\| \leq R,\ R > 0\}.$$

By this take a function $\mathbf{A} : B \to B$ and applying (3.3) as

$$\mathbf{A}(\mathbf{Z}) = \mathbf{Z}_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \phi(\zeta, \mathbf{Z}(\zeta)) \mathrm{d}\zeta. \tag{3.5}$$

At any $\mathbf{Z} \in B$, follows

$$\begin{aligned} |\mathbf{A}(\mathbf{Z})(t)| &\leq |\mathbf{Z}_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} |\phi(\zeta, \mathbf{Z}(\zeta))| \mathrm{d}\zeta, \\ &\leq |\mathbf{Z}_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta^{\alpha-1} [C_\phi |\mathbf{Z}| + M_\phi] \mathrm{d}s, \\ &\leq |\mathbf{Z}_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} [C_\phi \|\mathbf{Z}\| + M_f], \end{aligned}$$

which implies that

$$\begin{aligned} \|\mathbf{A}(\mathbf{Z})\| &\leq |\mathbf{Z}_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} [C_\phi \|\mathbf{Z}\| + M_f], \\ &\leq R. \end{aligned} \tag{3.6}$$

From (3.6), one has implies that $\mathbf{Z} \in B$. Thus $\mathbf{A}(B) \subset B$. From this we say that operator $\mathbf{A}$ is closed and bounded. Next we go ahead to prove the result for completely continuous operator as:

Let $t_2 > t_1$ lies in $[0, T]$, and take

$$
\begin{aligned}
|\mathbf{A}(\mathbf{Z})(t_2) - \mathbf{A}(\mathbf{Z})(t_1)| = &\Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \zeta)^{\alpha-1}, \\
&\phi(s, \mathbf{Z}(\zeta))\mathrm{d}\zeta - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - \zeta)^{\alpha-1} \phi(\zeta, \mathbf{Z}(\zeta))\mathrm{d}\zeta \Bigg|, \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_2} (t_2 - \zeta)^{\alpha-1} - \int_0^{t_1} (t_1 - \zeta)^{\alpha-1} \right] (C_\phi R + M_\Phi)\mathrm{d}\zeta, \\
&\leq \frac{(C_\phi R + M_\phi)}{\Gamma(\alpha+1)} [t_2^\alpha - t_1^\alpha].
\end{aligned} \tag{3.7}
$$

Now from (3.7), on can observe that ast_1 approaches to t_2, Then right side also vanishes. So one concludes that $|\mathbf{A}(\mathbf{Z})(t_2) - \mathbf{A}(\mathbf{Z})(t_1)|$ tends to 0, as $t_1$tends to t_2.

So $\mathbf{A}$ is “equi- continuous operator”. By using “Arzelá-Ascoli theorem”, the operator $\mathbf{A}$ is completely continuous operator and also uniformly bounded proved already. By “Schauder's fixed point theorem” model (1.2) one or more than one solution. □

Further we proceeds for uniqueness as:

Theorem 3.2. *“Using (A1), system has unique or one solution if* $\frac{T^\alpha}{\Gamma(\alpha+1)} L_\Phi < 1$*”.*

Proof. Take $\mathbf{A} : E \to E$, consider $\mathbf{Z}$ and $\bar{\mathbf{Z}} \in E$ as

$$
\begin{aligned}
\|\mathbf{A}(\mathbf{Z}) - \mathbf{A}(\bar{\mathbf{Z}})\| = &\max_{t\in[0,T]} \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \Phi(\zeta, \mathbf{Z}(\zeta))\mathrm{d}\zeta - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \Phi(\zeta, \bar{\mathbf{Z}}(\zeta))\mathrm{d}\zeta \right|, \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} L_\Phi \|\mathbf{Z} - \bar{\mathbf{Z}}\|.
\end{aligned} \tag{3.8}
$$

From (3.8), follows

$$
\|\mathbf{A}(\mathbf{Z}) - \mathbf{A}(\bar{\mathbf{Z}})\| \leq \frac{2T^\alpha}{\Gamma(\alpha+1)} L_\Phi \|\mathbf{Z} - \bar{\mathbf{Z}}\|. \tag{3.9}
$$

Hence F is contraction. By “Banach contraction theorem” system has one solution. □

4. General series solution of the consider system (1.2)

To produce semi-analytical solution to the considered model, we first generate a general algorithm by using system (3.1). We apply, Laplace transform to (3.1) on both sides as

$$
\begin{aligned}
\mathscr{L}\left[\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} \right] &= \mathscr{L}\Big[\phi_1(x(t), y(t), t) \Big], \\
\mathscr{L}\left[\frac{{}^c d^\alpha y(t)}{\mathrm{d}t^\alpha} \right] &= \mathscr{L}\Big[\phi_2(x(t), y(t), t) \Big],
\end{aligned} \tag{4.1}
$$

on using initial condition, follows

$$\begin{aligned} s^{\alpha}\mathscr{L}\Big[x(t)\Big] &= s^{\alpha-1}x_0 + \mathscr{L}\Big[\phi_1(x(t),y(t),t)\Big],\\ s^{\alpha}\mathscr{L}\Big[y(t)\Big] &= s^{\alpha-1}y_0 + \mathscr{L}\Big[\phi_2(x(t),y(t),t)\Big], \end{aligned} \tag{4.2}$$

or

$$\begin{aligned} \mathscr{L}\Big[x(t)\Big] &= \frac{1}{s}x_0 + \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_1(x(t),y(t),t)\Big],\\ \mathscr{L}\Big[y(t)\Big] &= \frac{1}{s}y_0 + \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_2(x(t),y(t),t)\Big]. \end{aligned} \tag{4.3}$$

Let consider the needed solution as

$$x(t) = \sum_{n=0}^{\infty} x_n(t),\ \ y(t) = \sum_{n=0}^{\infty} y_n(t)$$

and the term due nonlinearity as xy may be expressed as $xy = \sum_{n=0}^{\infty} x_n(t)\sum_{n=0}^{\infty} y_n(t)$, then (4.3) becomes

$$\begin{aligned} \mathscr{L}\Big[\sum_{n=0}^{\infty} x_n(t)\Big] &= \frac{1}{s}x_0 + \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_1(t,\sum_{n=0}^{\infty} x_n(t),\sum_{n=0}^{\infty} y_n(t))\Big],\\ \mathscr{L}\Big[\sum_{n=0}^{\infty} y_n(t)\Big] &= \frac{1}{s}y_0 + \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_2(t,\sum_{n=0}^{\infty} x_n(t),\sum_{n=0}^{\infty} y_n(t))\Big]. \end{aligned} \tag{4.4}$$

On comparison, we have

$$\begin{aligned} \mathscr{L}\Big[x_0(t)\Big] &= \frac{1}{s}x_0,\\ \mathscr{L}\Big[y_0(t)\Big] &= \frac{1}{s}y_0,\\ \mathscr{L}\Big[x_1(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_1(x_0(t),y_0(t),t)\Big],\\ \mathscr{L}\Big[y_1(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_2(x_0(t),y_0(t),t)\Big],\\ \mathscr{L}\Big[x_2(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_1(x_1(t),y_1(t),t)\Big],\\ \mathscr{L}\Big[y_2(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_2(x_1(t),y_1(t),t)\Big],\\ &\vdots\\ \mathscr{L}\Big[x_{n+1}(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_1(x_n(t),y_n(t),t)\Big],\\ \mathscr{L}\Big[y_{n+1}(t)\Big] &= \frac{1}{s^{\alpha}}\mathscr{L}\Big[\phi_2(x_n(t),y_n(t),t)\Big]. \end{aligned}$$

Evaluating inverse Laplace transform, we have

$$
\begin{aligned}
x_0(t) &= x_0, \quad y_0(t) = y_0,\\
x_1(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_1(x_0(t), y_0(t), t)\Big]\right],\\
y_1(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_2(x_0(t), y_0(t), t)\Big]\right],\\
x_2(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_1(x_1(t), y_1(t), t)\Big]\right],\\
y_2(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_2(x_1(t), y_1(t), t)\Big]\right],\\
&\vdots\\
x_{n+1}(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_1(x_n(t), y_n(t), t)\Big]\right],\\
y_{n+1}(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\Big[\phi_2(x_n(t), y_n(t), t)\Big]\right].
\end{aligned}
\tag{4.5}
$$

Hence the required series solution of the considered model (1.2) will be received as

$$
\begin{cases}
x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots,\\
y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots.
\end{cases}
\tag{4.6}
$$

5. Approximate results and discussion

To compute the Approximate results, we now take some values for parameters of (1.1) under fractional order. Let us in community the the healthy population is at initial stage x_0 and infected be y_0 and taking various values for rates we discuss certain cases as bellow:

$$
\begin{aligned}
\frac{{}^c d^\alpha x(t)}{\mathrm{d}t^\alpha} &= \beta + ax(t) - bx(t)y(t) + (e-\delta)x(t),\\
\frac{{}^c d^\alpha y(t)}{\mathrm{d}t^\alpha} &= bx(t)y(t) + (c-d-e-\delta)y(t),\\
x(0) &= x_0,\ y(0) = y_0,
\end{aligned}
\tag{5.1}
$$

using the proposed algorithm to (5.1) as constructed in (4.5), analogously one has

$$
\begin{cases}
x_0(t) = x_0, \quad y_0(t) = y_0, \\
x_1(t) = \left[\beta + a x_0 - b x_0 y_0 + (e-\delta) x_0\right] \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
y_1(t) = \left[b x_0 y_0 + (c-d-e-\delta) y_0\right] \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
x_2(t) = \left[\beta + (a+e-\delta) x_{11}\right] \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left[b x_{11} y_{11}\right] \left(\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) \dfrac{t^{3\alpha}}{\Gamma 3\alpha+1}, \\
y_2(t) = \left[b x_{11} y_{11}\right] \left(\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) \dfrac{t^{3\alpha}}{\Gamma 3\alpha+1} + \left[(c-d-e-\delta) y_{11}\right] \dfrac{t^{2\alpha}}{\Gamma 2\alpha+1}, \\
x_3(t) = \beta + a x_{111} \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} - a y_{111} \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\
\qquad - b x_{111} y_{111} \dfrac{\Gamma(5\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} - b x_{111} y_{22} \dfrac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} \\
\qquad + b (y_{111})^2 \dfrac{\Gamma^2(2\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)} t^{6\alpha} + b y_{111} y_{22} \dfrac{1}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{5\alpha} \\
\qquad + e y_{111} \dfrac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2(\Gamma(4\alpha+1))} t^{4\alpha} + e y_{22} \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
y_3(t) = b x_{111} y_{111} \dfrac{\Gamma(5\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)} t^{6\alpha} + b x_{111} y_{22} \dfrac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} \\
\qquad - b (y_{111})^2 \dfrac{\Gamma^2(2\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)} t^{6\alpha} - b y_{111} y_{22} \dfrac{1}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{5\alpha} \\
\qquad + (c-d-e-\delta) y(111) \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)(\Gamma(4\alpha+1))} t^{4\alpha} + (c-d-e-\delta) y_{22} \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{cases}
\tag{5.2}
$$

and so on. Further terms can be calculated in same way. The unknown values in (5.2) are given as

$$
\begin{aligned}
x_{11} = \beta + a x_0 - b x_0 y_0 + (e-\delta) x_0, \quad y_{11} = b x_0 y_0 + (c-d-e-\delta) y_0, \\
x_{111} = (a+e-\delta) x_{11}, \; y_{111} = b x_{11} y_{11}, \; y_{22} = (c-d-e-\delta) y_{11}.
\end{aligned}
$$

Case-I: First of all we present the graphical presentation for steady state solution at integer order and taking $a = 0.001, \; b = 0.0003, \; c = 0.0001, \; d = 0.001, \; e = 0.05, \; \delta = 0.5, \; \beta = 0.0089$.

In Figure 1 we plotted the steady state solution of the integer order model (1.2) against the given date. Clearly in the presence of very small value of immigration rates a, c, the papulation of healthy class is growing while the infection population decays. Again in Figure 2, we have presented the numerical solution for various non-integer order up to first ten terms of the series solution using the same values $a = 0.001, \; b = 0.0003, \; c = 0.0001, \; d = 0.001, \; e = 0.05, \; \delta = 0.5, \; \beta = 0.0089$. using Matlab. We see that in the presence of very small values of immigration rate, the decline in infected papulation is fastest at small fractional order and similarly the raise in healthy papulation is high at lower order. As the fractional order α inclines to its integer value 1, the graph of fractional order also tends to the integer order graph.

Case-II: We simulate the results above for taking $\beta = 0.0089, \; a = 0.05, \; b = 0.03, \; c = 0.05, \; d = 0.02, \; e = 0.05, \; \delta = 0.5$. Here the immigration of both infected and healthy is taken slightly small than above Case I in Figure 3 as shown From Figure 3, we see that in the presence of slight small value of immigration rate that the dynamics of the corresponding population has shown at various fractional order. Here the papulation dynamics produce oscillations and the concerned oscillation is different at various arbitrary order.

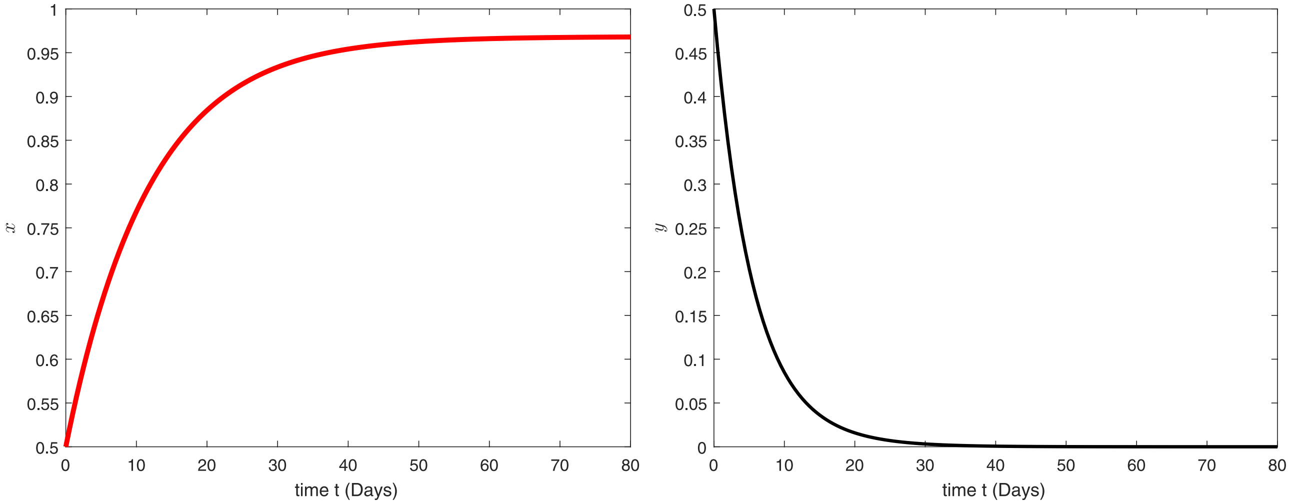

FIGURE 1. Graphical presentation of papulation dynamics at integer order for Model (1.2) at the given values of parameters.

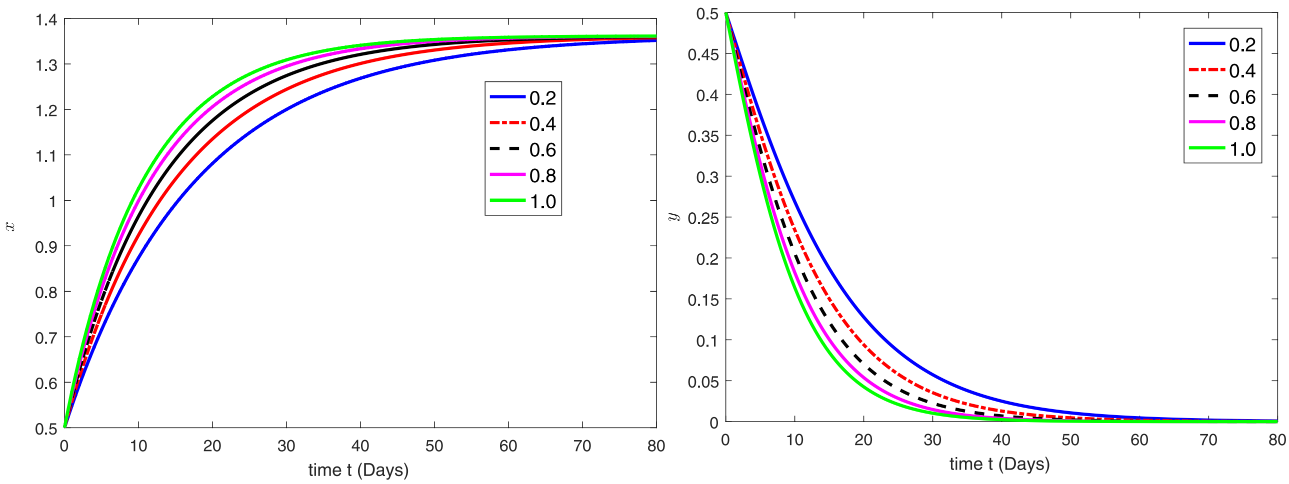

FIGURE 2. Graphical presentation of papulation dynamics at various fractional order for Model (1.2) at the given values of parameters.

Case-III: Next the simulation is provided for taking $\beta = 0.0089;\ a = 0.06,\ b = 0.0003,\ c = 0.06,\ d = 0.001,\ e = 0.05,\ \delta = 0.5$. Here the immigration of infected and healthy population also oscillates with maximum amplitude . The behavior of the approximate solutions up to initial ten terms corresponding to various fractional order has presented in Figure 4. The oscillation is very fast due to increasing in the values of immigration rates.

Remark 5.1. Clearly from (5.1), corresponding to the given values of parameters, the conditions of Theorem 3.1 and Theorem 3.2 are satisfied with $L_\Phi = 0.05,\ T = 4$, we see that $\frac{L_\Phi T^\alpha}{\Gamma(\alpha+1)} = \frac{0.20}{\Gamma(\alpha+1)} < 1$, for all $\alpha \in (0, 1]$. Similarly for other cases can be verified.

In the last we have provided the comparison for both quantities by two methods of laplace Adomian decomposition and modified Euler method in Figures 5 and 6 respectively.

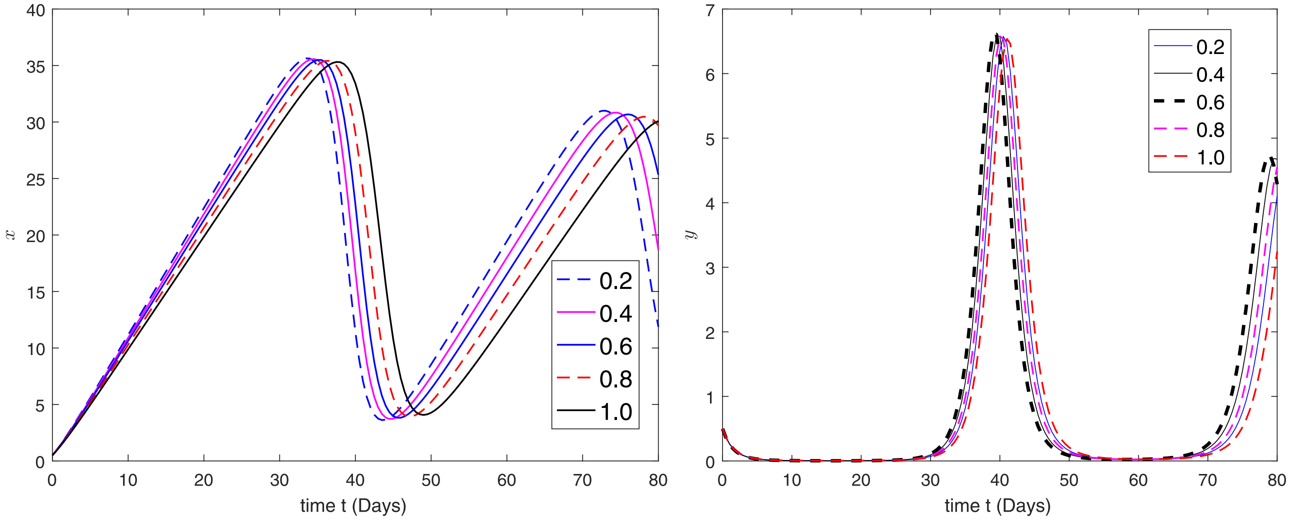

FIGURE 3. Dynamical behaviors of population dynamics in the absence of immigration corresponding to different fractional order.

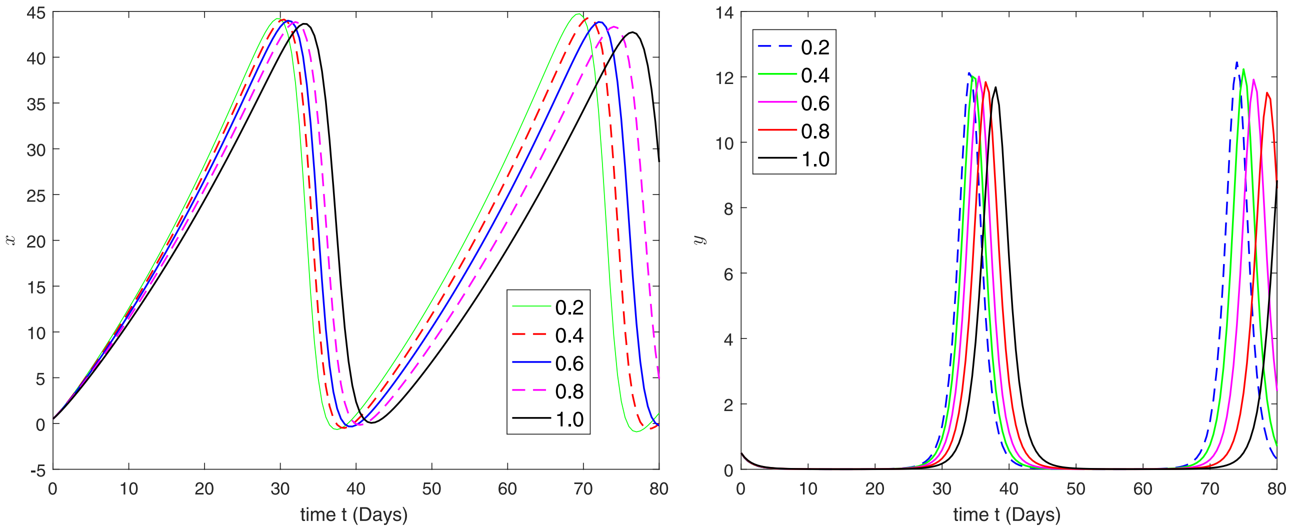

FIGURE 4. Dynamical behaviors of population dynamics in the absence of immigration corresponding to different fractional order.

6. Concluding remarks

Assigning different experimental values taken from [20] to a, b, c, d, and e, we can obtained the required comparable result as for integer order system of (1.1). It can be observe that by increasing rate of protection and cure and decreasing rate of the immigration, we can eventually decrease the number of infected individuals to minimum or towards stability. By studying such dynamical system one can know easily that how to control the populations of healthy from being infected and discourage of infected ones from immigration. This model can be applied to the population where social gathering occur locally or globally. Further such model exist in real world or represent real world problem as proved by using fixed point theory. While the rate of decaying and growth has been shown thorough fractional differentiation in global ways. Hence fractional calculus can be

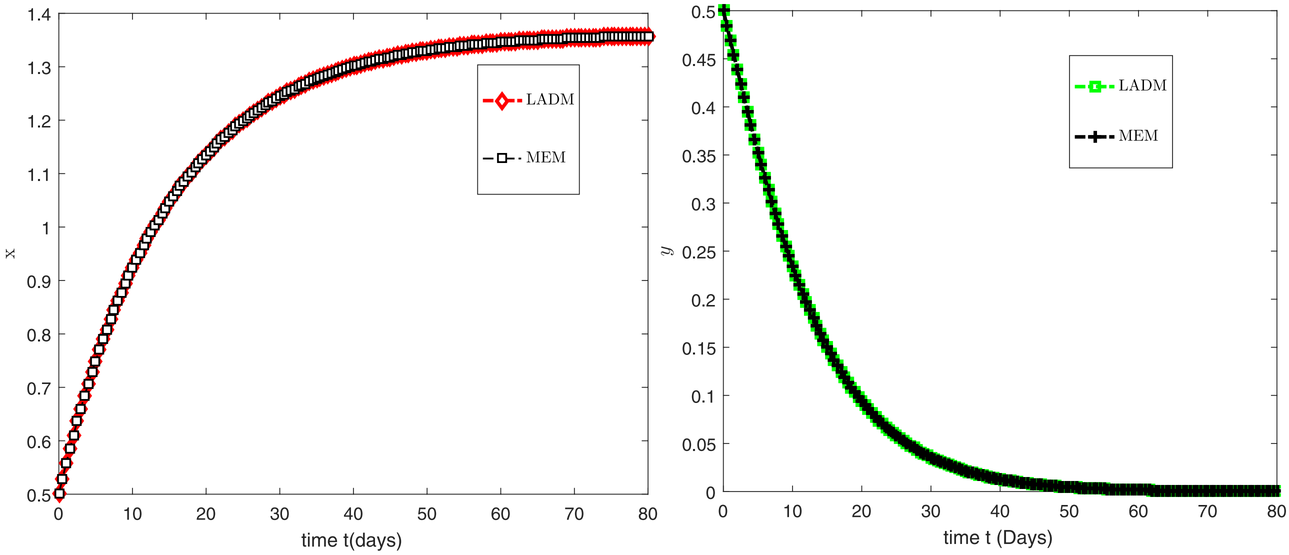

FIGURE 5. Comparison of population dynamics in the presence of immigration corresponding to LADM and MEM.

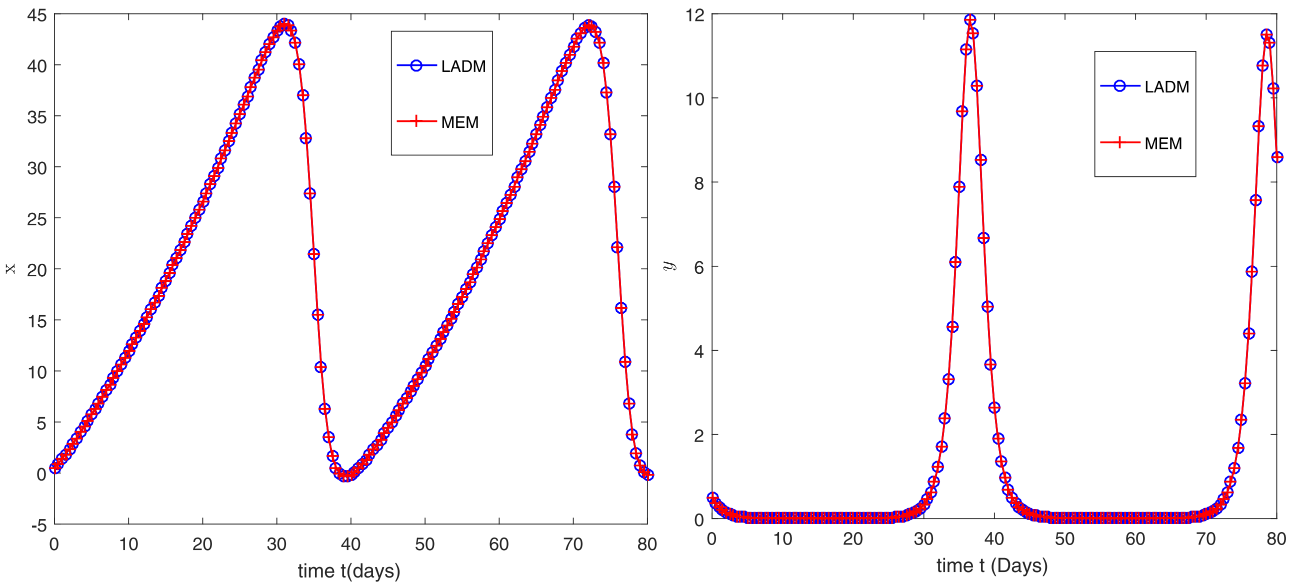

FIGURE 6. Comparison of population dynamics in the absence of immigration corresponding to LADM and MEM.

used to comprehensively explained the model. In future if recover class is involved, then we can further extend this model to see the immigration effect on recovered population also. The simulation shows the dynamics of healthy and infected individuals with some prediction in the near future in fractional form. This model is just an indication that how the aforementioned parameters are important in the transmission dynamics of the present novel coronavirus-19 disease.

Competing interest

There does not exists any kind of competing interest regarding this work.

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