OPEN ACCESS
Fractals, Vol. 28, No. 8 (2020) 2040027 (12 pages)

DOI: 10.1142/S0218348X20400277

NEW ESTIMATES OF INTEGRAL INEQUALITIES VIA GENERALIZED PROPORTIONAL FRACTIONAL INTEGRAL OPERATOR WITH RESPECT TO ANOTHER FUNCTION

SAIMA RASHID
Department of Mathematics, Government College University, Faisalabad, Pakistan
saimarashid@gcuf.edu.pk

ZAKIA HAMMOUCH
Faculty of Science and Techniques Moulay Ismail University of Meknes
52000 Errachidia, Morocco
Department of Mathematics and Science Education, Harran University
Sanliurfa, Turkey
z.hammouch@fste.umi.ac.ma

FAHD JARAD
Department of Mathematics, Çankaya University, 06790 Ankara, Turkey
fahd@cankaya.edu.tr

YU-MING CHU*
Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China
School of Mathematics and Statistics, Changsha University of Science & Technology
Changsha 410114, P. R. China
chuyuming2005@126.com

Received December 23, 2019
Accepted February 15, 2020
Published June 25, 2020

*Corresponding author.

Abstract

In this paper, the newly proposed concept of the generalized proportional fractional integral operator with respect to another function Φ has been utilized to generate integral inequalities using convex function. This new concept will have the option to reduce self-similitudes in the fractional attractors under investigation. We discuss the implications and other consequences of the integral inequalities concerning the generalized proportional fractional integral operator with respect to another function Φ are derived here and these outcomes permit us specifically to generalize some classical inequalities. Certain intriguing subsequent consequences of the fundamental hypotheses are also figured. It is to be supposed that this investigation will provide new directions in the quantum theory of capricious nature.

Keywords: Convex Functions; Generalized Proportional Fractional Integral Operator With Respect To Another Function; Integral Inequalities.

1. INTRODUCTION

Fractional calculus and its extensive utilities have lately been paid to a regularly expanding degree considerations. A fascinating specialty of this paper is that there are several fractional operators. The attractors with numerical simulations for varying values and this permits the readers to choose the most appropriate operator for demonstrating the issue under investigation. In addition, as a result of its effortlessness in applications, analysts have given more consideration to presently determined fractional operators without singular kernels,[13,15–17,23,25,27,36,37,49] and afterward numerous articles considering these sorts of fractional operators nowadays turn out to be noticeable. Almeida[9] explored a new fractional derivative called Caputo derivative with respect to another function Φ and Kilbas *et al.*[33] contemplated the concept of Riemann–Liouville fractional integrals with respect to another function Φ, which is the extraordinary generalization of all fractional integral operators. It is fairly well-known that there are a number of different definitions of fractional integrals and their applications. Each definition has its own advantages and appropriate for utilities to diverse kinds of issues. Currently, Atangana and Baleanu[11] provided one more direction to this investigation by introducing a new operator that depends on the generalized Mittag–Leffler function, since the Mittag–Leffler function is more reasonable in communicating nature than control work. Within the structure of applied science and mathematical modeling, there exists an outstanding kind of operator known as generalized proportional fractional integral operator in which the variable is a scaled according to proportionality index ς. This diversified operator was introduced by Rashid et al.,[47] to the conceivable role those physical problems for which classical physical law, for example, the well-known Mellin transform, Fourier transform and probability theory is suitable, such physical issue is accepted to be found on the fractional calculus and pertinent to the media of non-integral fractional operators.[2–5,19,26,30,32,34,46,48]

Integral inequalities are considered to be significant as these are helpful in the investigation of various classes of differential and integral equations.[1,6,12,14,20,22,24,31,35,38–41,51,54–64] Few decades ago, numerous scientists have acquired different fractional integral inequalities encompassing the diverse fractional differential and integral operators. There is countless fractional integral operators mentioned within the literature, however, due to their utilities in numerous areas of sciences, the Riemann–Liouville and Hadamard fractional integral operators have been contemplated broadly.[52,53]

Recently, the authors introduced numerous variants via fractional integral operator such as Hermite–Hadamard inequalities, Gruss type and certain Gronwall variants with applications by employing Riemann–Liouville, Hadamard fractional operators and generalized proportional Hadamard fractional integral.[7,8,10,42,44] In Ref. 21 Dahmani established certain classes of fractional integral inequalities by using a family of n positive functions. In Ref. 8 Aldhaifallah *et al.* derived several extended versions for a class of family of $n(n \in \mathbb{N})$ positive continuous and decreasing functions on an interval by introducing the concept of combination of two classes via fractional integral operators. Currently, the well-known fractional operators have taken considerable attention of several researchers for deriving remarkable speculations, modifications, exploring properties and utilities for several fractional integrals can be found in the literature.[18,43,45,50]

In this paper, a new concept of the generalized proportional fractional integral operator with respect to another function ψ is introduced. This new concept takes into account the fractional calculus. These novelties are a combination of convex functions based on several variants that correlate with the generalized proportional fractional integral operator with respect to another function ψ. New results are presented and new theorems are established. In addition to this, the numerical approximations for the new Definition 2.5 in fractional calculus are presented. The newly introduced numerical approximation is used to solve problems in integrodifferential equations, aerodynamics, and optimization theory. The new definition could open new doors of investigation toward convexity and fractional calculus.

2. PRELIMINARIES

This segment is dedicated to some recognized definitions and outcomes associated with the generalized proportional fractional integral operator with respect to another function Φ.

Definition 2.1 (Refs. 30 and 33). A function $\mathcal{U}(\xi)$ is said to be in $L_{q,r}[0,\infty]$ if

$$L_{q,r}[0,\infty) = \left\{\mathcal{U} : \|\mathcal{U}\|_{L_{q,r}[0,\infty)} = \left(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{U}(\lambda)|^q \lambda^r d\lambda\right)^{\frac{1}{q}} < \infty, \quad 1 \le q < \infty,\ r \ge 0\right\}.$$

For $r = 0$,

$$L_q[0,\infty) = \left\{\mathcal{U} : \|\mathcal{U}\|_{L_q[0,\infty)} = \left(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{U}(\lambda)|^q d\lambda\right)^{\frac{1}{q}} < \infty, \quad 1 \le q < \infty\right\}.$$

Definition 2.2 (Ref. 28). Let $\mathcal{U} \in L_1[0,\infty)$ and Ψ be an increasing and positive monotone function on $[0,\infty)$ and also derivative Ψ' is continuous on $[0,\infty)$ and $\Psi(0) = 0$. The space $\chi_\Psi^q(0,\infty)$ $(1 \le q < \infty)$ of those real-valued Lebesgue measureable functions $\mathcal{U}$ on $[0,\infty)$ for which

$$\|\mathcal{U}\|_{\chi_\Psi^q} = \left(\int_0^\infty |\mathcal{U}(\lambda)|^p \Psi'(\lambda) d\lambda\right)^{\frac{1}{p}} < \infty, \quad 1 \le q < \infty$$

and for the case $q = \infty$

$$\|\mathcal{U}\|_{\chi_\Psi^\infty} = \operatorname{ess} \sup_{0 \le \lambda < \infty} \left[\Psi'(\lambda)\mathcal{U}(\lambda)\right].$$

Specifically, when $\Psi(\xi) = \xi \quad (1 \le q < \infty)$ the space $\chi_\Psi^q(0,\infty)$ coinsides with the $L_q[0,\infty)$-space and also if we choose $\Psi(\xi) = \ln \xi \quad (1 \le q < \infty)$ the space $\chi_\Psi^q(0,\infty)$ coincides with $L_{q,r}[1,\infty)$-space.

Now, we present a new fractional operator which is known as the GPF-integral operator of a function in the sense of another function Ψ, which is mainly due to Rashid *et al.*[47]

Definition 2.3 (Ref. 47). Let $\mathcal{U} \in \chi_\Psi^q(0,\infty)$, there is an increasing, positive monotone function Ψ defined on $[0,\infty)$ having continuous derivative $\Psi'(\xi)$ on $[0,\infty)$ with $\Psi(0) = 0$. Then, the left-sided and right-sided GPF-integral operator of a function $\mathcal{U}$ in the sense of another function Ψ of order $\eta > 0$ are stated as follows:

$$({}^{\Psi}\mathcal{T}_{\upsilon_1}^{\eta,\varsigma}\mathcal{U})(\xi) = \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_{\upsilon_1}^{\xi} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\xi) - \Psi(\lambda))]}{(\Psi(\xi) - \Psi(\lambda))^{1-\eta}} \times \mathcal{U}(\lambda)\Psi'(\lambda) d\lambda, \quad \upsilon_1 < \xi \tag{2.1}$$

and

$$({}^{\Psi}\mathcal{T}_{\upsilon_2}^{\delta,\varsigma}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta \Gamma(\delta)} \int_{\xi}^{\upsilon_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\lambda) - \Psi(\xi))]}{(\Psi(\lambda) - \Psi(\xi))^{1-\delta}} \times \mathcal{U}(\lambda)\Psi'(\lambda) d\lambda, \quad \xi < \upsilon_2, \tag{2.2}$$

where the proportionality index $\varsigma \in (0,1], \delta \in \mathbb{C}, \Re(\delta) > 0$, and $\Gamma(\xi) = \int_0^\infty \lambda^{\xi-1} e^{-\lambda} d\lambda$ is the Gamma function.

Remark 2.4. Several existing fractional operators are just special cases of (2.1) and (2.2).

(1) Choosing $\Psi(\xi) = \xi$ in (2.1) and (2.2), then we acquire the left- and right-sided GPF operator proposed by Jarad *et al.*,[26] stated as follows:

$$(\mathcal{T}_{\upsilon_1}^{\delta,\varsigma}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta \Gamma(\delta)} \int_{\upsilon_1}^{\xi} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\xi - \lambda)]}{(\xi - \lambda)^{1-\delta}} \times \mathcal{U}(\lambda) d\lambda, \quad \upsilon_1 < \xi \tag{2.3}$$

and

$$(\mathcal{T}^{\delta,\varsigma}_{\upsilon_2}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_\xi^{\upsilon_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\lambda-\xi)]}{(\lambda-\xi)^{1-\delta}} \times \mathcal{U}(\lambda)d\lambda, \quad \xi < \upsilon_2. \tag{2.4}$$

(2) Choosing $\varsigma = 1$ in (2.1) and (2.2), then we acquire the left- and right-sided generalized RL-fractional integral operator introduced by Kilbas *et al.*,[33] stated as follows:

$$({}^\Psi\mathcal{T}^{\delta}_{\upsilon_1}\mathcal{U})(\xi) = \frac{1}{\Gamma(\delta)}\int_{\upsilon_1}^{\xi} \frac{\Psi'(\lambda)\mathcal{U}(\lambda)}{(\Psi(\xi)-\Psi(\lambda))^{1-\delta}}d\lambda, \quad \upsilon_1 < \xi \tag{2.5}$$

and

$$({}^\Psi\mathcal{T}^{\delta}_{\upsilon_2}\mathcal{U})(\xi) = \frac{1}{\delta\Gamma(\delta)}\int_{\xi}^{\upsilon_2} \frac{\mathcal{U}(\lambda)\Psi'(\lambda)}{(\Psi(\lambda)-\Psi(\xi))^{1-\delta}}d\lambda, \quad \xi < \upsilon_2. \tag{2.6}$$

(3) Choosing $\Psi(\xi) = \ln\xi$ in (2.1) and (2.2), then we acquire the left- and right-sided generalized proportional Hadamard fractional integral operator established by Rahman *et al.*,[42] stated as follows:

$$(\mathcal{T}^{\delta,\varsigma}_{\upsilon_1}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_{\upsilon_1}^{\xi} \frac{\exp[\frac{\varsigma-1}{\varsigma}\big(\ln\frac{\xi}{\lambda}\big)]}{\big(\ln\frac{\xi}{\lambda}\big)^{1-\delta}} \times \frac{\mathcal{U}(\lambda)}{\lambda}d\lambda, \quad \upsilon_1 < \xi \tag{2.7}$$

and

$$(\mathcal{T}^{\delta,\varsigma}_{\upsilon_2}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_{\xi}^{\upsilon_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}\big(\ln\frac{\lambda}{\xi}\big)]}{(\ln\frac{\lambda}{\xi})^{1-\delta}} \times \frac{\mathcal{U}(\lambda)}{\lambda}d\lambda, \quad \xi < \upsilon_2. \tag{2.8}$$

(4) Choosing $\Psi(\xi) = \ln\xi$ along with $\varsigma = 1$ in (2.1) and (2.2), then we acquire the left- and right-sided Hadamard fractional integral operator obtained by Kilbas *et al.*[33] and Samko *et al.*,[49] stated as follows:

$$\mathcal{T}^{\delta}_{\upsilon_1}\mathcal{U}(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_{\upsilon_1}^{\xi} \frac{\mathcal{U}(\lambda)}{\lambda\big(\ln\frac{\xi}{\lambda}\big)^{1-\delta}}d\lambda, \quad \upsilon_1 < \xi \tag{2.9}$$

and

$$\mathcal{T}^{\delta}_{\upsilon_2}\mathcal{U}(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_{\xi}^{\upsilon_2} \frac{\mathcal{U}(\lambda)}{\lambda(\ln\frac{\lambda}{\xi})^{1-\delta}}d\lambda, \quad \xi < \upsilon_2. \tag{2.10}$$

(5) Choosing $\Psi(\xi) = \frac{\xi^\rho}{\rho}$ $(\rho > 0)$ in (2.1) and (2.2), then we acquire the left- and right-sided generalized fractional integral operator obtained by Katugampola,[29] stated as follows:

$$\mathcal{T}^{\delta}_{\upsilon_1}\mathcal{U}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\upsilon_1}^{\xi} \Big(\frac{\xi^\rho-\lambda^\rho}{\rho}\Big)^{\delta-1} \times \mathcal{U}(\lambda)\frac{d\lambda}{\lambda^{1-\rho}}, \quad \upsilon_1 < \xi \tag{2.11}$$

and

$$\mathcal{T}^{\delta}_{\upsilon_2}\mathcal{U}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\xi}^{\upsilon_2} \Big(\frac{\lambda^\rho-\xi^\rho}{\rho}\Big)^{\delta-1} \times \mathcal{U}(\lambda)\frac{d\lambda}{\lambda^{1-\rho}}, \quad \xi < \upsilon_2. \tag{2.12}$$

(6) Choosing $\Psi(\xi) = \xi$ along with $\varsigma = 1$ in (2.1) and (2.2), then we get the left- and right-sided RL-fractional integral operator stated as follows:

$$\mathcal{T}^{\delta}_{\upsilon_1}\mathcal{U}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\upsilon_1}^{\xi} \frac{\mathcal{U}(\lambda)}{(\xi-\lambda)^{1-\delta}}d\lambda, \quad \upsilon_1 < \xi \tag{2.13}$$

and

$$\mathcal{T}^{\delta}_{\upsilon_2}\mathcal{U}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\xi}^{\upsilon_2} \frac{\mathcal{U}(\lambda)}{(\lambda-\xi)^{1-\delta}}d\lambda, \quad \xi < \upsilon_2. \tag{2.14}$$

Definition 2.5. Let $\mathcal{U} \in \chi^q_\Psi(0,\infty)$ and there is an increasing, positive monotone function Ψ defined on $[0,\infty)$ having continuous derivative $\Psi'(\xi)$ on $[0,\infty)$ with $\Psi(0) = 0$. Then, the one-sided GPF-integral operator of a function $\mathcal{U}$ in the sense of another function Ψ of order $\delta > 0$ and proportionality index $\varsigma \in [0,1]$ is stated as follows:

$$(\mathcal{T}^{\delta,\varsigma}_{0^+,\xi}\mathcal{U})(\xi) = \frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^{\xi} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\xi)-\Psi(\lambda)]}{(\Psi(\xi)-\Psi(\lambda))^{1-\delta}} \times \mathcal{U}(\lambda)\Psi'(\lambda)d\lambda, \quad \lambda > 0. \tag{2.15}$$

3. MAIN RESULTS

This section is devoted to establishing generalizations of some classical inequalities by employing *GPF* integral with respect to another function Ψ defined in (2.15).

Theorem 3.1. *For $\varsigma \in (0,1], \delta \in \mathcal{C}, \mathbb{R}(\delta) > 0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U} \le \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$, then for any convex function Θ having $\Theta(0) = 0$. Assume that, Φ*

is an increasing and positive monotone function on $[0,\infty)$, *derivative* Φ' *is continuous on* $[0,\infty)$ *with* $\Phi(0)=0$, *then the generalized proportional fractional integral operator with respect to another function* Φ *given by* (2.7) *satisfies the inequality*

$$\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left[\mathcal{U}(\xi)\right]}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left[\mathcal{V}(\xi)\right]} \geq \frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left[\Theta\left(\mathcal{U}(\xi)\right)\right]}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left[\Theta\left(\mathcal{V}(\xi)\right)\right]}. \tag{3.1}$$

Proof. Since $\mathcal{U}$ is increasing along with the function $\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}$, moreover, using convexity of Θ having $\Theta(0)=0$, and the function $\frac{\Theta(\mathcal{U}(\xi))}{\xi}$ is increasing.

Thus, for all $\lambda,\rho\in[0,\infty)$, we have

$$\left(\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}-\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\right)\left(\frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)}-\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}\right)\geq 0. \tag{3.2}$$

It follows that

$$\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)}+\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}-\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\frac{\mathcal{U}(\rho)}{\mathcal{V}(\rho)} -\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}\geq 0. \tag{3.3}$$

Multiplying (3.3) by $\mathcal{V}(\lambda)\mathcal{V}(\rho)$, we have

$$\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\mathcal{U}(\rho)\mathcal{V}(\lambda)+\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{U}(\lambda)\mathcal{V}(\rho) -\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\mathcal{V}(\lambda)-\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\rho)\geq 0. \tag{3.4}$$

Multiplying (3.4) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}$, which is positive because $\lambda\in(0,\xi),\ \xi>0$ and integrating with respect to λ from 0 to ξ, we have

$$\begin{aligned}
&\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\mathcal{U}(\rho)\mathcal{V}(\lambda)d\lambda\\
&+\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{U}(\lambda)\mathcal{V}(\rho)d\lambda\\
&-\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\mathcal{V}(\lambda)d\lambda\\
&-\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\rho)d\lambda\geq 0.
\end{aligned} \tag{3.5}$$

This follows that

$$\begin{aligned}
&\mathcal{U}(\rho)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta\left(\mathcal{U}(\xi)\right)}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right)\\
&+\left(\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{V}(\rho)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{U}(\xi)\right)\\
&-\left(\frac{\Theta\left(\mathcal{U}(\rho)\right)}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{V}(\xi)\right)\\
&-\mathcal{V}(\rho)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta\left(\mathcal{U}(\xi)\right)}{\mathcal{U}(\xi)}\mathcal{U}(\xi)\right)\geq 0.
\end{aligned} \tag{3.6}$$

Again, multiplying both sides of (3.6) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\rho))]\Phi'(\rho)}{(\Phi(\xi)-\Phi(\rho))^{1-\delta}}$, which is positive because $\rho\in(0,\xi),\ \xi>0$ and integrating with respect to ρ from 0 to ξ, we have

$$\begin{aligned}
&{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{U}(\xi)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta\left(\mathcal{U}(\xi)\right)}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right)\\
&+\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta\left(\mathcal{U}(\xi)\right)}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{U}(\xi)\right)\\
&\geq\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{V}(\xi)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\Theta(\mathcal{U}(\xi))\right)\\
&+\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\Theta(\mathcal{U}(\xi))\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{V}(\xi)\right).
\end{aligned} \tag{3.7}$$

It follows that

$$\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{U}(\xi)\right)}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\mathcal{V}(\xi)\right)}\geq\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\Theta\left(\mathcal{U}(\xi)\right)\right)}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\left(\mathcal{V}(\xi)\right)\right)}. \tag{3.8}$$

Now, since $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho\in[0,\xi)$, we have

$$\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\leq\frac{\Theta\left(\mathcal{V}(\lambda)\right)}{\mathcal{V}(\lambda)}. \tag{3.9}$$

Multiplying both sides of (3.9) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\mathcal{V}(\lambda)$, which is positive because $\lambda\in(0,\xi),\ \xi>0$ and integrating with respect to λ from 0 to ξ, we have

$$\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times\frac{\Theta\left(\mathcal{U}(\lambda)\right)}{\mathcal{U}(\lambda)}\mathcal{V}(\lambda)d\lambda$$

$$\leq \frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_{0}^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \times \frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}\mathcal{V}(\lambda)d\lambda, \quad (3.10)$$

which, in view of (2.7), can be written as

$$^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right) \leq\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\Theta(\mathcal{V}(\lambda))\right). \quad (3.11)$$

Hence from (3.8) and (3.11), we get (3.3). □

We give also the following result.

(I) Letting $\Phi(\lambda)=\lambda$, then we have a new result for generalized proportional fractional integral.

Corollary 3.2. *For $\varsigma\in(0,1], \delta\in\mathcal{C}, \mathbb{R}(\delta)>0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$, then for any convex function Θ having $\Theta(0)=0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]} \geq \frac{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))]}{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{V}(\xi))]}.$$

(II) Letting $\Phi(\lambda)=\lambda$ along with $\varsigma=1$, then we have a new result for Riemann–Liouville fractional integral.

Corollary 3.3. *For $\delta\in\mathcal{C}, \mathbb{R}(\delta)>0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$, then for any convex function Θ having $\Theta(0)=0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{U}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{V}(\xi)]} \geq \frac{\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{U}(\xi))]}{\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{V}(\xi))]}.$$

Remark 3.4. If we choose $\Phi(\lambda)=\ln\lambda$, then Theorem 3.1 in this paper reduces to Theorem 3.1 in Ref. 42 and choosing $\Phi(\lambda)=\ln\lambda$ along with $\varsigma=1$, then we get Theorem 3.1 in Ref. 18

Theorem 3.5. *For $\varsigma\in(0,1], \delta,\eta\in\mathcal{C}, \mathbb{R}(\delta),\mathbb{R}(\eta)>0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$ and Θ is a convex function having $\Theta(0)=0$. Assume that, Φ is an increasing and positive monotone function on $[0,\infty)$, derivative Φ' is continuous on $[0,\infty)$ with $\Phi(0)=0$, then the generalized proportional fractional integral operator with respect to another function Φ given by (2.7) satisfies the inequality*

$$\frac{^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{V}(\xi))]+\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{U}(\xi)]\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{V}(\xi))]}{^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{U}(\xi))]+\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{V}(\xi)]\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))]} \geq 1, \quad (3.12)$$

Proof. Since $\mathcal{U}$ is increasing along with the function $\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}$, moreover, using convexity of Θ having $\Theta(0)=0$, and the function $\frac{\Theta(\mathcal{U}(\xi))}{\xi}$ is increasing.

Thus, for all $\lambda,\rho\in[0,\xi)$. Multiplying (3.6) by $\frac{1}{\varsigma^{\eta}\Gamma(\eta)}\ \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\rho))]\Phi'(\rho)}{(\Phi(\xi)-\Phi(\rho))^{1-\eta}}$, which is positive because $\rho\in(0,\xi), \xi>0$ and integrating with respect to ρ from 0 to ξ, we have

$$\begin{aligned} &^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\mathcal{U}(\xi))\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right) \\ &\quad +\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right)\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi)) \\ &\geq\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{V}(\xi))\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\Theta(\mathcal{U}(\xi))) \\ &\quad +\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta(\mathcal{U}(\xi)))\ ^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\mathcal{V}(\xi)). \end{aligned} \quad (3.13)$$

Now, since $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho\in[0,\xi)$, we have

$$\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)} \leq \frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}. \quad (3.14)$$

Multiplying both sides of (3.14) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\ \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\mathcal{V}(\lambda)$, which is positive because $\lambda\in(0,\xi), \xi>0$ and integrating with respect to λ from 0 to ξ, we have

$$\begin{aligned} &\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_{0}^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\mathcal{V}(\lambda)d\lambda \\ &\leq \frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_{0}^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}} \frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}\mathcal{V}(\lambda)d\lambda, \end{aligned} \quad (3.15)$$

which, in view of (2.7), can be written as

$$^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\right) \leq\ ^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\Theta(\mathcal{V}(\xi))\right). \quad (3.16)$$

Hence from (3.11), (3.13) and (3.16), we get our desired result. □

We give the following generalizations of 3.5 as follows.

(I) Letting $\Phi(\lambda)=\lambda$, then we have a new result for generalized proportional fractional integral.

Corollary 3.6. *For $\varsigma \in (0,1], \delta, \eta \in \mathcal{C}$, $\mathbb{R}(\delta), \mathbb{R}(\eta) > 0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$, then for any convex function Θ having $\Theta(0)=0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]\,\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{V}(\xi))]+\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{U}(\xi)]\,\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{V}(\xi))]}{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]\,\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{U}(\xi))]+\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{V}(\xi)]\,\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))]}\geq 1.$$

(II) Letting $\Phi(\lambda)=\lambda$ along with $\varsigma=1$, then we have a new result for Riemann–Liouville fractional integral.

Corollary 3.7. *For $\delta,\eta \in \mathcal{C}$, $\mathbb{R}(\delta), \mathbb{R}(\eta) > 0$ and there are two positive continuous functions $\mathcal{U}$ and $\mathcal{V}$ defined on $[0,\infty)$ and $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$ then for any convex function Θ having $\Theta(0)=0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{U}(\xi)]\,\mathcal{T}_{0,\xi}^{\eta}[\Theta(\mathcal{V}(\xi))]+\mathcal{T}_{0,\xi}^{\eta}[\mathcal{U}(\xi)]\,\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{V}(\xi))]}{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{V}(\xi)]\,\mathcal{T}_{0,\xi}^{\eta}[\Theta(\mathcal{U}(\xi))]+\mathcal{T}_{0,\xi}^{\eta}[\mathcal{V}(\xi)]\,\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{U}(\xi))]}\geq 1.$$

Remark 3.8. If we choose $\Phi(\lambda)=\ln\lambda$, then Theorem 3.5 in this paper reduces to Theorem 3.2 in Ref. 42 and choosing $\Phi(\lambda)=\ln\lambda$ along with $\varsigma=1$, then we get Theorem 3.2 in Ref. 18 Also, if we consider $\delta=\eta$, then Theorem 3.5 will lead to Theorem 3.1.

We further have the following main theorem.

Theorem 3.9. *For $\varsigma \in (0,1], \delta \in \mathcal{C}, \mathbb{R}(\delta) > 0$ and there are three positive continuous functions $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ defined on $[0,\infty)$ and $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$ and Θ is a convex function having $\Theta(0)=0$. Assume that, Φ is an increasing and positive monotone function on $[0,\infty)$, derivative Φ' is continuous on $[0,\infty)$ with $\Phi(0)=0$, then the generalized proportional fractional integral operator with respect to another function Φ given by (2.7) satisfies the inequality*

$$\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]}\geq\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))\,\mathcal{W}(\xi)]}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{V}(\xi))\,\mathcal{W}(\xi)]}. \tag{3.17}$$

Proof. Since $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda,\rho \in [0,\xi)$, we have

$$\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\leq\frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}. \tag{3.18}$$

Multiplying both sides of (3.18) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(x)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(x)-\Phi(\lambda))^{1-\delta}}\mathcal{V}(\lambda)\mathcal{W}(\lambda)$, which is positive because $\lambda \in (0,\xi), \xi > 0$ and integrating with respect to λ from 0 to ξ, we get

$$\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\mathcal{V}(\lambda)\mathcal{W}(\lambda)d\lambda$$
$$\leq\frac{1}{\varsigma^{\delta}\Gamma(\delta)}\int_0^{\xi}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}\mathcal{V}(\lambda)\mathcal{W}(\lambda)d\lambda, \tag{3.19}$$

which, in view of (2.7), can be written as

$${}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right)\leq{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta(\mathcal{V}(\xi))\,\mathcal{W}(\xi)). \tag{3.20}$$

Also, since Θ is convex with $\Theta(0)=0$, the function $\frac{\Theta(\lambda)}{\lambda}$ is increasing. As $\mathcal{U}$ is increasing, so is the function $\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}$. Obviously, the function $\frac{\mathcal{U}(\lambda)}{\mathcal{V}(\lambda)}$ is decreasing for all $\lambda,\rho \in [0,\xi), \xi > 0$. We have

$$\left(\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}-\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\mathcal{W}(\rho)\right)(\mathcal{U}(\rho)\mathcal{V}(\lambda)-\mathcal{U}(\lambda)\mathcal{V}(\rho))\geq 0. \tag{3.21}$$

It follows that

$$\frac{\Theta(\mathcal{U}(\lambda))\,\mathcal{W}(\lambda)}{\mathcal{U}(\lambda)}\mathcal{U}(\rho)\mathcal{V}(\lambda)+\frac{\Theta(\mathcal{U}(\rho))\,\mathcal{W}(\rho)}{\mathcal{U}(\rho)}\mathcal{U}\lambda\mathcal{V}(\rho)-\frac{\Theta(\mathcal{U}(\rho))\,\mathcal{W}(\rho)}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\mathcal{V}(\lambda)-\frac{\Theta(\mathcal{U}(\lambda))\,\mathcal{W}(\lambda)}{\mathcal{U}(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\rho)\geq 0. \tag{3.22}$$

Multiplying (3.22) by $\frac{1}{\varsigma^\delta\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}$, which is positive because $\lambda\in(0,\xi), \xi>0$ and integrating with respect to λ from 0 to ξ, we have

$$\begin{aligned}&\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\\&\quad\times\mathcal{U}(\rho)\mathcal{V}(\lambda)\mathcal{W}(\lambda)d\lambda\\&+\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\\&\quad\times\mathcal{U}(\lambda)\mathcal{V}(\rho)\mathcal{W}(\rho)d\lambda\\&-\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\\&\quad\times\mathcal{U}(\rho)\mathcal{V}(\lambda)\mathcal{W}(\rho)d\lambda\\&-\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\\&\quad\times\mathcal{U}(\lambda)\mathcal{V}(\rho)\mathcal{W}(\lambda)d\lambda\geq 0. \qquad (3.23)\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\Theta(\mathcal{U}(\lambda))\mathcal{W}(\lambda)}{\mathcal{U}(\lambda)}\mathcal{U}(\rho)\mathcal{V}(\lambda)\\&+\frac{\Theta(\mathcal{U}(\rho))\mathcal{W}(\rho)}{\mathcal{U}(\rho)}\mathcal{U}(\lambda)\mathcal{V}(\rho)\\&-\frac{\Theta(\mathcal{U}(\rho))\mathcal{W}(\rho)}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\mathcal{V}(\lambda)\\&-\frac{\Theta(\mathcal{U}(\lambda))\mathcal{W}(\lambda)}{\mathcal{U}(\lambda)}\mathcal{U}(\lambda)\mathcal{V}(\rho)\geq 0. \qquad (3.24)\end{aligned}$$

Multiplying (3.22) by $\frac{1}{\varsigma^\delta\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}$, which is positive because $\lambda\in(0,\xi)$, $\xi>0$ and integrating with respect to λ from 0 to ξ, we have

$$\begin{aligned}&\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\\&\quad\times\mathcal{U}(\rho)\mathcal{V}(\lambda)\mathcal{W}(\lambda)d\lambda\\&+\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\\&\quad\times\mathcal{U}(\lambda)\mathcal{V}(\rho)\mathcal{W}(\rho)d\lambda\\&-\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^\xi\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\\&\quad\times\mathcal{U}(\rho)\mathcal{V}(\lambda)\mathcal{W}(\rho)d\lambda\\&-\frac{1}{\varsigma^\delta\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)}\\&\quad\times(\lambda)\mathcal{V}(\rho)\mathcal{W}(\lambda)d\lambda\geq 0. \qquad (3.25)\end{aligned}$$

This follows that

$$\begin{aligned}&\mathcal{U}(\rho)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right)\\&\quad+\left(\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\mathcal{V}(\rho)\mathcal{W}(\rho)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi))\\&\quad-\left(\frac{\Theta(\mathcal{U}(\rho))}{\mathcal{U}(\rho)}\mathcal{U}(\rho)\mathcal{W}(\rho)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{V}(\xi))\\&\quad-\mathcal{V}(\rho)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{U}(\xi)\mathcal{W}(\xi)\right)\geq 0. \qquad (3.26)\end{aligned}$$

Again, multiplying both sides of (3.26) by $\frac{1}{\varsigma^\delta\Gamma(\delta)}\frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\rho))]\Phi'(\rho)}{(\Phi(\xi)-\Phi(\rho))^{1-\delta}}$, which is positive because $\rho\in(0,\xi)$, $\xi>0$ and integrating with respect to ρ from 0 to ξ, we have

$$\begin{aligned}&{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right)\\&\quad+\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi))\\&\geq\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{V}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta\mathcal{U}(\xi)\mathcal{W}(\xi))\\&\quad+\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta\mathcal{U}(\xi)\mathcal{W}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{V}(\xi)). \qquad (3.27)\end{aligned}$$

It follows that

$$\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi))}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi))}\geq\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi))}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}(\mathcal{V}(\xi))\mathcal{W}(\xi)\right)}. \qquad (3.28)$$

Hence from (3.20) and (3.28), we get our required result. □

At the end, we give the following corollaries.

(I) Letting $\Phi(\lambda)=\lambda$, then we have a new result for generalized proportional fractional integral.

Corollary 3.10. *For $\varsigma\in(0,1], \delta\in\mathcal{C}, \mathbb{R}(\delta)>0$ and there are three positive continuous functions $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ defined on $[0,\infty)$ and $\mathcal{U}\leq\mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$, then for any convex function Θ having $\Theta(0)=0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]}\geq\frac{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{V}(\xi))\mathcal{W}(\xi)]}. \qquad (3.29)$$

(II) Letting $\Phi(\lambda) = \lambda$ along with $\varsigma = 1$, then we have a new result for Riemann–Liouville fractional integral.

Corollary 3.11. *For $\varsigma \in (0,1], \delta \in \mathcal{C}, \mathbb{R}(\delta) > 0$ and there are three positive continuous functions $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ defined on $[0,\infty)$ and $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$ then for any convex function Θ having $\Theta(0) = 0$, we have the inequality*

$$\frac{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{U}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta}[\mathcal{V}(\xi)]} \geq \frac{\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)]}{\mathcal{T}_{0,\xi}^{\delta}[\Theta(\mathcal{V}(\xi))\mathcal{W}(\xi)]}. \tag{3.30}$$

Remark 3.12. If we choose $\Phi(\lambda) = \ln\lambda$, then Theorem 3.9 in this paper reduces to Theorem 3.3 in Ref. 42 and choosing $\Phi(\lambda) = \ln\lambda$ along with $\varsigma = 1$, then we get Theorem 3.3 in Ref. 18

Theorem 3.13. *For $\varsigma \in (0,1]$, $\delta, \eta \in \mathcal{C}$, $\mathbb{R}(\delta), \mathbb{R}(\eta) > 0$ and there are three positive continuous functions $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ defined on $[0,\infty)$ and $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$. If $\frac{\mathcal{U}}{\mathcal{V}}$ is decreasing, $\mathcal{U}$ is increasing on $[0,\infty)$ and Θ is a convex function having $\Theta(0) = 0$. Assume that, Φ is an increasing and positive monotone function on $[0,\infty)$, derivative Φ' is continuous on $[0,\infty)$ with $\Phi(0) = 0$, then the generalized proportional fractional integral operator with respect to another function Φ given by (2.7) satisfies the inequality*

$$\frac{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{U}(\xi)]\,{}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)] + {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{U}(\xi)]\,{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)]}{{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\mathcal{V}(\xi)]\,{}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)] + {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}[\mathcal{V}(\xi)]\,{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}[\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)]} \geq 1. \tag{3.31}$$

Proof. Multiplying both sides of (3.26) by $\frac{1}{\varsigma^{\eta}\Gamma(\eta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\rho))]\Phi'(\rho)}{(\Phi(\xi)-\Phi(\rho))^{1-\eta}}$, which is positive because $\rho \in (0,\xi)$, $\xi > 0$ and integrating with respect to ρ from 0 to ξ, we have

$$\begin{aligned} &{}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\mathcal{U}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right) \\ &\quad + {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right)\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{U}(\xi)) \\ &\geq {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\mathcal{V}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi)) \\ &\quad + {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta(\mathcal{U}(\xi))\mathcal{W}(\xi))\ {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\mathcal{V}(\xi)). \end{aligned} \tag{3.32}$$

Now, since $\mathcal{U} \leq \mathcal{V}$ on $[0,\infty)$ and $\frac{\Theta(\xi)}{\xi}$ is an increasing function, for $\lambda, \rho \in [0,\xi)$, we have

$$\frac{\Theta(\mathcal{U}(\lambda))}{\mathcal{U}(\lambda)} \leq \frac{\Theta(\mathcal{V}(\lambda))}{\mathcal{V}(\lambda)}. \tag{3.33}$$

Multiplying both sides of (3.33) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\delta}}\mathcal{V}(\lambda)\mathcal{W}(\lambda)$, which is positive because $\lambda \in (0,\xi)$, $\xi > 0$ and integrating the obtained inequality with respect to λ from 0 to ξ, we get

$$\begin{aligned} &{}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right) \\ &\quad \leq {}^{\Phi}\mathcal{T}_{0,\xi}^{\delta,\varsigma}(\Theta(\mathcal{V}(\xi))\mathcal{W}(\xi)). \end{aligned} \tag{3.34}$$

Similarly, multiplying both sides of (3.33) by $\frac{1}{\varsigma^{\eta}\Gamma(\eta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Phi(\xi)-\Phi(\lambda))]\Phi'(\lambda)}{(\Phi(\xi)-\Phi(\lambda))^{1-\eta}}\mathcal{U}(\lambda)\mathcal{W}(\lambda)$, which is positive because $\lambda \in (0,\xi)$, $\xi > 0$ and integrating with respect to λ from 0 to ξ, we get

$$\begin{aligned} &{}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}\left(\frac{\Theta(\mathcal{U}(\xi))}{\mathcal{U}(\xi)}\mathcal{V}(\xi)\mathcal{W}(\xi)\right) \\ &\quad \leq {}^{\Phi}\mathcal{T}_{0,\xi}^{\eta,\varsigma}(\Theta(\mathcal{V}(\xi))\mathcal{W}(\xi)). \end{aligned} \tag{3.35}$$

Hence from (3.32), (3.34) and (3.35), we get our desired result. □

Similar results can be concluded by using Remark 2.4.

4. CONCLUSION

The endurance of any area of research, pure and applied mathematics relies upon the capability of the specialists progressing in the direction of yet to be addressed inquiries and to update the existing hypothesis and practice. Several generalizations are predominantly because of the fact that analysts might want to explore a new scheme of study, they have to comprehend its tendency, dissect and anticipate it well. The prediction requires its utilities in the real-world. The main aim of this paper is to introduce new variants related to the newly proposed operator for generalized proportional fractional integral with respect to another function Φ. It is intriguing to specify here that, at whatever point the generalized proportional fractional integral with respect to another function Φ converted to other-concerning operators (by appropriately picking the estimations of proportionality index ς), the outcomes become generally increasingly significant from the application perspective. several special cases for changing parametric values for ς are presented. These new investigations will be displayed in future research work being handled by authors of this paper. We close this paper with the comment that the fractional integral inequalities determined in Sec. 3 can productively be utilized in building up the uniqueness of solutions in fractional boundary value problems.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their valuable suggestions and comments. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11971142, 61673169, 11701176, 11626101, 11601485, 11871202).

REFERENCES

1. I. Abbas Baloch and Y.-M. Chu, Petrović-type inequalities for harmonic h-convex functions, *J. Funct. Spaces* **2020** (2020), Article ID: 3075390, 7 pp.
2. M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, *J. Inequal. Appl.* **2018** (2018), Article ID: 70, 14 pp.
3. M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals versions of Hermite–Hadamard inequalities and their generalizations, *J. Funct. Spaces* **2018** (2018), Article ID: 6928130, 9 pp.
4. M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalization of Hermite–Hadamard type inequalities via conformable fractional integrals, *J. Funct. Spaces* **2018** (2018), Article ID: 5357463, 12 pp.
5. M. Adil Khan, A. Iqbal, M. Suleman and Y.-M. Chu, Hermite–Hadamard type inequalities for fractional integrals via Green's function, *J. Inequal. Appl.* **2018** (2018), Article ID: 161, 15 pp.
6. M. Adil Khan, N. Mohammad, E. R. Nwaeze and Y.-M. Chu, Quantum Hermite–Hadamard inequality by means of a Green function, *Adv. Differ. Equ.* **2020** (2020), Article ID: 99, 20 pp.
7. R. P. Agarwal and A. Özbekler, Lyapunov type inequalities for mixed nonlinear Riemann–Liouville fractional differential equations with a forcing term, *J. Comput. Appl. Math.* **314** (2017) 69–78.
8. M. Aldhaifallah, M. Tomar, K. S. Nisar and S. D. Purohit, Some new inequalities for (k, s)-fractional integrals, *J. Nonlinear Sci. Appl.* **9**(9) (2016) 5374–5381.
9. R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. *Nonlinear Sci. Numer. Simul.* **44** (2017) 460–481.
10. J. Alzabut, T. Abdeljawad, F. Jarad and W. Sudsutad, A Gronwall inequality via the generalized proportional fractional derivative with applications, *J. Inequal. Appl.* **2019** (2019), Article ID: 101, 12 pp.
11. A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Thermal Sci.* **20** (2016) 763–769.
12. D. Bainov and P. Simeonov, *Integral Inequalities and Applications* (Kluwer Academic Publishers Group, Dordrecht, 1992).
13. D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus* (World Scientific Publishing Co., Hackensack, 2017).
14. L. Bougoffa, On Minkowski and Hardy integral inequalities, *J. Inequal. Pure Appl. Math.* **7**(2) (2006), Article ID: 60, 3 pp.
15. J.-F. Cheng and Y.-M. Chu, Solution to the linear fractional differential equation using Adomian decomposition method, *Math. Probl. Eng.* **2011** (2011,) Article ID: 587068, 14 pp.
16. J.-F. Cheng and Y.-M. Chu, On the fractional difference equations of order $(2, q)$, *Abstr. Appl. Anal.* **2011** (2011), Article ID: 497259, 16 pp.
17. J.-F. Cheng and Y.-M. Chu, Fractional difference equations with real variable, *Abstr. Appl. Anal.* **2012** (2012), Article ID: 918529, 24 pp.
18. V. L. Chinchane and D. B. Pachpatte, On new fractional integral inequalities involving convex functions using Hadamard fractional integral, *Bull. Allahabad Math. Soc.* **31**(2) (2016) 183–192.
19. Y.-M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, *J. Inequal. Appl.* **2017** (2017), Article ID: 93, 12 pp.
20. Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, *Proc. Indian Acad. Sci. Math. Sci.* **122**(1) (2012) 41–51.
21. Z. Dahmani, New classes of integral inequalities of fractional order, *Matematiche* **69**(1) (2014) 237–247.
22. Z. Dahmani, New inequalities in fractional integrals, *Int. J. Nonlinear Sci.* **9**(4) (2010) 493–497.
23. H. L. Gray and N.-F. Zhang, On a new definition of the fractional difference, *Math. Comp.* **50**(182) (1988) 513–529.
24. T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, *J. Inequal. Appl.* **2018** (2018), Article ID: 239, 11 pp.

25. A. Iqbal, M. Adil Khan, S. Ullah and Y.-M. Chu, Some New Hermite–Hadamard-type inequalities associated with conformable fractional integrals and their applications, *J. Funct. Spaces* **2020** (2020), Article ID: 9845407, 18 pp.
26. F. Jarad, T. Abdeljawad and J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.* **226** (2017) 3457–3471.
27. F. Jarad, E. Uğurlu, T. Abdeljawad and D. Baleanu, On a new class of fractional operators, *Adv. Differ. Equ.* **2017** (2017), Article ID: 247, 16 pp.
28. E. Kacar, Z. Kacar and H. Yildirim, Integral inequalities for Riemann–Liouville fractional integrals of a function with respect to another function, *Iran. J. Math. Sci. Inform.* **13**(1) (2018) 1–13.
29. U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.* **218**(3) (2011) 860–865.
30. R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264** (2014) 65–70.
31. S. Khan, M. Adil Khan and Y.-M. Chu, Converses of the Jensen inequality derived from the Green functions with applications in information theory, *Math. Methods Appl. Sci.* **43**(5) (2020) 2577–2587.
32. Y. Khurshid, M. Adil Khan, Y.-M. Chu and Z. A. Khan, Hermite–Hadamard–Fejér inequalities for conformable fractional integrals via preinvex functions, *J. Funct. Spaces* **2019** (2019), Article ID: 3146210, 9 pp.
33. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier Science B.V., Amsterdam, 2006).
34. V. Kiryakova, *Generalized Fractional Calculus and Applications* (John Wiley & Sons, New York, 1994).
35. M. A. Latif, S. Rashid, S. S. Dragomir and Y.-M. Chu, Hermite–Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications, *J. Inequal. Appl.* **2019** (2019), Article ID: 317, 33 pp.
36. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley & Sons, New York, 1993).
37. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
38. W.-M. Qian, Z.-Y. He and Y.-M. Chu, Approximation for the complete elliptic integral of the first kind, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **114**(2) (2020), Article ID: 57, 12pp., https://doi.org/10.1007/s13398-020-00784-9.
39. W.-M. Qian, Z.-Y. He, H.-W. Zhang and Y.-M. Chu, Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean, *J. Inequal. Appl.* **2019** (2019), Article ID: 168, 13 pp.
40. W.-M. Qian, Y.-Y. Yang, H.-W. Zhang and Y.-M. Chu, Optimal two-parameter geometric and arithmetic mean bounds for the Sándor–Yang mean, *J. Inequal. Appl.* **2019** (2019), Article ID: 287, 12 pp.
41. S. Rafeeq, H. Kalsoom, S. Hussain, S. Rashid and Y.-M. Chu, Delay dynamic double integral inequalities on time scales with applications, *Adv. Differ. Equ.* **2020** (2020), Article ID: 40, 12 pp.
42. G. Rahman, T. Abdeljawad, F. Jarad, A. Khan and K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.* **2019** (2019), Article ID: 454, 10 pp.
43. G. Rahman, A. Khan, T. Abdeljawad and K. S. Nisar, The Minkowski inequalities via generalized proportional fractional integral operators, *Adv. Differ. Equ.* **2019** (2019), Article ID: 287, 14 pp.
44. S. Rashid, T. Abdeljawad, F. Jarad and M. A. Noor, Some estimates for generalized Riemann–Liouville fractional integrals of exponentially convex functions and their applications, *Mathematics* **7**(9) (2019), Article ID: 807, 18 pp., https://doi.org/10.3390/math7090807.
45. S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor and K. I. Noor, Simpson's type integral inequalities for k-fractional integrals and their applications, *AIMS Math.* **4**(4) (2019) 1087–1100.
46. S. Rashid, F. Jarad, H. Kalsoom and Y.-M. Chu, On Pólya-Szegö and Čebyšev type inequalities via generalized k-fractional integrals, *Adv. Differ. Equ.* **2020** (2020), Article ID: 125, 18 pp.
47. S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom and Y.-M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics* **7**(12) (2020), Article ID: 1225, 18 pp., https://doi.org/10.3390/math7121225.
48. S. Rashid, M. A. Noor, K. I. Noor and Y.-M. Chu, Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions, *AIMS Math.* **5**(3) (2020) 2629–2645.
49. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach Science Publishers, Yverdon, 1993).
50. E. Set, M. E. Özdemir and S. S. Dragomir, On the Hermite–Hadamard inequality and other integral inequalities involving two functions, *J. Inequal. Appl.* **2010** (2010), Article ID: 148102, 9 pp.
51. Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah and Y.-M. Chu, Integral inequalities involving strongly convex functions, *J. Funct. Spaces* **2018** (2018), Article ID: 6595921, 8 pp.
52. S. Taf and K. Brahim, Some new results using Hadamard fractional integral, *Int. J. Nonlinear Anal. Appl.* **7**(1) (2016) 103–109.

53. J. Tariboon, S. K. Ntouyas and W. Sudsutad, Some new Riemann–Liouville fractional integral inequalities, *Int. J. Math. Math. Sci.* **2014** (2014), Article ID: 869434, 6 pp.
54. M.-K. Wang, H.-H. Chu and Y.-M. Chu, Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals, *J. Math. Anal. Appl.* **480**(2) (2019), Article ID: 123388, 9 pp., https://doi.org/10.1016/j.jmaa.2019.123388.
55. M.-K. Wang, Z.-Y. He and Y.-M. Chu, Sharp power mean inequalities for the generalized elliptic integral of the first kind, *Comput. Methods Funct. Theory* **20**(1) (2020) 111–124.
56. M.-K. Wang, M.-Y. Hong, Y.-F. Xu, Z.-H. Shen and Y.-M. Chu, Inequalities for generalized trigonometric and hyperbolic functions with one parameter, *J. Math. Inequal.* **14**(1) (2020) 1–21.
57. B. Wang, C.-L. Luo, S.-H. Li and Y.-M. Chu, Sharp one-parameter geometric and quadratic means bounds for the Sándor–Yang means, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **114**(1) (2020), Article ID: 7, 10 pp., https://doi.org/10.1007/s13398-019-00734-0.
58. M.-K. Wang, W. Zhang and Y.-M. Chu, Monotonicity, convexity and inequalities involving the generalized elliptic integrals, *Acta Math. Sci.* **39B**(5) (2019) 1440–1450.
59. S.-H. Wu and Y.-M. Chu, Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters, *J. Inequal. Appl.* **2019** (2019), Article ID: 57, 11 pp.
60. Z.-H. Yang, W.-M. Qian, W. Zhang and Y.-M. Chu, Notes on the complete elliptic integral of the first kind, *Math. Inequal. Appl.* **23**(1) (2020) 77–93.
61. S. Zaheer Ullah, M. Adil Khan and Y.-M. Chu, A note on generalized convex functions, *J. Inequal. Appl.* **2019** (2019), Article ID: 291, 10 pp.
62. S. Zaheer Ullah, M. Adil Khan, Z. A. Khan and Y.-M. Chu, Integral majorization type inequalities for the functions in the sense of strong convexity, *J. Funct. Spaces* **2019** (2019), Article ID: 9487823, 11 pp.
63. T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, *Abstr. Appl. Anal.* **2011** (2011), Article ID: 896483, 13 pp.
64. T.-H. Zhao, L. Shi, and Y.-M. Chu, Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **114**(2) (2020) Article ID: 96, 14pp., https://doi.org/10.1007/s13398-020-00825-3.