Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 9452350, 6 pages
https://doi.org/10.1155/2020/9452350

Research Article

New Fixed Point Results in $\mathcal{F}$-Quasi-Metric Spaces and an Application

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Received 28 February 2020; Accepted 23 May 2020; Published 18 June 2020

Academic Editor: Emilio Turco

The main goal of the present paper is to obtain several fixed point theorems in the framework of $\mathcal{F}$-quasi-metric spaces, which is an extension of $\mathcal{F}$-metric spaces. Also, a Hausdorff δ-distance in these spaces is introduced, and a coincidence point theorem regarding this distance is proved. We also present some examples for the validity of the given results and consider an application to the Volterra-type integral equation.

1. Introduction and Preliminaries

In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various spaces and is the proof of fixed point results in these spaces along with its applications in engineering science. One of these spaces is function weighted metric space introduced by Jleli and Samet [1]. This is a generalization of metric spaces.

Definition 1 [1, 2]. A function $f:(0,+\infty)\longrightarrow\mathbb{R}$ is named called a nondecreasing function if $f(s_1)\le f(s_2)$ for every $s_1,s_2\in(0,+\infty)$. Also, f is said to be logarithmic-like when every positive sequence $\{t_n\}$ satisfies $\lim_{n\longrightarrow\infty}t_n=0$ iff $\lim_{n\longrightarrow\infty}f(t_n)=-\infty$.

In the sequel, we apply $\mathcal{F}$ for the set of all nondecreasing functions that are logarithmic-like.

In 2019, some of researchers such as Alqahtani et al. [2], Aydi et al. [3], and Bera et al. [4] discussed on the structure of this space and on the fixed points of mappings satisfying in various contractive conditions.

Definition 2 [1]. Consider a mapping $\delta:X\times X\longrightarrow[0,+\infty)$, a constant $B\in[0,+\infty)$ and a $f\in\mathcal{F}$ so that

(η_1) $\delta(x_1,x_2)=0\Leftrightarrow x_1=x_2$ for all $x_1,x_2\in X$;

(η_2) $\delta(x_1,x_2)=\delta(x_1,x_2)$ for all $x_1,x_2\in X$; and

(η_3) $\delta(x_1,x_2)>0$ implies that $f(\delta(x_1,x_2))\le f(\sum_{i=1}^{N-1}\delta(v_i,v_{i+1}))+B$ for every $N\in\mathbb{N}$ with $N\ge2$, for every $x_1,x_2\in X$, and for all $(v_i)_{i=1}^{N}\subset X$ with $(v_1,v_N)=(x_1,x_2)$.

Then, the function δ is named as a function weighted metric or a $\mathcal{F}$-metric on X, and the pair (X,δ) is called a $\mathcal{F}$-metric space.

Definition 3 [5]. Consider a mapping $\delta_q:X\times X\longrightarrow[0,+\infty)$ satisfies the properties (η_1) and (η_3) from the definition of a $\mathcal{F}$-metric. Then, δ_q is named a $\mathcal{F}$-quasi-metric on X and (X,δ_q) is called a $\mathcal{F}$-quasi-metric space.

In [5], Karapinar et al. showed that δ_q generally induces other $\mathcal{F}$-quasi-metrics such as $\delta_q^{-1},\delta_q^{*}:X\times X\longrightarrow[0,+\infty)$ defined by $\delta_q^{-1}(s,t)=\delta_q(t,s)$ and $\delta_q^{*}(s,t)=\max\{\delta_q(s,t),\delta_q^{-1}(s,t)\}$. Regarding the discussion above, we conclude that any quasi-metric is a $\mathcal{F}$-quasi-metric by choosing $f(t)=\ln t$ for the axiom (η_3) with $B=0$.

Definition 4 [5]. Let (X, δ_q) be a $\mathcal{F}$-quasi-metric space. For $x \in X$, the right (left) centered ball at x and of radius $e > 0$ is the set $B_r(s, e) = \{t \in X : \delta_q(s, t) < e\}$ $(B_l(s, e) = \{t \in X : \delta_q(t, s) < e\})$.

Definition 5 [5]. Consider a $\mathcal{F}$-quasi-metric space (X, δ_q) with a sequence $\{x_n\}$ therein. Then, $\{x_n\}$ is said to be a right-convergent sequence (left-convergent sequence) to $x \in X$ if $\lim_{n \longrightarrow \infty} \delta_q(x, x_n) = 0$ ($\lim_{n \longrightarrow \infty} \delta_q(x_n, x) = 0$). Further, $\{x_n\}$ is said to be a biconvergent sequence (in summary, convergent sequence) when it is both right-convergent and left-convergent.

Proposition 6 [5]. *Consider a $\mathcal{F}$-quasi-metric space (X, δ_q) with a sequence $\{x_n\}$ therein. Also, let $\lim_{n \longrightarrow \infty} \delta_q(s, x_n) = \lim_{n \longrightarrow \infty} \delta_q(x_n, t) = 0$ for every $s, t \in X$. Then $s = t$.*

Definition 7 [5]. Consider a $\mathcal{F}$-quasi-metric space (X, δ_q) with a sequence $\{x_n\}$ therein. Then, $\{x_n\}$ is a right-Cauchy sequence (a left-Cauchy sequence) if $\lim_{n,m \longrightarrow \infty} \delta_q(x_n, x_m) = 0$ ($\lim_{n,m \longrightarrow \infty} \delta_q(x_m, x_n) = 0$). With this interpretation, $\{x_n\}$ is bi-Cauchy sequence (in summary, Cauchy sequence) if it is both left-Cauchy sequence and right-Cauchy sequence.

Now, a $\mathcal{F}$-quasi-metric space (X, δ_q) is named right-complete (left-complete) if every right-Cauchy sequence (left-Cauchy sequence) in X is a right-convergent sequence (left-convergent sequence) in X. Further, (X, δ_q) is bicomplete (in short, complete) if it is both left-complete and right-complete.

Example 8. Define $\delta_q : \mathbb{N} \times \mathbb{N} \longrightarrow [0, +\infty)$ by

$$\delta_q(s, t) = \begin{cases} 0 & \text{when } s = t \\ e^s + |s - t| & \text{otherwise,} \end{cases} \tag{1}$$

for every $s, t \in \mathbb{N}$. Evidently, $(\mathbb{N}, \delta_q)$ is a bicomplete $\mathcal{F}$-quasi-metric space.

On the other hand, Bhaskar and Lakshmikantham [6] defined the notion of coupled fixed point and presented several coupled fixed point propositions for a mixed monotone mapping in partially ordered matric spaces. Also, they studied the existence and uniqueness of a solution to a periodic boundary value problem. For more details on coupled, tripled, and n-tupled fixed point assertions, one can see [7] and references therein.

Definition 9 [8, 9]. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two optional mappings. An element $(u, v) \in X \times X$ is said to be a coupled coincidence point of F and g if $F(u, v) = gu$ and $F(v, u) = gv$. Further, an element $(u, v) \in X \times X$ is named a common fixed point of F and g if $F(u, v) = gu = u$ and $F(v, u) = gv = v$.

Note that if g is the identity mapping, then (x, y) is called a coupled fixed point of F [6].

Definition 10 [9]. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two optional mappings. Then F and g is said to be commutative if $F(gu, gv) = g(F(u, v))$ for every $u, v \in X$.

In this paper, we introduce several common fixed point and common coupled fixed point theorems in such spaces and prove them. In Section 2, we prove a common fixed point theorem and a common coupled fixed point result in this space. In Section 3, we obtain a coincidence point result for single-valued and multivalued mappings regarding a Hausdorff δ-distance. Ultimately, as an application of these results, the existence of solution of the Volterra-type integral equation is investigated in Section 4.

2. $\mathcal{F}$-Quasi-Metric Space and Fixed Point Theory

Theorem 11. *Let (X, δ_q) be a bicomplete $\mathcal{F}$-quasi-metric space. Also, let $g, T : X \longrightarrow X$ be two arbitrary mappings so that T and g are commutative, $T(X) \subset g(X)$, $g(X)$ is closed, and*

$$\delta_q(Tx, Ty) \le k\delta_q(gx, gy), \tag{2}$$

for every $x, y \in X$, where $k \in (0, 1)$. Then T and g contain a unique common fixed point in X.

Proof. Due to $T(X) \subset g(X)$, we select a point $x_1 \in X$ such that $Tx_0 = gx_1$ for a given $x_0 \in X$. By continuing this process, we can construct a sequence y_n in X by $y_n = Tx_n = gx_{n+1}$ for $n = 0, 1, \cdots$. First, note that T and g possess a unique coincidence point. On the contrary, assume that $u_1, v_1 \in X$ are two different coincidence points of T and g. Then, $\delta_q(u_2, v_2) > 0$ with $gu_1 = Tu_1 = u_2$ and $gv_1 = Tv_1 = v_2$. Now, by (2), we get

$$\begin{aligned} \delta_q(u_2, v_2) = \delta_q(Tu_1, Tv_1) &\le k\delta_q(gu_1, gv_1) \\ &= k\delta_q(u_2, v_2) < \delta_q(u_2, v_2), \end{aligned} \tag{3}$$

which is a contradiction.

Assume $(f, B) \in \mathcal{F} \times [0, +\infty)$ so that (η_3) is complied. For an arbitrary $\varepsilon > 0$ and because of (η_3), there exists $\gamma > 0$ such that

$$0 < t < \gamma \Longrightarrow f(t) < f(\varepsilon) - B. \tag{4}$$

Now, let $\{y_n\}$ be a sequence in X. Without loss of totality, suppose that $\delta_q(Tx_0, Tx_1) > 0$. Otherwise, x_1 is a coincidence point of T and g. Now, using (2), we obtain

$$\begin{aligned} \delta_q(Tx_n, Tx_{n+1}) &\le k\delta_q(gx_n, gx_{n+1}) = k\delta_q(Tx_{n-1}, Tx_n) \\ &\le k^2\delta_q(gx_{n-1}, gx_n), \end{aligned} \tag{5}$$

which implies by induction that

$$\delta_q(Tx_n, Tx_{n+1}) \leq k^n \delta_q(Tx_0, Tx_1) \quad (6)$$

for every n in $\mathbb{N}$. Hence, for every m and n in $\mathbb{N}$ so that $m > n$, we get

$$\sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1}) \leq \frac{k^n}{1-k} \delta_q(Tx_0, Tx_1). \quad (7)$$

Since

$$\lim_{n \longrightarrow \infty} \frac{k^n}{1-k} \delta_q(Tx_0, Tx_1) = 0, \quad (8)$$

there is some $N \in \mathbb{N}$ so that $0 < (k^n/1-k)\delta_q(Tx_0, Tx_1) < \gamma$ for every $n \geq N$. Hence, from (4) and (η_1), we observe that

$$f\left(\sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1})\right) \leq f\left(\frac{k^n}{1-k}\delta_q(Tx_0, Tx_1)\right) < f(\varepsilon) - B, \quad (9)$$

for $m > n \geq N$. Employing (η_3) together with (9), we obtain

$$\begin{aligned} \delta_q(Tx_n, Tx_m) > 0 \Longrightarrow & f\left(\delta_q(Tx_n, Tx_m)\right) \\ & \leq f\left(\sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1})\right) + B < f(\varepsilon). \end{aligned} \quad (10)$$

It follows that $\delta_q(Tx_n, Tx_m) < \varepsilon$. Therefore, $\{y_n\} = \{Tx_n\}$ is right-Cauchy. Similarly, by changing the order of the pairs (x_{i+1}, x_i) in the above process, we conclude that $\{y_n\}$ is also a left-Cauchy sequence. Hence, it is a Cauchy sequence. Now, since (X, δ_q) is a bicomplete space, there exists $z \in X$ such that $\{y_n\}$ is convergent to z. Since $\{Tx_n\} = \{gx_{n+1}\} \subset g(X)$ and $g(X)$ is closed, we have $\lim_{n \longrightarrow \infty} \delta(gx_n, gz) = 0$. As a next step, we show that z is a coincidence point of T and g. On the contrary, consider $\delta_q(Tz, gz) > 0$. Then we have

$$\begin{aligned} f\left(\delta_q(Tz, gz)\right) & \leq f\left(\delta_q(Tz, Tx_n) + \delta_q(Tx_n, gz)\right) + B \\ & \leq f\left(k\delta_q(gz, gx_n) + \delta_q(gx_{n+1}, gz)\right) + B. \end{aligned} \quad (11)$$

As $n \longrightarrow \infty$ in the inequality above, we obtain

$$\lim_{n \longrightarrow \infty} f\left(k\delta_q(gz, gx_n) + \delta_q(gx_{n+1}, gz)\right) + B = -\infty, \quad (12)$$

which is a contradiction. Hence, $\delta_q(Tz, gz) = 0$; that is, z is a unique coincidence point of T and g. Therefore, g and T contian a unique point of coincidence $w = gz = Tz$. By commutativity of the mapping T and g, we have $gw = g(gz) = gT(z) = Tg(z) = Tw$. Hence, gw is another point of coincidence of g and T. Now, by the uniqueness of the point of coincidence of g and T, we have $w = gw = Tw$; that is, g and T contain a unique common fixed point. This completes the proof.

In the sequel, denote for simplicity $X \times \cdots \times X$ by X^n, where X is a nonempty set and $n \in \mathbb{N}$.

Lemma 12. *Consider a $\mathcal{F}$-quasi-metric space (X, δ_q). Then, the following assertions hold:*

(1) (X^n, Δ_q) is a $\mathcal{F}$-quasi-metric space with

$$\begin{aligned} & \Delta_q((u_1, \cdots, u_n), (v_1, \cdots, v_n)) \\ & \quad = \max\left[\delta_q(u_1, v_1), \delta_q(u_2, v_2), \cdots, \delta_q(u_n, v_n)\right]. \end{aligned} \quad (13)$$

(2) The mapping $f : X^n \longrightarrow X$ and $g : X \longrightarrow X$ contain an n-tuple common fixed point iff the mapping $F : X^n \longrightarrow X^n$ and $G : X^n \longrightarrow X^n$ defined by

$$\begin{aligned} F(u_1, u_2, \cdots, u_n) = & (f(u_1, u_2, \cdots, u_n), f(u_2, \cdots, u_n, u_1), \cdots, \\ & f(u_n, u_1, \cdots, u_{n-1})) \\ G(u_1, u_2, \cdots, u_n) = & (gu_1, gu_2, \cdots, gu_n) \end{aligned} \quad (14)$$

possess a common fixed point in X^n.

(3) (X, δ_q) is bicomplete iff (X^n, Δ_q) is bicomplete.

Proof. Clearly, Δ_q satisfies in (η_1). We show that Δ_q satisfies in (η_3). For every $(x_{i,j}) \subset X$ for $1 \leq i \leq N$ and $1 \leq j \leq n$, consider $(x_{1j}, x_{N-1j}) = (u_j, v_j)$. Suppose that

$$\delta_q(u_j, v_j) = \max\left[\delta_q(u_1, v_1), \delta_q(u_2, v_2), \cdots, \delta_q(u_n, v_n)\right]. \quad (15)$$

Then, we have

$$f_j(\delta_q(u_j, v_j)) \leq f_j\left(\sum_{i=1}^{N-1} \delta_q(x_{i,j}, x_{i+1,j})\right) + B_j, \quad (16)$$

where $f_j \in \mathcal{F}$ and $B_j \in [0, +\infty)$. Therefore, we obtain

$$\begin{aligned} & f_j(\Delta_q((u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n))) \\ & \quad = f_j\left(\max\left[\delta_q(u_1, v_1), \delta_q(u_2, v_2), \cdots, \delta_q(u_n, v_n)\right]\right) \\ & \quad = f_j(\delta_q(u_j, v_j)) \leq f_j\left(\sum_{i=1}^{N-1} \delta_q(x_{i,j}, x_{i+1,j}) + B_j\right) \\ & \quad \leq f_j\left(\sum_{i=1}^{N-1} \Delta_q((x_{i,1}, x_{i,2}, \cdots, x_{i,n}), (x_{i+1,1}, x_{i+1,2}, \cdots, x_{i+1,n}))\right) + B_j. \end{aligned} \quad (17)$$

The proofs of (2) and (3) are straightforward and left to the reader.

Remember that Lemma 12 is a two-way relationship. Consequently, we can establish n-tuple fixed point propositions from fixed point assertions and conversely. Now, set $n=2$ in Lemma 12. Then, we have the following theorem.

Theorem 13. *Let (X, δ_q) be a bicomplete $\mathscr{F}$-quasi-metric space. Also, let $g: X \longrightarrow X$ and $T: X^2 \longrightarrow X$ be two arbitrary mappings so that T and g are commutative, $T(X^2) \subset g(X)$, $g(X)$ is closed, and*

$$\delta_q(T(x,y), T(x^*, y^*)) \leq \frac{k}{2}\left(\delta_q(gx, gx^*) + \delta_q(gy, gy^*)\right), \tag{18}$$

for all (x, y) and (x^*, y^*) in X^2, where $k \in (0, 1)$. Then, T and g contain a unique common coupled fixed point in X^2.

Proof. Let us define $\Delta_q: X^2 \times X^2 \longrightarrow [0, \infty)$ by $\Delta_q((u_1, u_2), (v_1, v_2)) = \max\left[\delta_q(u_1, v_1), \delta_q(u_2, v_2)\right]$ for all $u_1, u_2, v_1, v_2 \in X$. Further, we consider $F: X^2 \longrightarrow X^2$ by $F(x, y) = (T(x, y), T(y, x))$ and $G: X^2 \longrightarrow X^2$ by $G(x, y) = (gx, gy)$ for all $x, y \in X$. Using Lemma 12, (X^2, Δ_q) is a bicomplete $\mathscr{F}$-quasi-metric space. Also, $(x, y) \in X^2$ is a common coupled fixed point of T and g iff it is a common fixed point of F and G. On the other hand, from (18), we have either

$$\begin{aligned}
&\Delta_q(F(x,y), F(x^*, y^*)) \\
&\quad = \Delta_q((T(x,y), T(y,x)), (T(x^*, y^*), T(y^*, x^*))) \\
&\quad = \max\left[\delta_q(T(x,y), T(x^*, y^*)), \delta_q(T(y,x), T(y^*, x^*))\right] \\
&\quad = \delta_q(T(x,y), T(x^*, y^*)) \leq \frac{k}{2}\left(\delta_q(gx, gx^*) + \delta_q(gy, gy^*)\right) \\
&\quad \leq k \max\left[\delta_q(gx, gx^*), \delta_q(gy, gy^*)\right] \\
&\quad = k\Delta_q(G(x,y), G(x^*, y^*)),
\end{aligned} \tag{19}$$

or

$$\begin{aligned}
&\Delta_q(F(x,y), F(x^*, y^*)) \\
&\quad = \Delta_q((T(x,y), T(y,x)), (T(x^*, y^*), T(y^*, x^*))) \\
&\quad = \max\left[\delta_q(T(x,y), T(x^*, y^*)), \delta_q(T(y,x), T(y^*, x^*))\right] \\
&\quad = \delta_q(T(y,x), T(y^*, x^*)) \leq \frac{k}{2}\left(\delta_q(gy, gy^*) + \delta_q(gx, gx^*)\right) \\
&\quad \leq k \max\left[\delta_q(gy, gy^*), \delta_q(gx, gx^*)\right] \\
&\quad = k\Delta_q(G(y,x), G(y^*, x^*)).
\end{aligned} \tag{20}$$

Now, by Theorem 11, F and G have a common fixed point and by Lemma 12, T and g have a common coupled fixed point.

Example 14. Let $X = [0, 1]$. Define $\delta_q: X \times X \longrightarrow [0, \infty)$ by

$$\delta_q(s,t) = \begin{cases} 0 & \text{when } s = t \\ |s| + |s - t| & \text{otherwise,} \end{cases} \tag{21}$$

for every $x, y \in X$. Evidently, δ_q is a bicomplete $\mathscr{F}$-quasi-metric with $f(t) = \ln t$ and $B = 0$. Consider $T: X^2 \longrightarrow X$ by $T(x, y) = (x/2) + (y/2)$ and define $g: X \longrightarrow X$ by $g(x) = 2x$. Clearly T and g are commutative. Also, we have

$$\begin{aligned}
&\delta_q(T(x,y), T(x^*, y^*)) \\
&\quad = \left(\left|\frac{x}{2} + \frac{y}{2} - \left(\frac{x^*}{2} + \frac{y^*}{2}\right)\right|\right) + \left|\frac{x}{2} + \frac{y}{2}\right| \\
&\quad = \frac{1}{4}\left(|2x - 2x^* + (2y - 2y^*)| + |2x + 2y|\right) \\
&\quad \leq \frac{1}{4}\left(|2x - 2x^*| + |2y - 2y^*| + |2x| + |2y|\right) \\
&\quad = \frac{1}{4}\delta_q(gx, gx^*) + \frac{1}{4}\delta_q(gy, gy^*).
\end{aligned} \tag{22}$$

Therefore, by letting $k = 1/2$, all of the hypotheses of Theorem 13 hold. Thus, T and g possess a common coupled fixed point in X^2.

3. Fixed Point Theorem and Hausdorff δ_q-Distance

Let us start with the following definition:

Consider a $\mathscr{F}$-quasi-metric space (X, δ_q), and denote the family of all nonempty bounded closed subsets of X by $CB(X)$. Then, $H(\cdot, \cdot)$ is said to be a Hausdorff δ_q-distance on $CB(X)$, if

$$H_{\delta_q}(A, B) = \max\left\{\sup_{x \in A}\delta_q(x, B), \sup_{x \in B}\delta_q(A, x)\right\}, \tag{23}$$

where $\delta_q(x, B) = \inf\{\delta_q(x, y), y \in B\}$.

Definition 15 [10]. Let X be a nonempty set, $g: X \longrightarrow X$ be a single-valued mapping, and $T: X \longrightarrow CB(X)$ be a multivalued mapping. Also, let $w = gx \in Tx$ for some $x \in X$. Then w is said to be a point of coincidence of g and T, and x is said to be a coincidence point of g and T.

Theorem 16. *Let (X, δ_q) be a bicomplete $\mathscr{F}$-quasi-metric space. Also, let $g: X \longrightarrow X$ be a single-valued mapping and $T: X \longrightarrow CB(X)$ be a multivalued mapping so that $T(X) \subset g(X)$, $g(X)$ is closed, and g is continuous. Assume that there exists $k \in (0, 1)$ such that*

$$H_{\delta_q}(Tx, Ty) \leq k\delta_q(gx, gy), \tag{24}$$

for all $x, y \in X$. Then T and g have a coincidence point in X.

Proof. Due to $T(X) \subset g(X)$, we select a point $x_1 \in X$ such that $gx_1 \in Tx_0$ for a given $x_0 \in X$. By continuing this procedure, we can construct a sequence x_n in X such that $gx_{n+1} \in Tx_n$ for $n = 0, 1, \cdots$. Suppose that $(f, B) \in \mathscr{F} \times [0, +\infty)$ so that (η_3) holds. For an arbitrary $\varepsilon > 0$ and due to (η_3), there exists $\gamma > 0$ such that

$$0 < t < \gamma \Longrightarrow f(t) < f(\varepsilon) - B. \quad (25)$$

Consider the sequence $\{gx_n\} \subset X$. Now, without loss of generality, suppose that $H_{\delta_q}(Tx_0, Tx_1) > 0$. Otherwise, x_1 is a coincidence point of T and g. Now, from (24), we have

$$\begin{aligned}\delta_q(gx_{n+1}, gx_{n+2}) &\le H_{\delta_q}(Tx_n, Tx_{n+1}) \le k\delta_q(gx_n, gx_{n+1}) \\ &\le kH_{\delta_q}(Tx_{n-1}, Tx_n) \le k^2\delta_q(gx_{n-1}, gx_n).\end{aligned} \quad (26)$$

Hence, we have $\delta_q(gx_n, gx_{n+1}) \le k^n\delta_q(gx_0, x_1)$ for all $n \in \mathbb{N}$. Now, let m and n be two natural numbers with $m > n$. Then, we have

$$\sum_{i=n}^{m-1} \delta_q(gx_i, gx_{i+1}) \le \frac{k^n}{1-k}\delta_q(gx_0, gx_1). \quad (27)$$

On the other hand, since $\lim_{n\longrightarrow\infty} (k^n/1-k)\delta_q(gx_0, gx_1) = 0$, there exists $N \in \mathbb{N}$ so that

$$0 < \frac{k^n}{1-k}\delta_q(gx_0, gx_1) < \gamma, \quad (28)$$

for $n \ge N$. Hence, by (25) and (η_1), we have

$$f\left(\sum_{i=n}^{m-1} \delta_q(gx_i, gx_{i+1})\right) \le f\left(\frac{k^n}{1-k}\delta_q(gx_0, gx_1)\right) < f(\varepsilon) - B, \quad (29)$$

for all $m > n \ge N$. Employing (η_3) together with (29), we obtain

$$\begin{aligned}\delta_q(gx_n, gx_m) > 0 &\Longrightarrow f(\delta_q(gx_n, gx_m)) \\ &\le f\left(\sum_{i=n}^{m-1} \delta_q(gx_i, gx_{i+1})\right) + B < f(\varepsilon).\end{aligned} \quad (30)$$

Now, by (η_1), we have $\delta_q(gx_n, gx_m) < \varepsilon$. This proves that $\{gx_n\}$ is right-Cauchy. Similarly, by changing the order of the pairs (x_{i+1}, x_i) in the above process, we conclude that $\{gx_n\}$ is also a left-Cauchy sequence. Therefore, it is a Cauchy sequence. Note that (X, δ_q) is bicomplete and $g(X)$ is closed. Thus, there exists $x \in X$ such that $\lim_{n\longrightarrow\infty} gx_n = gx$. Now, we shall show that $gx \in Tx$. For this purpose, using (24), we have

$$\delta_q(gx_{n+1}, Tx) \le H_{\delta_q}(Tx_n, Tx) \le k\delta_q(gx_n, gx). \quad (31)$$

Thus,

$$\lim_{n\longrightarrow\infty} \delta_q(gx_{n+1}, Tx) = \delta_q(gx, Tx) = 0. \quad (32)$$

Hence, $gx \in Tx$. Consequently, T and g possess a point of coincidence.

Example 17. Consider $X = [0, 1]$. Define $T : X \longrightarrow CB(X)$ by $Tx = [0, 1/16x]$ and $g : X \longrightarrow X$ by $gx = x/2$. Also, define $\delta_q : X \times X \longrightarrow [0, \infty)$ by

$$\delta_q(s, t) = \begin{cases} 0 & \text{if } s = t \\ |s| + |s-t| & \text{otherwise.} \end{cases} \quad (33)$$

Clearly, δ_q is a bicomplete $\mathcal{F}$-quasi-metric with $f(t) = \ln t$ and $B = 0$. Also, evidently, $T(X) \subset g(X)$ and $g(X)$ is closed. First, let $x = y = 0$. Then

$$H_{\delta_q}(Tx, Ty) = 0 \le k\delta_q(gx, gy). \quad (34)$$

Therefore, we may consider x and y are not zero. Without loss of totality, suppose that $x \le y$. Then

$$\begin{aligned}H_{\delta_q}(Tx, Ty) &= H_{\delta_q}\left(\left[0, \frac{1}{16}x\right], \left[0, \frac{1}{16}y\right]\right) \\ &= \max \sup_{0\le a\le 1/16x} \delta_q\left(a, \left[0, \frac{1}{16}y\right]\right), \\ &\quad \cdot \sup_{0\le b\le 1/16y} \delta_q\left(\left[0, \frac{1}{16}x\right], b\right).\end{aligned} \quad (35)$$

Due to $x \le y$, we have $[0, 1/16x] \subset [0, 1/16y]$. For every $a \in [0, 1/16x]$, we have $\delta_q(a, [0, 1/16y]) = 0$. Also, for every $b \in [0, 1/16y]$, we obtain

$$\delta_q\left(\left[0, \frac{1}{16}x\right], b\right) = \begin{cases} 0 & b \le \dfrac{x}{16} \\ \left(|0| + \left|\dfrac{x}{8} - 2b\right|\right) & b \ge \dfrac{x}{16}. \end{cases} \quad (36)$$

This yields that

$$\sup_{0\le b\le 1/16y} \delta_q\left(\left[0, \frac{1}{16}x\right], b\right) = \left|\frac{x}{8} - \frac{y}{8}\right|. \quad (37)$$

We deduce that

$$H_{\delta_q}(Tx, Ty) = \left|\frac{x}{8} - \frac{y}{8}\right| \le \frac{1}{4}\left(\left|\frac{x}{2}\right| + \left|\frac{x}{2} - \frac{y}{2}\right|\right) = \frac{1}{4}\delta_q(gx, gy). \quad (38)$$

Obviously, all other hypotheses of Theorem 16 hold. Hence, g and T possess a coincidence point in X.

4. An Application

As an application of our results, we consider the following Volterra-integral equation:

$$x(t) = \int_0^t K(t, s, x(s))ds + v(t), \quad (39)$$

where $t \in I = [0, 1]$, $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$, and $v \in C(I, \mathbb{R})$.

Consider a Banach space of all real continuous functions defined on I $(C(I, \mathbb{R}))$ with norm $\|x\|_\infty = \max_{t\in I}|x(t)|$ for every $x \in C(I, \mathbb{R})$. Also, let $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ be the space of all continuous functions defined on $I \times I \times C(I, \mathbb{R})$. On the other hand, the Banach space $C(I, \mathbb{R})$ can be dedicated with the Bielecki norm $\|x\|_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}$ for all $x \in C(I, \mathbb{R})$ and $\tau > 0$, and the derived metric $\delta_B(x, y) = \|x - y\|_B$ for all $x, y \in C(I, \mathbb{R})$. Define $\delta_q : X \times X \longrightarrow [0, \infty)$ by

$$\delta_q(x, y) = \sup_{t\in I}\left\{|x(t)-y(t)|e^{-\|x\|_B t}\right\}. \tag{40}$$

Also, define $T : C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ by

$$Tx(t) = \int_0^t K(t, s, x(s))ds + v(t), \quad v \in C(I, \mathbb{R}). \tag{41}$$

Theorem 18. *Consider a bicomplete $\mathcal{F}$-quasi-metric space $(C(I, \mathbb{R}), \delta_B)$ with $f(t) = lnt$. Also, let T be an operator from $C(I, \mathbb{R})$ into $C(I, \mathbb{R})$ with $Tx(t) = \int_0^t K(t, s, x(s))ds + v(t)$, and let $gx = I(x)$. Assume that $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$is an operator such that*

(i) K is continuous;

(ii) $\int_0^t K(t, s, \cdot)$ for all $t, s \in I$ is increasing; and

(iii) for every x and y in $C(I, \mathbb{R})$, and t and s in I, we have

$$|K(t, s, x(s)) - K(t, s, y(s))| \le e^{-\|x\|_B}|x(s) - y(s)|. \tag{42}$$

Then, the integral equation (39) possesses an answer in $C(I, \mathbb{R})$.

Proof. By definition of T, we have

$$\begin{aligned}\delta_q(Tx, Ty) &= \sup_{t\in I}\left\{\left|\int_0^t K(t, s, x(s))ds - \int_0^t K(t, s, y(s))ds\right|e^{-\|x\|_B t}\right\}\\ &\le \sup_{t\in I}\left\{\int_0^t |K(t, s, x(s)) - K(t, s, y(s))|e^{-\|x\|_B t}\, ds\right\}\\ &\le \sup_{t\in I}\left\{\int_0^t e^{-\|x\|_B}|x(s) - y(s)|e^{-\|x\|_B t}\, ds\right\}\\ &\le (\|x - y\|_B)\sup_{t\in I}\left\{\int_0^t e^{-\|x\|_B} ds\right\} \le e^{-\|x\|_B}\delta_q(x, y).\end{aligned} \tag{43}$$

Now, we consider that the function $f(t) = \ln t$ for every $t \in I$, $B = 0$, and $k = e^{-\|x\|_B}$. Therefore, all assertions of Theorem 11 hold. As a result, Theorem 11 confirms the existence of fixed point of T so that this fixed point is the answer of the integral equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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