



*Research article*

## **New computations for the two-mode version of the fractional Zakharov-Kuznetsov model in plasma fluid by means of the Shehu decomposition method**

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**Abstract:** In this research, the Shehu transform is coupled with the Adomian decomposition method for obtaining the exact-approximate solution of the plasma fluid physical model, known as the Zakharov-Kuznetsov equation (briefly, ZKE) having a fractional order in the Caputo sense. The Laplace and Sumudu transforms have been refined into the Shehu transform. The action of weakly nonlinear ion acoustic waves in a plasma carrying cold ions and hot isothermal electrons is investigated in this study. Important fractional derivative notions are discussed in the context of Caputo. The Shehu decomposition method (SDM), a robust research methodology, is effectively implemented to generate the solution for the ZKEs. A series of Adomian components converge to the exact solution of the assigned task, demonstrating the solution of the suggested technique. Furthermore, the outcomes of this technique have generated important associations with the precise solutions to the problems being researched. Illustrative examples highlight the validity of the current process. The usefulness of the technique is reinforced via graphical and tabular illustrations as well as statistics theory.

**Keywords:** Shehu transform; Caputo fractional derivative; iterative transform method; Zakharov-Kuznetsov equation; analysis of variance

**Mathematics Subject Classification:** 26A33, 26A51, 26D07, 26D10, 26D15

## 1. Introduction

Fractional partial differential equations (PDEs) have gained prominence and recognition in recent years, owing to their verified applicability in a wide range of relatively diverse domains of science and engineering. For instance, considering the nonlinear oscillation of fractional derivatives can be employed to model earthquakes, fractional derivatives in a fluid-dynamic traffic model can be leveraged to alleviate the deficiency caused by the assumption of a continuous flow of traffic. Researchers including Coimbra, Davison and Essex, Riesz, Riemann and Liouville, Hadamard, Weyl, Jumarie, Caputo and Fabrizio, Atangana and Baleanu, Grünwald and Letnikov, Liouville and Caputo have proposed a variety of fractional operator formulations and conceptions. On the other hand, the Liouville-Caputo is the finest fractional filter. Furthermore, fractional PDEs are used to model a variety of physical phenomena, including chemical reaction and population dynamics, virology, image processing, bifurcation, thermodynamics, Levy statistics, porous media, physics, and engineering problems, (see [1–7]).

The Shehu transform (ST) was recently highlighted by Maitama and Zhao [8] as an interesting integral transformation. A modification of the Laplace and Sumudu transformation is the ST. However, we can retrieve the Laplace transform by replacing  $\varpi = 1$  in ST. This approach can be used to compress complex non-linear PDEs into simpler equations.

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The comprehensive evaluation of numerous advanced asymptotic approaches for the exploration of solitary solutions of nonlinear PDEs, and DEs has been presented, see [9–16]. For instance, the Adomian decomposition method (ADM) [17] for obtaining seven order Sawada-Kotera equations, pseudo-spectral method (PSM) [18] for finding the numerical solution of the Laxs 7th-order KdV equation, q-homotopy analysis method (q-HAM) for finding the convergence of special PDES [19], Lie symmetry analysis (LSA) [20] for dealing with the conservation laws and exact solutions of the seventh-order time fractional Sawada-Kotera-Ito equation, Laguerre wavelets collocation method (LWCM) and Haar wavelet for the numerical solution of the Benjamina-Bona-Mohany equations [21], a new Legendre Wavelets decomposition method (NLWDM) for solving PDEs [22], discrete Adomian decomposition method (DADM) [23] for constructing numerical solution of time fractional Navier-Stokes equation.

The ZKE was built in two dimensions to demonstrate nonlinear complex phenomena such as isotope waves in a massively magnetic flux uncompressed plasma [24, 25]. The SDM will be implemented to develop the major objectives of this research. The time-fractional ZKE is stated as:

$$\mathcal{D}_\zeta^\delta \mathcal{U} + a_1(\mathcal{U}^{\eta_1})_\phi + b_1(\mathcal{U}^{\eta_2})_{\phi\phi\phi} + b_1(\mathcal{U}^{\eta_3})_{\zeta\zeta\phi} = 0, \quad (1.1)$$

where  $\mathcal{U} = \mathcal{U}(\phi, \psi, \zeta)$  and  $\mathcal{D}_\zeta^\delta$  is the Caputo fractional derivative with order  $\delta$ ,  $0 < \delta \leq 1$ ,  $a_1$  and  $b_1$  are arbitrary constants and  $\eta_i$ ,  $i = 1, 2, 3$  are integers and  $\eta_i \neq 0$  ( $i = 1, 2, 3$ ) that demonstrates the characteristics of physical phenomena such as ion acoustic waves in a plasma consisting of cold ions and hot isothermal electrons in the framework of a balanced magnetic flux ([26, 27]). In [25], for example, the ZKEs were used to investigate shallowly nonlinear isotope ripples in significantly

magnetism impaired plasma in three dimensions.

In spite of the incredible improvement, the Adomian decomposition method (ADM) was contemplated by Gorge Adomian in 1980. The ADM, for example, has been effectively defined in numerous analytical structures of PDEs, especially in Burger's equation [28], time-fractional Kawahara equation [29], fuzzy heat-like and wave-like equations [30] and Lane-Emden-Fowler type equations [31]. The ADM was found to be strongly associated with a plethora of integral transforms, including ARA, Shehu, Fourier, Aboodh, Laplace and others. Presently, modified Laplace ADM [32] has been utilized to effectively resolve Volterra integral equations employing the noted numerical formulation, discrete ADM [23] has been used to solve the time-fractional Navier-Stokes equation, and Laplace ADM [33] has been considered to identify the approximate results of a fractional system of epidemic structures of a vector-borne disease, and so on.

Several of the aforesaid approaches have the disadvantage of being always stratified and necessitating a significant amount of algorithmic effort. To minimize the computing complexity and intricacy, we suggested the Shehu decomposition method (SDM), which is a composition of the ST and the decomposition method for solving the time-fractional ZKE, which is the main motivation for this research. The projected technique develops a convergent series as a solution. SDM has fewer parameters than other analytical methods. It is the preferred approach because it does not require discretion or linearization.

In this study, we first provided a fractional ZKE, followed by a description of the SDM, and then, a comparison characterization of the SDM presented with the existing methods. The graphical representations were then thoroughly explained in relation to the ZK problem. We presented an algorithm for SDM, discussed its estimation accuracy, and then showed two examples that demonstrate the effectiveness and stability of a novel approach so that their obtained simulations can be analyzed. Finally, as a part of our concluding remarks, we discussed the accumulated facts of our findings.

## 2. Prelude

In order to perform our research, we require various terminologies and postulate outcomes from the literature.

**Definition 2.1.** ([8]) Shehu transform (ST) for a mapping  $\mathcal{U}(\zeta)$  containing exponential order defined on the set of mappings is described as follows:

$$\mathbf{S} = \left\{ \mathcal{U}(\zeta) \mid \exists \mathcal{P}, p_1, p_2 > 0, |\mathcal{U}(\zeta)| < \mathcal{P} \exp(|\zeta|/p_j), \text{ if } \zeta \in (-1)^j \times [0, \infty), j = 1, 2; (\mathcal{P}, p_1, p_2 > 0) \right\}, \quad (2.1)$$

where  $\mathcal{U}(\zeta)$  is denoted by  $\mathbf{S}[\mathcal{U}(\zeta)] = \mathcal{S}(\xi, \varpi)$ , is stated as

$$\mathbf{S}[\mathcal{U}(\zeta)] = \int_0^{\infty} \mathcal{U}(\zeta) \exp\left(-\frac{\xi}{\varpi}\zeta\right) d\zeta = \mathcal{S}(\xi, \varpi), \quad \zeta \leq 0, \quad \varpi \in [\kappa_1, \kappa_2]. \quad (2.2)$$

The following is an example of a supportive ST:

$$\mathbf{S}[\zeta^\delta] = \int_0^{\infty} \exp\left(-\frac{\xi}{\varpi}\zeta\right) \zeta^\delta d\zeta = \Gamma(\delta + 1) \left(\frac{\varpi}{\xi}\right)^{\delta+1}. \quad (2.3)$$

**Definition 2.2.** ([8]) The inverse ST of a function  $\mathcal{U}(\zeta)$  is described as

$$\mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^{m\delta+1}\right] = \frac{\zeta^{m\delta}}{\Gamma(m\delta+1)}, \quad \Re(\delta) > 0, \quad \text{and } m > 0. \quad (2.4)$$

**Lemma 2.3.** Consider ST of  $\mathcal{U}_1(\zeta)$  and  $\mathcal{U}_2(\zeta)$  are  $\mathcal{P}(\xi, \varpi)$  and  $\mathcal{Q}(\xi, \varpi)$ , respectively [8],

$$\begin{aligned} \mathbf{S}[\gamma_1\mathcal{U}_1(\zeta) + \gamma_2\mathcal{U}_2(\zeta)] &= \mathbf{S}[\gamma_1\mathcal{U}_1(\zeta)] + \mathbf{S}[\gamma_2\mathcal{U}_2(\zeta)] \\ &= \gamma_1\mathcal{P}(\xi, \varpi) + \gamma_2\mathcal{Q}(\xi, \varpi), \end{aligned} \quad (2.5)$$

where  $\gamma_1$  and  $\gamma_2$  are unspecified terms.

**Lemma 2.4.** ([8]) For order  $\delta > 0$ , the Caputo fractional derivative (CFD) of ST is defined as

$$\mathbf{S}[\mathcal{D}_\zeta^\delta \mathcal{U}(\zeta)] = \left(\frac{\xi}{\varpi}\right)^\delta \mathbf{S}[\mathcal{U}(\phi, \zeta)] - \sum_{\kappa=0}^{m-1} \left(\frac{\xi}{\varpi}\right)^{\delta-\kappa-1} \mathcal{U}^{(\kappa)}(\phi, 0), \quad m-1 \leq \delta \leq m, \quad m \in \mathbb{N}. \quad (2.6)$$

### 3. Description of the SDM

Considering the nonlinear partial differential equation:

$$\mathcal{D}_\zeta^\delta \mathcal{U}(\phi, \zeta) + \mathcal{L}\mathcal{U}(\phi, \zeta) + \check{\mathcal{N}}\mathcal{U}(\phi, \zeta) = \mathcal{F}(\phi, \zeta), \quad \zeta > 0, \quad 0 < \delta \leq 1, \quad (3.1)$$

subject to the condition

$$\mathcal{U}(\phi, 0) = \mathcal{G}(\phi), \quad (3.2)$$

where  $\mathcal{D}_\zeta^\delta = \frac{\partial^\delta \mathcal{U}(\phi, \zeta)}{\partial \zeta^\delta}$  indicates the CFD with  $0 < \delta \leq 1$  while  $\mathcal{L}$  and  $\check{\mathcal{N}}$  are linear/nonlinear factors and the source term refers to  $\mathcal{F}(\phi, \zeta)$ .

Implementing the ST to (3.1), and we attain

$$\mathbf{S}[\mathcal{D}_\zeta^\delta \mathcal{U}(\phi, \zeta) + \mathcal{L}\mathcal{U}(\phi, \zeta) + \check{\mathcal{N}}\mathcal{U}(\phi, \zeta)] = \mathbf{S}[\mathcal{F}(\phi, \zeta)].$$

Applying the differentiation property of ST, yields

$$\frac{\xi^\delta}{\varpi^\delta} \mathcal{U}(\xi, \varpi) = \sum_{\kappa=0}^{m-1} \left(\frac{\xi}{\varpi}\right)^{\delta-\kappa-1} \mathcal{U}^{(\kappa)}(0) + \mathbf{S}[\mathcal{L}\mathcal{U}(\phi, \zeta) + \check{\mathcal{N}}\mathcal{U}(\phi, \zeta)] + \mathbf{S}[\mathcal{F}(\phi, \zeta)]. \quad (3.3)$$

The inverse ST of (3.3) provides

$$\mathcal{U}(\phi, \zeta) = \mathbf{S}^{-1}\left[\sum_{\kappa=0}^{m-1} \left(\frac{\xi}{\varpi}\right)^{\delta-\kappa-1} \mathcal{U}^{(\kappa)}(0) + \frac{\varpi^\delta}{\xi^\delta} \mathbf{S}[\mathcal{F}(\phi, \zeta)]\right] - \mathbf{S}^{-1}\left[\frac{\varpi^\delta}{\xi^\delta} \mathbf{S}[\mathcal{L}\mathcal{U}(\phi, \zeta) + \check{\mathcal{N}}\mathcal{U}(\phi, \zeta)]\right]. \quad (3.4)$$

The infinite series representation of SDM is denoted by the mapping  $\mathcal{U}(\phi, \zeta)$  as follows:

$$\mathcal{U}(\phi, \zeta) = \sum_{m=0}^{\infty} \mathcal{U}_m(\phi, \zeta). \quad (3.5)$$

Thus, the nonlinearity  $\check{\mathcal{N}}(\phi, \zeta)$  can be estimated by the ADM represented as

$$\check{\mathcal{N}}\mathcal{U}(\phi, \zeta) = \sum_{m=0}^{\infty} \tilde{A}_m(\mathcal{U}_0, \mathcal{U}_1, \dots), \quad m = 0, 1, \dots, \quad (3.6)$$

where

$$\tilde{A}_m(\mathcal{U}_0, \mathcal{U}_1, \dots) = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \check{\mathcal{N}} \left( \sum_{j=0}^{\infty} \lambda^j \mathcal{U}_j \right) \right]_{\lambda=0}, \quad m > 0.$$

Substituting (3.5) and (3.6) into (3.4), we have

$$\sum_{m=0}^{\infty} \mathcal{U}_m(\phi, \zeta) = \mathcal{G}(\phi) + \tilde{\mathcal{G}}(\phi) - \mathbf{S}^{-1} \left[ \frac{\varpi^\delta}{\xi^\delta} \mathbf{S} [\mathcal{L}\mathcal{U}(\phi, \zeta) + \sum_{m=0}^{\infty} \tilde{A}_m] \right]. \quad (3.7)$$

Consequently, the following is the recursive methodology for (3.7):

$$\begin{aligned} \mathcal{U}_0(\phi, \zeta) &= \mathcal{G}(\phi) + \tilde{\mathcal{G}}(\phi), \quad m = 0, \\ \mathcal{U}_{m+1}(\phi, \zeta) &= -\mathbf{S}^{-1} \left[ \frac{\varpi^\delta}{\xi^\delta} \mathbf{S} [\mathcal{L}(\mathcal{U}_m(\phi, \zeta)) + \sum_{m=0}^{\infty} \tilde{A}_m] \right], \quad m \geq 1. \end{aligned} \quad (3.8)$$

#### 4. Illustrative examples

**Example 4.1.** Assume the following time-dependent fractional-order Zakharov-Kuznetsov equation:

$$\mathcal{D}_\zeta^\delta \mathcal{U} + \frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8} \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^2}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3} \right] = 0, \quad (4.1)$$

subject to the initial condition

$$\mathcal{U}(\phi, \psi, 0) = \frac{4}{3} \theta \sinh^2(\phi + \psi), \quad (4.2)$$

where  $\theta$  is an arbitrary constant.

*Proof.* Applying the ST on both sides of (4.1), we find

$$\begin{aligned} \mathbf{S} \left[ \frac{\partial^\delta \mathcal{U}}{\partial \zeta^\delta} \right] &= -\mathbf{S} \left[ \frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8} \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^2}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3} \right] \right], \\ \left( \frac{\xi}{\varpi} \right)^\delta \mathbf{S} [\mathcal{U}(\phi, \psi, \zeta)] - \sum_{\kappa=0}^{n_1-1} \left( \frac{\xi}{\varpi} \right)^{\delta-\kappa-1} \frac{\partial^\kappa \mathcal{U}^{(\kappa)}(\phi, 0)}{\partial \zeta^\kappa} &= -\mathbf{S} \left[ \frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8} \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^2}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3} \right] \right]. \end{aligned} \quad (4.3)$$

Employing the inverse ST, we have

$$\mathcal{U}(\phi, \psi, \zeta) = \mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \sum_{\kappa=0}^{n_1-1} \left( \frac{\xi}{\varpi} \right)^{\delta-\kappa-1} \frac{\partial^\kappa \mathcal{U}^{(\kappa)}(\phi, 0)}{\partial \zeta^\kappa} - \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8} \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^2}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3} \right] \right] \right]. \quad (4.4)$$

It follows that

$$\begin{aligned}\mathcal{U}(\phi, \psi, \zeta) &= \mathbf{S}^{-1}\left[\frac{\varpi}{\xi}\mathcal{U}(\phi, \psi, 0)\right] - \mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[\frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8}\left[\frac{\partial}{\partial \phi}\left(\frac{\partial^2 \mathcal{U}^2}{\partial \psi^2}\right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3}\right]\right]\right], \\ \mathcal{U}(\phi, \psi, \zeta) &= \mathbf{S}^{-1}\left[\frac{4}{3}\frac{\varpi}{\xi}\theta \sinh^2(\phi + \psi)\right] - \mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[\frac{\partial \mathcal{U}^2}{\partial \phi} + \frac{1}{8}\left[\frac{\partial}{\partial \phi}\left(\frac{\partial^2 \mathcal{U}^2}{\partial \psi^2}\right) + \frac{\partial^3 \mathcal{U}^2}{\partial \phi^3}\right]\right]\right].\end{aligned}\quad (4.5)$$

Utilizing the Shehu's decomposition method, we get

$$\sum_{j=0}^{\infty} \mathcal{U}_j(\phi, \psi, \zeta) = \frac{4}{3}\theta \sinh^2(\phi + \psi) - \mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[\check{\mathcal{N}}(\mathcal{U})_\phi + \frac{1}{8}\left[\check{\mathcal{N}}(\mathcal{U})_{\phi\phi\phi} + \check{\mathcal{N}}(\mathcal{U})_{\phi\psi\psi}\right]\right]\right],\quad (4.6)$$

where  $\check{\mathcal{N}}(\mathcal{U})$  is the Admoian's polynomial describing nonlinear term appearing in the above mentioned equations.

$$\check{\mathcal{N}}(\mathcal{U}) = \mathcal{U}^2 = \sum_{j=0}^{\infty} \mathcal{H}_j(\mathcal{U}).\quad (4.7)$$

First few Admoian's polynomials are presented as follows:

$$\begin{aligned}\mathcal{H}_0 &= \mathcal{U}_0^2, \\ \mathcal{H}_1 &= 2\mathcal{U}_0\mathcal{U}_1, \\ \mathcal{H}_2 &= 2\mathcal{U}_0\mathcal{U}_2 + \mathcal{U}_1^2, \\ \mathcal{U}_0(\phi, \psi, \zeta) &= \frac{4}{3}\theta \sinh^2(\phi + \psi), \\ \mathcal{U}_{j+1}(\phi, \psi, \zeta) &= -\mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[\left(\sum_{j=0}^{\infty} \mathcal{H}_j(\mathcal{U})\right)_\phi + \frac{1}{8}\left(\sum_{j=0}^{\infty} \mathcal{H}_j(\mathcal{U})\right)_{\phi\phi\phi} + \frac{1}{8}\left(\sum_{j=0}^{\infty} \mathcal{H}_j(\mathcal{U})\right)_{\phi\psi\psi}\right]\right],\end{aligned}$$

for  $j = 0, 1, 2, \dots$

$$\begin{aligned}\mathcal{U}_1(\phi, \psi, \zeta) &= -\mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[\left(\mathcal{U}_0^2\right)_\phi + \frac{1}{8}\left(\mathcal{U}_0^2\right)_{\phi\phi\phi} + \frac{1}{8}\left(\mathcal{U}_0^2\right)_{\phi\psi\psi}\right]\right] \\ &= \left(-\frac{224}{9}\theta^2 \sinh^2(\phi + \psi) \cosh(\phi + \psi) - \frac{32}{3}\theta^2 \sinh(\phi + \psi) \cosh^3(\phi + \psi)\right) \mathbf{S}^{-1}\left(\left(\frac{\varpi}{\xi}\right)^{\delta+1}\right) \\ &= \left(-\frac{224}{9}\theta^2 \sinh^2(\phi + \psi) \cosh(\phi + \psi) - \frac{32}{3}\theta^2 \sinh(\phi + \psi) \cosh^3(\phi + \psi)\right) \frac{\zeta^\delta}{\Gamma(\delta + 1)}.\end{aligned}$$

Accordingly, we can derive the remaining terms as follows

$$\begin{aligned}\mathcal{U}_2(\phi, \psi, \zeta) &= -\mathbf{S}^{-1}\left[\left(\frac{\varpi}{\xi}\right)^\delta \mathbf{S}\left[2\mathcal{U}_0\mathcal{U}_1 + \frac{1}{8}(2\mathcal{U}_0\mathcal{U}_1)_{\phi\phi\phi} + \frac{1}{8}(2\mathcal{U}_0\mathcal{U}_1)_{\phi\psi\psi}\right]\right] \\ &= \frac{128}{27}\theta^3\left(1200 \cosh^6(\phi + \psi) - 2080 \cosh^4(\phi + \psi)\right. \\ &\quad \left.+ 968 \cosh^2(\phi + \psi) - 79\right) \frac{\zeta^{2\delta}}{\Gamma(2\delta + 1)},\end{aligned}\quad (4.8)$$

$$\begin{aligned}
\mathcal{U}_3(\phi, \psi, \zeta) &= -\mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ (2\mathcal{U}_0\mathcal{U}_2 + \mathcal{U}_1^2)_\phi + \frac{1}{8}(2\mathcal{U}_0\mathcal{U}_2 + \mathcal{U}_1^2)_{\phi\phi\phi} + \frac{1}{8}(2\mathcal{U}_0\mathcal{U}_2 + \mathcal{U}_1^2)_{\phi\psi\psi} \right] \right] \\
&= -\frac{2048}{81} \theta^4 \sinh(\phi + \psi) \cosh(\phi + \psi) (88,4000 \cosh^6(\phi + \psi) - 160,200 \cosh^4(\phi + \psi) \\
&\quad + 85,170 \cosh^2(\phi + \psi) - 11,903) \frac{\zeta^{3\delta}}{\Gamma(3\delta + 1)}. \tag{4.9}
\end{aligned}$$

The approximate-analytical SDM solution is

$$\begin{aligned}
\mathcal{U}(\phi, \psi, \zeta) &= \mathcal{U}_0(\phi, \psi, \zeta) + \mathcal{U}_1(\phi, \psi, \zeta) + \mathcal{U}_2(\phi, \psi, \zeta) + \mathcal{U}_3(\phi, \psi, \zeta) + \dots, \\
\mathcal{U}(\phi, \psi, \zeta) &= \frac{4}{3} \theta \sinh^2(\phi + \psi) - \left( \frac{224}{9} \theta^2 \sinh^2(\phi + \psi) \cosh(\phi + \psi) \right. \\
&\quad \left. + \frac{32}{3} \theta^2 \sinh(\phi + \psi) \cosh^3(\phi + \psi) \right) \frac{\zeta^\delta}{\Gamma(\delta + 1)} + \frac{128}{27} \theta^3 (1200 \cosh^6(\phi + \psi) \\
&\quad - 2080 \cosh^4(\phi + \psi) + 968 \cosh^2(\phi + \psi) - 79) \frac{\zeta^{2\delta}}{\Gamma(2\delta + 1)} \\
&\quad - \frac{2048}{81} \theta^4 \sinh(\phi + \psi) \cosh(\phi + \psi) (88,4000 \cosh^6(\phi + \psi) - 160,200 \cosh^4(\phi + \psi) \\
&\quad + 85,170 \cosh^2(\phi + \psi) - 11,903) \frac{\zeta^{3\delta}}{\Gamma(3\delta + 1)} + \dots \tag{4.10}
\end{aligned}$$

The exact solution for  $\delta = 1$  is presented by

$$\mathcal{U}(\phi, \psi, \zeta) = \frac{4}{3} \theta \sinh^2(\phi + \psi - \theta\zeta). \tag{4.11}$$

Tables 1 and 2 show the comparison results for exact, SDM, and absolute error of  $\mathcal{U}_{abs} = \|\mathcal{U}^E - \mathcal{U}^{SDM}\|$  solution for (4.1), when  $\theta = 0001$  and for various fractional orders  $\delta = 0.67, 0.75, 1$ . It can be seen that the proposed method closely corresponds the exact, VIM [34], VIA [35] and RPSM [35].

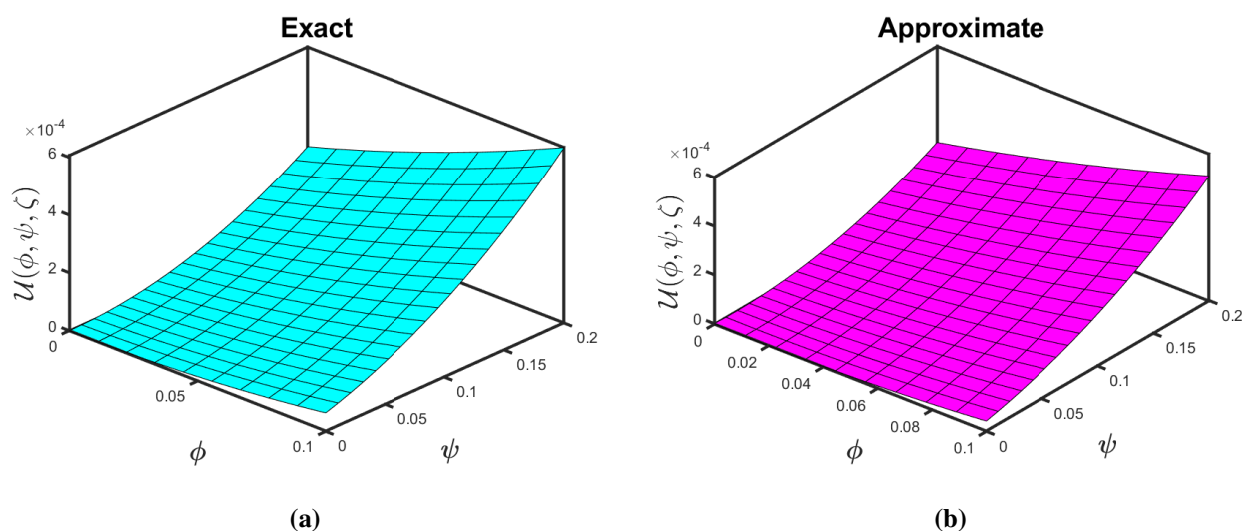
**Table 1.** Exact ( $\mathcal{U}_E$ ) and SDM-approximate ( $\mathcal{U}_{SDM}$ ) solution with absolute error ( $\mathcal{U}_{abs}$ ) in comparison derived by VIM ( $\mathcal{U}_{VIM}$ ) [34], PIA ( $\mathcal{U}_{PIA}$ ) [35] and RPSM ( $\mathcal{U}_{RPSM}$ ) [35] for Example 4.1 at  $\theta = 0.001, \delta = 1, 0.67$  and  $0.75$ .

| $\phi$ | $\psi$ | $\zeta$ | $\mathcal{U}_E$        | $\mathcal{U}_{SDM}$    | $\mathcal{U}_{abs}$    | $\mathcal{U}_{\delta=0.67}$ | $\mathcal{U}_{\delta=0.75}$ | $\mathcal{U}_{VIM}$ [34] |
|--------|--------|---------|------------------------|------------------------|------------------------|-----------------------------|-----------------------------|--------------------------|
| 0.1    | 0.1    | 0.2     | $5.394 \times 10^{-5}$ | $5.331 \times 10^{-5}$ | $6.313 \times 10^{-7}$ | $5.341 \times 10^{-5}$      | $5.328 \times 10^{-5}$      | $5.356 \times 10^{-5}$   |
|        |        | 0.3     | $7.668 \times 10^{-4}$ | $7.562 \times 10^{-4}$ | $1.055 \times 10^{-5}$ | $7.488 \times 10^{-4}$      | $7.507 \times 10^{-4}$      | $7.570 \times 10^{-4}$   |
|        |        | 0.4     | $5.383 \times 10^{-5}$ | $5.308 \times 10^{-5}$ | $7.541 \times 10^{-7}$ | $5.419 \times 10^{-5}$      | $5.375 \times 10^{-5}$      | $5.410 \times 10^{-5}$   |
| 0.6    | 0.6    | 0.2     | $7.668 \times 10^{-4}$ | $7.562 \times 10^{-4}$ | $1.055 \times 10^{-5}$ | $7.488 \times 10^{-4}$      | $7.507 \times 10^{-4}$      | $7.570 \times 10^{-4}$   |
|        |        | 0.3     | $7.665 \times 10^{-4}$ | $7.513 \times 10^{-4}$ | $1.522 \times 10^{-5}$ | $7.447 \times 10^{-4}$      | $7.461 \times 10^{-4}$      | $7.531 \times 10^{-4}$   |
|        |        | 0.4     | $7.663 \times 10^{-4}$ | $7.468 \times 10^{-4}$ | $1.948 \times 10^{-5}$ | $7.417 \times 10^{-4}$      | $7.425 \times 10^{-4}$      | $7.501 \times 10^{-4}$   |
| 0.9    | 0.9    | 0.2     | $1.840 \times 10^{-3}$ | $1.801 \times 10^{-3}$ | $3.993 \times 10^{-5}$ | $1.772 \times 10^{-3}$      | $1.779 \times 10^{-3}$      | $1.803 \times 10^{-3}$   |
|        |        | 0.3     | $1.740 \times 10^{-3}$ | $1.882 \times 10^{-3}$ | $5.803 \times 10^{-5}$ | $1.755 \times 10^{-3}$      | $1.761 \times 10^{-3}$      | $1.788 \times 10^{-3}$   |
|        |        | 0.4     | $1.840 \times 10^{-3}$ | $1.765 \times 10^{-3}$ | $7.487 \times 10^{-5}$ | $1.743 \times 10^{-3}$      | $1.747 \times 10^{-3}$      | $1.775 \times 10^{-3}$   |

**Table 2.** Other comparison of the projected scheme with PIA and RPSM for Example 4.1 at  $\theta = 0.001$  having different fractional-order  $\delta = 0.67$  and  $\delta = 0.75$ .

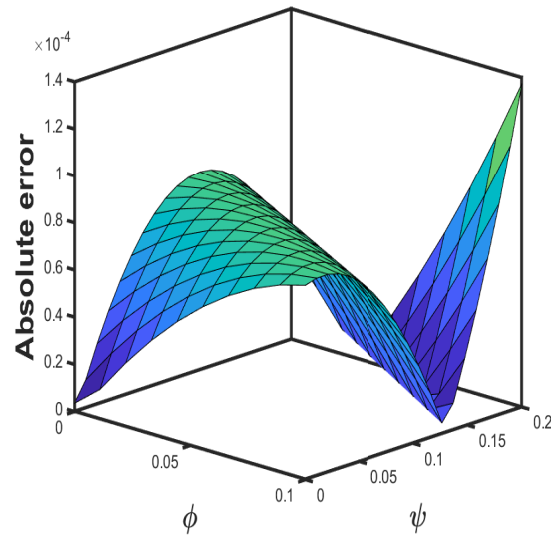
| $\phi$ | $\psi$ | $\zeta$ | $\mathcal{U}_{PIA}$ [35] | $\mathcal{U}_{RPSM}$ [35] | $\mathcal{U}_{PIA}$ [35] | $\mathcal{U}_{RPSM}$ [35] |
|--------|--------|---------|--------------------------|---------------------------|--------------------------|---------------------------|
| 0.1    | 0.1    | 0.2     | $5.3307 \times 10^{-5}$  | $6.3129 \times 10^{-7}$   | $5.3285 \times 10^{-5}$  | $5.3562 \times 10^{-5}$   |
|        |        | 0.3     | $5.28631 \times 10^{-5}$ | $5.28410 \times 10^{-5}$  | $5.29757 \times 10^{-5}$ | $5.29675 \times 10^{-5}$  |
|        |        | 0.4     | $5.25777 \times 10^{-5}$ | $5.25897 \times 10^{-5}$  | $5.27039 \times 10^{-5}$ | $5.27119 \times 10^{-5}$  |
| 0.6    | 0.6    | 0.2     | $2.95493 \times 10^{-3}$ | $2.95185 \times 10^{-3}$  | $2.96356 \times 10^{-3}$ | $2.96251 \times 10^{-3}$  |
|        |        | 0.3     | $2.92662 \times 10^{-3}$ | $2.92709 \times 10^{-3}$  | $2.93717 \times 10^{-3}$ | $2.93780 \times 10^{-3}$  |
|        |        | 0.4     | $2.90307 \times 10^{-3}$ | $2.90522 \times 10^{-3}$  | $2.91448 \times 10^{-3}$ | $2.91561 \times 10^{-3}$  |
| 0.9    | 0.9    | 0.2     | $1.06822 \times 10^{-2}$ | $1.05522 \times 10^{-2}$  | $1.07716 \times 10^{-2}$ | $2.91561 \times 10^{-2}$  |
|        |        | 0.3     | $1.04487 \times 10^{-2}$ | $1.01199 \times 10^{-2}$  | $1.05488 \times 10^{-2}$ | $1.03695 \times 10^{-2}$  |
|        |        | 0.4     | $9.02777 \times 10^{-2}$ | $9.60606 \times 10^{-2}$  | $1.03736 \times 10^{-2}$ | $9.96743 \times 10^{-2}$  |

Taking  $\theta = 0.005$  and  $\delta = 1$ , we exhibit the approximate-analytical solution of the fractional KZEs equation up to 4 components in Figure 1 (a and b). Furthermore, we establish absolute errors at  $\delta = 1$  for the exact-approximate solutions in the accompanying Figure 2. Also, we have seen how different fractional orders perform in surface plots and 2D plots in Figure 3 and some  $\delta_1 \delta_2$  - slice (a and b) solutions are presented in 4 when  $\theta = 0.005$  and  $\zeta = 0.5$ . As a result of this behaviour, we might conclude that the approximation solution tends to be a precise solution. Accordingly, as the iteration increases, the absolute inaccuracy decreases. Consequently, as the number of terms grows, the SDM findings approach the exact result.

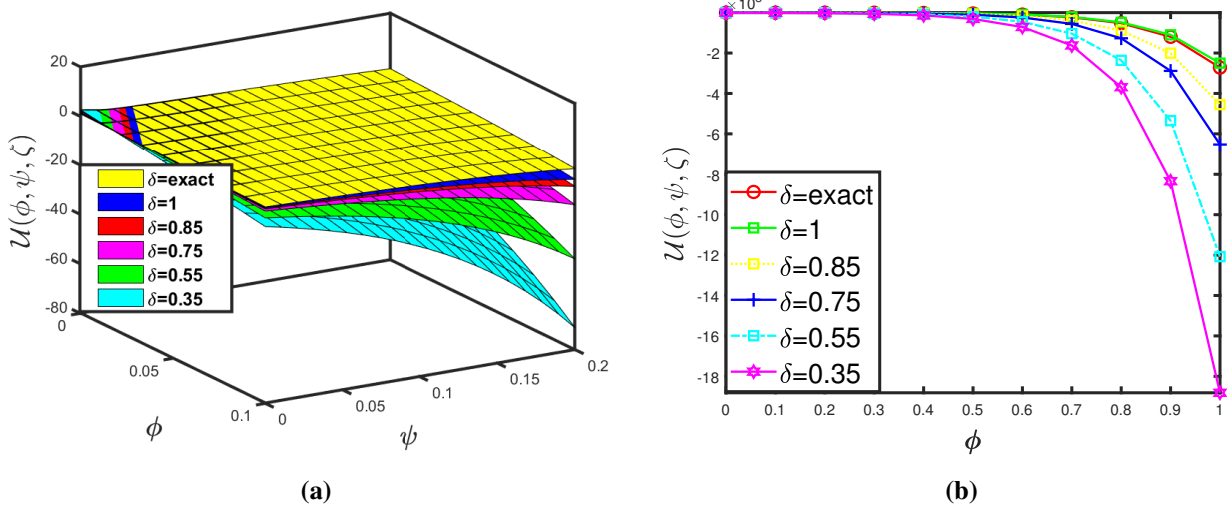


**Figure 1.** Numerical behavior of exact and approximate solution to the  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.1 when the parameters are  $\theta = 0.0005$ ,  $\delta = 1$ , and  $\zeta = 0.5$ .



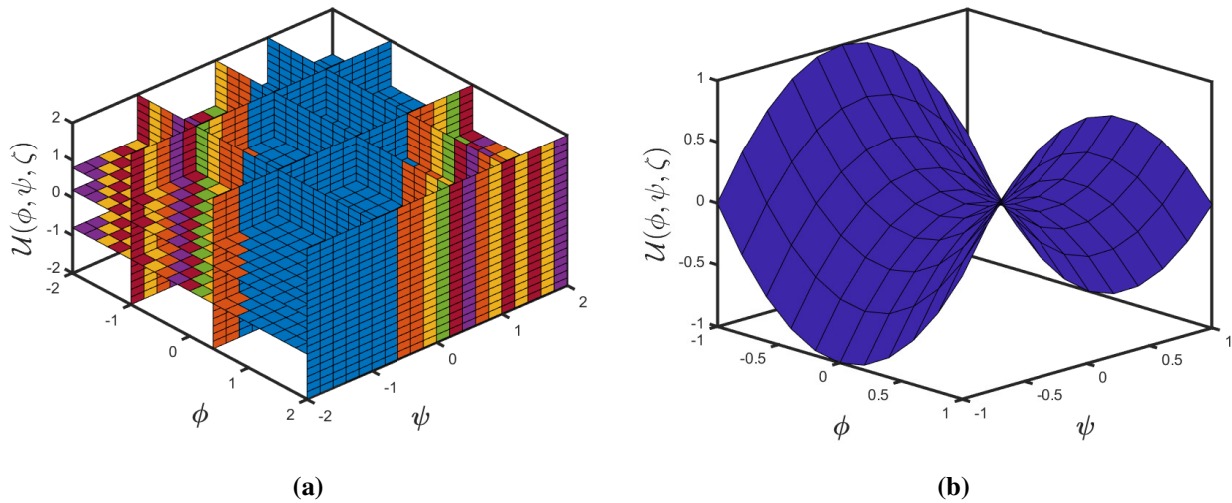


**Figure 2.** Surface representation of  $\mathcal{U}(\phi, \psi, \zeta)$  absolute error plot for Example 4.1 at  $\theta = 0.005$  and  $\delta = 1$ .



**Figure 3.** Numerical behavior of different fractional orders to the function  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.1 when the parameters are  $\theta = 0.05$ , and  $\zeta = 0.9$ .

The graphs in Figures 1–4 assist us to comprehend the behaviour of fractional orders when space and time scale variables fluctuate. Additionally, the findings of this study will aid scientists connected to pattern formation theory, optical designs, or mathematical modelling in comprehending the structural phenomena of the ANOVA-test. Furthermore, the efficiency of the projected method can be boosted by getting additional approximate solution expressions.



**Figure 4.** Numerical behavior of  $\delta_1\delta_2$ -slice solution to the  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.1 (a) exact and (b) approximate when the parameters are  $\theta = 0.0005$ ,  $\delta = 1$ , and  $\zeta = 0.5$ .

□

**Example 4.2.** Assume the following time-dependent fractional-order Zakharov-Kuznetsov equation:

$$\mathcal{D}_\zeta^\delta \mathcal{U} + \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] = 0, \quad (4.12)$$

subject to the initial condition

$$\mathcal{U}(\phi, \psi, 0) = \frac{3}{2} \theta \sinh \left[ \frac{1}{6} (\phi + \psi) \right], \quad (4.13)$$

where  $\theta$  is an arbitrary constant.

*Proof.* Applying the ST on both sides of (4.12), we find

$$\begin{aligned} \mathbf{S} \left[ \frac{\partial^\delta \mathcal{U}}{\partial \zeta^\delta} \right] &= -\mathbf{S} \left[ \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] \right], \\ \left( \frac{\xi}{\varpi} \right)^\delta \mathbf{S} [\mathcal{U}(\phi, \psi, \zeta)] - \sum_{\kappa=0}^{n_1-1} \left( \frac{\xi}{\varpi} \right)^{\delta-\kappa-1} \frac{\partial^\kappa \mathcal{U}^{(k)}(\phi, 0)}{\partial \zeta^\kappa} &= -\mathbf{S} \left[ \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] \right]. \end{aligned} \quad (4.14)$$

Employing the inverse ST, we have

$$\mathcal{U}(\phi, \psi, \zeta) = \mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \sum_{\kappa=0}^{n_1-1} \left( \frac{\xi}{\varpi} \right)^{\delta-\kappa-1} \frac{\partial^\kappa \mathcal{U}^{(k)}(\phi, 0)}{\partial \zeta^\kappa} - \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] \right] \right]. \quad (4.15)$$

It follows that

$$\mathcal{U}(\phi, \psi, \zeta) = \mathbf{S}^{-1} \left[ \frac{\varpi}{\xi} \mathcal{U}(\phi, \psi, 0) \right] - \mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] \right] \right],$$

$$\mathcal{U}(\phi, \psi, \zeta) = \mathbf{S}^{-1} \left[ \frac{3}{2} \frac{\varpi}{\xi} \theta \sinh \left[ \frac{1}{6}(\phi + \psi) \right] \right] - \mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \frac{\partial \mathcal{U}^3}{\partial \phi} + 2 \left[ \frac{\partial}{\partial \phi} \left( \frac{\partial^2 \mathcal{U}^3}{\partial \psi^2} \right) + \frac{\partial^3 \mathcal{U}^3}{\partial \phi^3} \right] \right] \right] \quad (4.16)$$

Utilizing the Shehu's decomposition method, we get

$$\sum_{j=0}^{\infty} \mathcal{U}_j(\phi, \psi, \zeta) = \frac{3}{2} \theta \sinh \left[ \frac{1}{6}(\phi + \psi) \right] - \mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \check{\mathcal{N}}(\mathcal{U})_\phi + \frac{1}{8} \left[ \check{\mathcal{N}}(\mathcal{U})_{\phi\phi\phi} + \check{\mathcal{N}}(\mathcal{U})_{\phi\psi\psi} \right] \right] \right], \quad (4.17)$$

where  $\check{\mathcal{N}}(\mathcal{U})$  is the Admoian's polynomial describing nonlinear term appearing in the above mentioned equations.

$$\check{\mathcal{N}}(\mathcal{U}) = \mathcal{U}^3 = \sum_{j=0}^{\infty} \mathcal{G}_j(\mathcal{U}). \quad (4.18)$$

First few Admoian's polynomials are presented as follows:

$$\begin{aligned} \mathcal{G}_0 &= \mathcal{U}_0^3, \\ \mathcal{G}_1 &= 3\mathcal{U}_0^2\mathcal{U}_1, \\ \mathcal{G}_2 &= 3\mathcal{U}_0^2\mathcal{U}_2 + 3\mathcal{U}_0^2\mathcal{U}_1^2, \\ \mathcal{U}_0(\phi, \psi, \zeta) &= \frac{3}{2} \theta \sinh \left[ \frac{1}{6}(\phi + \psi) \right], \\ \mathcal{U}_{j+1}(\phi, \psi, \zeta) &= -\mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ \left( \sum_{j=0}^{\infty} \mathcal{G}_j(\mathcal{U}) \right)_\phi + 2 \left( \sum_{j=0}^{\infty} \mathcal{G}_j(\mathcal{U}) \right)_{\phi\phi\phi} + 2 \left( \sum_{j=0}^{\infty} \mathcal{G}_j(\mathcal{U}) \right)_{\phi\psi\psi} \right] \right], \end{aligned}$$

for  $j = 0, 1, 2, \dots$

$$\begin{aligned} \mathcal{U}_1(\phi, \psi, \zeta) &= -\mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ (\mathcal{U}_0^3)_\phi + 2(\mathcal{U}_0^3)_{\phi\phi\phi} + 2(\mathcal{U}_0^3)_{\phi\psi\psi} \right] \right] \\ &= \left( -3\theta^3 \sinh^2 \left[ \frac{1}{6}(\phi + \psi) \right] \cosh \left[ \frac{1}{6}(\phi + \psi) \right] + \frac{3}{8}\theta^3 \cosh^3 \left[ \frac{1}{6}(\phi + \psi) \right] \right) \mathbf{S}^{-1} \left( \left( \frac{\varpi}{\xi} \right)^{\delta+1} \right) \\ &= \left( -3\theta^3 \sinh^2 \left[ \frac{1}{6}(\phi + \psi) \right] \cosh \left[ \frac{1}{6}(\phi + \psi) \right] + \frac{3}{8}\theta^3 \cosh^3 \left[ \frac{1}{6}(\phi + \psi) \right] \right) \frac{\zeta^\delta}{\Gamma(\delta + 1)}. \end{aligned}$$

Accordingly, we can derive the remaining terms as follows

$$\begin{aligned} \mathcal{U}_2(\phi, \psi, \zeta) &= -\mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ (3\mathcal{U}_0^2\mathcal{U}_1)_\phi + 2(3\mathcal{U}_0^2\mathcal{U}_1)_{\phi\phi\phi} + 2(3\mathcal{U}_0^2\mathcal{U}_1)_{\phi\psi\psi} \right] \right] \\ &= \frac{3}{32} \theta^5 \sinh \left[ \frac{1}{6}(\phi + \psi) \right] \left[ 765 \cosh^4 \left[ \frac{1}{6}(\phi + \psi) \right] \right. \\ &\quad \left. - 729 \cosh^2 \left[ \frac{1}{6}(\phi + \psi) \right] + 91 \right] \frac{\zeta^{2\delta}}{\Gamma(2\delta + 1)}, \end{aligned}$$

$$\mathcal{U}_3(\phi, \psi, \zeta) = -\mathbf{S}^{-1} \left[ \left( \frac{\varpi}{\xi} \right)^\delta \mathbf{S} \left[ (3\mathcal{U}_0^2\mathcal{U}_2 + 3\mathcal{U}_0^2\mathcal{U}_1^2)_\phi \right] \right]$$

$$\begin{aligned}
& +2(3\mathcal{U}_0^2\mathcal{U}_2 + 3\mathcal{U}_0^2\mathcal{U}_1^2)_{\phi\phi\phi} + 2(3\mathcal{U}_0^2\mathcal{U}_2 + 3\mathcal{U}_0^2\mathcal{U}_1^2)_{\phi\psi\psi} \Big] \\
& = -\frac{3}{128} \cosh\left[\frac{1}{6}(\phi + \psi)\right] \left[ 171,738 \cosh^6\left[\frac{1}{6}(\phi + \psi)\right] - 349,884 \cosh^4\left[\frac{1}{6}(\phi + \psi)\right] \right. \\
& \quad \left. + 215,496 \cosh^2\left[\frac{1}{6}(\phi + \psi)\right] - 36,907 \right] \frac{\zeta^{3\delta}}{\Gamma(3\delta + 1)}.
\end{aligned}$$

The approximate-analytical SDM solution is

$$\begin{aligned}
\mathcal{U}(\phi, \psi, \zeta) &= \mathcal{U}_0(\phi, \psi, \zeta) + \mathcal{U}_1(\phi, \psi, \zeta) + \mathcal{U}_2(\phi, \psi, \zeta) + \mathcal{U}_3(\phi, \psi, \zeta) + \dots, \\
\mathcal{U}(\phi, \psi, \zeta) &= \frac{3}{2}\theta \sinh\left[\frac{1}{6}(\phi + \psi)\right] - \left( 3\theta^3 \sinh^2\left[\frac{1}{6}(\phi + \psi)\right] \cosh\left[\frac{1}{6}(\phi + \psi)\right] \right. \\
& \quad \left. + \frac{3}{8}\theta^3 \cosh^3\left[\frac{1}{6}(\phi + \psi)\right] \right) \frac{\zeta^\delta}{\Gamma(\delta + 1)} \\
& \quad + \frac{3}{32}\theta^5 \sinh\left[\frac{1}{6}(\phi + \psi)\right] \left[ 765 \cosh^4\left[\frac{1}{6}(\phi + \psi)\right] - 729 \cosh^2\left[\frac{1}{6}(\phi + \psi)\right] \right. \\
& \quad \left. + 91 \right] \frac{\zeta^{2\delta}}{\Gamma(2\delta + 1)} - \frac{3}{128} \cosh\left[\frac{1}{6}(\phi + \psi)\right] \left[ 171,738 \cosh^6\left[\frac{1}{6}(\phi + \psi)\right] \right. \\
& \quad \left. - 349,884 \cosh^4\left[\frac{1}{6}(\phi + \psi)\right] \right. \\
& \quad \left. + 215,496 \cosh^2\left[\frac{1}{6}(\phi + \psi)\right] - 36,907 \right] \frac{\zeta^{3\delta}}{\Gamma(3\delta + 1)} + \dots.
\end{aligned} \tag{4.19}$$

The exact solution for  $\delta = 1$  is presented by

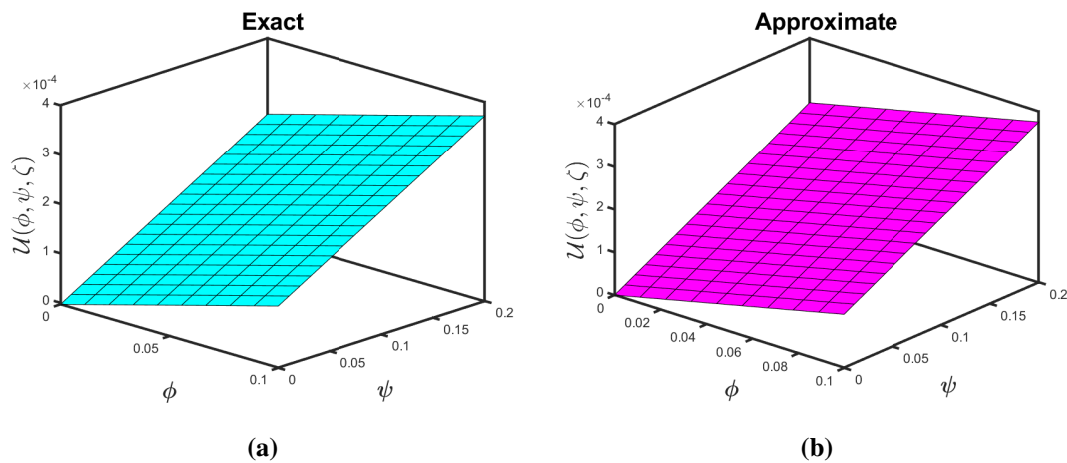
$$\mathcal{U}(\phi, \psi, \zeta) = \frac{3}{2}\theta \sinh\left[\frac{1}{6}(\phi + \psi - \theta\zeta)\right]. \tag{4.20}$$

Table 3 show the comparison results for exact, SDM, and absolute error of  $\mathcal{U}_{abs} = \|\mathcal{U}^E - \mathcal{U}^{SDM}\|$  solution for (4.12), when  $\theta = 0001$  and for various fractional orders  $\delta = 0.67, 0.75, 1$ . It can be seen that the proposed method closely matches the exact, and VIM [34].

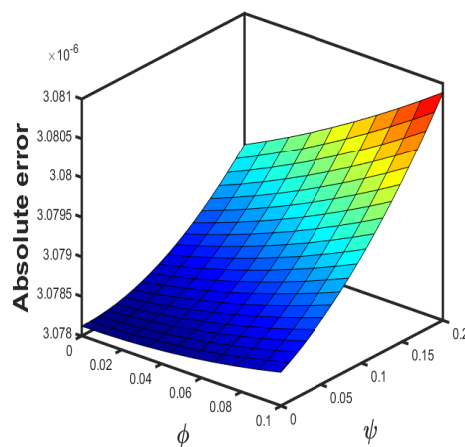
**Table 3.** Exact ( $\mathcal{U}_E$ ) and SDM-approximate ( $\mathcal{U}_{SDM}$ ) solution with absolute error ( $\mathcal{U}_{abs}$ ) in comparison derived by VIM ( $\mathcal{U}_{VIM}$ ) [34], for Example 4.2 at  $\theta = 0.001$ ,  $\delta = 1, 0.67$  and  $0.75$ .

| $\phi$ | $\psi$ | $\zeta$ | $\mathcal{U}_E$        | $\mathcal{U}_{SDM}$    | $\mathcal{U}_{abs}$    | $\mathcal{U}_{0.67}$   | $\mathcal{U}_{0.75}$   | $\mathcal{U}_{VIM}$    |
|--------|--------|---------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.1    | 0.1    | 0.2     | $4.996 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $4.988 \times 10^{-8}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ |
|        |        | 0.3     | $4.993 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $7.481 \times 10^{-8}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ |
|        |        | 0.4     | $4.991 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $9.975 \times 10^{-8}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ | $5.001 \times 10^{-5}$ |
| 0.6    | 0.6    | 0.2     | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $5.079 \times 10^{-8}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ |
|        |        | 0.3     | $3.019 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $7.619 \times 10^{-8}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ |
|        |        | 0.4     | $3.019 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $1.016 \times 10^{-7}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ | $3.020 \times 10^{-4}$ |
| 0.9    | 0.9    | 0.2     | $4.567 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $5.198 \times 10^{-8}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ |
|        |        | 0.3     | $4.567 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $7.797 \times 10^{-7}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ |
|        |        | 0.4     | $4.567 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $1.040 \times 10^{-7}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ | $4.568 \times 10^{-4}$ |

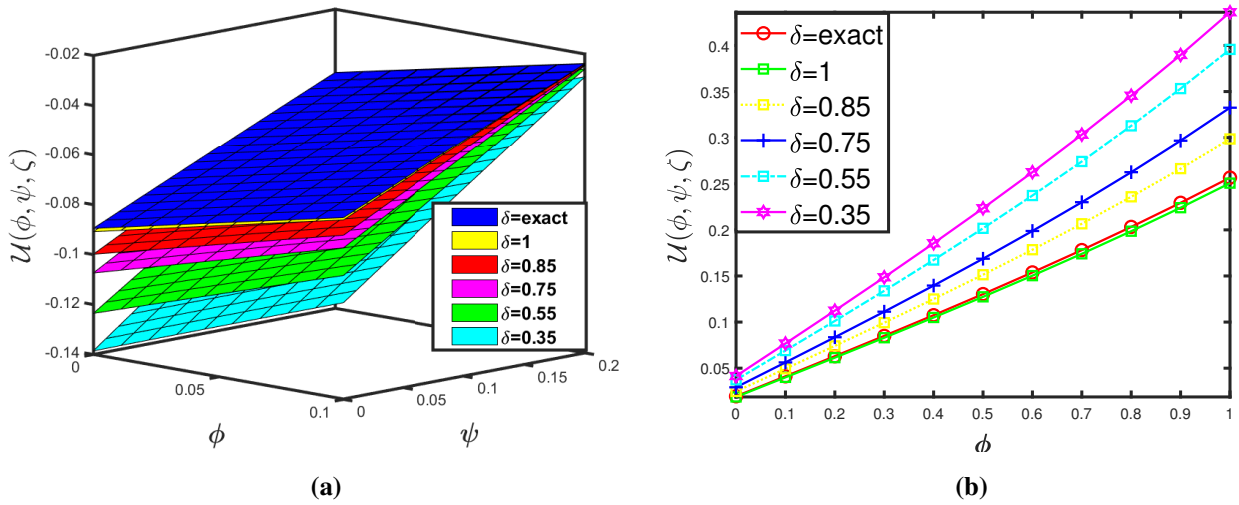
Taking  $\theta = 0.0005$  and  $\delta = 1$ , we exhibit the approximate-analytical solution of the fractional KZEs equation up to 4 components in Figure 5 (a and b), respectively. Furthermore, we established absolute errors at different values of  $\delta$  for the exact-approximate solutions in the accompanying Figure 6. Also, we have seen how different fractional orders perform in 2D and 3D plots in Figure 7 in (a and b) behaves. Also, Figure 8 denotes the  $\delta_1\delta_2$ -slice solutions for the exact and approximate solutions (a and b), respectively. As a result of this behaviour, we might conclude that the approximation solution tends to actual solution. Accordingly, as iteration increases, the absolute inaccuracy decreases. Consequently, as the iterations expands, the SDM findings approaches the exact result. The graphs in Figures 5–8 assist us to comprehend the behaviour of fractional orders when space and time scale variables fluctuate. Additionally, the findings of this study will aid scientists connecting in pattern formation theory, optical designs, or mathematical modelling in comprehending the structural phenomena of the ANOVA-test. Furthermore, the efficiency of the projected method can be boosted by getting additional approximate solution expressions.



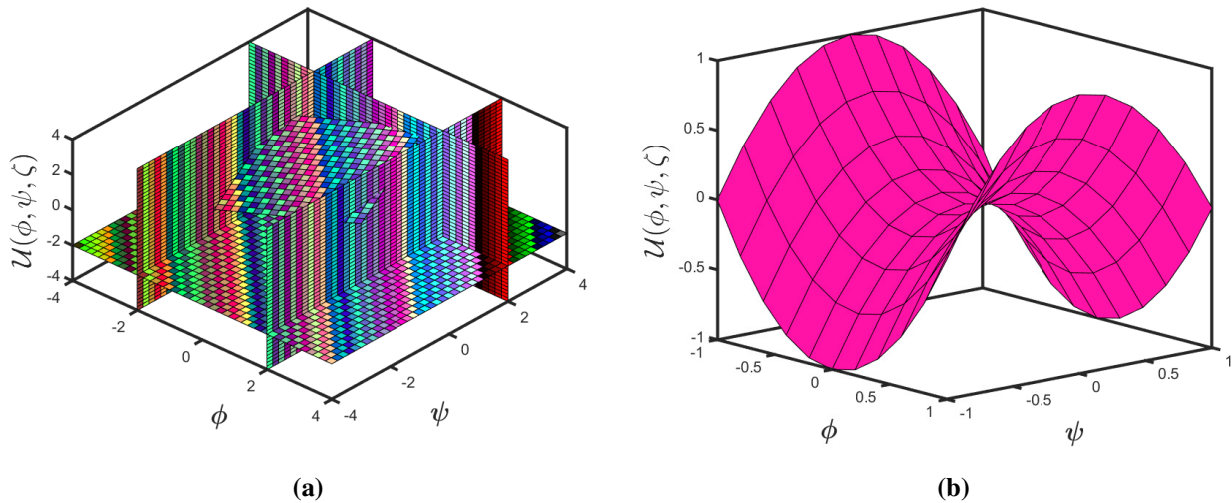
**Figure 5.** Numerical behavior of exact and approximate solution to the  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.2 when the parameters are  $\theta = 0.0005$ ,  $\delta = 1$ , and  $\zeta = 0.5$ .



**Figure 6.** Surface representation of  $\mathcal{U}(\phi, \psi, \zeta)$  absolute error plot for Example 4.2 at  $\theta = 0.005$  and  $\delta = 1$ .



**Figure 7.** Numerical behavior of  $\delta_1\delta_2$ -slice solution to the  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.2 (a) exact and (b) approximate when the parameters are  $\theta = 0.0005, \delta = 1,$  and  $\zeta = 0.5$ .



**Figure 8.** Numerical behavior of  $\delta_1\delta_2$ -slice solution to the  $\mathcal{U}(\phi, \psi, \zeta)$  for Example 4.1 (a) exact and (b) approximate when the parameters are  $\theta = 0.0005, \delta = 1,$  and  $\zeta = 0.5$ .

□

### 5. Conclusions

In this paper, the Shehu decomposition method (SDM) is effectively implemented for solving nonlinear time-fractional ZKEs. The proposed findings illustrate that there is a strong correlation between the projected method and the closed form solutions. Moreover, the governed approach is reliable and pragmatic for solving other diverse linear and nonlinear PDEs appearing in various disciplines of physics and mathematics. However, this methodology does not necessitate the condition

matrix, Lagrange multiplier, or costly integration calculations, so the findings are noise-free, which addresses the drawbacks of earlier techniques. It is worth mentioning that the proposed methods are pragmatic analytical tools for identifying approximate-analytical solutions to complicated nonlinear PDEs. Moreover, we deduce that this approach will be used to deal with other non-linear fractional order systems of equations that are extremely complex.

## Acknowledgments

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## Conflict of interest

The authors declare that they have no conflict of interest.

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