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# New fractional inequalities of Hermite–Hadamard type involving the incomplete gamma functions

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## Abstract

A specific type of convex functions is discussed. By examining this, we investigate new Hermite–Hadamard type integral inequalities for the Riemann–Liouville fractional operators involving the generalized incomplete gamma functions. Finally, we expose some examples of special functions to support the usefulness and effectiveness of our results.

**MSC:** 26D07; 26D10; 26D15

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## 1 Introduction

In the past two decades, fractional calculus has received much attention. The fast interest in the topic is due to its extensive applications in various fields such as biochemistry, physics, viscoelasticity, fluid mechanics, computer modeling, and engineering, see [1–7] for further details. Most of the studies have been devoted to the existence and uniqueness of solutions for fractional differential or difference equations; see e.g. [8–12].

A fractional differential equation needs a certain inequality for existence and uniqueness of solution. For this reason, a huge number of mathematicians have competed to seek such inequalities; see e.g. [13–25].

As always, it is important and necessary to specify which model or definition of fractional calculus is being used because there are many different ways of defining fractional operators (integrals and derivatives). To further facilitate the discussion of this model, we present here the definition which is most commonly used for fractional operators, namely the Riemann–Liouville (RL) definition.

**Definition 1.1** ([1, 2]) For any  $L^1$  function  $\bar{w}(x)$  on an interval  $[\varepsilon_3, \varepsilon_4]$  with  $x \in [\varepsilon_3, \varepsilon_4]$ , the  $\kappa$ th left-RL fractional integral of  $\bar{w}(x)$  is given by

$${}^{\text{RL}}J_{\varepsilon_3+}^{\kappa} \bar{w}(x) := \frac{1}{\Gamma(\kappa)} \int_{\varepsilon_3}^x (x - \varsigma)^{\kappa-1} \bar{w}(\varsigma) d\varsigma \quad (1.1)$$

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for  $\text{Re}(\kappa) > 0$ . Also, the  $\kappa$ th right-RL fractional integral of  $\bar{w}(x)$  is given by

$${}^{\text{RL}}\mathbf{J}_{\varepsilon_4^-}^{\kappa} \bar{w}(x) := \frac{1}{\Gamma(\kappa)} \int_x^{\varepsilon_4} (\zeta - x)^{\kappa-1} \bar{w}(\zeta) \, d\zeta. \tag{1.2}$$

Before starting the main findings, we review some definitions, notations, theorems which will be necessary later to proceed.

**Definition 1.2** ([26]) We say that the function  $\bar{w} : \mathcal{J} \subseteq \mathcal{R} \rightarrow \mathcal{R}$  is convex on  $\mathcal{J}$  if

$$\bar{w}(\zeta \varepsilon_3 + (1 - \zeta) \varepsilon_4) \leq \zeta \bar{w}(\varepsilon_3) + (1 - \zeta) \bar{w}(\varepsilon_4) \tag{1.3}$$

holds for every  $\varepsilon_3, \varepsilon_4 \in \mathcal{J}$  and  $\zeta \in [0, 1]$ .

**Definition 1.3** ([27]) We say that the function  $\bar{w} : \mathcal{J} \subseteq \mathcal{R} \rightarrow \mathcal{R}$  is exponential type convex (or briefly exp-convex function) on  $\mathcal{J}$  if

$$\bar{w}(\zeta \varepsilon_3 + (1 - \zeta) \varepsilon_4) \leq (e^{\zeta} - 1) \bar{w}(\varepsilon_3) + (e^{1-\zeta} - 1) \bar{w}(\varepsilon_4) \tag{1.4}$$

holds for every  $\varepsilon_3, \varepsilon_4 \in \mathcal{J}$  and  $\zeta \in [0, 1]$ .

The well-known integral inequality of Hermite–Hadamard type (HH-type) for such a convex function (1.3) is given by

$$\bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \leq \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} \bar{w}(x) \, dx \leq \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2}. \tag{1.5}$$

On the same convex function (1.3), in 2013, Sarikaya et al. [28] generalized the HH-inequality (1.5) to fractional integrals of RL type, which is as follows:

$$\bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \leq \frac{\Gamma(\kappa + 1)}{2(\varepsilon_4 - \varepsilon_3)^{\kappa}} \left[ {}^{\text{RL}}\mathbf{J}_{\varepsilon_3^+}^{\kappa} \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\varepsilon_4^-}^{\kappa} \bar{w}(\varepsilon_3) \right] \leq \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2}, \tag{1.6}$$

where  $\kappa > 0$  and  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  is supposed to be an  $L^1$  convex function. One year later, Sarikaya and Yildirim [29] found a new version of the above inequality, which is as follows:

$$\begin{aligned} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq \frac{2^{\kappa-1} \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^{\kappa}} \left[ {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^{\kappa} \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^{\kappa} \bar{w}(\varepsilon_3) \right] \\ &\leq \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2}. \end{aligned} \tag{1.7}$$

Again, one can note that this result is valid for any  $L^1$  convex function  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  and for each  $\kappa > 0$ .

In 2020, Kadakal and İşcan obtained the new refinement of the classical HH-inequality (1.5) on the exp-convex function (1.4), which is as follows:

$$\frac{1}{2(e^{\frac{1}{2}} - 1)} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \leq \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} \bar{w}(x) \, dx \leq (e - 2) [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \tag{1.8}$$

A huge number of generalizations and modifications of classical HH-inequality (1.5) have been established by means of fractional operators (1.1) and (1.2); e.g. see [20, 26, 30–44].

In this study, we follow the line of result mentioned above to investigate a new integral inequality, namely, the RL version of the new HH-type inequality (1.8). The rest of the attempt is designed as follows: in Sect. 2.1 we prove the HH inequalities of trapezoidal type by using differintegrals starting from the endpoints of the interval. In Sect. 2.2, we prove the HH inequalities of midpoint type by using differintegrals starting from the midpoint of the interval for the RL-fractional operators. Finally, some applications on special functions are exposed in Sect. 4.

## 2 Main results

Our main results are split into two subsections. The following facts will be needed in establishing our main results.

*Remark 2.1* For  $\text{Re}(\kappa) > 0$ , the following identities can hold:

$$\int_0^1 \varsigma^{\kappa-1} e^\varsigma \, d\varsigma = (-1)^\kappa \gamma(\kappa, -1); \tag{2.1}$$

$$\int_0^1 \varsigma^{\kappa-1} e^{1-\varsigma} \, d\varsigma = e \gamma(\kappa, 1), \tag{2.2}$$

where  $\gamma(\cdot, \cdot)$  is the lower incomplete gamma function [45]:

$$\gamma(\kappa, x) = \int_0^x \varsigma^{\kappa-1} e^{-\varsigma} \, d\varsigma, \quad x \in \mathbb{C}. \tag{2.3}$$

*Proof* By making change of the variable  $u := -\varsigma$  in the first integral, we get

$$\int_0^1 \varsigma^{\kappa-1} e^\varsigma \, d\varsigma = (-1)^\kappa \int_0^{-1} u^{\kappa-1} e^{-u} \, du = (-1)^\kappa \gamma(\kappa, -1),$$

which ends identity (2.1).

Identity (2.2) can be directly obtained from the original definition (2.3). □

*Remark 2.2* For  $\text{Re}(\kappa) > 0$ , the following identities can hold:

$$\int_0^1 \varsigma^{\kappa-1} e^{\frac{\varsigma}{2}} \, d\varsigma = (-2)^\kappa \gamma\left(\kappa, -\frac{1}{2}\right); \tag{2.4}$$

$$\int_0^1 \varsigma^{\kappa-1} e^{1-\frac{\varsigma}{2}} \, d\varsigma = e 2^\kappa \gamma\left(\kappa, \frac{1}{2}\right). \tag{2.5}$$

*Proof* We can use the same method used for Remark 2.1 to produce the results for Remark 2.2. □

**Lemma 2.1** ([28]) *If  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  is  $L^1[\varepsilon_3, \varepsilon_4]$  with  $0 < \varepsilon_3 < \varepsilon_4$  and  $\kappa > 0$ , then we have*

$$\begin{aligned} & \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2} - \frac{\Gamma(\kappa + 1)}{2(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right] \\ &= \frac{\varepsilon_4 - \varepsilon_3}{2} \int_0^1 \left[ (1 - \varsigma)^\kappa - \varsigma^\kappa \right] \bar{w}'(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4) \, d\varsigma. \end{aligned}$$

**Lemma 2.2** ([29]) *If  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  is  $L^1[\varepsilon_3, \varepsilon_4]$  with  $0 < \varepsilon_3 < \varepsilon_4$  and  $\kappa > 0$ , then we have*

$$\begin{aligned} & \frac{2^{\kappa-1} \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \bar{w}(\varepsilon_3) \right] - \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \\ &= \frac{\varepsilon_4 - \varepsilon_3}{4} \left[ \int_0^1 \varsigma^\kappa \bar{w}'\left(\frac{\varsigma}{2} \varepsilon_3 + \frac{2 - \varsigma}{2} \varepsilon_4\right) \, d\varsigma - \int_0^1 \varsigma^\kappa \bar{w}'\left(\frac{2 - \varsigma}{2} \varepsilon_3 + \frac{\varsigma}{2} \varepsilon_4\right) \, d\varsigma \right]. \end{aligned}$$

**2.1 Trapezoidal inequalities**

**Proposition 2.1** *Suppose that  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  is an  $L^1$  and exp-convex function. Then we have, for  $\kappa > 0$ ,*

$$\begin{aligned} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq \frac{(e^{\frac{1}{2}} - 1) \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right] \\ &\leq \kappa (e^{\frac{1}{2}} - 1) \left( e \gamma(\kappa, 1) + (-1)^\kappa \gamma(\kappa, -1) - \frac{2}{\kappa} \right) [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \end{aligned} \tag{2.6}$$

*Proof* By the exp-convexity of  $f$ , we have

$$\begin{aligned} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &= \bar{w}\left(\frac{[\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4] + [(1 - \varsigma)\varepsilon_3 + \varsigma \varepsilon_4]}{2}\right) \\ &\leq (e^{\frac{1}{2}} - 1) \bar{w}(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4) + (e^{\frac{1}{2}} - 1) \bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma \varepsilon_4). \end{aligned}$$

Multiplying by  $\varsigma^{\kappa-1}$  on both sides and then integrating over  $[0, 1]$ , we get

$$\begin{aligned} \frac{1}{\kappa} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq (e^{\frac{1}{2}} - 1) \int_0^1 \varsigma^{\kappa-1} \bar{w}(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4) \, d\varsigma \\ &\quad + (e^{\frac{1}{2}} - 1) \int_0^1 \varsigma^{\kappa-1} \bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma \varepsilon_4) \, d\varsigma. \end{aligned}$$

Multiplying by  $\kappa > 0$  on both sides and making the change of variables in the last inequality, we obtain

$$\begin{aligned} & \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \\ &\leq \kappa (e^{\frac{1}{2}} - 1) \left[ \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_4}^{\varepsilon_3} \left(\frac{\varepsilon_4 - u}{\varepsilon_4 - \varepsilon_3}\right)^{\kappa-1} \bar{w}(u) \, du + \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} \left(\frac{v - \varepsilon_3}{\varepsilon_4 - \varepsilon_3}\right)^{\kappa-1} \bar{w}(v) \, dv \right] \\ &= \frac{\Gamma(\kappa + 1)(e^{\frac{1}{2}} - 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right]. \end{aligned} \tag{2.7}$$

On the other hand, we have by exp-convexity

$$\bar{w}(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4) \leq (e^\varsigma - 1) \bar{w}(\varepsilon_3) + (e^{1-\varsigma} - 1) \bar{w}(\varepsilon_4);$$

$$\bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma\varepsilon_4) \leq (e^{1-\varsigma} - 1)\bar{w}(\varepsilon_3) + (e^\varsigma - 1)\bar{w}(\varepsilon_4).$$

Adding both inequalities, we get

$$\bar{w}(\varsigma\varepsilon_3 + (1 - \varsigma)\varepsilon_4) + \bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma\varepsilon_4) \leq (e^\varsigma + e^{1-\varsigma} - 2)[\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)].$$

Multiplying by  $\varsigma^{\kappa-1}$  on both sides and then integrating over  $[0, 1]$ , we get

$$\begin{aligned} & \int_0^1 \varsigma^{\kappa-1} \bar{w}(\varsigma\varepsilon_3 + (1 - \varsigma)\varepsilon_4) \, d\varsigma + \int_0^1 \varsigma^{\kappa-1} \bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma\varepsilon_4) \, d\varsigma \\ & \leq [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)] \int_0^1 \varsigma^{\kappa-1} (e^\varsigma + e^{1-\varsigma} - 2) \, d\varsigma. \end{aligned}$$

By making the change of variables and Remark 2.1, we get

$$\begin{aligned} & \frac{\Gamma(\kappa)}{(\varepsilon_4 - \varepsilon_3)^\kappa} [\text{RL}\mathcal{J}_{\varepsilon_3+}^\kappa \bar{w}(\varepsilon_4) + \text{RL}\mathcal{J}_{\varepsilon_4-}^\kappa \bar{w}(\varepsilon_3)] \\ & \leq \left( e\gamma(\kappa, 1) + (-1)^\kappa \gamma(\kappa, -1) - \frac{2}{\kappa} \right) [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \end{aligned}$$

Multiplying by positive constants  $\kappa > 0$  and  $(e^{\frac{1}{2}} - 1) > 0$  on both sides, we get

$$\begin{aligned} & \frac{(e^{\frac{1}{2}} - 1)\Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} [\text{RL}\mathcal{J}_{\varepsilon_3+}^\kappa \bar{w}(\varepsilon_4) + \text{RL}\mathcal{J}_{\varepsilon_4-}^\kappa \bar{w}(\varepsilon_3)] \\ & \leq \kappa (e^{\frac{1}{2}} - 1) \left( e\gamma(\kappa, 1) + (-1)^\kappa \gamma(\kappa, -1) - \frac{2}{\kappa} \right) [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \end{aligned} \tag{2.8}$$

Both of inequalities (2.7) and (2.8) rearrange to the required result. □

*Remark 2.3* The expression  $(-1)^\kappa \gamma(\kappa, -1)$  occurring in inequality (2.6) may not be clear for the readers, and they will imagine that this value is complex, or does it make sense? Actually, the complex part coming from  $(-1)^\kappa$  cancels out the complex part coming from the incomplete gamma  $\gamma(\kappa, -1)$ . Furthermore, this value came from the integral formula (2.1): from looking at the integral we can clearly see that it is real (and positive). Therefore, the answer is yes, it does make sense; the overall expression is real and positive.

On the other hand, we can clarify the above expression by using the Taylor expansion for the integral formula (2.1):

$$\begin{aligned} (-1)^\kappa \gamma(\kappa, -1) &= \int_0^1 \varsigma^{\kappa-1} e^\varsigma \, d\varsigma = \int_0^1 \varsigma^{\kappa-1} \left( 1 + \varsigma + \frac{\varsigma^2}{2!} + \dots \right) \, d\varsigma \\ &= \frac{1}{\kappa} + \frac{1}{\kappa + 1} + \frac{1}{2!(\kappa + 2)!} + \frac{1}{3!(\kappa + 3)} + \dots \\ &= \sum_{l=0}^\infty \frac{1}{l!(\kappa + l)}. \end{aligned}$$

This formula confirms that  $(-1)^\kappa \gamma(\kappa, -1) > 0$  for  $\kappa > 0$ .

*Remark 2.4* Inequality (2.6) with  $\kappa = 1$  becomes inequality (1.8).

**Theorem 2.1** Let  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  be  $L^1[\varepsilon_3, \varepsilon_4]$  with  $0 < \varepsilon_3 < \varepsilon_4$  and  $\kappa > 0$ . If  $|\bar{w}'|$  is an exp-convex function, then we have

$$\begin{aligned} & \left| \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2} - \frac{\Gamma(\kappa + 1)}{2(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right] \right| \\ & \leq \frac{\varepsilon_4 - \varepsilon_3}{2} \left( [\delta_0(\kappa, \bar{h}_0) + \delta_1(\kappa, \bar{h}_1)] |\bar{w}'(\varepsilon_3)| + [\delta_0(\kappa, \bar{h}_1) + \delta_1(\kappa, \bar{h}_0)] |\bar{w}'(\varepsilon_4)| \right. \\ & \quad \left. + [\delta_1(\kappa, \bar{h}_0) + \delta_0(\kappa, \bar{h}_1)] |\bar{w}'(\varepsilon_3)| + [\delta_1(\kappa, \bar{h}_1) + \delta_0(\kappa, \bar{h}_0)] |\bar{w}'(\varepsilon_4)| \right) \\ & = (\varepsilon_4 - \varepsilon_3) \frac{|\bar{w}'(\varepsilon_3)| + |\bar{w}'(\varepsilon_4)|}{2} \sum_{i=0}^1 \sum_{j=0}^1 \delta_i(\kappa, \bar{h}_j), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \delta_0(\kappa, \bar{h}_j) &= (-1)^j \bar{h}_j \left[ \gamma\left(\kappa + 1, (-1)^j\right) - \gamma\left(\kappa + 1, \frac{(-1)^j}{2}\right) \right] + (-1)^{j+1} \frac{1}{\kappa + 1} \left[ 1 - \left(\frac{1}{2}\right)^{\kappa+1} \right]; \\ \delta_1(\kappa, \bar{h}_j) &= \frac{1}{\kappa + 1} \left(\frac{1}{2}\right)^{\kappa+1} + (-1)^{j+1} \bar{h}_j \gamma\left(\kappa + 1, \frac{(-1)^j}{2}\right), \end{aligned}$$

and

$$\bar{h}_j = \begin{cases} e, & \text{if } j = 0, \\ (-1)^\kappa, & \text{if } j = 1. \end{cases}$$

*Proof* From Lemma 2.1, we have

$$\begin{aligned} & \left| \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2} - \frac{\Gamma(\kappa + 1)}{2(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right] \right| \\ & \leq \frac{\varepsilon_4 - \varepsilon_3}{2} \int_0^1 |(1 - \varsigma)^\kappa - \varsigma^\kappa| |\bar{w}'(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4)| \, d\varsigma \\ & = \frac{\varepsilon_4 - \varepsilon_3}{2} \left[ \int_0^{\frac{1}{2}} [(1 - \varsigma)^\kappa - \varsigma^\kappa] |\bar{w}'(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4)| \, d\varsigma \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 [\varsigma^\kappa - (1 - \varsigma)^\kappa] |\bar{w}'(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4)| \, d\varsigma \right]. \end{aligned}$$

By using the exp-convexity of  $|\bar{w}'|$ , it follows that

$$\begin{aligned} & \left| \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2} - \frac{\Gamma(\kappa + 1)}{2(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\varepsilon_3^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\varepsilon_4^-}^\kappa \bar{w}(\varepsilon_3) \right] \right| \\ & \leq \frac{\varepsilon_4 - \varepsilon_3}{2} \left[ \int_0^{\frac{1}{2}} [(1 - \varsigma)^\kappa - \varsigma^\kappa] [(e^\varsigma - 1) |\bar{w}'(\varepsilon_3)| + (e^{1-\varsigma} - 1) |\bar{w}'(\varepsilon_4)|] \, d\varsigma \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 [\varsigma^\kappa - (1 - \varsigma)^\kappa] [(e^\varsigma - 1) |\bar{w}'(\varepsilon_3)| + (e^{1-\varsigma} - 1) |\bar{w}'(\varepsilon_4)|] \, d\varsigma \right] \\ & = \frac{\varepsilon_4 - \varepsilon_3}{2} \left( [\delta_0(\kappa, \bar{h}_0) + \delta_1(\kappa, \bar{h}_1)] |\bar{w}'(\varepsilon_3)| + [\delta_0(\kappa, \bar{h}_1) + \delta_1(\kappa, \bar{h}_0)] |\bar{w}'(\varepsilon_4)| \right) \end{aligned}$$

$$\begin{aligned}
 &+ [\delta_1(\kappa, \hbar_0) + \delta_0(\kappa, \hbar_1)]|\bar{w}'(\varepsilon_3)| + [\delta_1(\kappa, \hbar_1) + \delta_0(\kappa, \hbar_0)]|\bar{w}'(\varepsilon_4)| \\
 &= \frac{\varepsilon_4 - \varepsilon_3}{2} [\delta_0(\kappa, \hbar_0) + \delta_1(\kappa, \hbar_1) + \delta_0(\kappa, \hbar_1) + \delta_1(\kappa, \hbar_0)](|\bar{w}'(\varepsilon_3)| + |\bar{w}'(\varepsilon_4)|),
 \end{aligned}$$

where the following identities are used:

$$\begin{aligned}
 \delta_0(\kappa, \hbar_0) &= \int_0^{\frac{1}{2}} (1 - \varsigma)^\kappa (e^\varsigma - 1) \, d\varsigma = \int_{\frac{1}{2}}^1 \varsigma^\kappa (e^{1-\varsigma} - 1) \, d\varsigma; \\
 \delta_0(\kappa, \hbar_1) &= \int_0^{\frac{1}{2}} (1 - \varsigma)^\kappa (e^{1-\varsigma} - 1) \, d\varsigma = \int_{\frac{1}{2}}^1 \varsigma^\kappa (e^\varsigma - 1) \, d\varsigma; \\
 \delta_1(\kappa, \hbar_0) &= - \int_0^{\frac{1}{2}} \varsigma^\kappa (e^{1-\varsigma} - 1) \, d\varsigma = - \int_{\frac{1}{2}}^1 (1 - \varsigma)^\kappa (e^\varsigma - 1) \, d\varsigma; \\
 \delta_1(\kappa, \hbar_1) &= - \int_0^{\frac{1}{2}} \varsigma^\kappa (e^\varsigma - 1) \, d\varsigma = - \int_{\frac{1}{2}}^1 (1 - \varsigma)^\kappa (e^{1-\varsigma} - 1) \, d\varsigma.
 \end{aligned}$$

Thus, our proof is completed. □

*Remark 2.5* Inequality (2.9) with  $\kappa = 1$  becomes the following inequality:

$$\begin{aligned}
 &\left| \frac{\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)}{2} - \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} \bar{w}(x) \, dx \right| \\
 &\leq (\varepsilon_4 - \varepsilon_3) \left( 4e^{\frac{1}{2}} - e - \frac{7}{2} \right) \frac{|\bar{w}'(\varepsilon_3)| + |\bar{w}'(\varepsilon_4)|}{2},
 \end{aligned} \tag{2.10}$$

which was obtained by Kadakal and İşcan [27].

### 2.2 Midpoint inequalities

**Proposition 2.2** *If  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  is an  $L^1$  and exp-convex function and  $\kappa > 0$ , then we have*

$$\begin{aligned}
 \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq \frac{(e^{\frac{1}{2}} - 1)2^\kappa \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \bar{w}(\varepsilon_3) \right] \\
 &\leq \kappa (e^{\frac{1}{2}} - 1) \left[ e 2^\kappa \gamma\left(\kappa, \frac{1}{2}\right) + (-2)^\kappa \gamma\left(\kappa, -\frac{1}{2}\right) - \frac{2}{\kappa} \right] [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)].
 \end{aligned} \tag{2.11}$$

*Proof* By the exp-convexity of  $f$ , we have

$$\begin{aligned}
 \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &= \bar{w}\left(\frac{\left[\frac{\varsigma}{2}\varepsilon_3 + \frac{2-\varsigma}{2}\varepsilon_4\right] + \left[\frac{2-\varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right]}{2}\right) \\
 &\leq (e^{\frac{1}{2}} - 1)\bar{w}\left(\frac{\varsigma}{2}\varepsilon_3 + \frac{2-\varsigma}{2}\varepsilon_4\right) + (e^{\frac{1}{2}} - 1)\bar{w}\left(\frac{2-\varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right).
 \end{aligned}$$

Multiplying by  $\varsigma^{\kappa-1}$  on both sides and then integrating over  $[0, 1]$ , we get

$$\begin{aligned}
 \frac{1}{\kappa} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq (e^{\frac{1}{2}} - 1) \int_0^1 \varsigma^{\kappa-1} \bar{w}\left(\frac{\varsigma}{2}\varepsilon_3 + \frac{2-\varsigma}{2}\varepsilon_4\right) \, d\varsigma \\
 &\quad + (e^{\frac{1}{2}} - 1) \int_0^1 \varsigma^{\kappa-1} \bar{w}\left(\frac{2-\varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right) \, d\varsigma.
 \end{aligned}$$

Multiplying by  $\kappa > 0$  on both sides and making the change of variables, we get

$$\begin{aligned} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &\leq \kappa(e^{\frac{1}{2}} - 1) \left[ \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_4}^{\frac{\varepsilon_3 + \varepsilon_4}{2}} \left(\frac{\varepsilon_4 - u}{\varepsilon_4 - \varepsilon_3}\right)^{\kappa-1} \bar{w}(u) \, du \right. \\ &\quad \left. + \frac{1}{\varepsilon_4 - \varepsilon_3} \int_{\varepsilon_3}^{\frac{\varepsilon_3 + \varepsilon_4}{2}} \left(\frac{v - \varepsilon_3}{\varepsilon_4 - \varepsilon_3}\right)^{\kappa-1} \bar{w}(v) \, dv \right] \\ &= \frac{(e^{\frac{1}{2}} - 1)2^\kappa \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \bar{w}(\varepsilon_3) \right]. \end{aligned} \tag{2.12}$$

On the other hand, we have by exp-convexity

$$\begin{aligned} \bar{w}\left(\frac{\varsigma}{2}\varepsilon_3 + \frac{2 - \varsigma}{2}\varepsilon_4\right) &\leq (e^{\frac{\varsigma}{2}} - 1)\bar{w}(\varepsilon_3) + (e^{1 - \frac{\varsigma}{2}} - 1)\bar{w}(\varepsilon_4); \\ \bar{w}\left(\frac{2 - \varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right) &\leq (e^{1 - \frac{\varsigma}{2}} - 1)\bar{w}(\varepsilon_3) + (e^{\frac{\varsigma}{2}} - 1)\bar{w}(\varepsilon_4). \end{aligned}$$

Adding both inequalities, we get

$$\bar{w}\left(\frac{\varsigma}{2}\varepsilon_3 + \frac{2 - \varsigma}{2}\varepsilon_4\right) + \bar{w}\left(\frac{2 - \varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right) \leq (e^{\frac{\varsigma}{2}} + e^{1 - \frac{\varsigma}{2}} - 2)[\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)].$$

Multiplying by  $\varsigma^{\kappa-1}$  on both sides and then integrating over  $[0, 1]$ , we get

$$\begin{aligned} &\int_0^1 \varsigma^{\kappa-1} \bar{w}\left(\frac{\varsigma}{2}\varepsilon_3 + \frac{2 - \varsigma}{2}\varepsilon_4\right) \, d\varsigma + \int_0^1 \varsigma^{\kappa-1} \bar{w}\left(\frac{2 - \varsigma}{2}\varepsilon_3 + \frac{\varsigma}{2}\varepsilon_4\right) \, d\varsigma \\ &\leq [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)] \int_0^1 \varsigma^{\kappa-1} (e^{\frac{\varsigma}{2}} + e^{1 - \frac{\varsigma}{2}} - 2) \, d\varsigma. \end{aligned}$$

By making the change of variables and Remark 2.2, we get

$$\begin{aligned} &\frac{2^\kappa \Gamma(\kappa)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \bar{w}(\varepsilon_3) \right] \\ &\leq \left[ e^{2^\kappa} \gamma\left(\kappa, \frac{1}{2}\right) + (-2)^\kappa \gamma\left(\kappa, -\frac{1}{2}\right) - \frac{2}{\kappa} \right] [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \end{aligned}$$

Multiplying by positive constants  $\kappa > 0$  and  $(e^{\frac{1}{2}} - 1) > 0$  on both sides, we get

$$\begin{aligned} &\frac{(e^{\frac{1}{2}} - 1)2^\kappa \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \bar{w}(\varepsilon_3) \right] \\ &\leq \kappa(e^{\frac{1}{2}} - 1) \left[ e^{2^\kappa} \gamma\left(\kappa, \frac{1}{2}\right) + (-2)^\kappa \gamma\left(\kappa, -\frac{1}{2}\right) - \frac{2}{\kappa} \right] [\bar{w}(\varepsilon_3) + \bar{w}(\varepsilon_4)]. \end{aligned} \tag{2.13}$$

Both of inequalities (2.12) and (2.13) rearrange to the required result. □



**Theorem 2.2** Let  $\bar{w} : [\varepsilon_3, \varepsilon_4] \rightarrow \mathcal{R}$  be  $L^1[\varepsilon_3, \varepsilon_4]$  with  $0 < \varepsilon_3 < \varepsilon_4$  and  $\kappa > 0$ . If  $|\bar{w}'|$  is an exp-convex function, then we have

$$\begin{aligned} & \left| \frac{2^{\kappa-1}\Gamma(\kappa+1)}{(\varepsilon_4-\varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{(\frac{\varepsilon_3+\varepsilon_4}{2})_+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{(\frac{\varepsilon_3+\varepsilon_4}{2})_-}^\kappa \bar{w}(\varepsilon_3) \right] - \bar{w}\left(\frac{\varepsilon_3+\varepsilon_4}{2}\right) \right| \\ & \leq (\varepsilon_4-\varepsilon_3) \frac{|\bar{w}'(\varepsilon_3)| + |\bar{w}'(\varepsilon_4)|}{4} \sum_{j=0}^1 \bar{\delta}(\kappa, \hbar_j), \end{aligned} \tag{2.14}$$

where  $\hbar_j$  is as before, and

$$\bar{\delta}(\kappa, \hbar_j) = (-1)^j \hbar_j 2^{\kappa+1} \gamma\left(\kappa+1, \frac{(-1)^j}{2}\right) - \frac{1}{\kappa+1}.$$

*Proof* With the help of Lemma 2.2 and the exp-convexity of  $|\bar{w}'|$ , we have

$$\begin{aligned} & \left| \frac{2^{\kappa-1}\Gamma(\kappa+1)}{(\varepsilon_4-\varepsilon_3)^\kappa} \left[ {}^{\text{RL}}\mathcal{J}_{(\frac{\varepsilon_3+\varepsilon_4}{2})_+}^\kappa \bar{w}(\varepsilon_4) + {}^{\text{RL}}\mathcal{J}_{(\frac{\varepsilon_3+\varepsilon_4}{2})_-}^\kappa \bar{w}(\varepsilon_3) \right] - \bar{w}\left(\frac{\varepsilon_3+\varepsilon_4}{2}\right) \right| \\ & \leq \frac{\varepsilon_4-\varepsilon_3}{4} \int_0^1 \zeta^\kappa \left[ \left| \bar{w}'\left(\frac{\zeta}{2}\varepsilon_3 + \frac{2-\zeta}{2}\varepsilon_4\right) \right| + \left| \bar{w}'\left(\frac{2-\zeta}{2}\varepsilon_3 + \frac{\zeta}{2}\varepsilon_4\right) \right| \right] d\zeta \\ & \leq \frac{\varepsilon_4-\varepsilon_3}{4} \left\{ \int_0^1 \zeta^\kappa \left[ (e^{\frac{\zeta}{2}}-1)|\bar{w}'(\varepsilon_3)| + (e^{\frac{2-\zeta}{2}}-1)|\bar{w}'(\varepsilon_4)| \right] d\zeta \right. \\ & \quad \left. + \int_0^1 \zeta^\kappa \left[ (e^{\frac{2-\zeta}{2}}-1)|\bar{w}'(\varepsilon_3)| + (e^{\frac{\zeta}{2}}-1)|\bar{w}'(\varepsilon_4)| \right] d\zeta \right\} \\ & = \frac{\varepsilon_4-\varepsilon_3}{4} \left\{ \bar{\delta}(\kappa, \hbar_1)|\bar{w}'(\varepsilon_3)| + \bar{\delta}(\kappa, \hbar_0)|\bar{w}'(\varepsilon_4)| + \bar{\delta}(\kappa, \hbar_0)|\bar{w}'(\varepsilon_3)| + \bar{\delta}(\kappa, \hbar_1)|\bar{w}'(\varepsilon_4)| \right\}, \end{aligned}$$

where the following identities are used:

$$\begin{aligned} \bar{\delta}(\kappa, \hbar_0) &= \int_0^1 \zeta^\kappa (e^{\frac{\zeta}{2}}-1) d\zeta; \\ \bar{\delta}(\kappa, \hbar_1) &= \int_0^1 \zeta^\kappa (e^{\frac{2-\zeta}{2}}-1) d\zeta. \end{aligned}$$

Thus, our proof is completed. □

*Remark 2.6* Inequality (2.14) with  $\kappa = 1$  becomes the following inequality:

$$\left| \frac{1}{\varepsilon_4-\varepsilon_3} \int_{\varepsilon_3}^{\varepsilon_4} \bar{w}(x) dx - \bar{w}\left(\frac{\varepsilon_3+\varepsilon_4}{2}\right) \right| \leq (\varepsilon_4-\varepsilon_3)(3+4e-8e^{\frac{1}{2}}) \frac{|\bar{w}'(\varepsilon_3)| + |\bar{w}'(\varepsilon_4)|}{4}. \tag{2.15}$$

### 3 He’s inequality

This section deals with the HH-inequality in the sense of He’s fractional derivatives as introduced in Definition 3.1. As we discussed before, there are many definitions on fractional derivatives in the literature. Herewith we recall the fractional derivatives by the variational iteration method [46, 47]. A complete review on variational iteration method and its application and development are available in references [48, 49].

Let us recall the following fractional derivative introduced by He [47].

**Definition 3.1** For any  $L^1$  function  $\bar{w}$  on an interval  $[0, s]$ , the  $\kappa$ th He's fractional derivative of  $\bar{w}(s)$  is defined by

$$D_s^\kappa \bar{w}(s) = \frac{1}{\Gamma(n - \kappa)} \frac{d^n}{ds^n} \int_0^s (\varsigma - s)^{n-\kappa-1} \bar{w}(\varsigma) d\varsigma.$$

Now, by making use of exp-convexity of  $\bar{w}$ , we have

$$\begin{aligned} \bar{w}\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) &= \bar{w}\left(\frac{[\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4] + [(1 - \varsigma)\varepsilon_3 + \varsigma \varepsilon_4]}{2}\right) \\ &\leq (e^{\frac{1}{2}} - 1) [\bar{w}(\varsigma \varepsilon_3 + (1 - \varsigma)\varepsilon_4) + \bar{w}((1 - \varsigma)\varepsilon_3 + \varsigma \varepsilon_4)]. \end{aligned} \tag{3.1}$$

Taking  $\varepsilon_3 = 0$  and  $\varepsilon_4 > 0$  for all  $s \in (0, 1)$ , multiplying by  $\frac{(\varsigma-s)^{n-\kappa-1}}{\Gamma(n-\kappa)}$  on both sides of (3.1), and integrating with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} &\frac{1}{\Gamma(n - \kappa)} \bar{w}\left(\frac{\varepsilon_4}{2}\right) \int_0^s (\varsigma - s)^{n-\kappa-1} d\varsigma \\ &\leq \frac{(e^{\frac{1}{2}} - 1)}{\Gamma(n - \kappa)} \left[ \int_0^s (t - s)^{n-\kappa-1} \bar{w}((1 - \varsigma)\varepsilon_4) d\varsigma + \int_0^s (\varsigma - s)^{n-\kappa-1} \bar{w}(\varsigma \varepsilon_4) d\varsigma \right]. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{(-1)^{n-\kappa-1} s^{n-\kappa}}{\Gamma(n - \kappa)} \bar{w}\left(\frac{\varepsilon_4}{2}\right) \\ &\leq \frac{(e^{\frac{1}{2}} - 1)}{\Gamma(n - \kappa)} \left[ \int_0^s (\varsigma - s)^{n-\kappa-1} \bar{w}(\varsigma \varepsilon_4) d\varsigma + \int_0^s (\varsigma - s)^{n-\kappa-1} \bar{w}((1 - \varsigma)\varepsilon_4) d\varsigma \right]. \end{aligned} \tag{3.2}$$

After getting the  $n$ -th derivatives on both sides of (3.2) with respect to  $s$  and using Definition 3.1, we obtain

$$(-1)^{n-\kappa-1} \bar{w}\left(\frac{\varepsilon_4}{2}\right) \leq \frac{(e^{\frac{1}{2}} - 1) s^\kappa}{\varepsilon_4^{n-\kappa}} [D_{sb}^\kappa \bar{w}(sb) + (-1)^{n-\kappa-1} D_{(1-s)b}^\kappa \bar{w}((1-s)b)]. \tag{3.3}$$

### 4 Examples

In this section, some examples in the frame of special functions, matrices, and fractional Zakharov–Kuznetsov functions are selected to fulfil the applicability of obtained results.

*Example 4.1* Let the function  $\mathcal{J}_{\bar{\rho}} : \mathcal{R} \rightarrow [1, \infty)$  be defined by [50]

$$\mathcal{J}_{\bar{\rho}}(z) = 2^{\bar{\rho}} \Gamma(\bar{\rho} + 1) z^{-\nu} I_{\bar{\rho}}(z), \quad z \in \mathcal{R}.$$

In our attempt, we consider the first kind modified Bessel function  $I_{\bar{\rho}}$ , given by [50]

$$I_{\bar{\rho}}(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{\bar{\rho}+2n}}{n! \Gamma(\bar{\rho} + n + 1)}.$$

Then, the first and second order derivatives of  $\mathcal{J}_{\bar{\rho}}(z)$  are given as follows:

$$\begin{aligned} \mathcal{J}'_{\bar{\rho}}(z) &= \frac{z}{2(\bar{\rho} + 1)} \mathcal{J}_{\bar{\rho}+1}(z), \\ \mathcal{J}''_{\bar{\rho}}(z) &= \frac{1}{4(\bar{\rho} + 1)} \left[ \frac{z^2}{\bar{\rho} + 2} \mathcal{J}_{\bar{\rho}+2}(z) + 2\mathcal{J}_{\bar{\rho}+1}(z) \right]. \end{aligned} \tag{4.1}$$

Let  $\bar{w}(z) := \mathcal{J}'_{\bar{\rho}}(z)$ . Then, with the help of Remark 2.5 and the two identities in (4.1), we can deduce

$$\begin{aligned} & \left| \frac{\varepsilon_3 \mathcal{J}_{\bar{\rho}+1}(\varepsilon_3) + \varepsilon_4 \mathcal{J}_{\bar{\rho}+1}(\varepsilon_4)}{4(\bar{\rho} + 1)} - \frac{\mathcal{J}_{\bar{\rho}}(\varepsilon_4) - \mathcal{J}_{\bar{\rho}}(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} \right| \\ & \leq (\varepsilon_4 - \varepsilon_3) \left( 4e^{\frac{1}{2}} - e - \frac{7}{2} \right) \left\{ \frac{\varepsilon_3^2 \mathcal{J}_{\bar{\rho}+2}(\varepsilon_3) + \varepsilon_4^2 \mathcal{J}_{\bar{\rho}+2}(\varepsilon_4)}{8(\bar{\rho} + 1)(\bar{\rho} + 2)} + \frac{\mathcal{J}_{\bar{\rho}+1}(\varepsilon_3) + \mathcal{J}_{\bar{\rho}+1}(\varepsilon_4)}{4(\bar{\rho} + 1)} \right\}. \end{aligned}$$

Also, from Remark 2.6 and the two identities in (4.1), we can deduce

$$\begin{aligned} & \left| \frac{\mathcal{J}_{\bar{\rho}}(\varepsilon_4) - \mathcal{J}_{\bar{\rho}}(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} - \mathcal{J}_{\bar{\rho}+1} \left( \frac{\varepsilon_3 + \varepsilon_4}{2} \right) \right| \\ & \leq (\varepsilon_4 - \varepsilon_3) (3 + 4e - 8e^{\frac{1}{2}}) \left\{ \frac{\varepsilon_3^2 \mathcal{J}_{\bar{\rho}+2}(\varepsilon_3) + \varepsilon_4^2 \mathcal{J}_{\bar{\rho}+2}(\varepsilon_4)}{16(\bar{\rho} + 1)(\bar{\rho} + 2)} + \frac{\mathcal{J}_{\bar{\rho}+1}(\varepsilon_3) + \mathcal{J}_{\bar{\rho}+1}(\varepsilon_4)}{8(\bar{\rho} + 1)} \right\} \end{aligned}$$

for  $\bar{\rho} > -1$ ,  $\varepsilon_3, \varepsilon_4 \in \mathcal{R}$  with  $0 < \varepsilon_3 < \varepsilon_4$ .

*Example 4.2* In this example, we deal with the second kind modified Bessel function  $\mathcal{K}_{\bar{\rho}}$ , given by [50]

$$\mathcal{K}_{\bar{\rho}}(z) = \frac{\pi}{2} \frac{\mathcal{J}_{-\bar{\rho}}(z) + \mathcal{J}_{\bar{\rho}}(z)}{\sin(\bar{\rho}\pi)}.$$

Let  $\bar{w}(z) := -\left(\frac{\mathcal{K}_{\bar{\rho}}(z)}{z^{\bar{\rho}}}\right)'$  with  $\bar{\rho} \in \mathcal{R}$ . Following [50], we have the following integral representation:

$$\mathcal{K}_{\bar{\rho}}(z) = \int_0^\infty e^{-z \cosh t} \cosh(\bar{\rho}t) dt, \quad z > 0.$$

One can easily observe that the function  $z \mapsto \mathcal{K}_{\bar{\rho}}(z)$  is completely monotonic on  $(0, \infty)$  for each  $\bar{\rho} \in \mathcal{R}$ . Also, we know that the product of two completely monotonic functions is also completely monotonic, then  $z \mapsto \bar{w}(z)$  is a strictly completely monotonic function on  $(0, \infty)$  for each  $\bar{\rho} > 1$ . Thus, the function

$$\begin{aligned} \bar{w}(z) &= -\left(\frac{\mathcal{K}_{\bar{\rho}}(z)}{z^{\bar{\rho}}}\right)' = \frac{\mathcal{K}_{\bar{\rho}+1}(z)}{z^{\bar{\rho}}}; \\ \bar{w}'(z) &= \frac{\mathcal{K}_{\bar{\rho}+2}(z)}{z^{\bar{\rho}}} - \frac{\mathcal{K}_{\bar{\rho}+1}(z)}{z^{\bar{\rho}+1}} \end{aligned} \tag{4.2}$$

becomes strictly completely monotonic on  $(0, \infty)$  for each  $\bar{\rho} > 1$ . Hence  $\bar{w}(z)$  becomes a convex function. Then, with the help of identities (4.2), and Remark 2.5 and Remark 2.6,

respectively, we can deduce

$$\begin{aligned} & \left| \frac{\varepsilon_3^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_4) + \varepsilon_4^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_3)}{2} + \frac{\varepsilon_3^{\bar{\rho}} \mathcal{K}_{\bar{\rho}}(\varepsilon_4) - \varepsilon_4^{\bar{\rho}} \mathcal{K}_{\bar{\rho}}(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} \right| \\ & \leq (\varepsilon_4 - \varepsilon_3) \left( 4e^{\frac{1}{2}} - e - \frac{7}{2} \right) \\ & \quad \times \left\{ \frac{\varepsilon_4^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_3) + \varepsilon_3^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_4)}{2} + \frac{\varepsilon_4^{\bar{\rho}+1} \mathcal{K}_{\bar{\rho}+2}(\varepsilon_3) + \varepsilon_3^{\bar{\rho}+1} \mathcal{K}_{\bar{\rho}+2}(\varepsilon_4)}{2\varepsilon_3 \varepsilon_4} \right\} \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{\varepsilon_3^{\bar{\rho}} \mathcal{K}_{\bar{\rho}}(\varepsilon_4) - \varepsilon_4^{\bar{\rho}} \mathcal{K}_{\bar{\rho}}(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} + \left( \frac{2\varepsilon_3 \varepsilon_4}{\varepsilon_3 + \varepsilon_4} \right)^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1} \left( \frac{\varepsilon_3 + \varepsilon_4}{2} \right) \right| \\ & \leq (\varepsilon_4 - \varepsilon_3) (3 + 4e - 8e^{\frac{1}{2}}) \\ & \quad \times \left\{ \frac{\varepsilon_4^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_3) + \varepsilon_3^{\bar{\rho}} \mathcal{K}_{\bar{\rho}+1}(\varepsilon_4)}{4} + \frac{\varepsilon_4^{\bar{\rho}+1} \mathcal{K}_{\bar{\rho}+2}(\varepsilon_3) + \varepsilon_3^{\bar{\rho}+1} \mathcal{K}_{\bar{\rho}+2}(\varepsilon_4)}{4\varepsilon_3 \varepsilon_4} \right\} \end{aligned}$$

for each  $\bar{\rho} > 1$  and  $\varepsilon_3, \varepsilon_4 \in \mathcal{R}$  with  $0 < \varepsilon_3 < \varepsilon_4$ .

*Example 4.3* In this example, we deal with the  $q$ -digamma function  $\Psi_\rho$ , given by [50]

$$\begin{aligned} \Psi_\rho(z) &= -\ln(1 - \rho) + \ln(\rho) \sum_{t=0}^{\infty} \frac{\rho^{t+z}}{1 - \rho^{t+z}} \\ &= -\ln(1 - \rho) + \ln(\rho) \sum_{t=1}^{\infty} \frac{\rho^{tz}}{1 - \rho^{tz}} \quad \text{for } 0 < \rho < 1, \end{aligned}$$

or equivalently,

$$\begin{aligned} \Psi_\rho(z) &= -\ln(\rho - 1) + \ln(\rho) \left( z - \frac{1}{2} - \sum_{t=0}^{\infty} \frac{\rho^{-(t+z)}}{1 - \rho^{-(t+z)}} \right) \\ &= -\ln(\rho - 1) + \ln(\rho) \left( z - \frac{1}{2} - \sum_{t=1}^{\infty} \frac{\rho^{-tz}}{1 - \rho^{-tz}} \right) \quad \text{for } \rho > 1 \text{ and } z > 0. \end{aligned}$$

From the above definitions, we can observe that the function  $z \mapsto \Psi'_\rho(z)$  is completely monotonic on  $(0, \infty)$  for each  $\rho > 0$ . Consequently, we see that  $z \mapsto \Psi'_\rho(z)$  is a convex function on  $(0, \infty)$ .

Set  $\bar{w}(z) := \Psi'_\rho(z)$  with  $\rho > 0$ , then we see that  $\bar{w}'(z) := \Psi''_\rho(z)$  is completely monotonic on  $(0, \infty)$ . Then, with the help of Remark 2.5, we can obtain

$$\Psi'_\rho \left( \frac{\varepsilon_3 + \varepsilon_4}{2} \right) \leq \left| \frac{\Psi_\rho(\varepsilon_4) - \Psi_\rho(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} \right| \leq \frac{\Psi'_\rho(\varepsilon_3) + \Psi'_\rho(\varepsilon_4)}{2}. \tag{4.3}$$

Combining inequalities (1.5) and (2.10), we obtain

$$\left| \frac{\Psi'_\rho(\varepsilon_3) + \Psi'_\rho(\varepsilon_4)}{2} - \frac{\Psi_\rho(\varepsilon_4) - \Psi_\rho(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} \right| \leq \frac{\varepsilon_4 - \varepsilon_3}{2} \left( 4e^{\frac{1}{2}} - e - \frac{7}{2} \right) (|\Psi''_\rho(\varepsilon_3)| + |\Psi''_\rho(\varepsilon_4)|).$$

Also, combining inequalities (1.5) and (2.15), we obtain

$$\left| \frac{\Psi_\rho(\varepsilon_4) - \Psi_\rho(\varepsilon_3)}{\varepsilon_4 - \varepsilon_3} - \Psi'_\rho\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right) \right| \leq \frac{\varepsilon_4 - \varepsilon_3}{4} (3 + 4e - 8e^{\frac{1}{2}}) (|\Psi''_\rho(\varepsilon_3)| + |\Psi''_\rho(\varepsilon_4)|).$$

*Example 4.4* We denote by  $\mathbb{C}^n$  the set of  $n \times n$  complex matrices, by  $\mathbb{M}_n$  the algebra of  $n \times n$  complex matrices, and by  $\mathbb{M}_n^+$  the strictly positive matrices in  $\mathbb{M}_n$ . That is,  $A \in \mathbb{M}_n^+$  if  $\langle Ax, x \rangle > 0$  for all nonzero  $x \in \mathbb{C}^n$ .

In [51], Sababheh proved that the function

$$f(v) = \|A^v XB^{1-v} + A^{1-v} XB^v\|, \quad A, B \in \mathbb{M}_n^+, X \in \mathbb{M}_n$$

is convex for all  $v \in [0, 1]$ . Then, from [27], this nonnegative function is exp-convex on  $[0, 1]$ .

Then, by using Propositions 2.1 and 2.2 with  $A, B \in M_n^+, X \in M_n$ , respectively, we have

$$\begin{aligned} & \|A^{\frac{\varepsilon_3 + \varepsilon_4}{2}} XB^{1 - \frac{\varepsilon_3 + \varepsilon_4}{2}} + A^{1 - \frac{\varepsilon_3 + \varepsilon_4}{2}} XB^{\frac{\varepsilon_3 + \varepsilon_4}{2}}\| \\ & \leq \frac{(e^{\frac{1}{2}} - 1)\Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \left[ \mathbf{J}_{\varepsilon_3^+}^\kappa \|A^{\varepsilon_4} XB^{1 - \varepsilon_4} + A^{1 - \varepsilon_4} XB^{\varepsilon_4}\| + \mathbf{J}_{\varepsilon_4^-}^\kappa \|A^{\varepsilon_3} XB^{1 - \varepsilon_3} + A^{1 - \varepsilon_3} XB^{\varepsilon_3}\| \right] \\ & \leq \kappa (e^{\frac{1}{2}} - 1) \left( e \gamma(\kappa, 1) + (-1)^\kappa \gamma(\kappa, -1) - \frac{2}{\kappa} \right) \\ & \quad \times \left[ \|A^{\varepsilon_3} XB^{1 - \varepsilon_3} + A^{1 - \varepsilon_3} XB^{\varepsilon_3}\| + \|A^{\varepsilon_4} XB^{1 - \varepsilon_4} + A^{1 - \varepsilon_4} XB^{\varepsilon_4}\| \right] \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} & \|A^{\frac{\varepsilon_3 + \varepsilon_4}{2}} XB^{1 - \frac{\varepsilon_3 + \varepsilon_4}{2}} + A^{1 - \frac{\varepsilon_3 + \varepsilon_4}{2}} XB^{\frac{\varepsilon_3 + \varepsilon_4}{2}}\| \\ & \leq \frac{(e^{\frac{1}{2}} - 1)2^\kappa \Gamma(\kappa + 1)}{(\varepsilon_4 - \varepsilon_3)^\kappa} \\ & \quad \times \left[ \mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^+}^\kappa \|A^{\varepsilon_4} XB^{1 - \varepsilon_4} + A^{1 - \varepsilon_4} XB^{\varepsilon_4}\| + \mathbf{J}_{\left(\frac{\varepsilon_3 + \varepsilon_4}{2}\right)^-}^\kappa \|A^{\varepsilon_3} XB^{1 - \varepsilon_3} + A^{1 - \varepsilon_3} XB^{\varepsilon_3}\| \right] \\ & \leq \kappa (e^{\frac{1}{2}} - 1) \left( e 2^\kappa \gamma\left(\kappa, \frac{1}{2}\right) + (-2)^\kappa \gamma\left(\kappa, -\frac{1}{2}\right) - \frac{2}{\kappa} \right) \\ & \quad \times \left[ \|A^{\varepsilon_3} XB^{1 - \varepsilon_3} + A^{1 - \varepsilon_3} XB^{\varepsilon_3}\| + \|A^{\varepsilon_4} XB^{1 - \varepsilon_4} + A^{1 - \varepsilon_4} XB^{\varepsilon_4}\| \right] \end{aligned} \tag{4.5}$$

for all  $\varepsilon_3, \varepsilon_4 \in [0, 1]$ , where  $\varepsilon_3 < \varepsilon_4$  and  $\kappa > 0$ .

The following two examples are dedicated to Sect. 3.

*Example 4.5* Consider the fractional Zakharov–Kuznetsov ZK(2, 2, 2) equation [52]:

$$D_s^\kappa u + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0. \tag{4.6}$$

Denote  $f = u(x, y, T)$ , where  $T = \frac{t^\kappa}{\Gamma(1 + \kappa)}$ . Also, suppose that  $\varepsilon_4 > 0$  for all  $s \in (0, 1)$ . From equation (4.6), we have

$$D_s^\kappa u = - \left[ (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} \right]. \tag{4.7}$$

Applying inequality (3.3), we get

$$\begin{aligned}
 & (-1)^{n-\kappa-1} u\left(x, y, \frac{\varepsilon_4}{2}\right) \\
 & \leq \frac{(e^{\frac{1}{2}} - 1)s^\kappa}{\varepsilon_4^{n-\kappa}} \left[ D_{s\varepsilon_4}^\kappa u(x, y, s\varepsilon_4) + (-1)^{n-\kappa-1} D_{(1-s)\varepsilon_4}^\kappa u(x, y, (1-s)\varepsilon_4) \right], \tag{4.8}
 \end{aligned}$$

where  $D_\zeta^\kappa u$  is defined by (4.7).

*Example 4.6* Consider the fractional Zakharov–Kuznetsov ZK(3, 3, 3) equation [52]

$$D_\zeta^\kappa w + (w^3)_x + 2(w^3)_{xxx} + 2(w^3)_{yyx} = 0. \tag{4.9}$$

Let us denote  $f = w(x, y, T)$ , where  $T = \frac{t^\kappa}{\Gamma(1+\kappa)}$ . Also, suppose that  $\varepsilon_4 > 0$  for all  $s \in (0, 1)$ . From equation (4.9), we have

$$D_\zeta^\kappa w = -[(w^3)_x + 2(w^3)_{xxx} + 2(w^3)_{yyx}]. \tag{4.10}$$

Applying inequality (3.3), we obtain

$$\begin{aligned}
 & (-1)^{n-\kappa-1} w\left(x, y, \frac{\varepsilon_4}{2}\right) \\
 & \leq \frac{(e^{\frac{1}{2}} - 1)s^\kappa}{\varepsilon_4^{n-\kappa}} \left[ D_{s\varepsilon_4}^\kappa w(x, y, s\varepsilon_4) + (-1)^{n-\kappa-1} D_{(1-s)\varepsilon_4}^\kappa w(x, y, (1-s)\varepsilon_4) \right], \tag{4.11}
 \end{aligned}$$

where  $D_\zeta^\kappa w$  is defined by (4.10).

### 5 Conclusion

The study dealt with investigating new inequalities of HH-type for the new type of convex functions, namely the exp-convex function. The new results are established via the Riemann–Liouville fractional operators. Finally, we have applied our findings on special functions. By examining this, we can see the usefulness and efficiency of our results.

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