OPEN ACCESS
Fractals, Vol. 28, No. 8 (2020) 2040003 (11 pages)

DOI: 10.1142/S0218348X20400034

NEW GENERALIZATIONS IN THE SENSE OF THE WEIGHTED NON-SINGULAR FRACTIONAL INTEGRAL OPERATOR

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Received January 5, 2020
Accepted February 23, 2020
Published June 25, 2020

Abstract

In this paper, we propose a new fractional operator which is based on the weight function for Atangana–Baleanu ($\mathcal{AB}$)-fractional operators. A motivating characteristic is the generalization of classical variants within the weighted $\mathcal{AB}$-fractional integral. We aim to establish Minkowski

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and reverse Hölder inequalities by employing weighted $\mathcal{AB}$-fractional integral. The consequences demonstrate that the obtained technique is well-organized and appropriate.

Keywords: Weighted $\mathcal{AB}$-Fractional Operator; Minkowski Inequality; Reverse Hölder Inequality.

1. INTRODUCTION

These days the fractional calculus has a significant job in assorted logical fields because of its fertile utilities in dynamical issues regarding engineering, information technology, image processing, signal processing, fluid dynamics, relativity theory, porous media and numerous others.[1–18] Many researchers focused on theories of earlier outcomes regarding the generalizations, descriptions, characterizations, design and so forth. The nonlocal fractional derivatives are mainly of two types, for example, the classical ones consisting of singular kernels (the Riemann–Liouville and Caputo derivatives) and the other ones have been presented with nonsingular kernels (the Atangana–Baleanu and Caputo–Fabrizio derivatives). Despite the fact that there are no concrete numerical avocations of innovative types of fractional operators, they acquired the attraction of numerous analysts in light of their advent in various issues, for recent modifications of fractional derivatives with nonsingular kernels, we recommend Refs. 19–27.

A decade ago, the study used to generalize integrals and derivatives of complex orders, specifically, integrals including variants depend on fractional calculus. Recently, a new formula with the aid of fractional operators was proposed by Dumitru and Fernandez[28] by inserting Mittag-Leffler as kernels, specifically, integrals including inequalities. The higher-order differential equations and fractional operators with several kinds of kernels ($\mathcal{AB}$-fractional operators) have been investigated in regards of increasingly hypothetical ideas, while this new scheme of study played a crucial role to establish the semigroup property.[19,29] Rashid *et al.*[30] derived variants for a class of exponentially convex function by means of the extended Mittag-Leffler functions and also extended this concept in preinvex functions. Jarad *et al.*[31] established a class of ordinary differential equations in the frame of $\mathcal{AB}$-fractional derivative. In Ref. 32, the authors derived Gronwall inequality and discussed its applications to fractional-order $\mathcal{AB}$-differential equation.

The multifaceted nature of uses informs scientists to expand the characterizations concerning the fractional operators. Consequently, numerous scientists have proposed the novel weighted versions of fractional derivatives. The hypothesis and utilizations for the weighted Caputo and Riemann–Liouville derivatives have been investigated in Refs. 33–40. Additionally, researchers are compelled to utilize the weighted fractional derivatives for the exploration of various kinds of variants in an exquisite manner.[33] In Ref. 22, authors contemplated the linear and nonlinear fractional differential equations by proposing the weighted Caputo–Fabrizio fractional operators and concentrated its properties in Laplace transform. The development of classical variants in the sense of fractional operators is supposed as an intriguing part of the information theory. Numerous variants with various fractional operators for the existence and uniqueness of the solutions of fractional shörodinger equations having kernels with singular and nonsingular have been established. The Mittag-Leffler functions as kernel have been used as a generalization of the classical inequalities. Additional associated work can be searched in Refs. 41–48.

Inequality is an indispensable tool in all branches of mathematics,[49–55] it has wide applications in many other natural and human social sciences.[56–62]

This paper means to stretch out the investigation to the $\mathcal{AB}$-fractional operators. We present the weighted $\mathcal{AB}$-fractional operators and the new approach take into account for establishing the generalizations of several kind of inequalities. The novelties are a combination of the Minkowski inequality and reverse Hölder inequalities. Our outcomes are more broad and pertinent than the existing results. There are numerous descriptions of fractional operators, for instance, Riemann–Liouville, Hadamard, Liouville, Weyl, Erdelyi–Kober, and Katugampola, which can be supposed for acquiring similar outcomes.

Next, we demonstrate some preliminaries concerning to the $\mathcal{AB}$-fractional operator and weighted $\mathcal{AB}$-fractional operator.

Definition 1.1 (Ref. 63). For $0 < \beta < 1$, then the left-sided $\mathcal{AB}$-Caputo fractional derivative of a function $\mathcal{F} \in C^*(x, y)$ is stated as

$$ {}^{\mathcal{ABC}}\mathcal{D}_x^{\beta,\lambda}\mathcal{F}(\lambda) = \frac{\mathbb{M}(\beta)}{1-\beta}\int_x^\lambda \mathcal{F}'(s)E_\beta \times \left[\frac{-\beta(\lambda-s)^\beta}{1-\beta}\right] ds, \quad (1.1) $$

where the normalization function $\mathbb{M}(\beta) > 0$ satisfies the conditions $\mathbb{M}(0) = \mathbb{M}(1) = 1$ and E_β stands for the Mittag-Leffler function.

Definition 1.2 (Ref. 64). For $0 < \beta < 1$, then the left-sided $\mathcal{AB}$-fractional integral of a function $\mathcal{F} \in C^*[x, y]$ is stated as

$$ {}^{\mathcal{AB}}\mathcal{T}_x^{\beta,\lambda}\mathcal{F}(\lambda) = \frac{1-\beta}{\mathbb{M}(\beta)}\mathcal{F}(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)} \times \int_x^\lambda \mathcal{F}(s)(\lambda - s)^{\beta-1}ds, \quad (1.2) $$

where the normalization function $\mathbb{M}(\beta) > 0$ satisfies the conditions $\mathbb{M}(0) = \mathbb{M}(1) = 1$.

We start the novel methodology and initiate the concept of the left $\mathcal{AB}$-fractional derivative of a function $\mathcal{F}(\lambda)$ in the sense of weight function $\omega(\lambda)$.

Definition 1.3. For $0 < \beta < 1$, then the left-sided weighted $\mathcal{AB}$-Caputo fractional derivative of a function $\mathcal{F} \in C^*[x, y]$ in the sense of weight function $\omega(\lambda)$ is stated as

$$ {}^{\mathcal{AB}}\mathcal{D}_{x,\omega}^{\beta,\lambda}\mathcal{F}(\lambda) = \frac{\mathbb{M}(\beta)}{1-\beta}\frac{1}{\omega(\lambda)}\int_x^\lambda E_\beta\left[-\frac{\beta}{1-\beta}(\lambda - s)^\beta\right] \times \frac{d}{ds}(\omega\mathcal{F}(s))ds, \quad (1.3) $$

where the normalization function $\mathbb{M}(\beta) > 0$ satisfies the conditions $\mathbb{M}(0) = \mathbb{M}(1) = 1$ and $\omega(\lambda) \in C^*[x, y]$ such that $\mathcal{F}' \in L[x, y]$.

Definition 1.4. For $0 < \beta < 1$, then the weighted $\mathcal{AB}$-fractional integral of a function $\mathcal{F} \in C^*[x, y]$ in the sense of weight function $\omega(\lambda)$ is stated as

$$ {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mathcal{F}(\lambda) = \frac{1-\beta}{\mathbb{M}(\beta)}\mathcal{F}(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)}\frac{1}{\omega(\lambda)} \times \int_x^\lambda \omega(s)\mathcal{F}(s)(\lambda - s)^{\beta-1}ds, \quad (1.4) $$

where the normalization function $\mathbb{M}(\beta) > 0$ satisfies the conditions $\mathbb{M}(0) = \mathbb{M}(1) = 1$.

Remark 1.1. If we choose $\omega(\lambda) = 1$, then the weighted $\mathcal{AB}$-fractional integral, concurs with the ordinary $\mathcal{AB}$-fractional integral.

2. MINKOWSKI INEQUALITY IN THE FRAME OF WEIGHTED $\mathcal{AB}$-FRACTIONAL

Theorem 2.1. *For $0 < \beta < 1$ and $\alpha \geq 1$, and suppose that two positive functions $\mu, \nu \in \mathbb{C}_\beta[x, y]$ defined on $[0, \infty)$ with respect to the weight function w such that ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda) < \infty$ and ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda) < \infty$, $\lambda > x$. If $0 < \delta \leq \frac{\mu(\lambda)}{\nu(\lambda)} \leq \gamma$ for some $\delta, \gamma \in \mathbb{R}_+$ and $\forall\lambda \in [x, y]$, then the following weighted $\mathcal{AB}$-fractional integral inequality holds:*

$$ \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^\alpha(\lambda)\right)^{\frac{1}{\alpha}} + \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^\alpha(\lambda)\right)^{\frac{1}{\alpha}} \leq \mathcal{H}_1\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda) + \nu(\lambda)\right)^\alpha\right]^{\frac{1}{\alpha}}, \quad (2.1) $$

where $\mathcal{H}_1 = \frac{\gamma(\delta+1)+(\gamma+1)}{(1+\delta)(1+\gamma)}$.

Proof. Under the assumption $\frac{\mu(\lambda)}{\nu(\lambda)} \leq \gamma$, one obtains

$$ \mu(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)\left(\mu(\lambda) + \nu(\lambda)\right). \quad (2.2) $$

Applying the αth power on both sides of (2.2), we have

$$ \mu^\alpha(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^\alpha\left(\mu(\lambda) + \nu(\lambda)\right)^\alpha. \quad (2.3) $$

Taking product on both sides of (2.3) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$ \frac{1-\beta}{\mathbb{M}(\beta)}\mu^\alpha(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^\alpha\frac{1-\beta}{\mathbb{M}(\beta)}\left(\mu(\lambda) + \nu(\lambda)\right)^\alpha. \quad (2.4) $$

Further, replacing λ by s in (2.3) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to the variable s, we get

$$ \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda - s)^{\beta-1}\omega(s)\mu^\alpha(s)ds \leq \left(\frac{\gamma}{\gamma+1}\right)^\alpha\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda - s)^{\beta-1} \times \omega(s)\left(\mu(s) + \nu(s)\right)^\alpha ds. \quad (2.5) $$

Summing up (2.4) and (2.5), we obtain

$$ \frac{1-\beta}{\mathbb{M}(\beta)}\mu^\alpha(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \times \int_x^\lambda(\lambda - s)^{\beta-1}\omega(s)\mu^\alpha(s)ds \leq \left(\frac{\gamma}{\gamma+1}\right)^\alpha\left[\frac{1-\beta}{\mathbb{M}(\beta)}\left(\mu(\lambda) + \nu(\lambda)\right)^\alpha + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda - s)^{\beta-1} \times \omega(s)\left(\mu(s) + \nu(s)\right)^\alpha ds\right]. $$

This implies

$$ {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha}(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}. \tag{2.6} $$

Taking the $\frac{1}{\alpha}$th power of both sides of (2.6), we find

$$ \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha}(\lambda)\right)^{\frac{1}{\alpha}} \leq \left(\frac{\gamma}{\gamma+1}\right)\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}\right]^{\frac{1}{\alpha}}. \tag{2.7} $$

On the other hand, by using the condition $0<\delta\leq\frac{\mu(\lambda)}{\nu(\lambda)}$, we get

$$ \nu^{\alpha}(\lambda) \leq \frac{1}{(1+\delta)^{\alpha}}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}. \tag{2.8} $$

Taking product on both sides of (2.8) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$ \frac{1-\beta}{\mathbb{M}(\beta)}\nu^{\alpha}(\lambda) \leq \frac{1}{(1+\delta)^{\alpha}}\frac{1-\beta}{\mathbb{M}(\beta)}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}. \tag{2.9} $$

Further, replacing λ by s in (2.9) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$ \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^{\lambda}(\lambda-s)^{\beta-1}\omega(s)\nu^{\alpha}(s)ds \leq \frac{1}{(1+\delta)^{\alpha}}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^{\lambda}(\lambda-s)^{\beta-1} \times\omega(s)\left(\mu(s)+\nu(s)\right)^{\alpha}ds. \tag{2.10} $$

Summing up (2.9) and (2.10), one obtains

$$ \frac{1-\beta}{\mathbb{M}(\beta)}\nu^{\alpha}(\lambda)+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \times\int_x^{\lambda}(\lambda-s)^{\beta-1}\omega(s)\nu^{\alpha}(s)ds \leq \frac{1}{(1+\delta)^{\alpha}}\left[\frac{1-\beta}{\mathbb{M}(\beta)}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha} + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^{\lambda}(\lambda-s)^{\beta-1} \times\omega(s)\left(\mu(s)+\nu(s)\right)^{\alpha}ds\right], $$

which can be expressed as

$$ {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha}(\lambda) \leq \left(\frac{1}{\delta+1}\right)^{\alpha} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}. \tag{2.11} $$

Applying the $\frac{1}{p}$th power on both sides of (2.11), we find

$$ \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha}(\lambda)\right)^{\frac{1}{\alpha}} \leq \frac{1}{\delta+1}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}\right]^{\frac{1}{\alpha}}. \tag{2.12} $$

Adding (2.7) and (2.11) we obtain

$$ \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha}(\lambda)\right)^{\frac{1}{\alpha}}+\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha}(\lambda)\right)^{\frac{1}{\alpha}} \leq \mathcal{H}_1\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha}\right]^{\frac{1}{\alpha}}. $$

This completes the proof. □

3. HÖLDER-TYPE INEQUALITIES FOR WEIGHTED $\mathcal{AB}$-FRACTIONAL INTEGRAL OPERATOR

Theorem 3.1. *For $0<\beta<1$, $\alpha_1,\alpha_2>1$, with $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$ and suppose that two positive functions $\mu,\nu\in\mathbb{C}_{\beta}[x,y]$ defined on $[0,\infty)$ with respect to the weight function w such that ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)<\infty$ and ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)<\infty$ for all $\lambda>x$. If $0<\delta\leq\frac{\mu(\lambda)}{\nu(\lambda)}\leq\gamma$ for some $\delta,\gamma\in\mathbb{R}_+$ and for all $\lambda\in[x,y]$, then the following weighted $\mathcal{AB}$-fractional integral inequality holds*:

$$ \left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\right)^{\frac{1}{\alpha_1}}\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)\right)^{\frac{1}{\alpha_2}} \leq \mathcal{H}_2\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right)\right], \tag{3.1} $$

where $\mathcal{H}_2=\left(\frac{\gamma}{\delta}\right)^{\frac{1}{\alpha_1\alpha_2}}$.

Proof. Utilizing the supposition $\frac{\mu(\lambda)}{\nu(\lambda)}\leq\gamma$, we get

$$ \mu^{\frac{1}{\alpha_2}}(\lambda)\leq\gamma^{\frac{1}{\alpha_2}}\nu^{\frac{1}{\alpha_2}}(\lambda). \tag{3.2} $$

Taking product (3.2) by $\mu^{\frac{1}{\alpha_1}}$ and using the assumption $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$, we have

$$ \mu(\lambda)\leq\gamma^{\frac{1}{\alpha_2}}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda). \tag{3.3} $$

Taking product on both sides by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$ \frac{1-\beta}{\mathbb{M}(\beta)}\mu(\lambda)\leq\gamma^{\frac{1}{\alpha_2}}\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda). \tag{3.4} $$

Again, replacing λ by s in (3.3) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\mu(s)ds \leq \gamma^{\frac{1}{\alpha_2}}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1} \times\omega(s)\mu^{\frac{1}{\alpha_1}}(s)\nu^{\frac{1}{\alpha_2}}(s)ds. \tag{3.5}$$

Now, by adding (3.4) and (3.5), we find

$$\frac{1-\beta}{\mathbb{M}(\beta)}\mu(\lambda)+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\mu(s)ds \leq \gamma^{\frac{1}{\alpha_2}}\left[\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1} \times\omega(s)\mu^{\frac{1}{\alpha_1}}(s)\nu^{\frac{1}{\alpha_2}}(s)ds\right].$$

This implies

$${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\leq\gamma^{\frac{1}{\alpha_2}}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right]. \tag{3.6}$$

Applying the $\frac{1}{\alpha_1}$th power on both sides of (3.6), we get

$$\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\right)^{\frac{1}{\alpha_1}} \leq\gamma^{\frac{1}{\alpha_1\alpha_2}}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right]^{\frac{1}{\alpha_1}}. \tag{3.7}$$

Now, by using the supposition $\delta\leq\frac{\mu(\lambda)}{\nu(\lambda)}$, we have

$$\nu^{\frac{1}{\alpha_1}}(\lambda)\leq\delta^{\frac{-1}{\alpha}}\mu^{\frac{1}{\alpha_1}}(\lambda). \tag{3.8}$$

Taking product on both sides of (3.8) by $\nu^{\frac{1}{\alpha_2}}$, we get

$$\nu(\lambda)\leq\delta^{\frac{-1}{\alpha}}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}. \tag{3.9}$$

Again, taking product on both sides of (3.9) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$\frac{1-\beta}{\mathbb{M}(\beta)}\nu(\lambda)\leq\delta^{\frac{-1}{\alpha}}\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda). \tag{3.10}$$

Replacing λ by s in (3.9), then taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\nu(s)ds \leq\delta^{\frac{-1}{\alpha}}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1} \times\omega(s)\mu^{\frac{1}{\alpha_1}}(s)\nu^{\frac{1}{\alpha_2}}(s)ds. \tag{3.11}$$

Now, summing up (3.10) and (3.12) we find

$$\frac{1-\beta}{\mathbb{M}(\beta)}\nu(\lambda)+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\nu(s)ds \leq\delta^{\frac{-1}{\alpha}}\left[\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \times\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\mu^{\frac{1}{\alpha_1}}(s)\nu^{\frac{1}{\alpha_2}}(s)ds\right].$$

This implies

$${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)\leq\delta^{-\frac{1}{\alpha_1}}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right]. \tag{3.12}$$

Applying the $\frac{1}{q}$th power on both sides of (3.12), we have

$$\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)\right)^{\frac{1}{\alpha_2}} \leq\delta^{-\frac{1}{\alpha_1\alpha_2}}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right]^{\frac{1}{\alpha_2}}. \tag{3.13}$$

Finally, conducting product between (3.7) and (3.13), we obtain

$$\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\right)^{\frac{1}{\alpha_1}}\left({}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)\right)^{\frac{1}{\alpha_2}} \leq\mathcal{H}_2\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\frac{1}{\alpha_1}}(\lambda)\nu^{\frac{1}{\alpha_2}}(\lambda)\right]^{\frac{1}{\alpha_2}}. \tag{3.14}$$

□

Theorem 3.2. *For $0<\beta<1$, $\alpha_1,\alpha_2>1$, with $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$ and suppose that two positive functions $\mu,\nu\in\mathbb{C}_\beta[x,y]$ are defined on $[0,\infty)$ with respect to the weight function w such that ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)<\infty$ and ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)<\infty$, $\lambda>x$. If $0<\delta\leq\frac{\mu(\lambda)}{\nu(\lambda)}\leq\gamma$ for some $\delta,\gamma\in\mathbb{R}_+$ and $\forall\lambda\in[x,y]$, then the following weighted $\mathcal{AB}$-fractional integral inequality holds*:

$${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right) \leq\mathcal{H}_3\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu^{\alpha_1}(\lambda)+\nu^{\alpha_1}(\lambda)\right) +\mathcal{H}_4\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu^{\alpha_2}(\lambda)+\nu^{\alpha_2}(\lambda)\right), \tag{3.15}$$

where $\mathcal{H}_3=\frac{2^{\alpha_1-1}\gamma^{\alpha_1}}{\alpha_1(\gamma+1)^{\alpha_1}}$ and $\mathcal{H}_4=\frac{2^{\alpha_2-1}}{\alpha_2(\delta+1)^{\alpha_2}}$.

Proof. Utilizing the assumption $\frac{\mu(\lambda)}{\nu(\lambda)}\leq\gamma$, one obtains

$$\mu(\lambda)\leq\left(\frac{\gamma}{\gamma+1}\right)\left(\mu(\lambda)+\nu(\lambda)\right). \tag{3.16}$$

Taking the α_1th power of both sides of (3.16), we have

$$\mu^{\alpha_1}(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} (\mu(\lambda)+\nu(\lambda))^{\alpha_1}. \quad (3.17)$$

Taking product on both sides of (3.17) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\alpha_1}(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} \frac{1-\beta}{\mathbb{M}(\beta)} (\mu(\lambda)+\nu(\lambda))^{\alpha_1}. \quad (3.18)$$

Further, replacing λ by s in (3.17) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\mu^{\alpha_1}(s)ds \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1} \times \omega(s)(\mu(s)+\nu(s))^{\alpha_1} ds. \quad (3.19)$$

Summing up (3.18) and (3.19), one obtains

$$\frac{1-\beta}{\mathbb{M}(\beta)}\mu^{\alpha_1}(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\mu^{\alpha_1}(s)ds \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} \left[\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)+\nu(\lambda))^{\alpha_1} + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1} \times \omega(s)(\mu(s)+\nu(s))^{\alpha_1} ds\right].$$

This implies

$$^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha_1}(\lambda) \leq \left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_1}. \quad (3.20)$$

Multiplying (3.20) by the constant $\frac{1}{\alpha_1}$, we find

$$\frac{1}{\alpha_1} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha_1}(\lambda) \leq \frac{1}{\alpha_1}\left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_1}. \quad (3.21)$$

On the other hand, under the supposition $0<\delta\leq \frac{\mu(\lambda)}{\nu(\lambda)}$, we acquire

$$\nu^{\alpha_2}(\lambda) \leq \frac{1}{(\delta+1)^{\alpha_2}}(\mu(\lambda)+\nu(\lambda))^{\alpha_2}. \quad (3.22)$$

Multiplying (3.22) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$\frac{1-\beta}{\mathbb{M}(\beta)}\nu^{\alpha_2}(\lambda) \leq \frac{1}{(\delta+1)^{\alpha_2}}\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)+\nu(\lambda))^{\alpha_2}. \quad (3.23)$$

Further, replacing λ by s in (3.22) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\nu^{\alpha_2}(s)ds \leq \frac{1}{(\delta+1)^{\alpha_2}}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1} \times \omega(s)(\mu(s)+\nu(s))^{\alpha_2} ds. \quad (3.24)$$

Summing up (3.23) and (3.24), one obtains

$$\frac{1-\beta}{\mathbb{M}(\beta)}\nu^{\alpha_2}(\lambda) + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\nu^{\alpha_2}(s)ds \leq \frac{1}{(\delta+1)^{\alpha_2}} \left[\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)+\nu(\lambda))^{\alpha_2} + \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \int_x^\lambda (\lambda-s)^{\beta-1} \times \omega(s)(\mu(s)+\nu(s))^{\alpha_2} ds\right].$$

This implies

$$^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha_2}(\lambda) \leq \left(\frac{1}{\delta+1}\right)^{\alpha_2} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_2}. \quad (3.25)$$

Multiplying (3.25) by $\frac{1}{\alpha_2}$, we have

$$\frac{1}{\alpha_2} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha_2}(\lambda) \leq \frac{1}{\alpha_2}\left(\frac{1}{\delta+1}\right)^{\alpha_2} \times {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_2}. \quad (3.26)$$

Summing up (3.21) and (3.26), we get

$$\frac{1}{\alpha_1} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha_1}(\lambda) + \frac{1}{\alpha_2} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha_2}(\lambda) \leq \frac{1}{\alpha_1}\left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_1} + \frac{1}{\alpha_2}\left(\frac{1}{\delta+1}\right)^{\alpha_2} {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}(\mu(\lambda)+\nu(\lambda))^{\alpha_2}. \quad (3.27)$$

Taking into account Young's inequality, we have

$$\mu(\lambda)\nu(\lambda) \leq \frac{\mu^{\alpha_1}(\lambda)}{\alpha_1} + \frac{\nu^{\alpha_2}(\lambda)}{\alpha_2}. \quad (3.28)$$

Multiplying (3.28) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we get

$$\frac{1-\beta}{\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda) \leq \frac{1-\beta}{\mathbb{M}(\beta)}\left(\frac{\mu^{\alpha_1}(\lambda)}{\alpha_1}+\frac{\nu^{\alpha_2}(\lambda)}{\alpha_2}\right). \tag{3.29}$$

Again, replacing λ by s in (3.28) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\begin{aligned}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\mu(s)\nu(s)ds &\leq \int_x^\lambda \frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\alpha_1\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\mu^{\alpha_1}(s)ds\\ &\quad+\int_x^\lambda \frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\alpha_2\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\nu^{\alpha_2}(s)ds.\end{aligned} \tag{3.30}$$

Summing up (3.29) and (3.30), we obtain

$$\begin{aligned}&\frac{1-\beta}{\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda)+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\times\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)\mu(s)\nu(s)ds\\ &\leq \frac{1-\beta}{\mathbb{M}(\beta)}\left(\frac{\mu^{\alpha_1}(\lambda)}{\alpha_1}+\frac{\nu^{\alpha_2}(\lambda)}{\alpha_2}\right)+\int_x^\lambda \frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\alpha_1\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\mu^{\alpha_1}(s)ds\\ &\quad+\int_x^\lambda \frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\alpha_2\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\nu^{\alpha_2}(s)ds.\end{aligned} \tag{3.31}$$

This implies that

$$ {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\nu(\lambda) \leq \frac{1}{\alpha_1}\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu^{\alpha_1}(\lambda)+\frac{1}{\alpha_2}\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu^{\alpha_2}(\lambda). \tag{3.32}$$

Utilizing (3.27) and (3.32), we have

$$\begin{aligned}&{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)\nu(\lambda)\\ &\leq \frac{1}{\alpha_1}\left(\frac{\gamma}{\gamma+1}\right)^{\alpha_1}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha_1}\right]\\ &\quad+\frac{1}{\alpha_2}\left(\frac{1}{\delta+1}\right)^{\alpha_2}\left[{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha_2}\right].\end{aligned} \tag{3.33}$$

Using the inequality $(a_1+a_2)^p \leq 2^{p-1}(a_1^p+a_2^p)$, $a_1,a_2\geq 0$, $p>1$, we have

$$(\mu(\lambda)+\nu(\lambda))^{\alpha_1} \leq 2^{\alpha_1-1}(\mu(\lambda)^{\alpha_1}+\nu(\lambda)^{\alpha_1}). \tag{3.34}$$

Multiplying (3.34) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we find

$$\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)+\nu(\lambda))^{\alpha_1} \leq 2^{\alpha_1-1}\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)^{\alpha_1}+\nu(\lambda)^{\alpha_1}). \tag{3.35}$$

Again, replacing λ by s in (3.34) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\begin{aligned}&\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)(\mu(s)+\nu(s))^{\alpha_1}ds\\ &\leq 2^{\alpha_1-1}\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\times\omega(s)(\mu(s)^{\alpha_1}+\nu(s)^{\alpha_1})ds.\end{aligned} \tag{3.36}$$

Summing up (3.35) and (3.36), we obtain

$$\begin{aligned}&\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)+\nu(\lambda))^{\alpha_1}+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\omega(s)(\mu(s)+\nu(s))^{\alpha_1}ds\\ &\leq 2^{\alpha_1-1}\left[\frac{1-\beta}{\mathbb{M}(\beta)}(\mu(\lambda)^{\alpha_1}+\nu(\lambda)^{\alpha_1})+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda(\lambda-s)^{\beta-1}\times\omega(s)(\mu(s)^{\alpha_1}+\nu(s)^{\alpha_1})\right]ds.\end{aligned}$$

This implies

$${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha_1} \leq 2^{\alpha_1-1}\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(s)^{\alpha_1}+\nu(s)^{\alpha_1}\right). \tag{3.37}$$

Adopting the same procedure, we get

$${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(\lambda)+\nu(\lambda)\right)^{\alpha_2} \leq 2^{\alpha_2-1}\,{}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\left(\mu(s)^{\alpha_2}+\nu(s)^{\alpha_2}\right). \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.33) leads to the desired inequality (3.15). This completes the proof. □

Theorem 3.3. *For $0<\beta<1$, $\alpha_1,\alpha_2>1$, with $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$ and suppose that two positive functions $\mu,\nu\in\mathbb{C}_\beta[x,y]$ defined on $[0,\infty)$ with respect to the weight function w such that ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\mu(\lambda)<\infty$ and ${}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\nu(\lambda)<\infty$, $\lambda>x$. If $0<\delta\leq\frac{\mu(\lambda)}{\nu(\lambda)}\leq\gamma$ for* *some $\delta, \gamma \in \mathbb{R}_+$ and $\forall \lambda \in [x, y]$, then the following weighted $\mathcal{AB}$-fractional integral inequality holds*:

$$\begin{aligned}\frac{1}{\gamma}\, {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)\nu(\lambda)) &\le {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)+\nu(\lambda))^2 \\ &\le \frac{1}{\delta}\, {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)\nu(\lambda)). \end{aligned} \tag{3.39}$$

Proof. Utilizing the assumption

$$0 < \delta \le \frac{\mu(\lambda)}{\nu(\lambda)} \le \gamma, \tag{3.40}$$

we conclude that

$$\delta + 1\nu(\lambda) \le \mu(\lambda) + \nu(\lambda) \le 1 + \gamma\nu(\lambda). \tag{3.41}$$

Also,

$$\frac{\gamma+1}{\gamma}\mu(\lambda) \le \mu(\lambda) + \nu(\lambda) \le \frac{1+\alpha}{\alpha}\mu(\lambda). \tag{3.42}$$

By (3.41) and (3.42), we obtain

$$\frac{1}{\gamma}\mu(\lambda)\nu(\lambda) \le \frac{(\mu(\lambda)+\nu(\lambda))^2}{(1+\delta)(1+\gamma)} \le \frac{1}{\delta}\mu(\lambda)\nu(\lambda). \tag{3.43}$$

Multiplying (3.43) by $\frac{1-\beta}{\mathbb{M}(\beta)}$, we have

$$\begin{aligned}\frac{1-\beta}{\gamma\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda) &\le \frac{1-\beta}{\mathbb{M}(\beta)}\frac{(\mu(\lambda)+\nu(\lambda))^2}{(1+\delta)(1+\gamma)} \\ &\le \frac{1-\beta}{\delta\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda). \end{aligned} \tag{3.44}$$

Now, replacing λ by s in (3.43) and taking product on both sides by $\frac{\beta(\lambda-s)^{\beta-1}\omega(s)}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}$, and integrating the resultant inequality with respect to s, we get

$$\begin{aligned}&\frac{\gamma\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\mu(s)\nu(s)ds \\ &\quad\le \frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda (\lambda-s)^{\beta-1} \\ &\qquad\times\omega(s)\frac{(\mu(s)+\nu(s))^2}{(1+\delta)(1+\gamma)}ds \\ &\quad\le \frac{\beta}{\delta\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda (\lambda-s)^{\beta-1} \\ &\qquad\times\omega(s)\mu(s)\nu(s)ds. \end{aligned} \tag{3.45}$$

Summing up (3.44) and (3.45), we have

$$\begin{aligned}&\frac{1-\beta}{\gamma\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda) + \frac{\gamma\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \\ &\quad\times\int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\mu(s)\nu(s)ds \\ &\le \frac{1-\beta}{\mathbb{M}(\beta)}\frac{(\mu(\lambda)+\nu(\lambda))^2}{(1+\delta)(1+\gamma)} \\ &\quad+\frac{\beta}{\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)}\int_x^\lambda (\lambda-s)^{\beta-1} \\ &\quad\times\omega(s)\frac{(\mu(s)+\nu(s))^2}{(1+\delta)(1+\gamma)}ds \\ &\le \frac{1-\beta}{\delta\mathbb{M}(\beta)}\mu(\lambda)\nu(\lambda) + \frac{\beta}{\delta\mathbb{M}(\beta)\Gamma(\beta)\omega(\lambda)} \\ &\quad\times\int_x^\lambda (\lambda-s)^{\beta-1}\omega(s)\mu(s)\nu(s)ds, \end{aligned}$$

we conclude that

$$\begin{aligned}\frac{1}{\gamma}\, {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)\nu(\lambda)) &\le {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)+\nu(\lambda))^2 \\ &\le \frac{1}{\delta}\, {}^{\mathcal{AB}}\mathcal{T}_{x,\omega}^{\beta,\lambda}\,(\mu(\lambda)\nu(\lambda)). \end{aligned} \tag{3.46}$$

This completes the proof. □

4. CONCLUSION

In this work, we have employed the newly defined weighted $\mathcal{AB}$-fractional operators for Minkowski's and reverse Hölder inequalities. To illustrate the applicability and effectiveness of the presented operator, the established variants are the generalizations of classical inequalities under certain conditions. On account of the sort of the kernel, it is notable that dealing with $\mathcal{AB}$-fractional operators is quite troublesome than managing the Caputo–Fabrizio operators. Subsequently, the issue of presenting and considering the weighted $\mathcal{AB}$-fractional operators in the sense of the weight function $\omega(\lambda)$ with their highlights is as yet open. With the aid of this study, we derived more general variants than in the classical cases. For conceivable futuristic research, we propose applying the acquired inequalities to demonstrate the existence of solutions of fractional differential equations.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their valuable suggestions and comments. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11971142, 61673169, 11701176, 11626101, 11601485 and 11871202).

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