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New interpolation spaces and strict Hölder regularity for fractional abstract Cauchy problem

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Abstract

We know that interpolation spaces in terms of analytic semigroup have a significant role into the study of strict Hölder regularity of solutions of classical abstract Cauchy problem (ACP). In this paper, we first construct interpolation spaces in terms of solution operators in fractional calculus and characterize these spaces. Then we establish strict Hölder regularity of mild solutions of fractional order ACP.

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1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space. First, we take into consideration of the following abstract Cauchy problem:

$$\begin{cases} \frac{dv}{dt} = Av(t) + \mathfrak{F}(t), & 0 < t \leq T, \\ v(0) = v_0, \end{cases} \quad (1.1)$$

where $\mathfrak{F} : [0, T] \rightarrow X$, $v_0 \in X$, and $A : D(A) \subset X \rightarrow X$ is densely defined operator satisfying following conditions:

(i) $\sum_\theta := \{\mu \in \mathbb{C} : \mu \neq 0, |\arg(\mu)| < \theta\} \subset \rho(A)$,

(ii) $\|(\mu I - A)^{-1}\|_{L(X)} \leq \frac{N}{|\mu|}, \forall \mu \in \sum_\theta$,

for some $\theta \in (\frac{\pi}{2}, \pi)$, $N > 0$. The sets $\rho(A)$, $L(X)$ are denoted as resolvent set of A and the set of all bounded linear operators on X , respectively. Such operator A , is known as sectorial operator, and it generates the analytic semigroup $\{T(t)\}_{t \geq 0}$ on X (cf. [1]). The study of strict solution of (1.1) is very well known. For instance, we refer to [1, Chap. 4], [2] and the references therein. In these cited papers, the authors proved strict Hölder regularity of the mild solution of (1.1) under suitable space and time regularity of the initial datum. More precisely, if $\mathfrak{F} \in C^\theta([0, T]; X)$, where $0 < \theta < 1$, $v_0 \in D(A)$ with $Av_0 + \mathfrak{F}(0) \in D_A(\theta, \infty)$, an interpolation space, then (1.1) has strict solution. The results have been obtained by

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using the advantage of representation of interpolation space $D_A(\theta, \infty)$ in terms of analytic semigroup. Here we are interested in the investigation of similar results for the following fractional order ACP on X :

$$\begin{cases} {}^c D_t^\alpha v(t) = Av(t) + \mathfrak{F}(t), & 0 < t \leq T, \\ v(0) = v_0, \end{cases} \tag{1.2}$$

where ${}^c D_t^\alpha$ represents Caputo fractional differential operator of order $\alpha \in (0, 1)$.

Over the last few decades, the study in the area of fractional calculus has influenced the researchers owing to its generous applicability in the branch of science and engineering. Specifically, we refer the work [3] for some substantial applications of fractional differential equations.

There have been intensive investigation on existence, uniqueness and regularity of solution for variety of generalized model of (1.2). For the enthusiastic reader, we refer to [4–6] for linear autonomous case, [7, 8] for semilinear autonomous case, [9] and some work cited therein for the non-autonomous case with delay. But the study of strict regularity of the problem (1.2) is very rare for the case $\alpha \in (0, 1)$. Bazhlekova [10], in her precious thesis, studied strict L^p -regularity by introducing the concept of solution operators and using the resolvent representation of classical interpolation space $D_A(\theta, p)$ for $1 < p < \infty$. The case $p = \infty$ (i.e., strict Hölder regularity case) has been studied first by Ph. Clément et al. in their pioneer work [11]. Using some suitable transformation, the author split the problem (1.2) into two abstract problems, one is having homogeneous force function (i.e., $\mathfrak{F} = 0$) but with inhomogeneous initial data, and another is having inhomogeneous force function $\mathfrak{F}(t) - \mathfrak{F}(0)$ but with homogeneous initial data. The author used method of sum to investigate the regularity of the latter one, in which the problem is converted to an operator equation of the form $Bu + Cu = v$ on the space $\tilde{X} = \{v \in C([0, T]; X) : v(0) = 0\}$ and applied a suitable theorem [11, Theorem 8]. Whereas, the regularity of the first one is solved by exploiting resolvent representation of the interpolation space $D_A(\theta, \infty)$. Finally, combining the both results, the author obtained the strict Hölder regularity of the problem (1.2). Also, the work of Li liu et al. [12] is devoted to examining maximal regularity property of the weighted Hölder space $C_0^\theta([0, T]; X)$ for the problem (1.2) by utilizing the concept of α -times resolvent families introduced in [13], when $v(0) = v_0 \in D(A)$.

In contrast to this above-mentioned work, for the first time we are going to prove that the classical real interpolation space $D_A(\theta, p)$ can also be represented in terms of $S_\alpha(t)$, $T_\alpha(t)$, named as “solution operators” of fractional ACP. Guswanto, in [14], explicitly introduced these pair of operators on X , defined as follows:

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Upsilon e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda, \quad t > 0, \tag{1.3}$$

$$T_\alpha(t) = \frac{1}{2\pi i} \int_\Upsilon e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, \quad t > 0, \tag{1.4}$$

where $\Upsilon \subset \rho(A)$, an anticlockwise oriented path. He also demonstrated the topological properties of these operators which are somewhat similar to those in classical case; see [15]. These similarities create a great advantage to study the existence, uniqueness and regularity properties of solution of (1.2) in the same fashion as of classical case. However,

there are some disadvantages as these families do not satisfy semigroup properties and the operator $T_\alpha(t)$ has singularity near $t = 0$.

On the other hand, the study of strict solutions come into effect when one wants to get the differentiability of the solutions up to $t = 0$. In fact, it is known that the α -Caputo derivative of classical solutions of (1.2) cannot be extended up to $t = 0$, even under the conditions $v_0 \in D(A)$, $\mathfrak{F} \in C^\theta([0, T]; X)$ (see, e.g., [16]). Thanks to interpolation spaces which play crucial role to fill this gap between classical solutions and strict solutions. It is evident that interpolation spaces can be expressed in terms of semigroup. However, no such expression in terms of $S_\alpha(t)$, $T_\alpha(t)$ is available in the literature so far. Motivated by this, we analyze the following points:

- (I) constructing interpolation spaces in terms of solution operators $S_\alpha(t)$, $T_\alpha(t)$ of fractional ACP and characterize these spaces,
- (II) establishing the strict Hölder regularity (or maximal Hölder regularity) results for the problem (1.2) utilizing this new representation of interpolation space.

It is worth to mention that this new finding creates not only a great opportunity to study the strict regularity results of the problem (1.2) in the similar fashion as of classical case, but also provide a new sight in the area of fractional calculus.

This paper is organised as follows. In Sect. 2, we present preliminary results on some Hardy-type inequalities, a short introduction of real interpolation spaces and properties of the solution operators. In Sect. 3, we construct interpolation spaces in terms of solution operators and characterize these spaces, which is one of the goals of this paper. In the last section, we establish the sufficient conditions to investigate the strict Hölder regularity of mild solutions of (1.2), and provide an example to illustrate the results.

2 Some notations, preliminary results and interpolation spaces

Let $I \subset \mathbb{R}$ be an interval. Traditionally, for a compact interval I , $C(I; X)$ represents the space of X -valued continuous functions on I endowed with the usual supremum norm. The Hölder space $C^\theta(I; X)$ is defined by

$$C^\theta(I; X) := \left\{ g \in C(I; X) : [g]_\theta = \sup_{s, t \in I, s \neq t} \frac{\|g(t) - g(s)\|}{|t - s|^\theta} < \infty \right\},$$

assigned with the norm $\|g\|_{C^\theta(I; X)} = \sup_{t \in I} \|g(t)\| + [g]_\theta$, where $\theta \in (0, 1)$.

The space $L_*^p(I) := L^p(I, \frac{dt}{t})$, equipped with norm

$$\|g\|_{L_*^p(I)} = \left(\int_I |g(s)|^p \frac{ds}{s} \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty,$$

$$\|g\|_{L_*^\infty(I)} = \text{ess sup}_{s \in I} |g(s)|, \quad \text{if } p = \infty.$$

Definition 2.1 Let $f \in L^1([a, b]; X)$. The operator defined by

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) ds, \quad \text{a.e } t \in [a, b],$$

is known as *Riemann–Liouville integral* of order $\alpha \in (0, 1)$.

For $\alpha > 0$, we consider

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

The convolution between $f : [0, T] \rightarrow X$ and g_α is defined by

$$(g_\alpha * f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t \in (0, T].$$

One can verify that, for $\alpha, \beta \in (0, 1)$, $g_{\alpha+\beta} = g_\alpha * g_\beta$.

Definition 2.2 Let $f \in L^1([a, b]; X)$ be such that $I^{1-\alpha}(f(\cdot) - f(a)) \in W^{1,1}([a, b]; X)$. Then the *Caputo derivative* of order $\alpha \in (0, 1)$ is defined as

$$\begin{aligned} {}^c D_t^\alpha f(t) &= \frac{d}{dt} (I^{1-\alpha}(f(\cdot) - f(a)))(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} (f(s) - f(a)) ds \quad \text{a.e } t \in [a, b]. \end{aligned}$$

Definition 2.3 Let $\mathfrak{F} \in L^1((0, T); X)$. By a *mild solution* of (1.2), we mean a function $v \in C([0, T]; X)$ which satisfies the following integral equations:

$$v(t) = S_\alpha(t)v_0 + \int_0^t T_\alpha(t-s)\mathfrak{F}(s) ds, \quad t \in (0, T].$$

Definition 2.4 A function $v \in C([0, T]; X)$ satisfying:

- (i) $v \in C((0, T]; D(A))$,
- (ii) $t \mapsto g_{1-\alpha} * (v(\cdot) - v_0)(t) \in C^1((0, T]; X)$,
- (iii) Eq. (1.2),

is called a *classical solution* of (1.2).

Definition 2.5 A function $v \in C([0, T]; D(A))$ satisfying:

- (i) $t \mapsto g_{1-\alpha} * (v(\cdot) - v_0)(t) \in C^1([0, T]; X)$,
- (ii) Eq. (1.2),

is called a *strict solution* of (1.2).

Lemma 2.6 (Hardy-type inequalities involving Riemann Liouville integral [17]) *Let $1 < p \leq q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and u, v be non-negative weight functions. Then the boundedness of the linear operator $I^\alpha : L^p((0, \infty); v(t) dt) \rightarrow L^q((0, \infty); u(t) dt)$ holds if and only if for some $C > 0$*

$$\left(\int_{2R}^\infty t^{(\alpha-1)q} u(t) dt \right)^{\frac{1}{q}} \left(\int_0^R (v(t))^{1-p'} dt \right)^{\frac{1}{p'}} \leq C \quad \text{for all } R > 0$$

and $\tilde{I}_\alpha : L^p((0, \infty); v(t) dt) \rightarrow L^q((0, \infty); u(t) dt)$ defined by $\tilde{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} f(s) ds$ is bounded.

In particular, if $p = q$, $u(t) = t^{\eta-1}$ and $v(t) = t^{\nu-1}$, $t > 0$. Then boundedness of $I^\alpha : L^p((0, \infty); v(t) dt) \rightarrow L^p((0, \infty); u(t) dt)$ holds if and only if

$$\nu < p, \quad \eta < (1 - \alpha)p \quad \text{and} \quad \alpha + \frac{\eta}{p} = \frac{\nu}{p}. \tag{2.1}$$

Next, we give Hardy-type inequality involving Riemann–Liouville integral for the case $p = q = 1$ (cf. [18, Theorem 2.3]).

Lemma 2.7 *Let u, v be non-negative weight functions. If there exist $\beta \in [0, 1], C > 0$ such that*

$$\left(\int_r^\infty (t - r)^{(\alpha-1)\beta} u(t) dt \right) \left(\sup_{0 < t < r} (r - t)^{(\alpha-1)(1-\beta)} [v(t)]^{-1} \right) \leq C \quad \text{for all } r > 0. \tag{2.2}$$

Then there exists constant $B > 0$ such that $\int_0^\infty |(I^\alpha f)(t)| u(t) dt \leq B \int_0^\infty |f(t)| v(t) dt$.

Lemma 2.8 (Hardy–Young inequality [19, p. 245-246]) *Let $0 < b \leq \infty$ and $p \geq 1$. Then, for any measurable function $\Psi : (0, b) \rightarrow \mathbb{R}^+$ and $\gamma > 0$, the following inequality holds:*

$$\int_0^b t^{-\gamma p} \left(\int_0^t \Psi(s) \frac{ds}{s} \right)^p \frac{dt}{t} \leq \frac{1}{\gamma^p} \int_0^b s^{-\gamma p} \Psi(s)^p \frac{ds}{s}. \tag{2.3}$$

Before constructing interpolation spaces in terms of $S_\alpha(t)$, $T_\alpha(t)$, we draft a base by recalling the classical real interpolation spaces. For Banach spaces X and Y , $L(X; Y)$ stands for the set of all bounded linear operators from X to Y . For $X = Y$, $L(X; X)$ is indicated as $L(X)$.

2.1 Real interpolation spaces

In this subsection, we recall interpolation space for the case $Y \hookrightarrow X$, though for general theory of interpolation spaces, we direct [20] for the interested reader. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces with $Y \hookrightarrow X$. A Banach space E is called an *intermediate space* if $Y \hookrightarrow E \hookrightarrow X$. An intermediate space E is said to be an *interpolation space* if for every $T \in L(X)$ such that $T|_Y \in L(Y)$, we have $T|_E \in L(E)$. A classical method known as K -method to produce a class of real interpolation spaces is recalled now.

For $x \in X, t > 0$, set

$$K(t, x; X, Y) := \inf \{ \|x_1\|_X + t \|x_2\|_Y : x = x_1 + x_2, x_1 \in X, x_2 \in Y \}.$$

Let $0 < \theta < 1$. For $1 \leq p \leq \infty$, define the following spaces:

$$(X, Y)_{\theta, p} = \{ x \in X : t \mapsto \phi(t) = t^{-\theta} K(t, x; X, Y) \in L^p_*(0, \infty) \},$$

with the norm $\|x\|_{(X, Y)_{\theta, p}} = \|\phi\|_{L^p_*(0, \infty)}$.

Then $(X, Y)_{\theta, p}$ are known as real interpolation spaces. One important thing is to note that $K(t, x; X, Y) \leq \|x\|_X$ as $Y \hookrightarrow X$. Hence, to prove $x \in (X, Y)_{\theta, p}$, it is adequate to prove that $t \mapsto \phi(t) \in L^p_*(0, a)$, and then $\|x\|_{(X, Y)_{\theta, p}}$ becomes equivalent to $\|x\|_X + \|\phi\|_{L^p_*(0, a)}$, for any fixed $a > 0$.

A well-known result is that interpolation space $(X, D(A))_{\theta, p}$ can also be represented in terms of semigroup as follows:

$$(X, D(A))_{\theta, p} = \{x \in X : t \mapsto \psi(t) = t^{1-\theta} \|AT(t)x\|_X \in L^p_*(0, 1)\} := D_A(\theta, p), \tag{2.4}$$

with the equivalent norm $\|x\|_{\theta, p} = \|x\|_X + \|\psi\|_{L^p_*(0,1)}$ (see, [1, p. 46]).

Definition 2.9 Let $\theta \in (0, 1)$. An intermediate space E is of J_θ class if $\exists c > 0$ s.t.

$$\|x\|_E \leq c \|x\|_X^{1-\theta} \|x\|_Y^\theta \quad \text{for all } x \in Y.$$

In such a case, it is written as $E \in J_\theta(X, Y)$.

Definition 2.10 Let $\theta \in (0, 1)$. An intermediate space E is of K_θ class if $\exists k > 0$ s.t.

$$K(t, x; X, Y) \leq kt^\theta \|x\|_E \quad \text{for all } x \in E, t > 0.$$

In such case, it is written as $E \in K_\theta(X, Y)$.

One can observe that $(X, Y)_{\theta, p}$ belong to the class $J_\theta(X, Y) \cap K_\theta(X, Y)$ for every $p \in [1, \infty]$. For any $\theta \in (0, 1)$, the domain of fractional power of the sectorial operator, $D(A^\theta)$ is not an interpolation space but belongs to the class $J_\theta(X, D(A)) \cap K_\theta(X, D(A))$. However, in a Hilbert space X , if A is densely defined positive self-adjoint operator, then $D(A^\theta)$ is an interpolation space (cf. [1]).

Now, we recall the properties of the solution operators $S_\alpha(t)$, $T_\alpha(t)$ in fractional calculus in order to find the motivation behind constructing new interpolation spaces.

Lemma 2.11 ([14]) *Let $S_\alpha(t)$ be defined as (1.3). Then the following results hold.*

- (i) $S_\alpha(t) \in L(X)$ and $\exists B_1 = B_1(\alpha) > 0$ such that $\|S_\alpha(t)\|_{L(X)} \leq B_1, \forall t > 0$.
- (ii) For each $t > 0$, $S_\alpha(t) \in L(X; D(A))$, and $AS_\alpha(t)x = S_\alpha(t)Ax$ if $x \in D(A)$. Moreover, $\exists B_2 = B_2(\alpha) > 0$ s.t. $\|AS_\alpha(t)\|_{L(X)} \leq B_2t^{-\alpha}, \forall t > 0$.
- (iii) $t \mapsto S_\alpha(t) \in C^\infty((0, \infty); L(X))$ and $\exists M_n = M_n(\alpha) > 0$ such that $\|S_\alpha^{(n)}(t)\|_{L(X)} \leq M_n t^{-n}, \forall t > 0$. Moreover, $S_\alpha(t)$ can be continued analytically to the sector $\Sigma_{\theta-\frac{\pi}{2}}$.
- (iv) For each $x \in X$, $\lim_{t \rightarrow 0} \|S_\alpha(t)x - x\|_X = 0$.
- (v) $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S_\alpha(s)x ds \in D(A), \forall x \in D(A)$, and

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS_\alpha(s)x ds = S_\alpha(t)x - x. \tag{2.5}$$

Remark 2.12 Since A is densely defined, closed operator, therefore (2.5) holds for all $x \in X$. Thus, the operator equation $(I^\alpha AS_\alpha)(t) = S_\alpha(t) - I$ is valid on X .

Lemma 2.13 ([14]) *Let $T_\alpha(t)$ be defined as (1.4). Then the following results hold.*

- (i) $T_\alpha(t) \in L(X)$ and $\exists C_1 = C_1(\alpha) > 0$ such that $\|T_\alpha(t)\|_{L(X)} \leq C_1 t^{\alpha-1}, \forall t > 0$.
- (ii) For each $t > 0$, $T_\alpha(t) \in L(X; D(A))$, and $AT_\alpha(t)x = T_\alpha(t)Ax$ if $x \in D(A)$. Moreover, $\exists C_2 = C_2(\alpha) > 0$ such that $\|AT_\alpha(t)\|_{L(X)} \leq C_2 t^{-1}, \forall t > 0$.

- (iii) $t \mapsto T_\alpha(t) \in C^\infty((0, \infty); L(X))$, and $\exists K_n = K_n(\alpha) > 0$ such that $\|T_\alpha^{(n)}(t)\|_{L(X)} \leq K_n t^{\alpha-n-1}, \forall t > 0$. Moreover, $T_\alpha(t)$ can be continued analytically to the sector $\Sigma_{\theta-\frac{\pi}{2}}$.
- (iv) $S'_\alpha(t)x = AT_\alpha(t)x$ for all $x \in X, t > 0$.
- (v) $S_\alpha(t)x = (g_{1-\alpha} * T_\alpha)(t)x$ for all $x \in X, t > 0$.

Note that $S_\alpha(t)$ is bounded near zero. Hence, by defining $S_\alpha(0) = I$, the identity operator on X and taking note of the property given in Lemma 2.11 (iv), we can say that the family of bounded linear operators $\{S_\alpha(t)\}_{t \geq 0}$ is continuous for $t \geq 0$ in the strong operator topology.

Lemma 2.14 *Let $A : D(A) \subset X \rightarrow X$ be a densely defined sectorial operator and $S_\alpha(t), T_\alpha(t)$ be the fractional solutions operator defined as in (1.3), (1.4), respectively. Then the following identity holds:*

$$S''_\alpha(t)x = AT'_\alpha(t)x \quad \text{for all } t > 0, x \in X.$$

Proof Consider the following representations of $S_\alpha(t), T_\alpha(t)$ (see, [5, p. 213]):

$$S_\alpha(t)x = \int_0^\infty M_\alpha(s)T(st^\alpha)x ds,$$

$$T_\alpha(t)x = t^{\alpha-1} \int_0^\infty \alpha s M_\alpha(s)T(st^\alpha)x ds, \quad t > 0, x \in X,$$

where M_α is a probability density function having the property $\int_0^\infty s^r M_\alpha(s) ds = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}, r > -1$.

Since $t \mapsto T_\alpha(t) \in C^\infty((0, \infty); L(X))$, using the Lebesgue dominated convergence theorem, we have the following integral representation of $T'_\alpha(t)$:

$$T'_\alpha(t)x = (\alpha - 1)t^{-1}T_\alpha(t)x + t^{2(\alpha-1)} \int_0^\infty \alpha^2 s^2 M_\alpha(s)T'(st^\alpha)x ds.$$

Now, $\alpha^2 s^2 M_\alpha(s)\|AT'(st^\alpha)\|_{L(X)} \leq Ct^{-2\alpha}M_\alpha(s)$, which is integrable over $s \in (0, \infty)$. Hence, by [20, Proposition A.5.], we conclude that $T'_\alpha(t)x \in D(A)$ for all $x \in X, t > 0$.

Finally, using the identity $T^{(n)}(t) = A^n T(t), t > 0, n \in \mathbb{N}$, analyticity of $t \mapsto S_\alpha(t)$ and the Lebesgue dominated convergence theorem, we get

$$S''_\alpha(t)x = AT'_\alpha(t)x \quad \text{for all } t > 0, x \in X. \quad \square$$

3 Construction and characterization of the interpolation spaces in terms of $S_\alpha(t), T_\alpha(t)$

In this section, we introduce two classes of interpolation spaces in terms of solution operators. We show that these interpolation spaces are identical with the classical real interpolation space.

Let $0 < \theta < 1, 1 \leq p \leq \infty$. We define the following classes:

$$T_\alpha D_A(\theta, p) := \{y \in X : t \mapsto \psi_\alpha(t) = t^{1-\alpha\theta} \|AT_\alpha(t)y\|_X \in L^p_*(0, 1)\}, \tag{3.1}$$

equipped with the norm ${}^{T\alpha}\|y\|_{\theta,p} = \|y\|_X + \|\psi_\alpha\|_{L^p_*(0,1)}$.

$${}^{S\alpha}D_A(\theta,p) := \{y \in X : t \mapsto \phi_\alpha(t) = t^{\alpha(1-\theta)}\|AS_\alpha(t)y\|_X \in L^p_*(0,1)\}, \tag{3.2}$$

equipped with the norm ${}^{S\alpha}\|y\|_{\theta,p} = \|y\|_X + \|\phi_\alpha\|_{L^p_*(0,1)}$.

Remark 3.1 We observe that, for any $0 < a < b < \infty$, the two maps $t \mapsto \psi_\alpha(t)$ and $t \mapsto \phi_\alpha(t)$ belong to $L^p_*(a,b)$. Indeed, $\|\psi_\alpha\|_{L^p_*(a,b)} \leq C_1(\alpha,a,b,\theta,p)\|y\|_X$ and $\|\phi_\alpha\|_{L^p_*(a,b)} \leq C_2(\alpha,a,b,\theta,p)\|y\|_X$ for some $C_1, C_2 > 0$. Therefore, one can replace the interval $(0,1)$ by any intervals of the form $(0,T]$ and check that the corresponding norms will also be equivalent to the norms mentioned in (3.1), (3.2), respectively.

From now onward, we will use C as a positive constant appeared in any estimation instead of mentioning it precisely and the norm $\|\cdot\|$ in place of the norm $\|\cdot\|_X$ on X .

Theorem 3.2 *Let $\theta \in (0,1), p \in [1,\infty]$. Then*

$$D_A(\theta,p) = {}^{T\alpha}D_A(\theta,p) = {}^{S\alpha}D_A(\theta,p),$$

and the respective norms defined on these spaces are equivalent.

Proof Let $x \in D_A(\theta,p) := (X,D(A))_{\theta,p}$ with $x = x_1 + x_2$, where $x_1 \in X$ and $x_2 \in D(A)$. By the properties in Lemma 2.13, $\|AT_\alpha(t)x\| \leq \|AT_\alpha(t)x_1\| + \|AT_\alpha(t)x_2\| \leq C(t^{-1}\|x_1\| + t^{\alpha-1}\|Ax_2\|)$.

This implies that

$$\psi_\alpha(t) = t^{1-\alpha\theta}\|AT_\alpha(t)x\| \leq Ct^{-\alpha\theta}(\|x_1\| + t^\alpha\|x_2\|_{D(A)}) = Ct^{-\alpha\theta}K(t^\alpha,x;X,D(A)). \tag{3.3}$$

By the change of variable $t \mapsto t^\alpha$, the R.H.S of (3.3) belongs to $L^p_*(0,1)$, which concludes that the map $t \mapsto \psi_\alpha(t) \in L^p_*(0,1)$, and ${}^{T\alpha}\|x\|_{\theta,p} \leq C\|x\|_{\theta,p}$.

Hence we proved that

$$D_A(\theta,p) \hookrightarrow {}^{T\alpha}D_A(\theta,p). \tag{3.4}$$

Now, let $x \in {}^{T\alpha}D_A(\theta,p)$. By Lemma 2.13(v) and the closedness of A , we have

$$\begin{aligned} \|AS_\alpha(t)x\| &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\|AT_\alpha(s)x\| ds \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}s^{\alpha\theta-1}\psi_\alpha(s) ds, \quad \text{for } 0 < t < 1. \end{aligned}$$

If $p = \infty$, then $\|AS_\alpha(t)x\| \leq \frac{1}{\Gamma(1-\alpha)}t^{\alpha(\theta-1)}B(1-\alpha,\alpha\theta)\|\psi_\alpha\|_{L^\infty(0,1)}$, where B is the beta function defined as $B(m,n) = \int_0^1(1-s)^{m-1}s^{n-1} ds, m, n > 0$.

Thus, the map $t \mapsto \phi_\alpha(t) \in L^\infty_*(0,1)$, and ${}^{S\alpha}\|x\|_{\theta,\infty} \leq C{}^{T\alpha}\|x\|_{\theta,\infty}$.

For $1 < p < \infty$, one can easily check the validity of the inequality (2.1) with the functions $u(t) = t^{\alpha(1-\theta)p-1}, v(t) = t^{(1-\alpha\theta)p-1}$. Thus, by Lemma 2.6 and noting Lemma 2.13(ii), (v) we

get

$$\begin{aligned}
 & \int_0^1 t^{\alpha(1-\theta)p} \|AS_\alpha(t)x\|^p \frac{dt}{t} \\
 & \leq \int_0^\infty t^{\alpha(1-\theta)p} \|I^{1-\alpha}(AT_\alpha)(t)x\|^p \frac{dt}{t} \\
 & \leq C \int_0^\infty t^{(1-\alpha\theta)p} \|AT_\alpha(t)x\|^p \frac{dt}{t} \\
 & \leq C \left(\int_0^1 t^{(1-\alpha\theta)p} \|AT_\alpha(t)x\|^p \frac{dt}{t} + \int_1^\infty t^{(1-\alpha\theta)p-1} t^{-p} \|x\|^p dt \right) \\
 & \leq C \left(\int_0^1 t^{(1-\alpha\theta)p} \|AT_\alpha(t)x\|^p \frac{dt}{t} + \|x\|^p \right) \\
 & = C (\|\psi_\alpha\|_{L^p_{t^*(0,1)}}^p + \|x\|^p),
 \end{aligned}$$

which resulted to $S_\alpha \|x\|_{\theta,p} \leq C^{T_\alpha} \|x\|_{\theta,p}$. Hence, $T_\alpha D_A(\theta,p) \hookrightarrow S_\alpha D_A(\theta,p)$.

For the case $p = 1$, the inequality (2.2) holds for $u(t) = t^{\alpha(1-\theta)-1}$, $v(t) = t^{-\alpha\theta}$ and $\beta = 1$. Again, by a similar procedure to the last case, we can obtain $S_\alpha \|x\|_{\theta,1} \leq C^{T_\alpha} \|x\|_{\theta,1}$. Hence, for all $1 \leq p \leq \infty$,

$$T_\alpha D_A(\theta,p) \hookrightarrow S_\alpha D_A(\theta,p). \tag{3.5}$$

Again, let $x \in S_\alpha D_A(\theta,p)$. Then, by Lemma 2.11(v), we can write

$$x = -(S_\alpha(t)x - x) + S_\alpha(t)x = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS_\alpha(s)x ds + S_\alpha(t)x.$$

Therefore,

$$K(t^\alpha, x; X, D(A)) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|AS_\alpha(s)x\| ds + t^\alpha \|S_\alpha(t)x\|_{D(A)}, \quad t > 0.$$

First, consider $p = \infty$:

$$\begin{aligned}
 K(t^\alpha, x; X, D(A)) & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|AS_\alpha(s)x\| ds + t^\alpha \|S_\alpha(t)x\|_{D(A)} \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha(\theta-1)} \|\phi_\alpha\|_{L^\infty_{t^*(0,1)}} ds + Ct^\alpha \|x\| \\
 & \quad + t^{\alpha(\theta-1)+\alpha} \|\phi_\alpha\|_{L^\infty_{t^*(0,1)}} \\
 & \leq Ct^{\alpha\theta} \|\phi_\alpha\|_{L^\infty_{t^*(0,1)}} + Ct^\alpha \|x\| + t^{\alpha\theta} \|\phi_\alpha\|_{L^\infty_{t^*(0,1)}},
 \end{aligned}$$

implying $t^{-\alpha\theta} K(t^\alpha, x; X, D(A)) \leq Ct^{\alpha-\alpha\theta} S_\alpha \|x\|_{\theta,\infty}$.

Therefore, noting the change of variable $t \mapsto t^\alpha$, we obtain

$$\|x\|_{D_A(\theta,\infty)} \leq C^{S_\alpha} \|x\|_{\theta,\infty}.$$

Now, for the case $1 \leq p < \infty$, we first note that

$$\int_0^1 t^{-\alpha\theta p} K(t^\alpha, x; X, D(A))^p \frac{dt}{t} \leq \int_0^1 t^{-\alpha\theta p} \|(I^\alpha AS_\alpha)(t)x\|^p \frac{dt}{t} + C\|x\|^p + \|\phi_\alpha\|_{L_*^p(0,1)}^p.$$

Now, to prove $\|x\|_{D_A(\theta,p)} \leq C^{S_\alpha} \|x\|_{\theta,p}$, it suffices to show that

$$\int_0^1 t^{-\alpha\theta p} \|(I^\alpha AS_\alpha)(t)x\|^p \frac{dt}{t} \leq C \int_0^1 t^{\alpha(1-\theta)p} \|AS_\alpha(t)x\|^p \frac{dt}{t}. \tag{3.6}$$

This inequality immediately follows from Lemmas 2.6, 2.7 with $u(t) = t^{-\alpha\theta p-1}$, $v(t) = t^{\alpha(1-\theta)p-1}$, $\beta = 1$ and using the procedure same as in the proof of the second inclusion (inclusion number (3.5)). Hence for all $1 \leq p \leq \infty$,

$${}^{S_\alpha}D_A(\theta, p) \hookrightarrow D_A(\theta, p). \tag{3.7}$$

Therefore, the inclusions (3.4), (3.5) and (3.7) validate the theorem. □

Next we show the another formation of ${}^{S_\alpha}D_A(\theta, p)$ in the next result.

Proposition 3.3 *Let $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then the following holds:*

$${}^{S_\alpha}D_A(\theta, p) = \{y \in X : t \mapsto \varphi_\alpha(t) = t^{-\alpha\theta} \|S_\alpha(t)y - y\| \in L_*^p(0, 1)\},$$

and the norm $y \mapsto {}^{S_\alpha}\|y\|_{\theta,p}$ is equivalent to the norm $y \mapsto \|y\| + \|\varphi_\alpha\|_{L_*^p(0,1)}$.

Proof Let $x \in {}^{S_\alpha}D_A(\theta, p)$, i.e., $x \in X$ such that $t \mapsto \phi_\alpha(t) = t^{\alpha(1-\theta)} \|AS_\alpha(t)x\| \in L_*^p(0, 1)$.

For $p = \infty$. By Lemma 2.11(v), we get

$$\begin{aligned} \|S_\alpha(t)x - x\| &= \|(I^\alpha AS_\alpha)(t)x\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha(\theta-1)} \phi_\alpha(s) ds \\ &\leq Ct^{\alpha\theta} \|\phi_\alpha\|_{L_*^\infty(0,1)} \quad \text{for all } t \in (0, 1). \end{aligned}$$

Then it is obvious that $\varphi_\alpha \in L_*^\infty(0, 1)$ and $\|\varphi_\alpha\|_{L_*^\infty(0,1)} \leq C\|\phi_\alpha\|_{L_*^\infty(0,1)}$.

For $1 \leq p < \infty$, it is adequate to prove that $\exists C > 0$ such that

$$\int_0^1 t^{-\theta\alpha p} \|(I^\alpha AS_\alpha)(t)x\|^p \frac{dt}{t} \leq C \int_0^1 t^{\alpha(1-\theta)p} \|AS_\alpha(t)x\|^p \frac{dt}{t}. \tag{3.8}$$

This inequality appeared in the proof of the third inclusion of Theorem 3.2 and has been proved by using Lemmas 2.6, 2.7 (see the inequality (3.6)).

Conversely, let $y \in X$ such that $t \mapsto \varphi_\alpha(t) = t^{-\alpha\theta} \|S_\alpha(t)y - y\| \in L_*^p(0, 1)$.

Now one can check the validity of the following identity for all $t > 0$:

$$\begin{aligned} AS_\alpha(t)y &= AS_\alpha(t) \frac{\alpha}{t^\alpha} \int_0^t (t-s)^{\alpha-1} (y - S_\alpha(s)y) ds \\ &\quad + AS_\alpha(t) \frac{\Gamma(\alpha+1)}{t^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S_\alpha(s)y ds. \end{aligned}$$

By using Lemma 2.11(v), we get

$$AS_\alpha(t)y = AS_\alpha(t) \frac{\alpha}{t^\alpha} \int_0^t (t-s)^{\alpha-1} (y - S_\alpha(s)y) ds + S_\alpha(t) \frac{\Gamma(\alpha+1)}{t^\alpha} (S_\alpha(t)y - y).$$

For the case $p = \infty$, we employ the properties in Lemma 2.11 and get

$$\begin{aligned} \|AS_\alpha(t)y\| &\leq Ct^{-2\alpha} \int_0^t (t-s)^{\alpha-1} s^{\alpha\theta} \varphi_\alpha(s) ds + Ct^{\alpha(\theta-1)} \varphi_\alpha(t) \\ &\leq 2Ct^{\alpha(\theta-1)} \|\varphi_\alpha\|_{L^\infty_*(0,1)}. \end{aligned}$$

So $t \mapsto \phi_\alpha(t) = t^{\alpha(1-\theta)} \|AS_\alpha(t)y\|$ belongs to $L^\infty_*(0,1)$, and $\|\phi_\alpha\|_{L^\infty_*(0,1)} \leq C \|\varphi_\alpha\|_{L^\infty_*(0,1)}$.

For the case $1 \leq p < \infty$, first note that

$$t^{\alpha(1-\theta)} \|AS_\alpha(t)y\| \leq C(t^{-\alpha(1+\theta)} \|(I^\alpha(y - S_\alpha(\cdot)y))(t)\| + \varphi_\alpha(t)).$$

Hence, to prove $\|\phi_\alpha\|_{L^p_*(0,1)} \leq C \|\varphi_\alpha\|_{L^p_*(0,1)}$, it is enough to prove that

$$\int_0^1 t^{-\alpha(1+\theta)p} \|(I^\alpha(y - S_\alpha(\cdot)y))(t)\|^p \frac{dt}{t} \leq C \int_0^1 t^{-\alpha\theta p} \|y - S_\alpha(t)y\|^p \frac{dt}{t}. \tag{3.9}$$

The above inequality is validated by the Lemmas 2.6, 2.7 for $u(t) = t^{-\alpha(1+\theta)p-1}$, $v(t) = t^{-\alpha\theta p-1}$ and $\beta = 1$. Hence, the proposition is proved. \square

4 Strict regularity of mild solutions

This section is devoted to establishing strict Hölder regularity (i.e., the $p = \infty$ case) results using the interpolation space constructed in the last section. Before that we shall present the necessary and sufficient condition for a mild solution to be a strict solution of the problem (1.2) in the case $0 < \alpha < 1$. For the case $1 < \alpha < 2$, we refer to [21].

Proposition 4.1 *Let $\mathfrak{F} \in C([0, T]; X)$, $v_0 \in D(A)$, and v be a mild solution of (1.2) on $[0, T]$. Let us consider*

$$w(t) = \int_0^t T_\alpha(t-s)\mathfrak{F}(s) ds, \quad t > 0.$$

Then the following are equivalent:

- (i) v is strict solution of (1.2).
- (ii) $w(t) \in D(A), \forall t \in [0, T]$, and $w \in C([0, T]; D(A))$.

Proof Let v be a strict solution. Then $v \in C([0, T]; D(A))$, $I^{1-\alpha}(v(\cdot) - v_0) \in C^1([0, T]; X)$ and v satisfies Eq. (1.2), and explicitly $v(t) = S_\alpha(t)v_0 + w(t)$. Since by Lemma 2.11 $S_\alpha(t)v_0 \in D(A), \forall t \in [0, T]$, we have $w(t) \in D(A)$ and

$$Aw(t) = Av(t) - AS_\alpha(t)v_0 = Av(t) - S_\alpha(t)Av_0.$$

Hence, it follows that $w \in C([0, T]; D(A))$.

Now, we assume that $w(t) \in D(A), \forall t \in [0, T]$, and $w \in C([0, T]; D(A))$.

Then we prove that $v \in C([0, T]; D(A))$, $I^{1-\alpha}(v(\cdot) - v_0) \in C^1([0, T]; X)$ and v satisfies Eq. (1.2).

Since v is mild solution of (1.2), $v(t) = S_\alpha(t)v_0 + w(t)$. Obviously $v \in C([0, T]; D(A))$.

Now

$$\begin{aligned} (g_{1-\alpha} * (v(\cdot) - v_0))(t) &= (g_{1-\alpha} * (S_\alpha(\cdot)v_0 - v_0))(t) + (g_{1-\alpha} * T_\alpha * \mathfrak{F})(t) \\ &= (g_{1-\alpha} * g_\alpha * AS_\alpha(\cdot)v_0)(t) + (S_\alpha * \mathfrak{F})(t) \\ &= (g_1 * AS_\alpha(\cdot)v_0)(t) + (S_\alpha * \mathfrak{F})(t). \end{aligned} \tag{4.1}$$

Also, by Remark 2.12 we have

$$\begin{aligned} (S_\alpha * \mathfrak{F})(t) &= A(g_\alpha * S_\alpha * \mathfrak{F})(t) + (g_1 * \mathfrak{F})(t) \\ &= A(g_\alpha * g_{1-\alpha} * T_\alpha * \mathfrak{F})(t) + (g_1 * \mathfrak{F})(t) \\ &= A(g_1 * T_\alpha * \mathfrak{F})(t) + (g_1 * \mathfrak{F})(t) \\ &= (g_1 * Aw)(t) + (g_1 * \mathfrak{F})(t). \end{aligned} \tag{4.2}$$

Since $Aw \in C([0, T]; X)$ and $\mathfrak{F} \in C([0, T]; X)$, it follows that $t \mapsto (S_\alpha * \mathfrak{F})(t)$ belongs to $C^1([0, T]; X)$. Also $AS_\alpha(\cdot)v_0 \in C([0, T]; X)$ as $v_0 \in D(A)$. So, $t \mapsto (g_1 * AS_\alpha(\cdot)v_0)(t)$ belongs to $C^1([0, T]; X)$. Consequently, $I^{1-\alpha}(v(\cdot) - v_0) = (g_{1-\alpha} * (v(\cdot) - v_0)) \in C^1([0, T]; X)$. Now using (4.1), (4.2),

$$\begin{aligned} {}^c D_t^\alpha v(t) &= \frac{d}{dt}(g_{1-\alpha} * (v(\cdot) - v_0))(t) \\ &= AS_\alpha(t)v_0 + \frac{d}{dt}(S_\alpha * \mathfrak{F})(t) \\ &= AS_\alpha(t)v_0 + Aw(t) + \mathfrak{F}(t) \\ &= Av(t) + \mathfrak{F}(t). \end{aligned} \quad \square$$

Proposition 4.2 Consider the integral defined as

$$w(t) = \int_0^t T_\alpha(t-s)\mathfrak{F}(s) ds, \quad \mathfrak{F} \in C([0, T]; X).$$

Then $w \in C^\alpha([0, T]; X)$.

Proof By Lemma 2.13, $\|w(t)\| \leq C\|\mathfrak{F}\|_{C([0, T]; X)} \int_0^t (t-s)^{\alpha-1} ds \leq C\|\mathfrak{F}\|_{C([0, T]; X)} t^\alpha$, for all $t \in [0, T]$, with $w(0) = 0$.

For $0 \leq s < t \leq T$,

$$\begin{aligned} \|w(t) - w(s)\| &\leq \int_0^s \| [T_\alpha(t-\tau) - T_\alpha(s-\tau)]\mathfrak{F}(\tau) \| d\tau + \int_s^t \| T_\alpha(t-\tau)\mathfrak{F}(\tau) \| d\tau \\ &\leq \int_0^s \int_{s-\tau}^{t-\tau} \| T'_\alpha(\sigma)\mathfrak{F}(\tau) \| d\sigma d\tau + C\|\mathfrak{F}\|_{C([0, T]; X)} \int_s^t (t-\tau)^{\alpha-1} d\tau \\ &\leq C\|\mathfrak{F}\|_{C([0, T]; X)} \left(\int_0^s \int_{s-\tau}^{t-\tau} \sigma^{\alpha-2} d\sigma d\tau + \int_s^t (t-\tau)^{\alpha-1} d\tau \right) \end{aligned}$$

$$\leq C \|\mathfrak{F}\|_{C([0,T];X)}(t-s)^\alpha.$$

Hence, $w \in C^\alpha([0, T]; X)$ and $[w]_\alpha \leq C \|\mathfrak{F}\|_{C([0,T];X)}$. □

Theorem 4.3 *Let $\mathfrak{F} \in C^\theta([0, T]; X)$, where $\theta \in (0, 1)$, and v be the mild solution of (1.2). Also, we assume that $v_0 \in D(A)$ such that $Av_0 + \mathfrak{F}(0) \in {}^{T_\alpha}D_A(\theta, \infty)$. Then v is strict solution on $[0, T]$. Indeed, $v \in C^{\alpha\theta}([0, T]; D(A))$ and consequently ${}^cD_t^\alpha v \in C^{\alpha\theta}([0, T]; X)$. Moreover, there exists $C > 0$ such that*

$$\begin{aligned} & \|{}^cD_t^\alpha v\|_{C^{\alpha\theta}([0,T];X)} + \|Av\|_{C^{\alpha\theta}([0,T];X)} \\ & \leq C(\|\mathfrak{F}\|_{C^\theta([0,T];X)} + \|v_0\|_{D(A)} + {}^{T_\alpha}\|Av_0 + \mathfrak{F}(0)\|_{\theta,\infty}). \end{aligned} \tag{4.3}$$

Proof We can write

$$\begin{aligned} v(t) &= S_\alpha(t)v_0 + \int_0^t T_\alpha(t-s)(\mathfrak{F}(s) - \mathfrak{F}(t)) ds + \int_0^t T_\alpha(t-s)\mathfrak{F}(t) ds \\ &= S_\alpha(t)v_0 + w_1(t) + (g_1 * T_\alpha)(t)\mathfrak{F}(t) \\ &= S_\alpha(t)v_0 + w_1(t) + (g_\alpha * S_\alpha)(t)\mathfrak{F}(t) \\ &= w_1(t) + w_2(t), \end{aligned}$$

where $w_1(t) = \int_0^t T_\alpha(t-s)(\mathfrak{F}(s) - \mathfrak{F}(t)) ds$, $w_2(t) = S_\alpha(t)v_0 + (g_\alpha * S_\alpha)(t)\mathfrak{F}(t)$.

Now, $\|AT_\alpha(t-s)(\mathfrak{F}(s) - \mathfrak{F}(t))\| \leq C[\mathfrak{F}]_\theta(t-s)^{\theta-1}$, which is integrable on $(0, t)$. By the closedness of A , we have

$$\|Aw_1(t)\| \leq C[\mathfrak{F}]_\theta t^\theta \quad \text{for all } t \in [0, T].$$

Hence, $w_1(t) \in D(A), \forall t \in [0, T]$.

For $0 \leq s < t \leq T$, we have

$$\begin{aligned} & \|Aw_1(t) - Aw_1(s)\| \\ & \leq \int_0^s \|(AT_\alpha(t-\sigma) - AT_\alpha(s-\sigma))(\mathfrak{F}(\sigma) - \mathfrak{F}(s))\| d\sigma \\ & \quad + \left\| \int_0^s AT_\alpha(t-\sigma)(\mathfrak{F}(s) - \mathfrak{F}(t)) d\sigma \right\| + \int_s^t \|AT_\alpha(t-\sigma)(\mathfrak{F}(\sigma) - \mathfrak{F}(t))\| d\sigma \\ & \leq \int_0^s \int_{s-\sigma}^{t-\sigma} \|AT'_\alpha(\tau)\|_{L(X)} \|\mathfrak{F}(\sigma) - \mathfrak{F}(s)\| d\tau d\sigma \\ & \quad + \left\| \int_{t-s}^t AT_\alpha(\tau)(\mathfrak{F}(s) - \mathfrak{F}(t)) d\tau \right\| + C[\mathfrak{F}]_\theta \int_s^t (t-\sigma)^{\theta-1} d\sigma \\ & \leq [\mathfrak{F}]_\theta \int_0^s (s-\sigma)^\theta \int_{s-\sigma}^{t-\sigma} \|S'_\alpha(\tau)\|_{L(X)} d\tau d\sigma + \left\| \int_{t-s}^t S'_\alpha(\tau)(\mathfrak{F}(s) - \mathfrak{F}(t)) d\tau \right\| \\ & \quad + C[\mathfrak{F}]_\theta (t-s)^\theta \\ & \leq C[\mathfrak{F}]_\theta \int_0^s \int_{s-\sigma}^{t-\sigma} \tau^{\theta-2} d\tau d\sigma + \|(S_\alpha(t) - S_\alpha(t-s))(\mathfrak{F}(s) - \mathfrak{F}(t))\| \end{aligned}$$

$$\begin{aligned}
 &+ C[\mathfrak{F}]_\theta(t-s)^\theta \\
 &\leq C[\mathfrak{F}]_\theta(t-s)^\theta + 2B_1[\mathfrak{F}]_\theta(t-s)^\theta + C[\mathfrak{F}]_\theta(t-s)^\theta \\
 &\leq C[\mathfrak{F}]_\theta(t-s)^\theta \leq CT^{\theta(1-\alpha)}[\mathfrak{F}]_\theta(t-s)^{\alpha\theta}.
 \end{aligned} \tag{4.4}$$

By Remark 2.12, we also note that

$$Aw_2(t) = S_\alpha(t)Av_0 + (g_\alpha * AS_\alpha)(t)\mathfrak{F}(t) = S_\alpha(t)(Av_0 + \mathfrak{F}(0)) + S_\alpha(t)(\mathfrak{F}(t) - \mathfrak{F}(0)) - \mathfrak{F}(t).$$

For $0 \leq s < t \leq T$,

$$\begin{aligned}
 \|(S_\alpha(t) - S_\alpha(s))(Av_0 + \mathfrak{F}(0))\| &\leq \int_s^t \|S'_\alpha(\tau)(Av_0 + \mathfrak{F}(0))\| d\tau \\
 &\leq \int_s^t \|AT_\alpha(\tau)(Av_0 + \mathfrak{F}(0))\| d\tau \\
 &\leq \int_s^t \tau^{\alpha\theta-1} T_\alpha \|Av_0 + \mathfrak{F}(0)\|_{\theta,\infty} d\tau \\
 &\leq C^{T_\alpha} \|Av_0 + \mathfrak{F}(0)\|_{\theta,\infty} (t-s)^{\alpha\theta}
 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
 &\|S_\alpha(t)(\mathfrak{F}(t) - \mathfrak{F}(0)) - S_\alpha(s)(\mathfrak{F}(s) - \mathfrak{F}(0))\| \\
 &\leq \|(S_\alpha(t) - S_\alpha(s))(\mathfrak{F}(s) - \mathfrak{F}(0))\| + \|S_\alpha(t)(\mathfrak{F}(t) - \mathfrak{F}(s))\| \\
 &\leq [\mathfrak{F}]_\theta \int_s^t \|S'_\alpha(\tau)\|_{L(X)} s^\theta d\tau + C[\mathfrak{F}]_\theta(t-s)^\theta \\
 &\leq C[\mathfrak{F}]_\theta \int_s^t \tau^{\theta-1} d\tau + C[\mathfrak{F}]_\theta(t-s)^\theta \\
 &\leq C[\mathfrak{F}]_\theta(t-s)^\theta \leq C[\mathfrak{F}]_\theta T^{\theta(1-\alpha)}(t-s)^{\alpha\theta}.
 \end{aligned}$$

Hence, by (4.5), it is clear that $Aw_2 \in C^{\alpha\theta}([0, T]; X)$ with $[Aw_2]_{\alpha\theta} \leq C([\mathfrak{F}]_\theta + T_\alpha \|Av_0 + \mathfrak{F}(0)\|_{\theta,\infty})$. Also, from (4.4), $Aw_1 \in C^{\alpha\theta}([0, T]; X)$ with $[Aw_1]_{\alpha\theta} \leq C[\mathfrak{F}]_\theta$.

Consequently, $Av \in C^{\alpha\theta}([0, T]; X)$.

By Proposition 4.2, it follows that $v \in C^{\alpha\theta}([0, T]; D(A))$ and the estimation (4.3) follows easily. □

We say \mathcal{B} , set of all X -valued function on $[0, T]$, satisfies *maximal regularity property* for the problem (1.2) if for every $\mathfrak{F} \in \mathcal{B}$, the solution v of (1.2) is such that both ${}^cD_t^\alpha v$ and Av belong to \mathcal{B} . Clearly, if $0 < \theta < \alpha$, then by Theorem 4.3, $\mathcal{B} = C^\theta([0, T]; X)$ enjoys the maximal regularity property for the problem (1.2) which is stated now as a corollary.

Corollary 4.4 *Let $\mathfrak{F} \in C^\theta([0, T]; X)$, $\theta \in (0, \alpha)$. Assume that $v_0 \in D(A)$ such that $Av_0 + \mathfrak{F}(0) \in {}^{T_\alpha}D_A(\frac{\theta}{\alpha}, \infty)$. Then a mild solution, say v , of (1.2) is a strict solution. Indeed, $v \in C^\theta([0, T]; D(A))$ and consequently ${}^cD_t^\alpha v \in C^\theta([0, T]; X)$. Moreover, there exists $C > 0$ such*

that

$$\begin{aligned} & \| {}^c D_t^\alpha v \|_{C^\theta([0,T];X)} + \| Av \|_{C^\theta([0,T];X)} \\ & \leq C (\| \mathfrak{F} \|_{C^\theta([0,T];X)} + \| v_0 \|_{D(A)} + {}^{T_\alpha} \| Av_0 + \mathfrak{F}(0) \|_{\frac{\theta}{\alpha}, \infty}). \end{aligned} \tag{4.6}$$

Corollary 4.5 *Let $\mathfrak{F} \in C^\theta([0, T]; X)$, $\theta \in (0, 1)$. Assume that $v_0 \in D(A)$ such that $Av_0 + \mathfrak{F}(0) = 0$. Then a mild solution, say v , of (1.2) is a strict solution. Indeed, $v \in C^\theta([0, T]; D(A))$ and consequently ${}^c D_t^\alpha v \in C^\theta([0, T]; X)$. Moreover, there exists $C > 0$ such that*

$$\| {}^c D_t^\alpha v \|_{C^\theta([0,T];X)} + \| Av \|_{C^\theta([0,T];X)} \leq C (\| \mathfrak{F} \|_{C^\theta([0,T];X)} + \| v_0 \|_{D(A)}). \tag{4.7}$$

Remark 4.6 Now, one may question the establishment of the strict regularity of mild solution of (1.2) in the interpolation space ${}^{T_\alpha} D_A(\theta, \infty)$ using the method presented in this paper. This is possible when $T_\alpha(t)y \in D(A^2), \forall t > 0, y \in X$. However, till now it is an open problem to find whether $T_\alpha(t)y \in D(A^2), \forall t > 0, y \in X$.

In [16], we studied that the assumptions $v_0 \in D(A)$ and $\mathfrak{F} \in C^\theta([0, T]; X)$ with $\mathfrak{F}(0) \neq 0$ do not guarantee that the mild solution is strict. But, in the next theorem, we shall see that the assumption $\mathfrak{F} \in C^\theta([0, T]; {}^{T_\alpha} D_A(\theta, \infty))$ ensures the strict Hölder regularity of a mild solution of (1.2). We shall use the following constant:

$$M = \sup_{0 < t \leq T+1} t^{1-\alpha\theta} \| AT_\alpha(t) \|_{L({}^{T_\alpha} D_A(\theta, \infty); X)}. \tag{4.8}$$

Theorem 4.7 *Let $\mathfrak{F} \in C^\theta([0, T]; {}^{T_\alpha} D_A(\theta, \infty))$, $\theta \in (0, 1)$, and $v_0 \in D(A)$. Then a mild solution, say v , of (1.2) is a strict solution. Additionally, if we assume that $Av_0 \in {}^{T_\alpha} D_A(\theta, \infty)$, then $v \in C^{\alpha\theta}([0, T]; D(A))$ and consequently ${}^c D_t^\alpha v \in C^{\alpha\theta}([0, T]; X)$. Moreover, there exists $C > 0$ such that*

$$\begin{aligned} & \| {}^c D_t^\alpha v \|_{C^{\alpha\theta}([0,T];X)} + \| Av \|_{C^{\alpha\theta}([0,T];X)} \\ & \leq C (\| \mathfrak{F} \|_{C^\theta([0,T]; {}^{T_\alpha} D_A(\theta, \infty))} + \| v_0 \|_{D(A)} + {}^{T_\alpha} \| Av_0 \|_{\theta, \infty}). \end{aligned} \tag{4.9}$$

Proof Let $w(t) = \int_0^t T_\alpha(t-s)\mathfrak{F}(s) ds, t > 0$. Now, for $s \in (0, t)$ we have,

$$\begin{aligned} \| AT_\alpha(t-s)\mathfrak{F}(s) \| & \leq \| AT_\alpha(t-s) \|_{L({}^{T_\alpha} D_A(\theta, \infty); X)} \| \mathfrak{F}(s) \|_{{}^{T_\alpha} D_A(\theta, \infty)} \\ & \leq M(t-s)^{\alpha\theta-1} \| \mathfrak{F} \|_{C^\theta([0,T]; {}^{T_\alpha} D_A(\theta, \infty))}, \end{aligned}$$

which is integrable on $(0, t)$. Using the closedness of A , we get

$$\| Aw(t) \| \leq \frac{M}{\alpha\theta} t^{\alpha\theta} \| \mathfrak{F} \|_{C^\theta([0,T]; {}^{T_\alpha} D_A(\theta, \infty))} < \infty, \forall t \in [0, T] \quad \text{with } Aw(0) = 0.$$

Therefore, $w(t) \in D(A)$ for all $t \in [0, T]$. Now let $0 < z < 1$ be such that $t, t+z \in [0, T]$. Then

$$\begin{aligned} & \| Aw(t+z) - Aw(t) \| \\ & = \left\| \int_0^{t+z} AT_\alpha(t+z-\sigma)\mathfrak{F}(\sigma) d\sigma - \int_0^t AT_\alpha(t-\sigma)\mathfrak{F}(\sigma) d\sigma \right\| \end{aligned}$$

$$\begin{aligned}
 &= \left\| \int_{-z}^t AT_\alpha(t-\sigma)\mathfrak{F}(\sigma+z) d\sigma - \int_0^t AT_\alpha(t-\sigma)\mathfrak{F}(\sigma) d\sigma \right\| \\
 &\leq \int_{-z}^0 \|AT_\alpha(t-\sigma)\mathfrak{F}(\sigma+z)\| d\sigma + \int_0^t \|AT_\alpha(t-\sigma)(\mathfrak{F}(\sigma+z) - \mathfrak{F}(\sigma))\| d\sigma \\
 &\leq M\|\mathfrak{F}\|_{C^\theta([0,T];T_\alpha D_A(\theta,\infty))} \left(\int_{-z}^0 (t-\sigma)^{\alpha\theta-1} d\sigma + z^\theta \int_0^t (t-\sigma)^{\alpha\theta-1} d\sigma \right) \\
 &\leq C\|\mathfrak{F}\|_{C^\theta([0,T];T_\alpha D_A(\theta,\infty))} z^{\alpha\theta}.
 \end{aligned}$$

Therefore, $Aw \in C^{\alpha\theta}([0, T]; X)$ and by Proposition 4.2, $w \in C^{\alpha\theta}([0, T]; D(A))$. Hence by Proposition 4.1, it is concluded that v is a strict solution of (1.2).

Now, if $Av_0 \in T_\alpha D_A(\theta, \infty)$, then by following the procedure to obtain (4.5) in Theorem 4.3, we can prove that $t \mapsto AS_\alpha(t)v_0 \in C^{\alpha\theta}([0, T]; X)$ with $[AS_\alpha(\cdot)v_0]_{\alpha\theta} \leq T_\alpha \|v_0\|_{\theta,\infty}$ and the estimation (4.9) follows easily. \square

Example 4.8 Consider the following time-fractional diffusion equation:

$$\begin{aligned}
 {}^c D_t^\alpha u(t, x) &= \Delta u(t, x) + f(t, x), \quad t \in (0, T], x \in \mathbb{R}^n, \alpha \in (0, 1), \\
 u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n.
 \end{aligned}$$

Consider the operator $A : D(A) \rightarrow X = C_b(\mathbb{R}^n)$ defined by

$$\begin{cases} D(A) = \{u \in \bigcap_{1 \leq p < \infty} W_{loc}^{2,p}(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) : \Delta u \in C_b(\mathbb{R}^n)\} \\ Au = \Delta u, \end{cases} \tag{4.10}$$

where $C_b(\mathbb{R}^n)$ is the space of all bounded continuous functions endowed with the usual supremum norm. The operator A generates an analytic semigroup e^{tA} and $\overline{D(A)} = BUC(\mathbb{R}^n) := X_0$, the space of all bounded uniformly continuous functions (cf. [1, p. 81]). Now, the part of A in X_0 is defined by

$$D(A_0) = \{u \in D(A) : Au \in X_0\}, \quad A_0 u := Au, u \in D(A_0). \tag{4.11}$$

Then $A_0 : D(A_0) \rightarrow X_0$ is densely defined sectorial operator which generates the bounded analytic semigroup e^{tA_0} . In fact, $e^{tA_0} = e^{tA}$ (cf. [1, Remark 2.1.5]). Now, using the setting $u(t, x) = u(t)(x)$ and $f(t, x) = f(t)(x)$, the problem (4.10) can be written as the following abstract Cauchy problem in the Banach space X_0 :

$$\begin{cases} {}^c D_t^\alpha u(t) = A_0 u(t) + f(t), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{4.12}$$

We denote $C_b^\theta(\mathbb{R}^n)$ as the space of all bounded and Hölder continuous functions endowed with the usual Hölder norm. We also define $C_b^{2+\theta}(\mathbb{R}^n) = \{f : D^\beta f \text{ exists, } D^\beta f \in C_b(\mathbb{R}^n) \forall \text{ multi-index } \beta \text{ with } |\beta| \leq 2, \text{ and } D^\beta f \in C_b^\theta(\mathbb{R}^n) \text{ for } |\beta| = 2\}$.

By Theorem 3.2, [1, p. 45] and [1, Theorem 3.1.12], we know that

$$T_\alpha D_A(\theta, \infty) = S_\alpha D_A(\theta, \infty) = D_A(\theta, \infty) = D_{A_0}(\theta, \infty) = C_b^{2\theta}(\mathbb{R}^n).$$

Therefore, $u_0 \in C_b^{2+2\theta}(\mathbb{R}^n)$ implies that $u_0 \in D(A_0)$ and $A_0 u_0 \in T_\alpha D_A(\theta, \infty)$.

Now, if $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded uniformly continuous function such that f is θ -Hölder continuous with respect to time variables, uniformly in x , then $f \in C^\theta([0, T]; X_0)$. Also $f(0, \cdot) \in C_b^{2\theta}(\mathbb{R}^n)$ implies $f(0) \in T_\alpha D_A(\theta, \infty)$. Therefore, applying Theorem 4.3, we have the following result.

Theorem 4.9 *Let $\theta \in (0, \frac{1}{2})$ and $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded uniformly continuous function such that f is θ -Hölder continuous with respect to the time variables, uniformly in x . Assume that $u_0 \in C_b^{2+2\theta}(\mathbb{R}^n)$ and $f(0, \cdot) \in C_b^{2\theta}(\mathbb{R}^n)$. Then any mild solution, say u , of (4.10) is a strict solution. Moreover, ${}^c D_t^\alpha u, \Delta u \in C^{\alpha\theta}([0, T]; BUC(\mathbb{R}^n))$.*

Again, let $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded function such that

$$\sup_{s \neq t, x \neq y} \frac{|f(t, x) - f(s, y)|}{|s - t|^\theta + |x - y|^{2\theta}} < \infty. \quad (4.13)$$

Then $f \in C^\theta([0, T]; T_\alpha D_A(\theta, \infty))$. Hence, by Theorem 4.7, we have the following result.

Theorem 4.10 *Let $\theta \in (0, \frac{1}{2})$ and $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded function satisfying the condition (4.13). Assume that $u_0 \in C_b^{2+2\theta}(\mathbb{R}^n)$. Then any mild solution, say u , of (4.10) is a strict solution. Moreover, ${}^c D_t^\alpha u, \Delta u \in C^{\alpha\theta}([0, T]; BUC(\mathbb{R}^n))$.*

5 Conclusion

We constructed two classes of interpolation spaces in terms of solution operators of fractional abstract Cauchy problem, and showed that these classes coincide with the real interpolation space. Moreover, it is found that these newly formulated interpolation spaces are useful to prove the strict Hölder regularity of a mild solution of a fractional ACP in the semigroup fashion as of the classical case.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MMA and SD contributed in conception and design of study. They also participated in the analysis and proofs of the study. Authors made substantial contribution in drafting the manuscript. SD and DB participated in revising the manuscript critically for important intellectual content. All authors read and approved the final manuscript.

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