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New estimates considering the generalized proportional Hadamard fractional integral operators

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Abstract

In the article, we describe the Grüss type inequality, provide some related inequalities by use of suitable fractional integral operators, address several variants by utilizing the generalized proportional Hadamard fractional (GPHF) integral operator. It is pointed out that our introduced new integral operators with nonlocal kernel have diversified applications. Our obtained results show the computed outcomes for an exceptional choice to the GPHF integral operator with parameter and the proportionality index. Additionally, we illustrate two examples that can numerically approximate these operators.

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1 Introduction

A revolution inside the discipline of differentiation and integration has been witnessed: classical differentiation has become extended by the use of nonlocal operators. The classical derivative was combined with a strength regulation sort of kernel and ultimately this provided the upward thrust to new calculus referred to as the fractional calculus. The newly proposed calculus permits one to depict progressively complex problems with various properties, for example, in thermal conduction where the heat is streaming inside a medium with two distinct properties and a new mathematical model of heat conduction, one considered isotropic generalized thermoelasticity, with a three-phase lag, this model being considered in terms of the methodology of fractional calculus. Several significant results have been obtained [1–10].

Fractional calculus in continuous and discrete operators has also been comprehensively utilized in numerous fields [11–30]. But the concept has been propagated and implemented in applied mathematics, physics and porous media as a mathematical model. Various recognized generalized fractional operators comprise the Hadamard operator, the Erdélyi–Kober operator, the Saigo operator, the Gaussian hypergeometric operator, the Marichev–Saigo–Maeda fractional integral operator and so on. Out of these operators,

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the Riemann–Liouville fractional integral operator has been broadly applied by scientists in research just from an application viewpoint. For more details concerning fractional calculus operators, one may also refer to the expositions by Miller and Ross [31], Mathai [32], Kiryakova [33] and Baleanu et al. [34]. A captivating feature of this study is that there are numerous fractional operators that have fertile utilities in numerous areas of pure and applied mathematics. There are ideas with various qualities, and this grants the users an opportunity to select the suitable operator for exhibiting the issue under consideration. Moreover, as a consequence of its smoothness in the real world, experts have provided much contemplation to currently resolve fractional operators without singular kernels [35–40], and presently, a while later, various articles dealing with these sorts of fractional operators have been published.

Recently, the generalized proportional fractional integral operator contemplated by Jarad et al. [41] has potential application in statistical theory and also there have been enthralling presentations in the theory of fractional Schrödinger equations [42, 43]. These kinds of speculations elevate future studies to establish novel ideas to modify the fractional operators and help us to attain integral inequalities via such generalized fractional operators. Integral inequalities and their utilities perform- a crucial job in the theory of differential and difference equations. An assortment of distinct kinds of classical variants and their modifications have been built up by employing the classical fractional integral, derivative operators and their speculative ideas in the matter [44–46]. Recently, the Gronwall and the Minkowski inequalities concerning to the generalized proportional fractional derivative and fractional integral were explored by Alzabut et al. [3] and Rahman et al. [43]. Approving this predilection, we present an adjusted form for many recognized Grüss type inequalities [47] in the sense of the generalized proportional Hadamard fractional integral that could be more proficient and much more pertinent than the current ones. The well-known Grüss inequality can be stated as follows.

Let $\mathcal{F}, \mathcal{G} : [\nu_1, \nu_2] \to (0, \infty)$ be two positive real-valued functions such that $m \leq \mathcal{F}(l) \leq \mathcal{M}$ and $n \leq \mathcal{G}(l) \leq \mathcal{N}$ for all $l \in [\nu_1, \nu_2]$. Then the inequality

$$\begin{aligned} &\left| \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \mathcal{F}(l)\mathcal{G}(l)\,dl - \frac{1}{(\nu_2 - \nu_1)^2} \int_{\nu_1}^{\nu_2} \mathcal{F}(l)\,dl \int_{\nu_1}^{\nu_2} \mathcal{G}(l)\,dl \right| \\ &\quad \leq \frac{1}{4}(\mathcal{M} - m)(\mathcal{N} - n) \end{aligned} \tag{1.1}$$

holds with the best possible constant 1/4.

Inequality (1.1) is a marvelous instrument for exploring various systematic areas of science comprising chaos, porous media, biotechnology, heat transfer, time scale analysis and so on. There has been a continuous growth of eagerness for such a field of research to address the problems of various usages of these generalizations. Such discoveries had been analyzed by various investigators who in this way used assorted techniques for exploring and proposing these variations [48–53].

In this paper a new concept of integration that takes into account the fractional operator and also several generalizations will be introduced. Another important problem that could be handled by the new operators is the Grüss type and several other generalizations. We present, in general, three numerical schemes (Young, weighted arithmetic and geometric mean) that can be used to find solutions via the generalized proportional Hadamard fractional integral. Interestingly, several existing results recaptured by the results presented are Hadamard fractional integral inequalities. Therefore, the concept is rather novel and appears to make it possible to explore new directions of research as regards distinct scientific areas in pure and applied mathematics. We observe that the GPHF integral is able to show some kind of self-similarities.

2 Prelude

Now, we demonstrate concisely some essential preliminaries on fractional calculus for the convenience of the reader. The basic information is given in the monograph [37].

The left and right sides generalized proportional integral operators were introduced by Jarad et al. [41], they are defined by

$$\big(\mathfrak{J}^{\gamma,\rho}_{\nu_1,l}\mathcal{F}\big)(l)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_{\nu_1}^{l}\frac{\exp[\frac{\rho-1}{\rho}(l-\vartheta)]}{(l-\vartheta)^{1-\gamma}}\mathcal{F}(\vartheta)\,d\vartheta\quad(\nu_1<l)\tag{2.1}$$

and

$$\big(\mathfrak{J}^{\gamma,\rho}_{\nu_2,l}\mathcal{F}\big)(l)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_{l}^{\nu_2}\frac{\exp[\frac{\rho-1}{\rho}(\vartheta-l)]}{(\vartheta-l)^{1-\gamma}}\mathcal{F}(\vartheta)\,d\vartheta\quad(l<\nu_2),\tag{2.2}$$

where the proportionality index $\rho\in(0,1]$, $\gamma\in\mathbf{C}$ with $\Re(\gamma)>0$, and $\Gamma(l)=\int_0^\infty\vartheta^{l-1}e^{-\vartheta}\,d\vartheta$ is the Euler gamma function [54–56].

Remark 2.1 Letting $\rho=1$. Then (2.1) and (2.2) reduce to the following left and right side Riemann–Liouville fractional integral operators:

$$\big(\mathfrak{J}^{\gamma}_{\nu_1,l}\mathcal{F}\big)(l)=\frac{1}{\Gamma(\gamma)}\int_{\nu_1}^{l}\frac{\mathcal{F}(\vartheta)}{(l-\vartheta)^{1-\gamma}}\,d\vartheta\quad(\nu_1<l)\tag{2.3}$$

and

$$\big(\mathfrak{J}^{\gamma}_{\nu_2,l}\mathcal{F}\big)(l)=\frac{1}{\Gamma(\gamma)}\int_{l}^{\nu_2}\frac{\mathcal{F}(\vartheta)}{(\vartheta-l)^{1-\gamma}}\,d\vartheta\quad(l<\nu_2).\tag{2.4}$$

Next, we recall the concept of GPHF integral operator, which was introduced by Rahman et al. in [49]

Definition 2.2 ([49]) Let $\gamma>0$ and the proportionality index $\rho\in(0,1]$. Then the left and right side GPHF integrals are defined by

$$\big(\mathfrak{J}^{\gamma,\rho}_{\nu_1,l}\mathcal{F}\big)(l)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_{\nu_1}^{l}\frac{\exp[\frac{\rho-1}{\rho}(\ln\frac{l}{\vartheta})]}{(\ln\frac{l}{\vartheta})^{1-\gamma}}\frac{\mathcal{F}(\vartheta)}{\vartheta}\,d\vartheta\quad(\nu_1<l)\tag{2.5}$$

and

$$\big(\mathfrak{J}^{\gamma,\rho}_{\nu_2,l}\mathcal{F}\big)(l)=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_{l}^{\nu_2}\frac{\exp[\frac{\rho-1}{\rho}(\ln\frac{\vartheta}{l})]}{(\ln\frac{\vartheta}{l})^{1-\gamma}}\frac{\mathcal{F}(\vartheta)}{\vartheta}\,d\vartheta\quad(l<\nu_2).\tag{2.6}$$

Definition 2.3 Let $\gamma > 0$ and the proportionality index $\rho \in (0,1]$. Then the one-sided GPHF integral is defined as

$$\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\vartheta}))]}{(\ln\frac{l}{\vartheta})^{1-\gamma}} \frac{\mathcal{F}(\vartheta)}{\vartheta}\, d\vartheta \quad (\vartheta > 1). \tag{2.7}$$

Remark 2.4 Letting $\rho = 1$. Then (2.5)–(2.7) become the Hadamard fractional integrals

$$\left(\mathfrak{J}_{\nu_1,l}^{\gamma,\rho}\mathcal{F}\right)(l) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_{\nu_1}^l \frac{\mathcal{F}(\vartheta)}{\vartheta(\ln l - \ln\vartheta)^{1-\gamma}}\, d\vartheta \quad (\nu_1 < l), \tag{2.8}$$

$$\left(\mathfrak{J}_{\nu_2,l}^{\gamma,\rho}\mathcal{F}\right)(l) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_l^{\nu_2} \frac{\mathcal{F}(\vartheta)}{\vartheta(\ln l - \ln\vartheta)^{1-\gamma}}\, d\vartheta \quad (l < \nu_2), \tag{2.9}$$

and

$$\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l) = \frac{1}{\rho^\gamma \Gamma(\gamma)} \int_1^l \frac{\mathcal{F}(\vartheta)}{\vartheta(\ln\vartheta - \ln l)^{1-\gamma}}\, d\vartheta \quad (\vartheta > 1). \tag{2.10}$$

For convenience, we give the semigroup property

$$\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l) = \left(\mathfrak{J}_{1^-,l}^{\gamma+\varrho,\rho}\mathcal{F}\right)(l), \tag{2.11}$$

which implies the commutative property,

$$\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l) = \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l). \tag{2.12}$$

Remark 2.5 If we choose $\rho = 1$, then (2.11) becomes the result of [32].

3 Main results

In the section, we will provide the refinements for some classical variants by utilizing the GPHF integral operator defined in (2.7).

Theorem 3.1 *Let $\rho \in (0,1]$, $\gamma > 0$, and $\mathcal{F}$ be a positive real-valued function defined on $[1,\infty)$. Assume that*

(I) There exist two integrable functions φ_1 and φ_2 defined on $[1,\infty)$ such that

$$\varphi_1(l) \le \mathcal{F}(l) \le \varphi_2(l) \tag{3.1}$$

for all $l \in [1,\infty)$.

Then one has

$$\begin{aligned} &\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\varphi_1\right)(l) \\ &\quad \ge \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\varphi_1\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l), \end{aligned} \tag{3.2}$$

for all $l > 1$, $\gamma > 0$ and $\varrho \in (0,1]$.

Proof Let $\theta \geq 1$ and $\varsigma \geq 1$. Then from (I) we have

$$\big(\varphi_2(\theta) - \mathcal{F}(\theta)\big)\big(\mathcal{F}(\varsigma) - \varphi_1(\varsigma)\big) \geq 0. \tag{3.3}$$

Therefore,

$$\varphi_2(\theta)\mathcal{F}(\varsigma) + \varphi_1(\varsigma)\mathcal{F}(\theta) \geq \varphi_1(\varsigma)\varphi_2(\theta) + \mathcal{F}(\theta)\mathcal{F}(\varsigma). \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{1}{\rho^\gamma \Gamma(\gamma)} \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}$ and integrating the obtained inequality from 1 to l, we get

$$\begin{aligned}
&\mathcal{F}(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\varphi_2(\theta)\,d\theta \\
&\quad + \varphi_1(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\,d\theta \\
&\quad \geq \varphi_1(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\varphi_2(\theta)\,d\theta \\
&\quad + \mathcal{F}(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\,d\theta,
\end{aligned} \tag{3.5}$$

that is,

$$\mathcal{F}(\varsigma)\big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l) + \varphi_1(\varsigma)\big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l) \geq \varphi_1(\varsigma)\big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l) + \mathcal{F}(\varsigma)\big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l). \tag{3.6}$$

Multiplying both sides of (3.6) by $\frac{1}{\rho^\varrho \Gamma(\varrho)} \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}$ and integrating the obtained results from 1 to l, we have

$$\begin{aligned}
&\big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\frac{1}{\rho^\varrho \Gamma(\varrho)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{F}(\varsigma)\,d\varsigma \\
&\quad + \big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\frac{1}{\rho^\varrho \Gamma(\varrho)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\varphi_1(\varsigma)\,d\varsigma \\
&\quad \geq \big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\frac{1}{\rho^\varrho \Gamma(\varrho)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\varphi_1(\varsigma)\,d\varsigma \\
&\quad + \big(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\frac{1}{\rho^\varrho \Gamma(\varrho)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{F}(\varsigma)\,d\varsigma.
\end{aligned} \tag{3.7}$$

From (3.7) we immediately get the desired inequality (3.2). □

Corollary 3.2 is a special case of Theorem 3.1.

Corollary 3.2 *Letting $\rho = 1$. Then Theorem 3.1 leads to the Hadamard fractional integrals inequality*

$$\big(\mathfrak{J}_{1^-,l}^{\gamma}\varphi_2\big)(l)\big(\mathfrak{J}_{1^-,l}^{\varrho}\mathcal{F}\big)(l) + \big(\mathfrak{J}_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\mathfrak{J}_{1^-,l}^{\varrho}\varphi_1\big)(l)$$

$$\geq \left(\mathfrak{J}_{1^-,l}^{\gamma}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho}\varphi_1\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho}\mathcal{F}\right)(l),$$

which was proved by Sudsutad et al. in [48].

Theorem 3.3 *Let $\rho \in (0,1]$, $\gamma, \varrho > 0$, and $\mathcal{F}$ and $\mathcal{G}$ be two positive real-valued functions defined on $[1,\infty)$ such that condition (I) given in Theorem 3.1 and condition (II) hold.*
(II) There exist two integrable functions ω_1 and ω_2 defined on $[1,\infty)$ such that

$$\omega_1(l) \leq \mathcal{G}(l) \leq \omega_2(l) \tag{3.8}$$

for all $l \in [1,\infty)$.
Then, for all $x, \gamma, \varrho > 0$, we have the following inequalities:

$$\begin{aligned}
(N_1)\quad & \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_1\right)(l)\\
&\quad \geq \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_1\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l),\\
(N_2)\quad & \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\varphi_1\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{G}\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l)\\
&\quad \geq \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\varphi_1\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\omega_2\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{G}\right)(l),\\
(N_3)\quad & \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l)\\
&\quad \geq \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_2\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_2\right)(l),\\
(N_4)\quad & \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_1\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_1\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l)\\
&\quad \geq \left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\varphi_1\right)(l)\left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\mathcal{G}\right)(l) + \left(\mathfrak{J}_{1^-,l}^{\varrho,\rho}\omega_1\right)(l)\left(\mathfrak{J}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l).
\end{aligned} \tag{3.9}$$

Proof We first prove (N_1). Let $l \in [1,\infty)$. Then it follows from (I) and (II) that

$$\left(\varphi_2(\theta) - \mathcal{F}(\theta)\right)\left(\mathcal{G}(\varsigma) - \omega_1(\varsigma)\right) \geq 0, \tag{3.10}$$

that is,

$$\varphi_2(\theta)\mathcal{G}(\varsigma) + \omega_1(\varsigma)\mathcal{F}(\theta) \geq \omega_1(\varsigma)\varphi_2(\theta) + \mathcal{G}(\varsigma)\mathcal{F}(\theta). \tag{3.11}$$

Multiplying both sides of (3.11) by $\frac{1}{\rho^\gamma \Gamma(\gamma)} \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}$ and integrating the obtained inequality from 1 to l lead to

$$\begin{aligned}
&\mathcal{G}(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\varphi_2(\theta)\,d\theta\\
&\quad + \omega_1(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\,d\theta\\
&\geq \omega_1(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\varphi_2(\theta)\,d\theta\\
&\quad + \mathcal{G}(\varsigma)\frac{1}{\rho^\gamma \Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\,d\theta.
\end{aligned} \tag{3.12}$$

Inequality (3.12) can be rewritten as

$$\mathcal{G}(\varsigma)\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)+\omega_1(\varsigma)\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\geq\omega_1(\varsigma)\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)+\mathcal{G}(\varsigma)\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l). \tag{3.13}$$

Multiplying both sides of (3.13) by $\frac{1}{\rho^\varrho\Gamma(\varrho)}\frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}$ and integrating the obtained inequality from 1 to l we get

$$\begin{aligned}
&\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\frac{1}{\rho^\varrho\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{G}(\varsigma)\,d\varsigma\\
&\quad+\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\frac{1}{\rho^\varrho\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\omega_1(\varsigma)\,d\varsigma\\
&\quad\geq\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\frac{1}{\rho^\varrho\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\omega_1(\varsigma)\,d\varsigma\\
&\quad+\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\frac{1}{\rho^\varrho\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{G}(\varsigma)\,d\varsigma,
\end{aligned}$$

which leads to the desired inequality (N_1)

$$\begin{aligned}
&\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\big(\Im_{1^-,l}^{\varrho,\rho}\mathcal{G}\big)(l)+\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho,\rho}\omega_1\big)(l)\\
&\quad\geq\big(\Im_{1^-,l}^{\gamma,\rho}\varphi_2\big)(l)\big(\Im_{1^-,l}^{\varrho,\rho}\omega_1\big)(l)+\big(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho,\rho}\mathcal{G}\big)(l).
\end{aligned}$$

Inequalities (N_2)–(N_4) can be proved by using the similar arguments as in the proof of inequality (N_1) and the facts that

$$\begin{aligned}
&\big(\omega_2(\theta)-\mathcal{G}(\theta)\big)\big(\mathcal{F}(\varsigma)-\varphi_1(\varsigma)\big)\geq 0,\\
&\big(\varphi_2(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{G}(\varsigma)-\omega_2(\varsigma)\big)\leq 0,\\
&\big(\varphi_1(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{G}(\varsigma)-\omega_1(\varsigma)\big)\leq 0.
\end{aligned}$$

□

As a special case of Theorem 3.3, we have Corollary 3.4.

Corollary 3.4 *If $\rho=1$, then Theorem 3.3 leads to the Hadamard fractional integrals inequalities* [48]

$$\begin{aligned}
(N_5)\quad&\big(\Im_{1^-,l}^{\gamma}\varphi_2\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l)+\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho}\omega_1\big)(l)\\
&\quad\geq\big(\Im_{1^-,l}^{\gamma}\varphi_2\big)(l)\big(\Im_{1^-,l}^{\varrho}\omega_1\big)(l)+\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l),\\
(N_6)\quad&\big(\Im_{1^-,l}^{\varrho}\varphi_1\big)(l)\big(\Im_{1^-,l}^{\gamma}\mathcal{G}\big)(l)+\big(\Im_{1^-,l}^{\varrho}\omega_2\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{F}\big)(l)\\
&\quad\geq\big(\Im_{1^-,l}^{\varrho}\varphi_1\big)(l)\big(\Im_{1^-,l}^{\gamma}\omega_2\big)(l)+\big(\Im_{1^-,l}^{\varrho}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\gamma}\mathcal{G}\big)(l),\\
(N_7)\quad&\big(\Im_{1^-,l}^{\varrho}\omega_2\big)(l)\big(\Im_{1^-,l}^{\gamma}\varphi_2\big)(l)+\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l)\\
&\quad\geq\big(\Im_{1^-,l}^{\gamma}\varphi_2\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l)+\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho}\omega_2\big)(l),\\
(N_8)\quad&\big(\Im_{1^-,l}^{\gamma}\varphi_1\big)(l)\big(\Im_{1^-,l}^{\varrho}\omega_1\big)(l)+\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l)\\
&\quad\geq\big(\Im_{1^-,l}^{\gamma}\varphi_1\big)(l)\big(\Im_{1^-,l}^{\varrho}\mathcal{G}\big)(l)+\big(\Im_{1^-,l}^{\varrho}\omega_1\big)(l)\big(\Im_{1^-,l}^{\gamma}\mathcal{F}\big)(l).
\end{aligned}$$

Theorem 3.5 *Let $\rho \in (0,1]$, $\gamma, \varrho > 0$, $p, q > 0$ with $1/p + 1/q = 1$, and $\mathcal{F}$ and $\mathcal{G}$ be two positive real-valued functions defined on $[1,\infty)$. Then the following inequalities hold for $l > 1$:*

$$\begin{aligned}
(N_9)\quad & \frac{1}{p}\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^p\big)(l) + \frac{1}{q}\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^q\big)(l) \\
& \quad \geq \big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\mathcal{G}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}\mathcal{F}\big)(l), \\
(N_{10})\quad & \frac{1}{p}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\big)(l) + \frac{1}{q}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^p\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^q\big)(l) \\
& \quad \geq \big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^{q-1}\mathcal{F}^{p-1}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\mathcal{G}\big)(l), \\
(N_{11})\quad & \frac{1}{p}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^2\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\big)(l) + \frac{1}{q}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^2\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^q\big)(l) \\
& \quad \geq \big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^{\frac{2}{q}}\mathcal{G}^{\frac{2}{p}}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\mathcal{G}\big)(l), \\
(N_{12})\quad & \frac{1}{p}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\big)(l) + \frac{1}{q}\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^p\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^2\big)(l) \\
& \quad \geq \big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^{p-1}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^{\frac{2}{p}}\mathcal{G}^{\frac{2}{q}}\big)(l).
\end{aligned} \tag{3.14}$$

Proof It follows from the Young inequality [38] that

$$\frac{1}{p}\mu^p + \frac{1}{q}\nu^q \geq \mu\nu \tag{3.15}$$

for all $\mu, \nu \geq 0$.

Let $\theta, \varsigma > 1$, $\mu = \mathcal{F}(\theta)\mathcal{G}(\varsigma)$ and $\nu = \mathcal{F}(\varsigma)\mathcal{G}(\theta)$. Then inequality (3.15) becomes

$$\frac{1}{p}\big(\mathcal{F}(\theta)\mathcal{G}(\varsigma)\big)^p + \frac{1}{q}\big(\mathcal{F}(\varsigma)\mathcal{G}(\theta)\big)^q \geq \big(\mathcal{F}(\theta)\mathcal{G}(\varsigma)\big)\big(\mathcal{F}(\varsigma)\mathcal{G}(\theta)\big). \tag{3.16}$$

Multiplying both sides of inequality (3.16) by $\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta\rho^\gamma\Gamma(\gamma)}$ and integrating the obtained inequality from 1 to l give

$$\begin{aligned}
& \frac{\mathcal{G}^p(\varsigma)}{p\rho^\gamma\Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}^p(\theta)\,d\theta \\
& \quad + \frac{\mathcal{F}^q(\varsigma)}{q\rho^\gamma\Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{G}^q(\theta)\,d\theta \\
& \geq \frac{\mathcal{G}(\varsigma)\mathcal{F}(\varsigma)}{\rho^\gamma\Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\mathcal{G}(\theta)\,d\theta.
\end{aligned} \tag{3.17}$$

Inequality (3.17) can be rewritten as

$$\frac{\mathcal{G}^p(\varsigma)}{p}\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\big)(l) + \frac{\mathcal{F}^q(\varsigma)}{q}\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^q\big)(l) \geq \mathcal{G}(\varsigma)\mathcal{F}(\varsigma)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\mathcal{G}\big)(l). \tag{3.18}$$

Multiplying both sides of inequality (3.18) by $\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma\rho^{\varrho}\Gamma(\varrho)}$ and integrating the obtained result from 1 to l, one has

$$\begin{aligned}&\frac{1}{p}\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^p\big)(l)\frac{1}{\rho^{\varrho}\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{G}^p(\varsigma)\,d\varsigma\\&\quad+\frac{1}{q}\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}^q\big)(l)\frac{1}{\rho^{\varrho}\Gamma(\varrho)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{F}^q(\varsigma)\,d\varsigma\\&\quad\ge\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\varsigma}\mathcal{G}(\varsigma)\mathcal{F}(\varsigma)\,d\varsigma.\end{aligned}\tag{3.19}$$

Inequality (3.19) leads to the conclusion that

$$\begin{aligned}&\frac{1}{p}\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^p\big)(l)\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{G}^p\big)(l)+\frac{1}{q}\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{F}^q\big)(l)\\&\quad\ge\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{G}\mathcal{F}\big)(l),\end{aligned}\tag{3.20}$$

which completes the proof of inequality (N_9).

Let

$$\begin{aligned}&\mu=\frac{\mathcal{F}(\theta)}{\mathcal{F}(\varsigma)},\qquad\nu=\frac{\mathcal{G}(\theta)}{\mathcal{G}(\varsigma)}\big(\mathcal{F}(\varsigma),\quad\mathcal{G}(\varsigma)\neq0\big),\\&\mu=\mathcal{F}(\theta)\mathcal{G}^{\frac{2}{p}}(\varsigma),\qquad\nu=\mathcal{F}^{\frac{2}{q}}(\varsigma)\mathcal{G}(\theta),\end{aligned}$$

and

$$\mu=\mathcal{F}^{\frac{2}{p}}(\theta)\mathcal{F}(\varsigma),\qquad\nu=\mathcal{G}^{\frac{2}{q}}(\theta)\mathcal{G}(\varsigma)\quad\big(\mathcal{F}(\varsigma),\mathcal{G}(\varsigma)\neq0\big),$$

in the Young inequality, respectively. Then inequalities (N_{10})–(N_{12}) can be proved by use of similar arguments to the proof of inequality (N_9). □

Corollary 3.6 *Let* $\rho=1$. *Then Theorem* 3.5 *leads to the Hadamard fractional integrals inequalities*

$$\begin{aligned}(N_{13})\quad&\frac{1}{p}\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^p\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^p\big)(l)+\frac{1}{q}\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^q\big)(l)\\&\quad\ge\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}\mathcal{F}\big)(l),\\(N_{14})\quad&\frac{1}{p}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^p\big)(l)+\frac{1}{q}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^p\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^q\big)(l)\\&\quad\ge\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^{q-1}\mathcal{F}^{p-1}\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l),\\(N_{15})\quad&\frac{1}{p}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^2\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^p\big)(l)+\frac{1}{q}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^2\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^q\big)(l)\\&\quad\ge\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^{\frac{2}{q}}\mathcal{G}^{\frac{2}{p}}\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l),\\(N_{16})\quad&\frac{1}{p}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^2\big)(l)+\frac{1}{q}\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^p\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^2\big)(l)\\&\quad\ge\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^{p-1}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^{\frac{2}{p}}\mathcal{G}^{\frac{2}{q}}\big)(l).\end{aligned}$$

Theorem 3.7 *Let $\rho \in (0,1]$, $\gamma, \varrho > 0$, $p,q > 1$ with $1/p + 1/q = 1$, and $\mathcal{F}$ and $\mathcal{G}$ be two integrable functions defined on $[1,\infty)$. Then the inequalities*

$$\begin{aligned}
(N_{17})\quad & p\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}\big)(l) + q\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}\big)(l) \\
&\quad \geq \big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^q\mathcal{G}^p\big)(l), \\
(N_{18})\quad & p\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^{p-1}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}\mathcal{G}^q\big)(l) + q\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^q\mathcal{G}\big)(l) \\
&\quad \geq \big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^p\big)(l), \\
(N_{19})\quad & p\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^{\frac{2}{p}}\big)(l) + q\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^{\frac{2}{q}}\big)(l) \\
&\quad \geq \big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^p\mathcal{G}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^q\mathcal{F}^2\big)(l), \\
(N_{20})\quad & p\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^{\frac{2}{p}}\mathcal{G}^q\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^{p-1}\big)(l) + q\big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{F}^{\frac{2}{q}}\mathcal{G}^p\big)(l) \\
&\quad \geq \big(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\big)(l)\big(\mathfrak{I}_{1^-,l}^{\varrho,\rho}\mathcal{G}^2\big)(l),
\end{aligned} \tag{3.21}$$

hold for $l > 1$.

Proof It follows from the well-known weighted arithmetic–geometric mean inequality that

$$p\mu + q\nu \geq \mu^p\nu^q \tag{3.22}$$

for $\mu, \nu \geq 0$.

Let $\varsigma, \theta > 1$, $\mu = \mathcal{F}(\theta)\mathcal{G}(\varsigma)$ and $\nu = \mathcal{F}(\varsigma)\mathcal{G}(\theta)$. Then inequality (3.22) leads to

$$p\mathcal{F}(\theta)\mathcal{G}(\varsigma) + q\mathcal{F}(\varsigma)\mathcal{G}(\theta) \geq \big(\mathcal{F}(\theta)\mathcal{G}(\varsigma)\big)^p\big(\mathcal{F}(\varsigma)\mathcal{G}(\theta)\big)^q. \tag{3.23}$$

Multiplying both sides of inequality (3.23) by

$$\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\theta\rho^\gamma\Gamma(\gamma)\varsigma\rho^\varrho\Gamma(\varrho)}$$

and integrating the obtained inequality from 1 to l, we have

$$\begin{aligned}
&\frac{p}{\rho^\gamma\Gamma(\gamma)\rho^\varrho\Gamma(\varrho)} \\
&\quad\times\int_1^l\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\theta\varsigma}\mathcal{F}(\theta)\mathcal{G}(\varsigma)\,d\theta\,d\varsigma \\
&\quad+\frac{q}{\rho^\gamma\Gamma(\gamma)\rho^\varrho\Gamma(\varrho)} \\
&\quad\times\int_1^l\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\theta\varsigma}\mathcal{F}(\varsigma)\mathcal{G}(\theta)\,d\theta\,d\varsigma \\
&\geq \frac{1}{\rho^\gamma\Gamma(\gamma)\rho^\varrho\Gamma(\varrho)}\int_1^l\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\varsigma})](\ln(\frac{l}{\varsigma}))^{\varrho-1}}{\theta\varsigma} \\
&\quad\times\big(\mathcal{F}(\theta)\mathcal{G}(\varsigma)\big)^p\big(\mathcal{F}(\varsigma)\mathcal{G}(\theta)\big)^q\,d\varsigma d\theta.
\end{aligned} \tag{3.24}$$

Inequality (3.24) can be rewritten as

$$p\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\big)(l)\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{G}\big)(l)+q\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{F}\big)(l)\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}\big)(l)$$
$$\geq \big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^p\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho,\rho}_{1^-,l}\mathcal{F}^q\mathcal{G}^p\big)(l), \tag{3.25}$$

which completes the proof of inequality (N_{17}).

Let

$$\mu=\frac{\mathcal{F}(\varsigma)}{\mathcal{F}(\theta)},\qquad \nu=\frac{\mathcal{G}(\theta)}{\mathcal{G}(\varsigma)}\quad\big(\mathcal{F}(\theta),\mathcal{G}(\varsigma)\neq 0\big),$$
$$\mu=\mathcal{F}(\theta)\mathcal{G}^{\frac{2}{p}}(\varsigma),\qquad \nu=\mathcal{F}^{\frac{2}{q}}(\varsigma)\mathcal{G}(\theta),$$
$$\mu=\frac{\mathcal{F}^{\frac{2}{p}}(\theta)}{\mathcal{G}(\varsigma)},\qquad \nu=\frac{\mathcal{F}^{\frac{2}{q}}(\varsigma)}{\mathcal{G}(\theta)}\quad\big(\mathcal{G}(\theta),\mathcal{G}(\theta)\neq 0\big),$$

in the arithmetic–geometric mean inequality, respectively. Then inequalities (N_{18})–(N_{20}) can be proved by using the similar arguments as in the proof of inequality (N_{17}). □

Corollary 3.8 *Let* $\rho=1$. *Then Theorem* 3.7 *leads to the Hadamard fractional integrals inequalities*

(N_{21}) $$p\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}\big)(l)+q\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}\big)(l) \geq \big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^p\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^q\mathcal{G}^p\big)(l),$$

(N_{22}) $$p\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^{p-1}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}\mathcal{G}^q\big)(l)+q\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^q\mathcal{G}\big)(l) \geq \big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^p\big)(l),$$

(N_{23}) $$p\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^{\frac{2}{p}}\big)(l)+q\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^{\frac{2}{q}}\big)(l) \geq \big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^p\mathcal{G}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^q\mathcal{F}^2\big)(l),$$

(N_{24}) $$p\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^{\frac{2}{p}}\mathcal{G}^q\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^{p-1}\big)(l)+q\big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{G}^{q-1}\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{F}^{\frac{2}{q}}\mathcal{G}^p\big)(l) \geq \big(\mathfrak{I}^{\gamma}_{1^-,l}\mathcal{F}^2\big)(l)\big(\mathfrak{I}^{\varrho}_{1^-,l}\mathcal{G}^2\big)(l).$$

Example 3.9 Let $l>1$, $\gamma,\varrho>0$, $\mathcal{F}$ and $\mathcal{G}$ be two integrable functions defined on $[1,\infty)$, and

$$\hbar=\min_{0\leq\theta\leq l}\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)},\qquad \mathcal{H}=\max_{0\leq\theta\leq l}\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}. \tag{3.26}$$

Then we have the following three inequalities:

(1) $$0\leq\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^2\big)(l)\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}^2\big)(l)\leq\frac{\hbar+\mathcal{H}}{4\hbar\mathcal{H}}\big(\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\big)^2,$$

(2) $$0\leq\sqrt{\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^2\big)(l)\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}^2\big)(l)}-\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\leq\frac{\sqrt{\mathcal{H}}-\sqrt{\hbar}}{2\sqrt{\hbar\mathcal{H}}}\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l),$$

(3) $$0\leq\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}^2\big)(l)\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{G}^2\big)(l)-\big(\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\big)^2\leq\frac{\mathcal{H}-\hbar}{4\hbar\mathcal{H}}\big(\big(\mathfrak{I}^{\gamma,\rho}_{1^-,l}\mathcal{F}\mathcal{G}\big)(l)\big)^2.$$

Proof It follows from (3.26) that

$$\left(\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}-\hbar\right)\left(\mathcal{H}-\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}\right)\mathcal{G}^2(\theta)\geq 0. \tag{3.27}$$

Inequality (3.27) can be rewritten as

$$\mathcal{F}^2(\theta)+\hbar\mathcal{H}\mathcal{G}^2(\theta)\leq(\hbar+\mathcal{H})\mathcal{F}(\theta)\mathcal{G}(\theta). \tag{3.28}$$

Multiplying both sides of inequality (3.28) by

$$\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta\rho^{\gamma}\Gamma(\gamma)}$$

and integrating the obtained inequality from 1 to l lead to

$$\begin{aligned}
&\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}^2(\theta)\,d\theta\\
&\quad+\hbar\mathcal{H}\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{G}^2(\theta)\,d\theta\\
&\quad\leq(\hbar+\mathcal{H})\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\int_1^l\frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\theta})](\ln(\frac{l}{\theta}))^{\gamma-1}}{\theta}\mathcal{F}(\theta)\mathcal{G}(\theta)\,d\theta.
\end{aligned} \tag{3.29}$$

Inequality (3.29) implies that

$$\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\right)(l)+\hbar\mathcal{H}\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^2\right)(l)\leq(\hbar+\mathcal{H})\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\mathcal{G}\right)(l). \tag{3.30}$$

Alternately, it follows from $\hbar\mathcal{H}>0$ and

$$\left(\sqrt{\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\right)(l)}-\sqrt{\hbar\mathcal{H}\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^2\right)(l)}\right)^2\geq 0 \tag{3.31}$$

that

$$2\sqrt{\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\right)(l)}\sqrt{\hbar\mathcal{H}\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^2\right)(l)}\leq\sqrt{\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}^2\right)(l)}+\sqrt{\hbar\mathcal{H}\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{G}^2\right)(l)}. \tag{3.32}$$

Therefore inequality (1) follows easily from inequalities (3.30) and (3.32). Similarly, we also can prove inequalities (2) and (3). □

Example 3.10 Let $l>1$, $\gamma,\varrho>0$, $p,q>1$ with $1/p+1/q=1$, $\mathcal{F}$ be an integrable function defined on $[1,\infty)$, and $\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}$ be the generalized proportional Hadamard fractional integral operator. Then we have

$$\left|\left(\mathfrak{I}_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\right|\leq\Omega\left\|\mathcal{F}(\theta)\right\|_{L_1(1,l)},$$

where

$$\Omega=\frac{1}{\rho^{\gamma}\Gamma(\gamma)}\left(\frac{\rho x^{1-p}}{[(p+\rho)-2\rho p]}\right)^{\frac{1}{p}}\Theta^{\frac{1}{p}}\left((\gamma-1)p+1,(p+\rho-2\rho p)\ln l\right)$$

and

$$\Theta(\gamma,l) = \int_0^l e^{-\theta}\theta^{\gamma-1}\,d\theta$$

is the incomplete gamma function [57–60].

Proof It follows from Definition 2.3 and the modulus property that

$$\left|\left(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\right| \le \frac{1}{\rho^\gamma\Gamma(\gamma)}\int_1^l \frac{\exp[\frac{\rho-1}{\rho}(\ln(\frac{l}{\vartheta})]}{(\ln\frac{l}{\vartheta})^{1-\gamma}}\frac{|\mathcal{F}(\vartheta)|}{\vartheta}\,d\vartheta$$

for $\vartheta > 1$.

Making use of the well-known Hölder inequality, we obtain

$$\left|\left(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\right| \le \frac{1}{\rho^\gamma\Gamma(\gamma)}\left(\int_1^l \frac{\exp p[\frac{\rho-1}{\rho}(\ln(\frac{l}{\vartheta}))]}{\vartheta^p(\ln(\frac{l}{\vartheta}))^{(1-\gamma)p}}\,d\vartheta\right)^{\frac{1}{p}}\left\|\mathcal{F}(\vartheta)\right\|_{L_1(1,l)}.$$

Let $v = \ln(\frac{l}{\vartheta})$. Then elaborated computations lead to

$$\begin{aligned}\left|\left(\Im_{1^-,l}^{\gamma,\rho}\mathcal{F}\right)(l)\right| &\le \frac{1}{\rho^\gamma\Gamma(\gamma)}\left(\frac{\rho x^{1-p}}{[(p+\rho)-2\rho p]}\right)^{\frac{1}{p}} \\ &\quad\times \Theta^{\frac{1}{p}}\big((\gamma-1)p+1,(p+\rho-2\rho p)\ln l\big)\left\|\mathcal{F}(\vartheta)\right\|_{L_1(1,l)}.\end{aligned}$$

□

4 Conclusion

In this paper, we have derived numerous inequalities in the framework of a novel proposed GPHF integral operator with proportionality index ρ. Our obtained results are refinements of the Grüss inequality. In the special case of $\rho = 1$, it is worth mentioning that this allows for recapturing some existing operators from the GPHF integral operator, therefore, the GPHF integral operator is superior to many existing operators. In addition, our new approach recaptures the Grüss type inequalities and their variants proposed by Sudsutad et al. [48]. Our ideas may lead to a lot of follow-up research.

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