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MORE NEW RESULTS ON INTEGRAL INEQUALITIES FOR GENERALIZED $\mathcal{K}$-FRACTIONAL CONFORMABLE INTEGRAL OPERATORS

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Abstract. This paper aims to investigate the several generalizations by newly proposed generalized $\mathcal{K}$-fractional conformable integral operator. Based on these novel ideas, we derived a novel framework to study for Čebyšev and Pólya-Szegö type inequalities by generalized $\mathcal{K}$-fractional conformable integral operator. Several special cases are apprehended in the light of generalized fractional conformable integral. This novel strategy captures several existing results in the relative literature. We also aim at showing important connections of the results here with those including Riemann-Liouville fractional integral operator.

1. **Introduction.** Fractional calculus is the generalization of derivatives and integrals of arbitrary non-integer order. This discipline has earned greater popularity because of its application in numerous areas [4, 6, 7, 14, 18, 28]. The contemporary studies have motivated on developing some of fractional integral operators and their applications in several areas of sciences. An extensive form of fractional operators and their generalizations have been started with the classical Riemann-Liouville fractional operators, (see, e.g., [17, 28, 47]). Amongst a wide range of the fractional operators evolved, due to their fertile applications in masses areas of sciences, the Riemann-Liouville fractional integral operator has been appreciably introduced. Integrations with kernels are applied in several mathematical complications consisting of quantum theory, spectral analysis, statistical analysis, and the concept of probability distributions.

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Conformable derivatives are nonlocal fractional derivatives. They can be called fractional since we can derive up to arbitrary order. However, since in the community of fractional calculus nonlocal fractional derivatives only are used to be called fractional, we prefer to replace conformable fractional by conformable (as a type of local fractional). Conformable derivatives and other types of local fractional derivatives or modified conformable derivatives in [5] can gain their importance by the ability of using them to generate more generalized nonlocal fractional derivatives with singular kernels (see, [1, 2, 13, 15, 16, 19, 21, 22, 26]).

Recently, the research in fractional calculus had been progressed to generalize the present variants through the progressive mind and innovative techniques. The fractional integral operators are the use of noteworthy significant strategies amongst researchers, see [33, 35, 36, 37, 38, 41, 43]. On account of their potential outcomes to be utilized for the presence of nontrivial and positive solutions of distinct kind of fractional differential equations, our findings concerning fractional integrals are appreciably essential.

An enormous heft of present literature comprises of generalizations of several variants by fractional integral operators and their applications [23, 32, 44, 45]. In [42], authors investigated the continuous research by showing the developed form of generalized Gr$\ddot{u}$ss type integral inequalities for generalized $\mathcal{K}$-fractional integrals. Certain Hermite-Hadamard type inequalities for generalized $\mathcal{K}$-fractional integrals are acquired by Agarwal et al. [3]. Set et al. [46] derived the $\breve{C}$eby$\breve{s}$ev type inequalities using generalized Katugampola integrals via Polya-Szego inequality. Rashid et al. [40] employed the generalized $\mathcal{K}$-fractional integral operators for deriving Polya-Szeg$\ddot{o}$ and $\breve{C}$eby$\breve{s}$ev type inequalities. Khan et al. [25] obtained a generalization of Hermite-Hadamard type inequalities via conformable fractional integrals. Nisar et al. [30] used modern technique to derive $\breve{C}$eby$\breve{s}$ev type inequalities via generalized fractional conformable integrals.

Many famous versions mentioned in the literature are direct effects of the numerous applications in optimizations and and transform theory. In this regard P$\acute{o}$lya-Szeg$\ddot{o}$ integral inequality is one of the most intensively studied inequality. This inequality was introduced by P$\acute{o}$lya-Szeg$\ddot{o}$ [29]:

$$\frac{\int_a^b f^2(\lambda)d\lambda \int_a^b g^2(\lambda)d\lambda}{\left(\int_a^b f(\lambda)g(\lambda)d\lambda\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{QR}{qr}}+\sqrt{\frac{qr}{QR}}\right)^2. \tag{1}$$

The constant $\frac{1}{4}$ is best feasible in (1) make the experience it cannot get replaced by a smaller constant.

By using P$\acute{o}$lya-Szeg$\ddot{o}$ inequality, Dragomir and Diamond [12] derived the functional as follows:

$$|\mathfrak{T}(f,g)| \leq \frac{(Q-q)(R-r)}{4(b-a)\sqrt{qrQR}}\int_a^b f(\lambda)d\lambda\int_a^b g(\lambda)d\lambda,$$

$\forall\ \lambda \in [a,b]$ and $q,Q,r,R \in \mathbb{R}$, where two positive function f and g on $[a,b]$ satisfying $0 < q \leq f(\lambda) \leq Q < \infty$ and $0 < r \leq g(\lambda) \leq R < \infty$.

It is extensively identified that P$\acute{o}$lya-Szeg$\ddot{o}$ and $\breve{C}$eby$\breve{s}$ev type inequalities in continuous and discrete cases play a considerable job in examining the qualitative conduct of differential and difference equations, respectively, further to numerous new branches of mathematics. Inspired by Pólya-Szegö and Čebyšev [9, 29], our intention is to show more general versions of Pólya-Szegö and Čebyšev type inequalities.

Čebyšev [9] introduced the well-known celebrated functional and is defined as follows:

$$\mathfrak{T}(f,g)=\frac{1}{b-a}\int_a^b f(\lambda)g(\lambda)d\lambda-\Big(\frac{1}{b-a}\int_a^b f(\lambda)d\lambda\Big)\Big(\frac{1}{b-a}\int_a^b g(\lambda)d\lambda\Big), \tag{2}$$

where f and g are two integrable functions on $[a,b]$. If f and g are synchronous, i,e.,

$$\big(f(\lambda)-f(\omega)\big)\big(g(\lambda)-g(\omega)\big)\geq 0,$$

for any $\lambda,\omega\in[a,b]$, then $\mathfrak{T}(f,g)\geq 0$.

The functional (2) has vast applications in probability, numerical analysis, quantum, and statistical theory. The main concern of this research is to obtain integral inequalities by generalized conformable $\mathcal{K}$-fractional integral can focus a predetermined number of complex problems on one hand and on the other hands their applications can likewise catch various sorts of complexities, in this manner assembling these generalizations can help us to comprehend the complexities of existing nature in a vastly improved manner. Fractional integrals inequalities have fascinated the attention of practically all scientists from various fields of science. It is noted that the generalized $\mathcal{K}$-fractional conformable estimate is able to appreciate some kind of self-similarities. Alongside facet with numerous applications, the functional (2) has been gained plenty of interest to yield a variety of fundamental inequalities (see, for example, [3, 5, 8, 9, 10, 11, 27, 32, 39, 46, 48, 49, 50]).

Firstly, the Čebyšev inequalities for fractional integral operators are established by Belarbi and Dahmani in [8]. They demonstrated the subsequent outcomes via Riemann-Liouville fractional integral operators.

Throughout the paper, for the results concerning to [9] and [29], it is assumed that all functions are integrable in the Riemann sense.

Theorem 1.1. *Assume that there are two synchronous functions f and g defined on $[0,\infty)$. Then*

$$\mathfrak{I}^{\delta}(fg)(\lambda)\geq\frac{\Gamma(\delta+1)}{\lambda^{\delta}}\mathfrak{I}^{\delta}f(\lambda)\mathfrak{I}^{\delta}g(\lambda), \tag{3}$$

for all $\lambda>0,\ \delta>0$.

Theorem 1.2. *Assume that there are two synchronous functions f and g defined on $[0,\infty)$. Then*

$$\frac{\lambda^{\delta}}{\Gamma(\delta+1)}\mathfrak{I}^{\gamma}(fg)(\lambda)+\frac{\lambda^{\gamma}}{\Gamma(\gamma+1)}\mathfrak{I}^{\delta}(fg)(\lambda)\geq\mathfrak{I}^{\gamma}g(\lambda)\mathfrak{I}^{\delta}f(\lambda)+\mathfrak{I}^{\gamma}f(\lambda)\mathfrak{I}^{\delta}g(\lambda), \tag{4}$$

for all $\lambda>0,\ \delta>0,\gamma>0$.

Theorem 1.3. *Assume that there are n positive increasing functions f_i ; $i=1,2,...,n$ defined on $[0,\infty)$. Then*

$$\mathfrak{I}^{\delta}\Big(\prod_{i=1}^{n}f_i\Big)(\lambda)\geq\big(\mathfrak{I}^{\delta}(1)\big)^{1-n}\prod_{i=1}^{n}\mathfrak{I}^{\delta}f_i(\lambda), \tag{5}$$

for all $\lambda>0,\ \delta>0$.

Theorem 1.4. *Assume that there are two functions f and g defined on $[0,\infty)$ such that f is increasing, g is differentiable, and there exists a real number $\mu := \inf_{\lambda\geq 0}\varphi'(\lambda)$. Then*

$$\mathfrak{I}^{\delta}(fg)(\lambda) \geq \left(\mathfrak{I}^{\delta}(1)\right)^{-1}\mathfrak{I}^{\delta}f(\lambda)\mathfrak{I}^{\delta}g(\lambda) - \frac{\mu\lambda}{\delta+1}\mathfrak{I}^{\delta}f(\lambda) + \mu\mathfrak{I}^{\delta}(\lambda g(\lambda)), \tag{6}$$

for all $\lambda > 0,\ \delta > 0$.

Here we demonstrate some preliminaries for fractional calculus.

We mention the right and left sided generalized conformable fractional derivative operators introduced by [24] as follows:

Let f be a conformable integrable function on the interval $[a,b]$. The right and left sided generalized conformable fractional derivative operators ${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{a+}$ and ${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{b-}$ of order $0<\delta<1, \varrho\in(0,1]$ with $a\geq 0$ are defined by:

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{a+}f(\lambda) = \frac{\lambda^{-\varrho}}{\Gamma(1-\delta)}T_{\zeta}\int\limits_{a}^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-x^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{-\delta}f(x)x^{\varrho}d_{\zeta}x, \quad \lambda > a \tag{7}$$

and

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{b-}f(\lambda) = \frac{\lambda^{-\varrho}}{\Gamma(1-\delta)}T_{\zeta}\int\limits_{\lambda}^{b}\Big(\frac{x^{\zeta+\varrho}-\lambda^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{-\delta}f(x)x^{\varrho}d_{\zeta}x, \quad b > \lambda \tag{8}$$

respectively, ${}^{\varrho}_{\zeta}\mathfrak{I}^{0}_{a+}f(\lambda) ={}^{\varrho}_{\zeta}\mathfrak{I}^{0}_{b-}f(\lambda) = f(\lambda)$ and Here T_{ϱ} denotes the conformable derivative of order ϱ and Γ denotes the gamma function given by $\Gamma(\delta) = \int\limits_{0}^{\infty}e^{-\lambda}\lambda^{\delta-1}d\lambda$.

The left and right generalized fractional conformable integral operator are presented respectively in [24] as follows:

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{a+}f(\lambda) = \frac{1}{\Gamma(\delta)}\int\limits_{a}^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-x^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta-\varrho}}dx, \quad \lambda > a \tag{9}$$

and

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}_{b-}f(\lambda) = \frac{1}{\Gamma(\delta)}\int\limits_{\lambda}^{b}\Big(\frac{x^{\zeta+\varrho}-\lambda^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta-\varrho}}dx, \quad \lambda < b, \tag{10}$$

where $\delta\in\mathbb{C},\ \mathcal{R}(\delta)>0,\ \zeta\in(0,1]$, $\varrho\in\mathbb{R}$ with $\zeta+\varrho\neq 0$, and Γ is the well-known gamma function.

Remark 1. In the above equations (9) and (10):

(i) If $\varrho = 0$, then we get the generalized left and right fractional integrals in the sense of Katugampola [20] are given respectively.

$${}_{\zeta}\mathfrak{I}^{\delta}_{a+}f(\lambda) = \frac{1}{\Gamma(\delta)}\int\limits_{a}^{\lambda}\Big(\frac{\lambda^{\zeta}-x^{\zeta}}{\zeta}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta}}dx, \quad \lambda > a \tag{11}$$

and

$$ {}_{\zeta}\mathfrak{J}^{\delta}_{b^-}f(\lambda)=\frac{1}{\Gamma(\delta)}\int_{\lambda}^{b}\Big(\frac{x^{\zeta}-\lambda^{\zeta}}{\zeta}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta}}dx,\quad \lambda<b, \tag{12} $$

where $\delta\in\mathbb{C},\ \mathcal{R}(\delta)>0,\ \zeta\in(0,1]$.

(ii) If $\varrho=0$ and $\zeta=1$, then we get the subsequent Riemann-Liouville type fractional operators:

$$ \mathfrak{J}^{\delta}_{a^+}f(\lambda)=\frac{1}{\Gamma(\delta)}\int_{a}^{\lambda}(\lambda-x)^{\delta-1}f(x)dx,\quad \lambda>a \tag{13} $$

and

$$ \mathfrak{J}^{\delta}_{b^-}f(\lambda)=\frac{1}{\Gamma(\delta)}\int_{\lambda}^{b}(x-\lambda)^{\delta-1}f(x)dx,\quad \lambda<b, \tag{14} $$

where $\delta\in\mathbb{C},\ \mathcal{R}(\delta)>0$.

We employ the more general form of the subsequent one-sided generalized $\mathcal{K}$-fractional conformable for conformable integrable function f :

$$ {}^{\varrho}_{\zeta}\mathfrak{J}^{\delta;\mathcal{K}}f(\lambda)=\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_{0}^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-x^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f(x)}{x^{1-\zeta-\varrho}}dx, \tag{15} $$

where $\mathcal{K}>0,\ \delta\in\mathbb{C},\mathcal{R}(\delta)>0,\ \zeta\in(0,1],\ \varrho\in\mathbb{R}$ with $\zeta+\varrho\neq0$, with $\Gamma_{\mathcal{K}}$ is the $\mathcal{K}$-gamma function given by

$$ \Gamma_{\mathcal{K}}(\delta):=\int_{0}^{\infty}\lambda^{\delta-1}e^{-\frac{\lambda^{k}}{k}}d\lambda,\ \Re(\delta)>0, $$

with the properties $\Gamma_{\mathcal{K}}(\delta+\mathcal{K})=\delta\Gamma_{\mathcal{K}}(\delta)$ and $\Gamma_{\mathcal{K}}(\mathcal{K})=1$.

Remark 2. In the above Equation (15):
(i) Letting $\mathcal{K}=1$, then (15) becomes the generalized fractional conformable integral operator:

$$ {}^{\varrho}_{\zeta}\mathfrak{J}^{\delta}f(\lambda)=\frac{1}{\Gamma(\delta)}\int_{0}^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-x^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta-\varrho}}dx. \tag{16} $$

(ii)Letting $\mathcal{K}=1$ along with $\varrho=0$, then (15) becomes the subsequent Riemann-Liouville type fractional conformable integral operator:

$$ {}_{\zeta}\mathfrak{J}^{\delta}f(\lambda)=\frac{1}{\Gamma(\delta)}\int_{0}^{\lambda}\Big(\frac{\lambda^{\zeta}-x^{\zeta}}{\zeta}\Big)^{\delta-1}\frac{f(x)}{x^{1-\zeta}}dx. \tag{17} $$

(iii) Letting $\mathcal{K}=1$ along with $\varrho=0$ and $\zeta=1$ then (15) reduces to the following Riemann-Liouville type fractional integral operator:

$$\mathfrak{I}^{\delta}f(\lambda)=\frac{1}{\Gamma(\delta)}\int\limits_{0}^{\lambda}\big(\lambda-x\big)^{\delta-1}f(x)dx. \tag{18}$$

Our present paper has been inspired by the resource of the above-defined work. The primary objective of this paper is to bulid up the novel idea of generalized conformable $\mathcal{K}$-fractional integral which is the generalized form of fractional operators reported in [24]. Moreover, we generalize some integral inequalities of Pólya-Szegö types and Čebyšev type for generalized $\mathcal{K}$-fractional conformable integral operator. The concept is relatively new and appears to have opened new doors of research towards different areas of science including meteorology, quantum mechanics, biosciences, chaos, image processing, power-law, biochemistry, physics, and several others.

2. **Pólya-Szegö types inequalities involving the generalized $\mathcal{K}$-fractional conformable integrals.** In this section, we shall derive certain Pólya-Szegö type integral inequalities for real-valued integrable functions via generalized $\mathcal{K}$-fractional conformable integral operator defined in (15).

Lemma 2.1. *For $\mathcal{K}>0$, $\delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose there two real-valued integrable functions f and g defined on $[0,\infty)$. Assume that there exist four positive integrable functions θ_1,θ_2,χ_1 and χ_2 on $[0,\infty)$ such that*

$$(I)\quad 0\le\theta_1(\tau)\le f(\lambda)\le\theta_2(\tau),\quad 0\le\chi_1(\tau)\le g(\lambda)\le\chi_2(\tau),\quad \big(\tau\in[0,\lambda],\lambda>0\big).$$

then for $\lambda>0$ and $\varrho>0$, the following inequality holds:

$$\frac{1}{4}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\big(\theta_1\chi_1+\theta_2\chi_2\big)fg\big](\lambda)\Big)^2\ge{}^{\varrho}_{\zeta}\,\mathfrak{I}^{\delta;\mathcal{K}}\big[\chi_1\chi_2f^2\big](\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\theta_1\theta_2g^2\big](\lambda), \tag{19}$$

where $\zeta\in(0,1]$, $\varrho\in\mathbb{R}$, $\zeta+\varrho\neq0$.

Proof. From Condition (I), for $\tau\in[0,\lambda],\lambda>0$, we have

$$\Big(\frac{\theta_2(\tau)}{\chi_1(\tau)}-\frac{f(\tau)}{g(\tau)}\Big)\ge0. \tag{20}$$

Analogously, we have

$$\Big(\frac{f(\tau)}{g(\tau)}-\frac{\theta_1(\tau)}{\chi_2(\tau)}\Big)\ge0. \tag{21}$$

Multiplying (20) and (21), we obtain

$$\big[\theta_1(\tau)\chi_1(\tau)+\theta_2(\tau)\chi_2(\tau)\big]f(\tau)g(\tau)\ge\chi_1(\tau)\chi_2(\tau)f^2(\tau)+\theta_1(\tau)\theta_2(\tau)g^2(\tau). \tag{22}$$

By taking product on both sides of (22) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\tau^{1-\zeta-\varrho}}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}$ and integrating the ensuing inequality w.r.t τ over $(0,\lambda)$, we get

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\big(\theta_1\chi_1+\theta_2\chi_2\big)fg\big](\lambda)\ge{}^{\varrho}_{\zeta}\,\mathfrak{I}^{\delta;\mathcal{K}}\big[\chi_1\chi_2f^2\big](\lambda)+{}^{\varrho}_{\zeta}\,\mathfrak{I}^{\delta;\mathcal{K}}\big[\theta_1\theta_2g^2\big](\lambda).$$

Applying the $AM-GM$ inequality, i.e., $\mu+\nu\ge2\sqrt{\mu\nu}$, $\quad\mu,\nu\in\mathbb{R}^{+}$, we have

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\big(\theta_1\chi_1+\theta_2\chi_2\big)fg\big](\lambda)\ge2\sqrt{{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\chi_1\chi_2f^2\big](\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\theta_1\theta_2g^2\big](\lambda)},$$

which leads to

$$\frac{1}{4}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[(\theta_1\chi_1+\theta_2\chi_2)fg\big](\lambda)\Big)^2 \geq {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\chi_1\chi_2 f^2\big](\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big[\theta_1\theta_2 g^2\big](\lambda).$$

Therefore, we obtain the inequality (23) as required. □

Corollary 1. *Let two real-valued integrable functions f and g defined on $[0,\infty)$, satisfying*

$$(II) \quad 0<q\leq f(\tau)\leq Q<\infty, \quad 0<r\leq f(\tau)\leq R<\infty, \quad \big(\tau\in[0,\lambda], \lambda>0\big).$$

Then for $\lambda>0$ and $\varrho>0$, we have

$$\frac{{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f^2(\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g^2(\lambda)}{{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg(\lambda)\big)^2} \leq \frac{1}{4}\bigg(\sqrt{\frac{qr}{QR}}+\sqrt{\frac{QR}{qr}}\bigg)^2.$$

Corollary 2. *If we choose $\mathcal{K}=1$, then under the assumption of Lemma 2.1 reduces to generalized fractional conformable integral inequality*

$$\frac{1}{4}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\big[(\theta_1\chi_1+\theta_2\chi_2)fg\big](\lambda)\Big)^2 \geq {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\big[\chi_1\chi_2 f^2\big](\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\big[\theta_1\theta_2 g^2\big](\lambda). \tag{23}$$

Remark 3. If we choose $\mathcal{K}=1$ along with $\varrho=0$ and $\zeta=1$, then under the assumption of Lemma 2.1 reduces to Lemma 3.1 in [31].

Lemma 2.2. *Assume all conditions of lemma 2.1 hold. Then for $\lambda>0$ and $\gamma,\delta>0$ the following inequality holds:*

$$\frac{{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1\theta_2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1\chi_2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f^2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g^2(\lambda)}{\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1 f(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1 g(\lambda)+{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_2 f(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_2 g(\lambda)\Big)^2} \leq \frac{1}{4}, \tag{24}$$

where $\zeta\in(0,1]$, $\varrho\in\mathbb{R}$, $\zeta+\varrho\neq 0$.

Proof. Applying condition (I) to prove (24), we get

$$\Big(\frac{\theta_2(\tau)}{\chi_1(\rho)}-\frac{f(\tau)}{g(\rho)}\Big)\geq 0$$

and

$$\Big(\frac{f(\tau)}{g(\rho)}-\frac{\theta_1(\tau)}{\chi_2(\rho)}\Big)\geq 0,$$

which imply that

$$\Big(\frac{\theta_1(\tau)}{\chi_2(\rho)}+\frac{\theta_2(\tau)}{\chi_1(\rho)}\Big)\frac{f(\tau)}{f(\rho)} \geq \frac{f^2(\tau)}{g^2(\tau)}+\frac{\theta_1(\tau)\theta_2(\tau)}{\chi_1(\rho)\chi_2(\rho)}. \tag{25}$$

Multiplying both sides of (25) by $\chi_1(\rho)\chi_2(\rho)g^2(\rho)$, we have

$$\theta_1(\tau)f(\tau)\chi_1(\rho)g(\rho)+\theta_2(\tau)f(\tau)\chi_2(\rho)g(\rho) \geq \chi_1(\rho)\chi_2(\rho)f^2(\tau)+\theta_1(\tau)\theta_2(\tau)g^2(\rho). \tag{26}$$

By taking product on both sides of (26) by $\frac{\left(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\delta}{\mathcal{K}}-1}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\tau^{1-\zeta-\varrho}\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)\rho^{1-\zeta-\varrho}}$ and integrating the ensuing inequality w.r.t τ and ρ over $(0,\lambda)$, we get

$$\begin{aligned}&\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1 f\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1 g\big)(\lambda)+\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_2 f\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_2 g\big)(\lambda)\\ &\geq \big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f^2\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1\chi_2\big)(\lambda)+\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g^2\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1\theta_2\big)(\lambda).\end{aligned}$$

Applying the $AM-GM$ inequality, we get

$$\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1 f\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1 g\big)(\lambda)+\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_2 f\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_2 g\big)(\lambda)$$

$$\geq 2\sqrt{\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f^2\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\chi_1\chi_2\big)(\lambda)+\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g^2\big)(\lambda)\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\theta_1\theta_2\big)(\lambda)},$$

which leads to the desired inequality in (24). The proof is completed. □

Corollary 3. *Let two real-valued integrable functions f and g defined on $[0,\infty)$ satisfying (II). Then for $\lambda>0$ and $\gamma,\delta>0$, we have*

$$\frac{{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f^2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g^2(\lambda)}{\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g(\lambda)\big)^2}\leq\frac{\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})\Gamma_{\mathcal{K}}(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\gamma+\delta}{\mathcal{K}}}-2}{4(\lambda)^{\frac{(\zeta+\delta)(\gamma+\delta)}{\mathcal{K}}}}\left(\sqrt{\frac{qr}{QR}}+\sqrt{\frac{QR}{qr}}\right)^2.$$

Corollary 4. *If we choose $\mathcal{K}=1$, then under the assumption of Theorem 2.2, we have a new inequality for generalized fractional conformable integral*

$$\frac{{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}\theta_1\theta_2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\chi_1\chi_2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}f^2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}g^2(\lambda)}{\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}\theta_1f(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\chi_1g(\lambda)+{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}\theta_2f(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\chi_2g(\lambda)\Big)^2}\leq\frac{1}{4}.$$

Remark 4. If we choose $\mathcal{K}=1$ along with $\varrho=0$ and $\zeta=1$, then Theorem 2.2 reduces to Lemma 3.3 in [31].

Theorem 2.3. *Suppose conditions of Lemma 2.1 are satisfied. Then for $\lambda>0$ and $\gamma,\delta>0$, the following inequality holds:*

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big(\frac{\theta_2fg}{\chi_1}\big)(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\big(\frac{\chi_2fg}{\theta_1}\big)(\lambda)\geq{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f^2(\lambda){}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}g^2(\lambda), \tag{27}$$

where $\zeta\in(0,1],\ \varrho\in\mathbb{R},\ \zeta+\varrho\neq0$.

Proof. Using condition (I), we have

$$\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{\theta_2(\tau)}{\tau^{1-\zeta-\varrho}\chi_1(\tau)}f(\tau)g(\tau)d\tau$$

$$\geq\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f^2(\tau)}{\tau^{1-\zeta-\varrho}}d\tau,$$

which implies

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\big(\frac{\rho_2fg}{\chi_1}\big)(\lambda)\geq{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f^2(\lambda). \tag{28}$$

Analogously, we obtain

$$\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\gamma}{\mathcal{K}}-1}\frac{\chi_2(\rho)}{\rho^{1-\zeta-\varrho}\theta_1(\rho)}f(\rho)g(\rho)d\rho$$

$$\geq\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\gamma}{\mathcal{K}}-1}\frac{g^2(\rho)}{\rho^{1-\zeta-\varrho}}d\rho,$$

from which one has

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}\big(\frac{\chi_2fg}{\rho_1}\big)(\lambda)\geq{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}g^2(\lambda). \tag{29}$$

Multiplying (28) and (29), we get the desired inequality (27). □

Corollary 5. *Let two real-valued integrable functions f and g defined on $[0,\infty)$ satisfying (II). Then for $\lambda>0$ and $\varrho,\delta>0$, we have*

$$\frac{{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f^2(\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}g^2(\lambda)}{{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}fg(\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg(\lambda)}\leq\frac{QR}{qr}.$$

Corollary 6. *If we choose $\mathcal{K}=1$, then under the assumption of Theorem 2.3, we have a new result for generalized fractional conformable integral*

$${}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}\big(\frac{\theta_2fg}{\chi_1}\big)(\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}\big(\frac{\chi_2fg}{\rho_1}\big)(\lambda)\geq\ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta}f^2(\lambda)\,{}^{\varrho}_{\zeta}\mathfrak{I}^{\gamma}g^2(\lambda).$$

Remark 5. If we choose $\mathcal{K}=1$ along with $\varrho=0$ and $\zeta=1$, then Theorem 2.3 reduces to Lemma 3.4 in [31].

3. *Čebyšev* **types inequalities involving the generalized $\mathcal{K}$-fractional conformable integrals.** In this part, we investigate the left and right generalized $\mathcal{K}$-fractional conformable integrals defined in (15), which generalize the Riemann-Liouville fractional integrals.

Theorem 3.1. *For $\mathcal{K}>0$, $\delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose there are two integrable functions f and g which are synchronous on $[0,\infty)$. Then*

$$\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)\geq\frac{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),\tag{30}$$

where $\zeta\in(0,1]$, $\varrho\in\mathbb{R}$, $\zeta+\varrho\neq0$.

Proof. Since f and g are synchronous on $[0,\infty)$, one obtains

$$\big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big)\geq0,\tag{31}$$

on the other hand

$$f(\tau)g(\tau)+f(\rho)g(\rho)\geq f(\tau)g(\rho)+g(\tau)f(\rho).\tag{32}$$

By taking product on both sides of(32) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\tau^{1-\zeta-\varrho}}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}$ and integrating the ensuing inequality w.r.t τ over $(0,\lambda)$, we get

$$\begin{aligned}&\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f(\tau)g(\tau)}{\tau^{1-\zeta-\varrho}}d\tau\\&+\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f(\rho)g(\rho)}{\tau^{1-\zeta-\varrho}}d\tau\\&\geq\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f(\tau)g(\rho)}{\tau^{1-\zeta-\varrho}}d\tau\\&+\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{g(\tau)f(\rho)}{\tau^{1-\zeta-\varrho}}d\tau.\end{aligned}$$

It follows that

$$\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)+f(\rho)g(\rho)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{d\tau}{\tau^{1-\zeta-\varrho}}$$

$$\geq g(\rho)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)+f(\rho)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda).$$

Thus, we obtain

$$\begin{aligned}&\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)+\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}f(\rho)g(\rho)\\&\geq g(\rho)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)+f(\rho)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),\end{aligned}\tag{33}$$

where

$$\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\tau^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{d\tau}{\tau^{1-\zeta-\varrho}}=\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}.$$

Again taking product on both sides of (33) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\rho^{1-\zeta-\varrho}}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}$ and integrating the ensuing inequality w.r.t ρ over $(0,\lambda)$, we get

$$\begin{aligned}&\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)\rho^{1-\zeta-\varrho}}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{d\rho}{\rho^{1-\zeta-\varrho}}\\&+\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}f(\rho)g(\rho)\frac{d\rho}{\rho^{1-\zeta-\varrho}}\\&\geq\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{g(\rho)d\rho}{\rho^{1-\zeta-\varrho}}\\&+\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\delta)}\int_0^{\lambda}\Big(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\Big)^{\frac{\delta}{\mathcal{K}}-1}\frac{f(\rho)d\rho}{\rho^{1-\zeta-\varrho}}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)+\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)\\&\geq\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda)+\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),\end{aligned}$$

establishes the desired result. □

Corollary 7. *For $\mathcal{K}>0$, $\delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose two integrable functions f and g which are synchronous on $[0,\infty)$. Then*

$$\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)\geq\frac{\Gamma(\delta+\mathcal{K})(\zeta)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta)\frac{\delta}{\mathcal{K}}}}\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),$$

for all $\zeta\in(0,1]$.

Proof. Letting $\varrho=0$ in Theorem 3.1, then we get the desired inequality associating conformable $\mathcal{K}$-fractional integral. □

Corollary 8. *For $\mathcal{K}>0$, $\delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose two integrable functions f and g which are synchronous on $[0,\infty)$. Then*

$$\Big(\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)\geq\frac{\Gamma(\delta+\mathcal{K})}{\lambda^{\frac{\delta}{\mathcal{K}}}}\Big(\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big(\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),$$

for all $\zeta \in (0,1]$.

Proof. Letting $\varrho = 0$ and $\zeta = 1$ in Theorem 3.1, then we get the inequality involving $\mathcal{K}-$fractional integral. □

Remark 6. If we choose $\mathcal{K} = 1$ in Theorem 3.1, then we get Theorem 2.1 in [30] and if we choose $\zeta = 0,\ \varrho = 0$ along with $\mathcal{K} = 1$ in Theorem 3.1, then we will attain Theorem 1.1 in [8].

Theorem 3.2. *For* $\mathcal{K} > 0,\ \delta, \gamma \in \mathbb{C}$ *with* $\mathcal{R}(\delta),\ \mathcal{R}(\gamma) > 0$. *Suppose there are two integrable functions* f *and* g *which are synchronous on* $[0, \infty)$. *Then*

$$\frac{\lambda^{(\zeta+\varrho)\frac{\gamma}{\mathcal{K}}}}{\Gamma(\gamma+\mathcal{K})(\zeta+\varrho)^{\frac{\gamma}{\mathcal{K}}}}\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}fg\right)(\lambda) + \frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}fg\right)(\lambda) \tag{34}$$

$$\geq \left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\right)(\lambda)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}g\right)(\lambda) + \left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}f\right)(\lambda)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}g\right)(\lambda), \tag{35}$$

where $\zeta \in (0,1],\ \varrho \in \mathbb{R},\ \zeta+\varrho \neq 0$.

Proof. Continuing the inequality (33) from Theorem 3.1, we have

$$\begin{aligned}&\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}fg\right)(\lambda) + \frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}f(\rho)g(\rho)\\ &\geq g(\rho)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\right)(\lambda) + f(\rho)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}g\right)(\lambda).\end{aligned} \tag{36}$$

By taking product on both sides of (36) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)\rho^{1-\zeta-\varrho}}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}$ and integrating the ensuing inequality w.r.t ρ over $(0, \lambda)$, we get

$$\begin{aligned}&\frac{\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}fg\right)(\lambda)}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{d\rho}{\rho^{1-\zeta-\varrho}}\\ &\quad+\frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{f(\rho)g(\rho)d\rho}{\rho^{1-\zeta-\varrho}}\\ &\geq \left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\right)(\lambda)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{g(\rho)d\rho}{\rho^{1-\zeta-\varrho}}\\ &\quad+\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}g\right)(\lambda)\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_0^{\lambda}\left(\frac{\lambda^{\zeta+\varrho}-\rho^{\zeta+\varrho}}{\zeta+\varrho}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{f(\rho)d\rho}{\rho^{1-\zeta-\varrho}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{\lambda^{(\zeta+\varrho)\frac{\gamma}{\mathcal{K}}}}{\Gamma(\gamma+\mathcal{K})(\zeta+\varrho)^{\frac{\gamma}{\mathcal{K}}}}\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}fg\right)(\lambda) + \frac{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}fg\right)(\lambda)\\ &\geq \left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\right)(\lambda)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}g\right)(\lambda) + \left({}_{\zeta}^{\varrho}\mathfrak{I}^{\gamma;\mathcal{K}}f\right)(\lambda)\left({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}g\right)(\lambda),\end{aligned}$$

establishes the desired result. □

Corollary 9. *For $\mathcal{K} > 0,\ \delta,\gamma \in \mathbb{C}$ with $\mathcal{R}(\delta),\ \mathcal{R}(\gamma) > 0$. Suppose there are two integrable functions f and g which are synchronous on $[0,\infty)$. Then*

$$\begin{aligned}&\frac{\lambda^{\frac{\zeta\gamma}{\mathcal{K}}}}{\Gamma(\gamma+\mathcal{K})(\zeta)^{\frac{\gamma}{\mathcal{K}}}}\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)+\frac{\lambda^{\frac{\zeta\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})(\zeta)^{\frac{\delta}{\mathcal{K}}}}\Big({}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}fg\Big)(\lambda)\\&\geq\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}g\Big)(\lambda)+\Big({}_{\zeta}\mathfrak{I}^{\gamma;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),\end{aligned}$$

for all $\zeta \in (0,1]$.

Proof. Letting $\varrho = 0$ in Theorem 3.2, then we get the desired inequality associating $\mathcal{K}$ conformable fractional integral. □

Corollary 10. *For $\mathcal{K} > 0,\ \delta,\gamma \in \mathbb{C}$ with $\mathcal{R}(\delta),\ \mathcal{R}(\gamma) > 0$. Suppose there are two integrable functions f and g which are synchronous on $[0,\infty)$. Then*

$$\begin{aligned}&\frac{\lambda^{\frac{\gamma}{\mathcal{K}}}}{\Gamma(\gamma+\mathcal{K})}\Big(\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda)+\frac{\lambda^{\frac{\delta}{\mathcal{K}}}}{\Gamma(\delta+\mathcal{K})}\Big(\mathfrak{I}^{\gamma;\mathcal{K}}fg\Big)(\lambda)\\&\geq\Big(\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big(\mathfrak{I}^{\gamma;\mathcal{K}}g\Big)(\lambda)+\Big(\mathfrak{I}^{\gamma;\mathcal{K}}f\Big)(\lambda)\Big(\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda),\end{aligned}$$

for all $\zeta \in (0,1]$.

Proof. If we choose $\varrho = 0$ and $\zeta = 1$ in Theorem 3.2, then we get the inequality involving $\mathcal{K}$-fractional integral. □

Remark 7. Letting $\mathcal{K} = 1$ in Theorem 3.2, then we get Theorem 2.2 in [30] and if we choose $\zeta = 0,\ \varrho = 0$ with $\mathcal{K} = 1$ in Theorem 3.2, then we will attain Theorem 1.2 in [8].

Theorem 3.3. *For $\mathcal{K} > 0,\ \delta \in \mathbb{C}$ with $\mathcal{R}(\delta) > 0$. Suppose there is n positive increasing functions on $[0,\infty)$ is $(f)_i,\ i = 1,2,...,n$. Then*

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n} f_i\Big)(\lambda) \geq \big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\big)^{1-n}\prod_{i=1}^{n}\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f_i\big)(\lambda), \tag{37}$$

where $\lambda > 0,\ \tau \in [0,1],\ \varrho \in \mathbb{R},\ \delta \in \mathbb{C}$.

Proof. To demonstrate this hypothesis, we employ the mathematical induction on n. Evidently, for $n = 1$, one has

$$\big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(f_1)\lambda\big) \geq \big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(f_1)\lambda\big) \quad \forall\lambda > 0$$

vailds. For $n = 2$, there are two positive and increasing functions f_1 and f_2, consequently we have

$$\big(f_1(\lambda) - f_1(\omega)\big)\big(f_2(\lambda) - f_2(\omega)\big) \geq 0.$$

Hence, by applying theorem 3.1, we obtain

$$\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(f_1f_2)(\lambda)\Big) \geq \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\Big)^{-1}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(f_1)\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(f_2)\Big)(\lambda).$$

By induction hypothesis

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n-1} f_i\Big)(\lambda) \geq \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\Big)^{2-n}\prod_{i=1}^{n-1}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f_i\Big)(\lambda). \tag{38}$$

Since $f_i;\ i=1,2,...,n$, are positive increasing functions on $\mathbb{R}^+$, therefore $\varphi := \prod_{i=1}^{n-1} f_i$ is increasing on $\mathbb{R}^+$. Let $\psi := f_n$. Utilizing Theorem 3.1 for the function f and g, we have

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n-1} f_i\Big)(\lambda) = {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(fg)(\lambda) \geq \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\Big)^{-1}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n-1} f_i\Big)\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f_n\Big)(\lambda). $$

By using (3.3), one obtains

$$ {}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n-1} f_i\Big)(\lambda) \geq \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\Big)^{-1}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\Big)^{2-n}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n-1} f_i\Big)\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f_n\Big)(\lambda), $$

establishes the result. □

Corollary 11. *For $\mathcal{K} > 0$, $\delta \in \mathbb{C}$ with $\mathcal{R}(\delta) > 0$. Suppose there is n positive increasing functions on $[0,\infty)$ are $(f)_i$, $i = 1,2,...,n$. Then*

$$ {}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n} f_i\Big)(\lambda) \geq \big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(1)\big)^{1-n}\prod_{i=1}^{n}\big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f_i\big)(\lambda), \tag{39} $$

where $\lambda > 0$, $\varrho \in \mathbb{R}$, and $\delta \in \mathbb{C}$.

Proof. Letting $\varrho = 0$ in Theorem 3.3, then we get the corollary involving conformable $\mathcal{K}$-fractional integral. □

Corollary 12. *For $\mathcal{K} > 0$, $\delta \in \mathbb{C}$ with $\mathcal{R}(\delta) > 0$. Suppose there is n positive increasing functions on $[0,\infty)$ are $(f)_i$, $i = 1,2,...,n$. Then*

$$ \mathfrak{I}^{\delta;\mathcal{K}}\Big(\prod_{i=1}^{n} f_i\Big)(\lambda) \geq \big(\mathfrak{I}^{\delta;\mathcal{K}}(1)\big)^{1-n}\prod_{i=1}^{n}\big(\mathfrak{I}^{\delta;\mathcal{K}} f_i\big)(\lambda), \tag{40} $$

where $\lambda > 0$, $\varrho \in \mathbb{R}$, and $\delta \in \mathbb{C}$.

Proof. Letting $\varrho = 0$ and $\zeta = 1$ in Theorem 3.3, then we get the corollary involving conformable $\mathcal{K}$ -fractional integral. □

Remark 8. Letting $\mathcal{K} = 1$ in Theorem 3.3, then we get Theorem 2.3 in [30] and if we take $\zeta = 0$, $\varrho = 0$ with $\mathcal{K} = 1$ in Theorem 3.3, then we will attain Theorem 1.3 in [8].

Theorem 3.4. *For $\mathcal{K} > 0$, $\delta \in \mathbb{C}$ with $\mathcal{R}(\delta) > 0$. Suppose there are two functions $f, g : \mathbb{R}_0^+ \to \mathbb{R}$ such that f is increasing and g is differentiable with g' bounded below, and let $\mu = \inf_{\lambda\in\mathbb{R}_0^+} g'(\lambda)$. Then*

$$ \begin{aligned} \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} fg\Big)(\lambda) \ &\geq \frac{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} g\Big)(\lambda) \\ &\quad - \frac{\mu\lambda}{\delta+\mathcal{K}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f\Big)(\lambda) + \mu\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} If\Big)(\lambda). \end{aligned} \tag{41} $$

where $I(\lambda)$ is the identity function.

Proof. Let $h(\lambda) = g(\lambda) - \mu\lambda^{\zeta+\varrho}$. We have to show that h is differentiable and increasing on $\mathbb{R}_0^+$. As simultaneously of Theorem 3.3, for simplicity, let $p(\lambda) := \mu\lambda^{\zeta+\varrho}$, one obtains

$$ \Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}\varphi(\psi-p)(\lambda)\Big) \geq \frac{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}} f\Big)(\lambda)\Big({}^{\varrho}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}(\psi-p)\Big)(\lambda) $$

$$= \frac{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda) - \frac{\Gamma(\delta+\mathcal{K})(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}}\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta;\mathcal{K}}p\Big)(\lambda). \tag{42}$$

We have

$$\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta,\mathcal{K}}\varphi(\psi-p)\Big)(\lambda) = \Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta,\mathcal{K}}\varphi fg\Big)(\lambda) - \mu\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta,\mathcal{K}}If\Big)(\lambda) \tag{43}$$

and

$$\Big({}_{\zeta}^{\varrho}\mathfrak{I}^{\delta,\mathcal{K}}p\Big)(\lambda) = \frac{\mu\lambda^{(\zeta+\varrho)\frac{\delta}{\mathcal{K}}}+1}{\Gamma(\delta+\mathcal{K}+1)(\zeta+\varrho)^{\frac{\delta}{\mathcal{K}}}}. \tag{44}$$

Finally using (43) and (44) in (46), we acquire the proof. ☐

Corollary 13. *For $\mathcal{K}>0,\ \delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose there are two functions $f,g:\mathbb{R}_0^+\to\mathbb{R}$ such that f is increasing and g is differentiable with g' bounded below, and let $\mu=\inf_{\lambda\in\mathbb{R}_0^+}g'(\lambda)$. Then*

$$\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda) \geq \frac{\Gamma(\delta+\mathcal{K})(\zeta)^{\frac{\delta}{\mathcal{K}}}}{\lambda^{(\zeta)\frac{\delta}{\mathcal{K}}}}\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda) - \frac{\mu\lambda}{\delta+\mathcal{K}}\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda) + \mu\Big({}_{\zeta}\mathfrak{I}^{\delta;\mathcal{K}}If\Big)(\lambda). \tag{45}$$

where $I(\lambda)$ is the identity function.

Proof. If we choose $\varrho=0$ in Theorem 3.4, then we get the desired corollary , which involves conformable $\mathcal{K}$-fractional integral. ☐

Corollary 14. *For $\mathcal{K}>0,\ \delta\in\mathbb{C}$ with $\mathcal{R}(\delta)>0$. Suppose there are two functions $f,g:\mathbb{R}_0^+\to\mathbb{R}$ such that f is increasing and g is differentiable with g' bounded below, and let $\mu=\inf_{\lambda\in\mathbb{R}_0^+}g'(\lambda)$. Then*

$$\Big(\mathfrak{I}^{\delta;\mathcal{K}}fg\Big)(\lambda) \geq \frac{\Gamma(\delta+\mathcal{K})}{\lambda}\Big(\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda)\Big(\mathfrak{I}^{\delta;\mathcal{K}}g\Big)(\lambda) - \frac{\mu\lambda}{\delta+\mathcal{K}}\Big(\mathfrak{I}^{\delta;\mathcal{K}}f\Big)(\lambda) + \mu\Big(\mathfrak{I}^{\delta;\mathcal{K}}If\Big)(\lambda). \tag{46}$$

where $I(\lambda)$ is the identity function.

Proof. Letting $\varrho=0$ and $\zeta=1$ in Theorem 3.4, then we get the desired corollary, which involves $\mathcal{K}-$fractional integral. ☐

Remark 9. Letting $\mathcal{K}=1$ in Theorem 3.4, then we get Theorem 2.4 in [30] and if we take $\zeta=0,\ \varrho=0$ with $\mathcal{K}=1$ in Theorem 3.4, then we will attain Theorem 1.4 in [8].

4. **Conclusion.** We have applied the left and right generalized $\mathcal{K}$-fractional conformable integrals and generalized several consequences to ones for our newly defined generalized $\mathcal{K}$-fractional conformable integrals related to a positive and decreasing function. Our findings consist of $\mathcal{K}$-analogs of many earlier outcomes in the literature. Moreover, numerous precise cases for other integral operators may be derived from our generalizations. The outcomes acquired can be applied to affirm the existence of nontrivial answers of fractional differential equations of various problems. Such a potential connection needs further investigation. We conclude that the results derived in this paper are general in character and give some contributions to inequality theory, some applications for establishing the uniqueness of solutions in fractional boundary value problems, modelling and simulation. We can formulate the several fractional versions of the postulates for special functions and derived

the rules for determining difference equations involving dirac delta. This interesting aspect of fractional calculus is worth further investigation. Our generalized $\mathcal{K}$-fractional conformable integrals in this paper generalize well-known fractional integral operators such as Riemann-Liouville fractional integral operators. Finally, we state that possible future works can be in proving new inequalities in the frame of new generalized integrals. The integrals correspond to certain fractional derivatives with nonsingular kernels, for example. See the papers [2, 24].

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