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NEW MULTI-FUNCTIONAL APPROACH FOR κTH-ORDER DIFFERENTIABILITY GOVERNED BY FRACTIONAL CALCULUS VIA APPROXIMATELY GENERALIZED $(\psi, \hbar)$-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract

This work addresses several novel classes of convex function involving arbitrary non-negative function, which is known as approximately generalized $(\psi, \hbar)$-convex and approximately ψ-quasiconvex function, with respect to Raina's function, which are elaborated in Hilbert space. To ensure the feasibility of the proposed concept and with the discussion of special cases, it is presented that these classes generate other classes of generalized $(\psi, \hbar)$-convex functions such as higher-order strongly (HOS) generalized $(\psi, \hbar)$-convex functions and HOS generalized ψ-quasiconvex function. The core of the proposed method is a newly developed Simpson's type of identity in the settings of Riemann–Liouville fractional integral operator. Based on the HOS generalized $(\psi, \hbar)$-convex function representation, we established several theorems and related novel consequences. The presented results demonstrate better performance for HOS generalized ψ-quasiconvex functions where we can generate several other novel classes for convex functions that exist in the relative literature. Accordingly, the assortment in this study aims at presenting a direction in the related fields.

Keywords: Convex Functions; Generalized $(\psi, \hbar)$-Convex Functions; κ-th Order Differentiability; Raina's Function; Breckner-Type Function; Godunova–Levin-Type Function.

1. INTRODUCTION

In the last three decades, the most intriguing and captivating subject of current research in mathematical sciences is the fractional calculus as well as derivatives and integrals of non-integer order are involved. The fractional operators of the said calculus is the most essential phenomenon in the real world, and it has been treated as the crucial and exceptionally enormous factor in precisely portraying the conduct of oscillators, medicine, mechanical devices, electrical systems, granular soils, circuits, and financial systems has been represented in the continuous composition, e.g. see Refs. 1–3. Subsequently, the dynamical developments of authentic physical structures have been even more fittingly investigated by fractional-order differential and/or difference equations rather than the integer-order ones. Given the way that the high constancy model of physical frameworks can be portrayed by fractional-order frameworks, the region has obtained a lot of enthusiasm for the control network which has concentrated on stability and control problems in systems represented by non-integer-order differential equations.[4,5]

On the other hand, the study of the fractional integral inequalities such as in the case of differential and difference equations is of importance in various subjects.[6–9] Inequalities that are used in structural applications are of practical importance, as they have proven that fractional behavior has a profound impact on the performance of fractional integral inequalities. For instance, Sarikaya *et al.*[10] reported the Hermite–Hadamard-type inequalities for Riemann–Liouville fractional operators. Set *et al.*[11] established valuable consequences by utilizing fractional integral operators. The inequality theory is of importance in many fields such as in the field of solid-state physics, materials and metallurgy sciences because it is causally related to different microscopic physics processes and also can be used as an experimental investigation of these procedures.[7–9] Fractional integral inequalities play a significant role in both pure and applied sciences because of their wide applications as well as many other natural and human social sciences, while convexity theory has remained as an important tool in the establishment of the theory of integral inequalities. Simpson's inequality, as a member of the family of integral inequalities, is a classic inequality that has long been fascinated by numerous mathematical researchers, which can be stated as follows:

$$\left|\frac{1}{6}\left[\mathcal{Q}(\omega_1)+4\mathcal{Q}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathcal{Q}(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathcal{Q}(z)dz\right| \le \frac{\|\mathcal{Q}^{(4)}\|_\infty(\omega_2-\omega_1)^4}{2880}, \tag{1}$$

where $\mathcal{Q}:[\omega_1,\omega_2]\to\mathbb{R}$ is fourth-order differentiable function on (ω_1,ω_2) with the condition that

$$\|\mathcal{Q}^{(4)}\|_\infty = \sup_{z\in(\omega_1,\omega_2)}|\mathcal{Q}(z)| < \infty.$$

In recent years, several successful attempts have been made in obtaining the variants and applications of Simpson's type of inequality. For example, Dragomir *et al.*[12] provided many interesting results on its applications in numerical integration. Rashid *et al.*[13] contemplated the novel Simpson's-type inequalities in the settings of fractional calculus with applications. In Ref. 14, Li *et al.* obtained several extensions of Simpson's-type inequalities via extended (s,m)-convex functions. For more developments, generalizations and variants for Simpson's type of inequality have been the subject of much research, see Refs. 15 and 16.

In Ref. 17, Varosanec proposed a class of convex functions amplify and unifies several existing ideas of classical convex functions, encompassing Breckner-type convex functions,[18] P-functions,[19] Godunova–Levin-type convex, and Q-functions.[20] We acknowledge that this class plays a dominant role in convex analysis and this class plays a significant contribution to convexity theory and provides assistance to explore numerous novel classes of a convex function, see Refs. 21 and 22 and the references therein.

In Ref. 23, Polyak introduced and studied a new class of functionals that has significant importance in machine learning models, optimization theory and many other related areas. The existence of nonlinear complementary problems can be determined by strong convexity.[24] The convergence criteria of iterative schemes for variational and equilibrium issues are contemplated by Zu and Marcotte.[25] Bynum[26] and Chen*et al.*[27] introduced the general characterization and utilization of the parallelogram laws for the Banach spaces. Xu[28] explored new attributes of p-uniform convexity and q-uniform smoothness of a Banach space utilizing $\|\cdot\|^p$ and $\|\cdot\|^q$, respectively. In Ref. 29, Nikodem and Pales explored the new and novel utilities of the characterization of the inner product space by considering strongly convex functions. Interestingly, the Polyak–Lojasiewicz condition is fulfilled with the assembly of stochastic slope descent for the class of functions based on strongly-convex functions too as a wide scope of non-convex functions incorporating those utilized in machine-learning applications.[30] In their recent work, authors[31] reported results for differentiable higher-order strongly (HOS) $\hbar$-convex functions. In Ref. 32, the authors established the predominating η-convex functions in a wide broadway and also studied several generalizations for predominating η-quasiconvex functions. For further interesting papers related to HOS convex functions, see Refs. 33–35.

In response to the existing approaches, this paper aims to create refinements of Simpson-type inequalities utilizing approximately generalized $(\psi,\hbar)$-convex, approximately generalized $(\psi,\hbar)$-quasiconvex, HOS generalized $(\psi,\hbar)$-convex and HOS generalized ψ-quasiconvex functions with respect to Raina's function that not only possess the key properties of the classical convex functions but also have exceptional parameters adjustment with and without fixing the non-negative arbitrary function $\hbar$. Moreover, our consequences are correlated with the auxiliary identity which associates with fractional calculus for κth-order differentiable functions. Numerous refinements of Simpson's type of inequality are derived that can be used to characterize the uniformly reflex Banach space. The significant advantage of these outcomes is that they used to comprehend the parallelogram laws for L^p spaces. Several remarkable cases are provided to exhibit the novelty of the results given herein.

2. PRELIMINARIES

Initially, suppose a non-empty set $\mathcal{K}_{\mathcal{F}}$ in a real Hilbert space $\mathcal{H}$. The inner product and norm are denoted by the symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively.

Moreover, there is an arbitrary non-negative function $\hbar:(0,1)\to\mathbb{R}$, $\eta=\{\eta(m)\}_{m=0}^{\infty}$ be a bounded sequence of real numbers and $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\cdot)$ $\upsilon_1,\upsilon_2>0$ denotes Raina's function. In Ref. 36, Raina explored a new class of functions stated as follows:

$$\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(z)=\mathcal{F}^{\eta(0),\eta(1),\dots}_{\upsilon_1,\upsilon_2}(z)=\sum_{m=0}^{\infty}\frac{\eta(m)}{\Gamma(\upsilon_1 m+\eta)}z^m, \tag{2}$$

where $\upsilon_1,\upsilon_2>0,|z|<\mathcal{R}$ and

$$\eta=(\eta(0),\dots,\eta(m),\dots)$$

is bounded sequence of positive real numbers. Notice that if we choose $\upsilon_1=1,\upsilon_2=0$ in Eq. (2), then

$$\eta(m)=\frac{(\delta_1)_m(\delta_2)_m}{(\delta_3)_m}\quad\text{for } m=0,1,2,\dots,$$

where δ_1,δ_2 and δ_3 are parameters which can take arbitrary real and complex values (provided that $\delta_3\neq 0,-1,-2,\dots,$) and we denotes the symbol $(b)_m$ by

$$(b)_m=\tfrac{\Gamma(b+m)}{\Gamma(b)}=b(b+1)\dots(b+m-1)\ ,\quad m=0,1,2,\dots,$$

and restrict its domain to $|z|\le 1$ (with $z\in\mathbb{C}$), then we have the classical hypergeometric function, which is defined as

$$\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(z)=F(\delta_1,\delta_2;\delta_3;z)=\sum_{m=0}^{\infty}\frac{(\delta_1)_m(\delta_2)_m}{m!(\delta_3)_m}z^m.$$

Also, if $\eta=(1,1,\dots)$ with $\varsigma=\delta,(\Re(\delta)>0),\eta=1$ and restricting its domain to $z\in\mathbb{C}$ in Eq. (2), then we have the classical Mittag–Leffler function:

$$E_{\delta_1}(z)=\sum_{m=0}^{\infty}\frac{1}{\Gamma(1+\delta_1 m)}z^{\kappa}.$$

Next, we evoke a new class of set and a new class of functions including Raina's functions.

Definition 1 (Ref. 37). A non-empty set $\mathcal{K}_{\mathcal{F}}$ is said to be generalized ψ-convex set, if

$$\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\in\mathcal{K}_{\mathcal{F}} \tag{3}$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}},\xi\in[0,1]$.

We now define the generalized ψ-convex function presented by Cortez *et al.*[37]

Definition 2 (Ref. 37). Let a set $\mathcal{K}_{\mathcal{F}}\subseteq\mathbb{R}$ and a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be generalized ψ-convex, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\le(1-\xi)\mathcal{Q}(\omega_1)+\xi\mathcal{Q}(\omega_2) \tag{4}$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}},\xi\in[0,1]$.

Next, we present a new class of generalized ψ-convex functions for an arbitrary non-negative function $\hbar$.

Definition 3. Let $\hbar:(0,1)\to\mathbb{R}$ be a real function and a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be generalized $(\psi,\hbar)$-convex function, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\le\hbar(1-\xi)\mathcal{Q}(\omega_1)+\hbar(\xi)\mathcal{Q}(\omega_2) \tag{5}$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}},\xi\in[0,1]$.

Further, we demonstrate several novel concepts of generalized ψ-convex functions with respect to an arbitrary non-negative function.

Definition 4. Let $\hbar:(0,1)\to\mathbb{R}$ be a real function and a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be approximately generalized $(\psi,\hbar)$-convex, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\le\hbar(1-\xi)\mathcal{Q}(\omega_1)+\hbar(\xi)\mathcal{Q}(\omega_2)+\mathbb{D}(\omega_1,\omega_2) \tag{6}$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}},\xi\in[0,1]$.

Some noteworthy cases of Definition 4 are demonstrated as follows:
(I) For some $c\ge 0,p>2$ and taking $\mathbb{D}(\omega_1,\omega_2)=-c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p$, then we have a new concept of HOS generalized $(\psi,\hbar)$-convex function with respect to an arbitrary non-negative function $\hbar$.

Definition 5. Let $\hbar:(0,1)\to\mathbb{R}$ be a real function and a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be HOS generalized $(\psi,\hbar)$-convex having $c\ge 0$, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\le\hbar(1-\xi)\mathcal{Q}(\omega_1)+\hbar(\xi)\mathcal{Q}(\omega_2)-c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\times\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p \tag{7}$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}},\xi\in[0,1]$.

(II) Taking $\mathbb{D}(\omega_1,\omega_2) = -c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p$ with $p=2$ and $c\geq 0$, then we have a new concept of strongly generalized $(\psi,\hbar)$-convex function with respect to an arbitrary non-negative function $\hbar$.

Definition 6. Let $\hbar:(0,1)\to\mathbb{R}$ be a real function and a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be strongly generalized $(\psi,\hbar)$-convex having $c\geq 0$, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)) \leq \hbar(1-\xi)\mathcal{Q}(\omega_1) + \hbar(\xi)\mathcal{Q}(\omega_2) - c\xi(1-\xi)\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^2 \quad (8)$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}}, \xi\in[0,1]$.

(III) For some $c\geq 0, p>2$ with $\hbar(\xi)=\xi$ and taking $\mathbb{D}(\omega_1,\omega_2) = -c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p$, then we have a new concept of HOS generalized ψ-convex function.

Definition 7. Let a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be HOS generalized ψ-convex having $c\geq 0$, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)) \leq (1-\xi)\mathcal{Q}(\omega_1) + \xi\mathcal{Q}(\omega_2) - c\xi(1-\xi)\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p \quad (9)$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}}, \xi\in[0,1]$.

We now introduce a new concept of generalized approximately ψ-quasiconvex functions.

Definition 8. Let a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be approximately generalized ψ-quasiconvex, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)) \leq \max\{\mathcal{Q}(\omega_1),\mathcal{Q}(\omega_2)\} + \mathbb{D}(\omega_1,\omega_2) \quad (10)$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}}, \xi\in[0,1]$.

We now mention some notable cases of Definition 8 as follows:

(I) For some $c\geq 0, p>2$ with $\hbar(\xi)=\xi$ and taking $\mathbb{D}(\omega_1,\omega_2) = -c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p$, then we have a new concept of HOS generalized ψ-quasiconvex function.

Definition 9. Let a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to HOS generalized ψ-quasi-convex having $c\geq 0$, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)) \leq \max\{\mathcal{Q}(\omega_1),\mathcal{Q}(\omega_2)\} - c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p \quad (11)$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}}, \xi\in[0,1]$.

(II) For some $c\geq 0, p=2$ and taking $\mathbb{D}(\omega_1,\omega_2) = -c\{\xi^p(1-\xi)+(1-\xi)^p\xi\}\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^p$, then we have a new concept of strongly generalized ψ-quasiconvex function.

Definition 10. Let a function $\mathcal{Q}:\mathcal{K}_{\mathcal{F}}\to\mathbb{R}$ is said to be strongly generalized ψ-quasiconvex having $c\geq 0$, if

$$\mathcal{Q}(\omega_1+\xi\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)) \leq \max\{\mathcal{Q}(\omega_1),\mathcal{Q}(\omega_2)\} - c\xi(1-\xi)\|\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\|^2 \quad (12)$$

for all $\omega_1,\omega_2\in\mathcal{K}_{\mathcal{F}}, \xi\in[0,1]$.

We end this section by presenting a notable fractional integral operators in the literature.

Definition 11. Let $\mathcal{Q}\in L_1[\omega_1,\omega_2]$, then the left- and right-sided Riemann–Liouville fractional integrals $\mathcal{J}^{\delta}_{\omega_1^+}$ and $\mathcal{J}^{\delta}_{\omega_2^-}$ are defined as

$$\mathcal{J}^{\delta}_{\omega_1^+}\mathcal{Q}(z) = \frac{1}{\Gamma(\delta)}\int_{\omega_1}^{z}(z-\xi)^{\delta-1}\mathcal{Q}(\xi)d\xi, \quad z<\omega_1, \delta>0$$

and

$$\mathcal{J}^{\delta}_{\omega_2^-}\mathcal{Q}(z) = \frac{1}{\Gamma(\delta)}\int_{z}^{\omega_2}(\xi-z)^{\delta-1}\mathcal{Q}(\xi)d\xi, \quad z>\omega_2, \delta>0,$$

respectively, $\Gamma(z) := \int_0^{\infty}\xi^{z-1}e^{-\xi}d\xi, \Re(z)>0$ is the Gamma function.

Some of our computations need incomplete beta and hypergeometric functions, which are, respectively, stated as

$$\mathbb{B}_z(\omega_1,\omega_2) = \int_0^z \xi^{\omega_1-1}(1-\xi)^{\omega_2-1}d\xi.$$

The integral representation of the hypergeometric function is

$${}_2\mathfrak{F}_1(\omega_1,\omega_2;c;z) = \frac{1}{\mathbb{B}(\omega_2,c-\omega_2)}\int_0^1 \xi^{\omega_2-1} \times (1-\xi)^{c-\omega_2-1}(1-z\xi)^{-\omega_1}d\xi$$

for $|z|<1, c>\omega_2>0$.

3. AN AUXILIARY RESULT FOR GENERALIZED ψ-CONVEX FUNCTIONS

This section deals with the investigation of an integral identity of κth-order differentiable functions for generalized ψ-convex functions.

Lemma 12. *For $n,\kappa \in \mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)] \to \mathbb{R}$ is defined on Ω° (the interior of Ω) such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1) > 0$ and $\mathcal{Q}^{(\kappa)} \in L_1[\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]$ (the Lebesgue space). Then the following equality holds:*

$$\begin{aligned}\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q}) &:= \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right) \\ &\quad\times \mathcal{Q}^{(\kappa)}\left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi \\ &\quad+ \int_0^1 \left(\frac{\xi^{\delta+\kappa-1}-2(1-\xi)^{\delta+\kappa-1}}{3}\right) \\ &\quad\times \mathcal{Q}^{(\kappa)}\left(\omega_1 + \frac{n-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi,\end{aligned}$$

where

$$\begin{aligned}\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q}) &:= \frac{1}{6}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa} \\ &\quad\times \left[\mathcal{J}^{\delta}_{(\omega_1)^+}\mathcal{Q}\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right. \\ &\quad+(-1)^{\kappa}\mathcal{J}^{\delta}_{(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-} \\ &\quad\left.\times \mathcal{Q}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right] \\ &\quad+\frac{1}{3}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa} \\ &\quad\times\left[\mathcal{J}^{\delta}_{(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-}\mathcal{Q}(\omega_1)\right. \\ &\quad+(-1)^{\kappa}\mathcal{J}^{\delta}_{(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^+} \\ &\quad\left.\times\mathcal{Q}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\right] \\ &\quad-\frac{1}{6}\sum_{q=1}^{\kappa}\frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)} \\ &\quad\times\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{q} \\ &\quad\times\left[(-1)^{q-1}\mathcal{Q}(\omega_1)+2(-1)^{q-1}\mathcal{Q}^{(\kappa-q)}\right. \\ &\quad\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\ &\quad+2\mathcal{Q}^{(\kappa-q)}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\ &\quad\left.+\mathcal{Q}^{\kappa-q}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\right].\end{aligned}$$

Proof. Let

$$\begin{aligned}&\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right) \\ &\quad\times\left[\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right. \\ &\quad\left.-\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{n-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right]d\xi \\ &=\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\mathcal{Q}^{(\kappa)} \\ &\quad\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi \\ &\quad-\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\mathcal{Q}^{(\kappa)} \\ &\quad\times\left(\omega_1+\frac{n-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi \\ &= I_1 - I_2.\end{aligned}$$

Now, integrating by parts, we have

$$\begin{aligned}&\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\mathcal{Q}^{(\kappa)} \\ &\quad\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi \\ &=\frac{2}{3}\int_0^1(1-\xi)^{\delta}\mathcal{Q}^{(\kappa)} \\ &\quad\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}\int_0^1 \xi^{\delta}\mathcal{Q}^{(\kappa)} \\
&\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi \\
&=\frac{2}{3}\left\{\left[-\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}(1-\xi)^{\delta+\kappa-1}\mathcal{Q}^{(\kappa-1)}\right.\right. \\
&\left.\left.\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right]\right|_0^1 \\
&-\frac{(n+1)(\delta+\kappa-1)}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\int_0^1(1-\xi)^{\delta+\kappa-2}\mathcal{Q}^{(\kappa-1)} \\
&\left.\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\right\} \\
&-\frac{1}{3}\left\{\left[-\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\xi^{\delta+\kappa-1}\mathcal{Q}^{(\kappa-1)}\right.\right. \\
&\left.\left.\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right]\right|_0^1 \\
&+\frac{(n+1)(\delta+\kappa-1)}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\int_0^1\xi^{\delta+\kappa-2}\mathcal{Q}^{(\kappa-1)} \\
&\left.\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\right\}.
\end{aligned}
$$

Again, through the integration by parts, we obtain

$$
\begin{aligned}
I_1&=\frac{2}{3}\left\{\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\mathcal{Q}^{(\kappa-1)}\right. \\
&\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\
&-\frac{(n+1)^2(\delta+\kappa-1)}{(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^2}\mathcal{Q}^{(\kappa-2)} \\
&\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\
&+\frac{(n+1)^2(\delta+\kappa-1)(\delta+\kappa-2)}{(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^2} \\
&\times\int_0^1(1-\xi)^{\delta+\kappa-3} \\
&\left.\times\mathcal{Q}^{(\kappa-2)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\right\} \\
&-\frac{1}{3}\left\{-\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\mathcal{Q}^{(\kappa-1)}\right. \\
&\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\
&+\frac{(n+1)^2(\delta+\kappa-1)}{(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^2}\mathcal{Q}^{(\kappa-2)} \\
&\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\
&-\frac{(n+1)^2(\delta+\kappa-1)(\delta+\kappa-2)}{(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^2} \\
&\times\int_0^1(1-\xi)^{\delta+\kappa-3} \\
&\left.\times\mathcal{Q}^{(\kappa-2)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\right\}.
\end{aligned}
$$

Further, integrating by parts successively upto κ-times, we obtain

$$
\begin{aligned}
I_1&=\frac{2}{3}\left\{\sum_{q=1}^{\kappa}\frac{(-1)^{q-1}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right. \\
&\times\mathcal{Q}^{(\kappa-q)}\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) \\
&\times\prod_{\theta=0}^{q-1}(\delta+\kappa-\theta) \\
&+\frac{(-1)^{\kappa}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\kappa}\prod_{\theta=0}^{\kappa}(\delta+\kappa-q) \\
&\times\int_0^1(1-\xi)^{\delta-1}\mathcal{Q} \\
&\left.\times\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\right\} \\
&-\frac{1}{3}\left\{\sum_{q=1}^{\kappa}\frac{(-1)^{q-2}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right. \\
&\times\mathcal{Q}^{(\kappa-q)}(\omega_1)\prod_{\theta=0}^{q-1}(\delta+\kappa-\theta) \\
&+\frac{(-1)^{\kappa+1}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\kappa} \\
&\times\prod_{\theta=0}^{\kappa}(\delta+\kappa-q)\int_0^1(1-\xi)^{\delta-1}\mathcal{Q}
\end{aligned}
$$

$$\times \left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) d\xi\Bigg\}$$

$$= \frac{2}{3}\left\{\sum_{q=1}^{\kappa} \frac{(-1)^{q-1}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right.$$

$$\times \mathcal{Q}^{(\kappa-q)}\left(\omega_1 + \frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)$$

$$\times \prod_{\theta=0}^{q-1}(\delta+\kappa-\theta)$$

$$+ \frac{(-1)^{\kappa}\Gamma(\delta+\kappa)}{\Gamma(\delta)}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\kappa}$$

$$\times \prod_{\theta=0}^{\kappa}(\delta+\kappa-q)\int_0^1 (1-\xi)^{\delta-1}\mathcal{Q}$$

$$\times \left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) d\xi\Bigg\}$$

$$- \frac{1}{3}\left\{\sum_{q=1}^{\kappa} \frac{(-1)^{q-2}}{\delta+\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right.$$

$$\times \mathcal{Q}^{(\kappa-q)}(\omega_1)\prod_{\theta=0}^{q-1}(\delta+\kappa-\theta)$$

$$+ \frac{(-1)^{\kappa+1}\Gamma(\delta+\kappa)}{\Gamma(\delta)}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\kappa}$$

$$\times \prod_{\theta=0}^{\kappa}(\delta+\kappa-q)\int_0^1 (1-\xi)^{\delta-1}\mathcal{Q}$$

$$\times \left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) d\xi\Bigg\}$$

$$= \frac{2}{3}\left\{\sum_{q=1}^{\kappa}(-1)^{q-1}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right.$$

$$\times \frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)}$$

$$\times \mathcal{Q}^{(\kappa-q)}\left(\omega_1 + \frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)$$

$$+ (-1)^{\kappa}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}$$

$$\times \mathcal{J}^{\delta}_{(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-}\mathcal{Q}(\omega_1)\Bigg\}$$

$$- \frac{1}{3}\left\{\sum_{q=1}^{\kappa}(-1)^{q-2}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\right.$$

$$\times \frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)}\mathcal{Q}^{(\kappa-q)}(\omega_1)$$

$$+ (-1)^{\kappa+1}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}$$

$$\times \mathcal{J}^{\delta}_{\omega_1^+}\mathcal{Q}\left(\omega_1 + \frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\Bigg\}.$$

Analogously, we have

$$I_2 = \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)\mathcal{Q}^{(\kappa)}$$

$$\times \left(\omega_1 + \frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) d\xi$$

$$= \frac{2}{3}\left\{-\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\mathcal{Q}^{(\kappa-1)}\right.$$

$$\times \left(\omega_1 + \frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)$$

$$- (\delta+\kappa-1)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^2$$

$$\times \mathcal{Q}^{(\kappa-2)}\left(\omega_1 + \frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)$$

$$- (\delta+\kappa-1)(\delta+\kappa-2)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^3$$

$$\times \mathcal{Q}^{(\kappa-3)}\left(\omega_1 + \frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)$$

$$+ (\delta+\kappa-1)(\delta+\kappa-2)(\delta+\kappa-3)$$

$$\times \left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^3\int_0^1 (1-\xi)^{\delta+\kappa-4}\mathcal{Q}^{(\kappa-3)}$$

$$\times \left(\omega_1 + \frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right) d\xi\Bigg\}$$

$$- \frac{1}{3}\left\{\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\mathcal{Q}^{(\kappa-1)}\right.$$

$$\times (\omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))$$

$$+ (\delta+\kappa-1)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^2$$

$$\times \mathcal{Q}^{(\kappa-2)}(\omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))$$

$$
\begin{aligned}
&+(\delta+\kappa-1)(\delta+\kappa-2)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^3\\
&\times \mathcal{Q}^{(\kappa-3)}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\\
&-(\delta+\kappa-1)(\delta+\kappa-2)(\delta+\kappa-3)\\
&\times\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^3\int_0^1 \xi^{\delta+\kappa-4}\mathcal{Q}^{(\kappa-3)}\\
&\times\left(\omega_1+\frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)d\xi\Bigg\}\\
&\vdots\\
=&\frac{2}{3}\Bigg\{-\sum_{q=1}^{\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\\
&\times\frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)}\\
&\times\mathcal{Q}^{(\kappa-q)}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\\
&+\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}\\
&\times\mathcal{J}^{\delta}_{(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^+}\\
&\times\mathcal{Q}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\Bigg\}\\
&-\frac{1}{3}\Bigg\{\sum_{q=1}^{\kappa}\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\\
&\times\frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)}\\
&\times\mathcal{Q}^{(\kappa-q)}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\\
&+\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}\\
&\times\mathcal{J}^{\delta}_{(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-}\\
&\times\mathcal{Q}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\Bigg\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&I_1-I_2\\
&\quad=\frac{1}{6}\sum_{q=1}^{\kappa}\frac{\Gamma(\delta+\kappa)}{\Gamma(\delta+\kappa-q+1)}\\
&\quad\times\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^q\\
&\quad\times\Big[(-1)^{q-1}\mathcal{Q}(\omega_1)+2(-1)^{q-1}\mathcal{Q}^{(\kappa-q)}\\
&\quad\times\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\\
&\quad-2\mathcal{Q}^{(\kappa-q)}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\\
&\quad-\mathcal{Q}^{\kappa-q}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))\Big]\\
&\quad+\frac{1}{6}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}\\
&\quad\times\left[\mathcal{J}^{\delta}_{(\omega_1)^+}\mathcal{Q}\left(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right.\\
&\quad+(-1)^{\kappa}\mathcal{J}^{\delta}_{(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-}\\
&\quad\left.\times\mathcal{Q}\left(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right]\\
&\quad+\frac{1}{3}\Gamma(\delta+\kappa)\left(\frac{n+1}{\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)}\right)^{\delta+\kappa}\\
&\quad\times[\mathcal{J}^{\delta}_{(\omega_1+\frac{1}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^-}\mathcal{Q}(\omega_1)\\
&\quad+(-1)^{\kappa}\mathcal{J}^{\delta}_{(\omega_1+\frac{n}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^+}\\
&\quad\times\mathcal{Q}(\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))].
\end{aligned}
\tag{13}
$$

This completes the proof of Lemma 12. □

4. CERTAIN ESTIMATES FOR κTH DIFFERENTIABLE FUNCTIONS

We propose some new generalizations of upper bounds for the mapping $\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})$ for approximately generalized $(\psi,\hbar)$-convex functions, our main consequences are described in the subsequent theorems.

Theorem 13. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|$ is an approximately generalized $(\psi,\hbar)$-convex* *on* Ω, *then*

$$\begin{aligned}|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| &\le \Phi_1(\delta,\kappa,\xi)[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|] \\ &\quad + \frac{2\mathbb{D}(\omega_1,\omega_2)}{3(\delta+\kappa)},\end{aligned}$$

where

$$\begin{aligned}\Phi_1(\delta,\kappa,\xi) := \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right) \\ \times \left(\hbar\left(\frac{1-\xi}{n+1}\right)+\hbar\left(\frac{n+\xi}{n+1}\right)\right)d\xi.\end{aligned}$$

Proof. According to Lemma 12, the triangular property and utilizing the fact of approximately generalized $(\psi,\hbar)$-convex functions, we obtain

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \\ &\le \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\Bigg| \\ &\quad \times \mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\Bigg| d\xi \\ &\quad + \int_0^1 \left(\frac{\xi^{\delta+\kappa-1}-2(1-\xi)^{\delta+\kappa-1}}{3}\right)\Bigg| \\ &\quad \times \mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{n-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\Bigg| d\xi \\ &\le \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right) \\ &\quad \times \left[\hbar\left(\frac{n+\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)| \right. \\ &\quad \left. +\hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_2)|+\mathbb{D}(\omega_1,\omega_2)\right] d\xi \\ &\quad + \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right) \\ &\quad \times \left[\hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)| \right. \\ &\quad \left. +\hbar\left(\frac{n+\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_2)|+\mathbb{D}(\omega_1,\omega_2)\right] d\xi \\ &\le \Phi_1(\delta,\kappa,\xi)[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|] \\ &\quad + \frac{2\mathbb{D}(\omega_1,\omega_2)}{3(\delta+\kappa)}. \qquad (14)\end{aligned}$$

This completes the proof. □

Now, we shall state some special cases of Theorem 13.

(I) Letting $\hbar(\xi)=1$, then we acquire a new result for HOS generalized ψ-P-convex functions.

Corollary 14. *For* $n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|$ *is a HOS generalized* ψ-P-*convex on* Ω, *then*

$$\begin{aligned}|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| &\le \frac{2}{3(\delta+\kappa)}\{[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|] \\ &\quad +\mathbb{D}(\omega_1,\omega_2)\}.\end{aligned}$$

(II) Letting $\hbar(\xi)=\xi$, then we acquire a new result for HOS generalized ψ-convex functions.

Corollary 15. *For* $n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|$ *is a HOS generalized* ψ-*convex on* Ω, *then*

$$\begin{aligned}|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| &\le \frac{1}{3(\delta+\kappa)}\{[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|] \\ &\quad +2\mathbb{D}(\omega_1,\omega_2)\}.\end{aligned}$$

(III) Letting $\hbar(\xi)=\xi^s$, then we acquire a new result for Breckner type of HOS generalized ψ-s-convex functions.

Corollary 16. *For* $n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|$ *is a Breckner type of HOS generalized* ψ-s-*convex on* Ω, *then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \\ &\le \frac{1}{3(n+1)^s}\left[\frac{2}{\delta+\kappa+s}+2n^s_2\mathfrak{F}_1 \right. \\ &\quad \times \left(-s;1;\delta+\kappa+s+1;-\frac{1}{n}\right)\end{aligned}$$

$$- n\,{}_2\mathfrak{F}_1^s\left(-s;1;\delta+\kappa+s+1;-\frac{1}{n}\right) - \mathbb{B}(\delta+\kappa,s+1)\Big]\left[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|\right] + \frac{2\mathbb{D}(\omega_1,\omega_2)}{3(\delta+\kappa)}.$$

(IV) Letting $\hbar(\xi)=\xi^{-s}$, then we acquire a new result for Godunova–Levin type of HOS generalized ψ-s-convex functions.

Corollary 17. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|$ is a Godunova–Levin type of HOS generalized ψ-s-convex on Ω, then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\le\frac{1}{3(n+1)^{-s}}\left[\frac{2}{(\delta+\kappa-s)}-\mathbb{B}(1-s,\delta+\kappa)\right.\\&\quad+2n\,{}_2^{-s}\mathfrak{F}_1\left(s;1;\delta+\kappa;-\frac{1}{n}\right)\\&\quad-n^{-s}(\delta+\kappa)\,{}_2\mathfrak{F}_1\\&\quad\times\left.\left(s;\delta+\kappa;\delta+\kappa+1;-\frac{1}{n}\right)\right]\\&\quad\times\left[|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|\right]\\&\quad+\frac{2\mathbb{D}(\omega_1,\omega_2)}{3(\delta+\kappa)}.\end{aligned}$$

(V) Letting $\hbar(\xi)=\xi$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\}(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p$, then we acquire HOS ψ-quasiconvex function.

Corollary 18. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|$ is a HOS generalized ψ-quasiconvex on Ω, then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\le\frac{1}{3(\delta+\kappa)}\{|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|\}\\&\quad-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\left\{2n(\delta+\kappa+1)\,{}_2\mathfrak{F}_1\right.\\&\quad\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\quad-n\mathbb{B}(\delta+\kappa,p+1)\,{}_2\mathfrak{F}_1\\&\quad\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\quad+2n\,{}_2^p\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)\\&\quad-n^p\mathbb{B}(\delta+\kappa,2)\,{}_2\mathfrak{F}_1\\&\quad\times\left.\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}.\end{aligned}$$

(VI) Letting $\hbar(\xi)=1$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\big)^p$, then we acquire HOS ψ-P-convex function.

Corollary 19. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|$ is a HOS generalized ψ-P-convex on Ω, then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\le\frac{2}{3(\delta+\kappa)}\{|\mathcal{Q}^{(\kappa)}(\omega_1)|+|\mathcal{Q}^{(\kappa)}(\omega_2)|\}\\&\quad-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\left\{2n(\delta+\kappa+1)\,{}_2\mathfrak{F}_1\right.\\&\quad\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\quad-n\mathbb{B}(\delta+\kappa,p+1)\,{}_2\mathfrak{F}_1\\&\quad\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\quad+2n\,{}_2^p\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)\\&\quad-n^p\mathbb{B}(\delta+\kappa,2)\,{}_2\mathfrak{F}_1\\&\quad\times\left.\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}.\end{aligned}$$

(VII) Letting $\hbar(\xi)=\xi^s$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\}(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p$, then we acquire a new result for Breakner type of HOS generalized ψ-s-convex functions.

Corollary 20. *For $n, \kappa \in \mathbb{N}$ with $\upsilon_1, \upsilon_2, \delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|$ is a Breakner type of HOS generalized ψ-s-convex on Ω, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \le \frac{1}{3(n+1)^s}\left[\frac{2}{\delta+\kappa+s} + 2n_2^s\mathfrak{F}_1\right. \\
&\qquad \times \left(-s; 1; \delta+\kappa+s+1; -\frac{1}{n}\right) - n_2^s\mathfrak{F}_1 \\
&\qquad \times \left(-s; 1; \delta+\kappa+s+1; -\frac{1}{n}\right) \\
&\qquad \left. - \mathbb{B}(\delta+\kappa, s+1)\right][|\mathcal{Q}^{(\kappa)}(\omega_1)| + |\mathcal{Q}^{(\kappa)}(\omega_2)|] \\
&\qquad - \frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\left\{2n(\delta+\kappa+1)_2\mathfrak{F}_1\right. \\
&\qquad \times/\left(-1; 1, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\qquad - n\mathbb{B}(\delta+\kappa, p+1)_2\mathfrak{F}_1 \\
&\qquad \times \left(-1; \delta+\kappa, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\qquad + 2n_2^p\mathfrak{F}_1\left(-p; 1, \delta+\kappa+2, -\frac{1}{n}\right) \\
&\qquad - n^p\mathbb{B}(\delta+\kappa, 2)_2\mathfrak{F}_1 \\
&\qquad \left.\times \left(-p; \delta+\kappa, \delta+\kappa+2, -\frac{1}{n}\right)\right\}.
\end{aligned}$$

(VIII) Letting $\hbar(\xi) = \xi^{-s}$ and $\mathbb{D}(\omega_1, \omega_2) = -\frac{c}{(n+1)^{2p}}\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\}(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p$, then we acquire a new result for Godunova–Levin-type HOS generalized ψ-s-convex functions.

Corollary 21. *For $n, \kappa \in \mathbb{N}$ with $\upsilon_1, \upsilon_2, \delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|$ is a Godunova–Levin type of HOS generalized ψ-s-convex on Ω, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \le \frac{1}{3(n+1)^{-s}}\left[\frac{2}{\delta+\kappa-s} + 2n_2^{-s}\mathfrak{F}_1\right. \\
&\qquad \times \left(s; 1; \delta+\kappa-s+1; -\frac{1}{n}\right) \\
&\qquad -n_2^{-s}\mathfrak{F}_1\left(s; 1; \delta+\kappa-s+1; -\frac{1}{n}\right) \\
&\qquad \left. - \mathbb{B}(\delta+\kappa, 1-s)\right][|\mathcal{Q}^{(\kappa)}(\omega_1)| + |\mathcal{Q}^{(\kappa)}(\omega_2)|] \\
&\qquad - \frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}} \\
&\qquad \times \{2n(\delta+\kappa+1)_2\mathfrak{F}_1 \\
&\qquad \times \left(-1; 1, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\qquad - n\mathbb{B}(\delta+\kappa, p+1)_2\mathfrak{F}_1 \\
&\qquad \times \left(-1; \delta+\kappa, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\qquad + 2n_2^p\mathfrak{F}_1\left(-p; 1, \delta+\kappa+2, -\frac{1}{n}\right) \\
&\qquad - n^p\mathbb{B}(\delta+\kappa, 2)_2\mathfrak{F}_1 \\
&\qquad \times \left(-p; \delta+\kappa, \delta+\kappa+2, -\frac{1}{n}\right)\bigg\}.
\end{aligned}$$

Theorem 22. *For $n, \kappa \in \mathbb{N}$ with $\upsilon_1, \upsilon_2, \delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is an approximately generalized $(\psi, \hbar)$-convex on Ω for $q_2 > 1$ with $q_1^{-1} + q_2^{-1} = 1$, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \le \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \\
&\qquad \times \{[\Phi_1^{**}(\delta, \kappa, n, \xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} \\
&\qquad + \Phi_2^{**}(\delta, \kappa, n, \xi) \\
&\qquad \times |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1, \omega_2)]^{\frac{1}{q_2}}
\end{aligned}$$

$$+ [\Phi_2^{**}(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \Phi_1^{**}(\delta,\kappa,n,\xi) \times |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\}, \tag{15}$$

where

$$\Phi_1^{**}(\delta,\kappa,n,\xi) := \int_0^1 \hbar\left(\frac{n+\xi}{n+1}\right) d\xi \tag{16}$$

and

$$\Phi_2^{**}(\delta,\kappa,n,\xi) := \int_0^1 \hbar\left(\frac{1-\xi}{n+1}\right) d\xi. \tag{17}$$

Proof. According to Lemma 12, the well-known Hölder inequality and utilizing the fact of approximately generalized $(\psi,\hbar)$-convex functions, we obtain

$$\begin{aligned}
&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \\
&\le \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \times \left(\int_0^1 \left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right|^{q_2} d\xi\right)^{\frac{1}{q_2}} \\
&+ \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \times \left(\int_0^1 \left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right|^{q_2} d\xi\right)^{\frac{1}{q_2}} \\
&\le \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \\
&\times \left\{\left[\int_0^1 \left(\hbar\left(\frac{n+\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)\right) d\xi\right]^{\frac{1}{q_2}}\right. \\
&+ \left.\left[\int_0^1 \left(\hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \hbar\left(\frac{n+\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)\right) \times \xi\right]^{\frac{1}{q_2}}\right\} \\
&\le \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \\
&\times \{[\Phi_1^{**}(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \Phi_2^{**}(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}} \\
&+ [\Phi_2^{**}(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \Phi_1^{**}(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\}.
\end{aligned} \tag{18}$$

This completes the proof. □

Some new special cases of Theorem 22 can be described as follows:

(I) Letting $\hbar(\xi) = 1$, then we acquire a new result for HOS generalized ψ-P-convex functions.

Corollary 23. *For $n,\kappa \in \mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS generalized $(\psi,\hbar)$-convex on Ω for $q_2 > 1$ with $q_1^{-1}+q_2^{-1} = 1$, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \\
&\le 2\left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \times [|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} + \mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}.
\end{aligned}$$

(II) Letting $\hbar(\xi) = \xi$, then we acquire a new result for HOS generalized ψ-convex functions.

Corollary 24. *For $n,\kappa \in \mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable* *mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is a HOS generalized* ψ*-convex on* Ω *for* $q_2>1$ *with* $q_1^{-1}+q_2^{-1}=1$, *then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\quad\times\left\{\left[\frac{2n+1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+\frac{1}{2(n+1)}\right.\right.\\&\quad\left.\times|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+\mathbb{D}(\omega_1,\omega_2)\right]^{\frac{1}{q_2}}\\&\quad+\left[\frac{1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+\frac{2n+1}{2(n+1)}\right.\\&\quad\left.\left.\times|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+\mathbb{D}(\omega_1,\omega_2)\right]^{\frac{1}{q_2}}\right\}.\end{aligned}$$

(III) Letting $\hbar(\xi)=\xi^s$, then we acquire a new result for Breckner type of HOS generalized ψ-s-convex functions.

Corollary 25. *For* $s\in(0,1],n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is a Breckner type of HOS generalized* ψ*-s-convex on* Ω *for* $q_2>1$ *with* $q_1^{-1}+q_2^{-1}=1$, *then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\quad\times\left\{\left[\frac{n^{s+1}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}}{(n+1)^s(s+1)}\right.\right.\\&\quad\left.+\mathbb{D}(\omega_1,\omega_2)\right]^{\frac{1}{q_2}}\\&\quad+\left[\frac{n^{s+1}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}}{(n+1)^s(s+1)}\right.\\&\quad\left.\left.+\mathbb{D}(\omega_1,\omega_2)\right]^{\frac{1}{q_2}}\right\}.\end{aligned}$$

(IV) Letting $\hbar(\xi)=\xi^s$, then we acquire a new result for Godunova-Levin type of HOS generalized ψ-s-convex functions.

Corollary 26. *For* $s\in(0,1],n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is a Godunova–Levin type of HOS generalized* ψ*-s-convex on* Ω *for* $q_2>1$ *with* $q_1^{-1}+q_2^{-1}=1$, *then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\quad\times\left\{\left[\frac{n^{1-s}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}}{(n+1)^{-s}(1-s)}\right.\right.\\&\quad\left.+\mathbb{D}(\omega_1,\omega_2)\right]^{\frac{1}{q_2}}\\&\quad+\left[\frac{n^{1-s}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}}{(n+1)^{-s}(1-s)}\right.\\&\quad\left.\left.+\mathbb{D}(\omega_1,\omega_2\right]^{\frac{1}{q_2}}\right\}.\end{aligned}$$

(V) Letting $\hbar(\xi)=1$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\big)^p$, then we acquire a new result for HOS generalized ψ-P-convex functions.

Corollary 27. *For* $n,\kappa\in\mathbb{N}$ *with* $\upsilon_1,\upsilon_2,\delta>0$ *and* $\eta=\{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is a HOS of generalized* $(\psi,\hbar)$*-convex on* Ω *for* $q_2>1$ *with* $q_1^{-1}+q_2^{-1}=1$, *then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le 2\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\end{aligned}$$

$$\times\Big[|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} - \frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\Big\{2n(\delta+\kappa+1)\,{}_2\mathfrak{F}_1\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right) - n\mathbb{B}(\delta+\kappa,p+1)\,{}_2\mathfrak{F}_1\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right) + 2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right) - n^p\mathbb{B}(\delta+\kappa,2)\,{}_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\Big]^{\frac{1}{q_2}}.$$

(VI) Letting $\hbar(\xi)=\xi$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\big)^p$, then we acquire a new result for HOS generalized ψ-convex functions.

Corollary 28. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS generalized ψ-convex on Ω for $q_2>1$ with $q_1^{-1}+q_2^{-1}=1$, then*

$$|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \le \left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}$$
$$\times\Bigg\{\Big[\frac{2n+1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \frac{1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} - \frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\times\Big\{2n(\delta+\kappa+1)\,{}_2\mathfrak{F}_1\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right) - n\mathbb{B}(\delta+\kappa,p+1)\,{}_2\mathfrak{F}_1\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right) + 2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right) - n^p\mathbb{B}(\delta+\kappa,2)\times{}_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\Big]^{\frac{1}{q_2}}$$
$$+\Big[\frac{1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2} + \frac{2n+1}{2(n+1)}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2} - \frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\Big\{2n(\delta+\kappa+1)\,{}_2\mathfrak{F}_1\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right) - n\mathbb{B}(\delta+\kappa,p+1)\,{}_2\mathfrak{F}_1\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right) + 2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right) - n^p\mathbb{B}(\delta+\kappa,2)\,{}_2\mathfrak{F}_1\times\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\Big]^{\frac{1}{q_2}}\Bigg\}.$$

(VII) Letting $\hbar(\xi)=\xi^s$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\big)^p$, then we acquire a new result for Breckner type of HOS generalized ψ-s-convex functions.

Corollary 29. *For $s\in(0,1],n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a Breckner type of HOS generalized ψ-s-convex on Ω for $q_2>1$ with $q_1^{-1}+q_2^{-1}=1$, then*

$$|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \le \left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}$$

$$\times\left\{\left[\frac{n^{s+1}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}}{(n+1)^s(s+1)}\right.\right.$$

$$-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}$$

$$\times\{2n(\delta+\kappa+1)_2\mathfrak{F}_1$$

$$\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$-n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1$$

$$\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$+2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)$$

$$-n^p\mathbb{B}(\delta+\kappa,2)_2\mathfrak{F}_1$$

$$\left.\times\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}\Bigg]^{\frac{1}{q_2}}$$

$$+\left[\frac{n^{s+1}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}}{(n+1)^s(s+1)}\right.$$

$$-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}$$

$$\times\{2n(\delta+\kappa+1)_2\mathfrak{F}_1$$

$$\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$-n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1$$

$$\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$+2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)$$

$$-n^p\mathbb{B}(\delta+\kappa,2)_2\mathfrak{F}_1$$

$$\left.\left.\times\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}\right]^{\frac{1}{q_2}}\Bigg\}.$$

(VIII) Letting $\hbar(\xi)=\xi^{-s}$ and $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\left(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)^p$, then we acquire a new result for Godunova–Levin-type of HOS generalized ψ-s-convex functions.

Corollary 30. *For $s\in(0,1], n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a Godunova–Levin type of HOS generalized ψ-s-convex on Ω for $q_2>1$ with $q_1^{-1}+q_2^{-1}=1$, then*

$$|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|$$

$$\leq\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}$$

$$\times\left\{\left[\frac{n^{1-s}|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}}{(n+1)^{-s}(1-s)}\right.\right.$$

$$-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}$$

$$\times\{2n(\delta+\kappa+1)_2\mathfrak{F}_1$$

$$\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$-n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1$$

$$\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$+2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)$$

$$-n^p\mathbb{B}(\delta+\kappa,2)$$

$$\left.\times_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}\Bigg]^{\frac{1}{q_2}}$$

$$+\left[\frac{n^{1-s}|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}}{(n+1)^{-s}(1-s)}\right.$$

$$-\frac{2c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}$$

$$\times\{2n(\delta+\kappa+1)_2\mathfrak{F}_1$$

$$\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$-n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1$$

$$\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)$$

$$+2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)$$

$$-n^p\mathbb{B}(\delta+\kappa,2)_2\mathfrak{F}_1$$

$$\left.\left.\times\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\right\}\right]^{\frac{1}{q_2}}\Bigg\}.$$

Theorem 31. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS generalized $(\psi,\hbar)$-convex on Ω for $q_2>1$ with $q_1^{-1}+q_2^{-1}=1$, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\
&\le\left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}}\{[\Phi_1^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}\\
&\quad+\Phi_2^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\\
&\quad+\mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\\
&\quad+[\Phi_2^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}\\
&\quad+\Phi_1^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\\
&\quad+\mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\},
\end{aligned}\tag{19}$$

where

$$\Phi_1^*(\delta,\kappa,n,\xi):=\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\times\hbar\left(\frac{n+\xi}{n+1}\right)d\xi\tag{20}$$

and

$$\Phi_2^*(\delta,\kappa,n,\xi):=\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\times\hbar\left(\frac{1-\xi}{n+1}\right)d\xi.\tag{21}$$

Proof. According to Lemma 12, the well-known power-mean inequality and utilizing the fact of approximately generalized $(\psi,\hbar)$-convex functions, we obtain

$$\begin{aligned}
&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\
&\le\left(\int_0^1\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}d\xi\right)^{1-\frac{1}{q_2}}\\
&\quad\times\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\right.\\
&\quad\times\left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}\right.\right.\\
&\quad\left.\left.\left.\times(\omega_2-\omega_1)\right)\right|^{q_2}d\xi\right)^{\frac{1}{q_2}}\\
&\quad+\left(\int_0^1\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}d\xi\right)^{1-\frac{1}{q_2}}\\
&\quad\times\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\right.\\
&\quad\times\left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}\right.\right.\\
&\quad\left.\left.\left.(\omega_2-\omega_1)\right)\right|^{q_2}d\xi\right)^{\frac{1}{q_2}}\\
&\le\left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}}\\
&\quad\times\left\{\left[\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\right.\right.\\
&\quad\times\left(\hbar\left(\frac{n+\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}\right.\\
&\quad+\hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\\
&\quad\left.\left.+\mathbb{D}(\omega_1,\omega_2)\right)d\xi\right]^{\frac{1}{q_2}}\\
&\quad+\left[\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)\right.\\
&\quad\times\left(\hbar\left(\frac{1-\xi}{n+1}\right)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+\hbar\left(\frac{n+\xi}{n+1}\right)\right.\\
&\quad\left.\left.\left.\times|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+\mathbb{D}(\omega_1,\omega_2)\right)d\xi\right]^{\frac{1}{q_2}}\right\}\\
&\le\left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}}\{[\Phi_1^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}\\
&\quad+\Phi_2^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+\mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\\
&\quad+[\Phi_2^*(\delta,\kappa,n,\xi)|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}+\Phi_1^*(\delta,\kappa,n,\xi)\\
&\quad\times|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}+\mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}\}.
\end{aligned}\tag{22}$$

This completes the proof. □

Remark 32. By applying the similar arguments as we did for Theorems 13 and 22, we can find several special cases for Theorem 31 by considering the approximately generalized $(\psi,\hbar)$-convex functions with the appropriate and exceptional selection of function $\hbar$.

5. NEW GENERALIZATIONS OF APPROXIMATELY GENERALIZED ψ-QUASICONVEX FUNCTIONS HAVING κth ORDER DIFFERENTIABILITY

We propose some new generalizations of upper bounds for the mapping $\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})$ for approximately generalized ψ-quasiconvex functions by considering Definition 8 and Lemma 12.

Theorem 33. *For $n, \kappa \in \mathbb{N}$ with $\upsilon_1, \upsilon_2, \delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|$ is an approximately generalized ψ-quasiconvex on Ω, then*

$$|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \le \left(\frac{1}{3(\delta+\kappa)}\right)\{\max[|\mathcal{Q}^{(\kappa)}(\omega_1)|, |\mathcal{Q}^{(\kappa)}(\omega_2)|] + \mathbb{D}(\omega_1, \omega_2)\}.$$

Proof. According to Lemma 12, the triangular property and utilizing the fact of approximately generalized ψ-quasiconvex functions, we obtain

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\le \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) \times \left|\mathcal{Q}^{(\kappa)}\left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)\right)\right| d\xi \\
&\quad + \int_0^1 \left(\frac{\xi^{\delta+\kappa-1} - 2(1-\xi)^{\delta+\kappa-1}}{3}\right) \times \left|\mathcal{Q}^{(\kappa)}\left(\omega_1 + \frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)\right)\right| d\xi \\
&\le \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) \times [\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|, |\mathcal{Q}^{(\kappa)}(\omega_2)|\} + \mathbb{D}(\omega_1, \omega_2)] d\xi \\
&\quad + \int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) \times [\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|, |\mathcal{Q}^{(\kappa)}(\omega_2)|\} + \mathbb{D}(\omega_1, \omega_2)] d\xi \\
&= \left(\frac{1}{3(\delta+\kappa)}\right)\{\max[|\mathcal{Q}^{(\kappa)}(\omega_1)|, |\mathcal{Q}^{(\kappa)}(\omega_2)|] + \mathbb{D}(\omega_1, \omega_2)\}. \qquad (23)
\end{aligned}$$

This completes the proof. □

Some remarkable cases of Theorem 33 can be discussed as follows:

(I) Letting $\mathbb{D}(\omega_1, \omega_2) = -\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi) + (n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)\big)^p$, then we acquire HOS ψ-quasiconvex function.

Corollary 34. *For $n, \kappa \in \mathbb{N}$ with $\upsilon_1, \upsilon_2, \delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|$ is a HOS generalized ψ-quasiconvex on Ω, then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\le \left(\frac{1}{3(\delta+\kappa)}\right)\Big\{\max[|\mathcal{Q}^{(\kappa)}(\omega_1)|, |\mathcal{Q}^{(\kappa)}(\omega_2)|] - \frac{c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1))^p}{3(n+1)^{2p}}\{2n(\delta+\kappa+1) \\
&\quad \times {}_2\mathfrak{F}_1\left(-1; 1, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad - n\mathbb{B}(\delta+\kappa, p+1) \times {}_2\mathfrak{F}_1\left(-1; \delta+\kappa, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad + 2n^p{}_2\mathfrak{F}_1\left(-p; 1, \delta+\kappa+2, -\frac{1}{n}\right) \\
&\quad - n^p\mathbb{B}(\delta+\kappa, 2){}_2\mathfrak{F}_1 \times \left(-p; \delta+\kappa, \delta+\kappa+2, -\frac{1}{n}\right)\Big\}\Big\}.
\end{aligned}$$

(II) Letting $\mathbb{D}(\omega_1, \omega_2) = -\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi) + (n+\xi)(1-\xi)^p\Big\}(\omega_2 - \omega_1)^p$ along with $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) = \omega_2 - \omega_1$, then we acquire HOS quasiconvex function.

Corollary 35. *For $n, \kappa \in \mathbb{N}$ with $\delta > 0$. Let a κth-order differentiable mapping $\mathcal{Q} : \Omega \to \mathbb{R}$ be defined* *on Ω°. If $|\mathcal{Q}^{(\kappa)}|$ is a HOS quasiconvex on Ω, then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le\left(\frac{1}{3(\delta+\kappa)}\right)\Big\{\max[|\mathcal{Q}^{(\kappa)}(\omega_1)|,|\mathcal{Q}^{(\kappa)}(\omega_2)|]\\&\qquad-\frac{c(\omega_2-\omega_1)^p}{3(n+1)^{2p}}\{2n(\delta+\kappa+1)_2\mathfrak{F}_1\\&\qquad\times\left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\qquad-n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1\\&\qquad\times\left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right)\\&\qquad+2n^p{}_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right)\\&\qquad-n^p\mathbb{B}(\delta+\kappa,2)\\&\qquad\times{}_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\Big].\end{aligned}$$

Theorem 36. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is an approximately generalized ψ-quasiconvex on Ω for $q_2>1$ with $q_1^{-1}+q_2^{-1}=1$, then*

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le 2\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\qquad\times\{\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2},|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\}\\&\qquad+\mathbb{D}(\omega_1,\omega_2)\}^{\frac{1}{q_2}}.\end{aligned}\tag{24}$$

Proof. According to Lemma 12, the well-known Hölder inequality and utilizing the fact of approximately generalized ψ-quasiconvex functions, we obtain

$$\begin{aligned}&|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})|\\&\quad\le\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\qquad\times\left(\int_0^1\left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}\right.\right.\right.\\&\qquad\left.\left.\left.\times(\omega_2-\omega_1)\right)\right|^{q_2}d\xi\right)^{\frac{1}{q_2}}\\&\qquad+\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\qquad\times\left(\int_0^1\left|\mathcal{Q}^{(\kappa)}\left(\omega_1+\frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}\right.\right.\right.\\&\qquad\left.\left.\left.\times(\omega_2-\omega_1)\right)\right|^{q_2}d\xi\right)^{\frac{1}{q_2}}\\&\quad\le\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\qquad\times\left\{\left[\int_0^1(\max[|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2},|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}]\right.\right.\\&\qquad\left.+\mathbb{D}(\omega_1,\omega_2))d\xi\right]^{\frac{1}{q_2}}+\left[\int_0^1(\max[|\mathcal{Q}^{(\kappa)}\right.\\&\qquad\left.\left.\times(\omega_1)|^{q_2},|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}]+\mathbb{D}(\omega_1,\omega_2))d\xi\right]^{\frac{1}{q_2}}\right\}\\&\quad\le 2\left(\int_0^1\left(\frac{2(1-\xi)^{\delta+\kappa-1}-\xi^{\delta+\kappa-1}}{3}\right)^{q_1}d\xi\right)^{\frac{1}{q_1}}\\&\qquad\times\{\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2},|\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\}\\&\qquad+\mathbb{D}(\omega_1,\omega_2)\}^{\frac{1}{q_2}}.\end{aligned}\tag{25}$$

This completes the proof. □

Some remarkable cases of Theorem 36 can be discussed as follows:

(I) Letting $\mathbb{D}(\omega_1,\omega_2)=-\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi)+(n+\xi)(1-\xi)^p\Big\}\big(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\big)^p$, then we acquire HOS ψ-quasiconvex function.

Corollary 37. *For $n,\kappa\in\mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta>0$ and $\eta=\{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q}:\Omega=[\omega_1,\omega_1+\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)]\to\mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)>0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS generalized ψ-quasiconvex on Ω*

for $q_2 > 1$ *with* $q_1^{-1} + q_2^{-1} = 1$, *then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \leq 2\left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \\
&\quad \times [\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} \\
&\quad - \frac{c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\{2n(\delta+\kappa+1)_2\mathfrak{F}_1 \\
&\quad \times \left(-1; 1, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad - n\mathbb{B}(\delta+\kappa, p+1)_2\mathfrak{F}_1 \\
&\quad \times \left(-1; \delta+\kappa, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad + 2n_2^p\mathfrak{F}_1\left(-p; 1, \delta+\kappa+2, -\frac{1}{n}\right) \\
&\quad - n^p\mathbb{B}(\delta+\kappa, 2)_2\mathfrak{F}_1 \\
&\quad \times \left(-p; \delta+\kappa, \delta+\kappa+2, -\frac{1}{n}\right)\Big\}\Big]^{\frac{1}{q_2}}.
\end{aligned}$$

(II) Letting $\mathbb{D}(\omega_1, \omega_2) = -\frac{c}{(n+1)^{2p}}\{(n+\xi)^p(1-\xi) + (n+\xi)(1-\xi)^p\}(\omega_2-\omega_1)^p$ along with $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2 - \omega_1) = \omega_2 - \omega_1$, then we acquire HOS quasiconvex function.

Corollary 38. *For* $n, \kappa \in \mathbb{N}$ *with* $\delta > 0$. *Let a* κ*th-order differentiable mapping* $\mathcal{Q} : \Omega \to \mathbb{R}$ *be defined on* Ω°. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is a HOS quasiconvex on* Ω *for* $q_2 > 1$ *with* $q_1^{-1} + q_2^{-1} = 1$, *then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \leq 2\left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)^{q_1} d\xi\right)^{\frac{1}{q_1}} \\
&\quad \times[\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} \\
&\quad - \frac{c(\omega_2-\omega_1)^p}{3(n+1)^{2p}}\Big\{2n(\delta+\kappa+1)_2\mathfrak{F}_1 \\
&\quad \times \left(-1; 1, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad - n\mathbb{B}(\delta+\kappa, p+1)_2\mathfrak{F}_1 \\
&\quad \times \left(-1; \delta+\kappa, \delta+\kappa+p+1, -\frac{1}{n}\right) \\
&\quad + 2n_2^p\mathfrak{F}_1\left(-p; 1, \delta+\kappa+2, -\frac{1}{n}\right) \\
&\quad - n^p\mathbb{B}(\delta+\kappa, 2)_2\mathfrak{F}_1 \\
&\quad \left(-p; \delta+\kappa, \delta+\kappa+2, -\frac{1}{n}\right)\Big\}\Big]^{\frac{1}{q_2}}.
\end{aligned}$$

Theorem 39. *For* $n, \kappa \in \mathbb{N}$ *with* $\upsilon_1, \upsilon_2, \delta > 0$ *and* $\eta = \{\eta(m)\}_{m=0}^{\infty}$ *a bounded sequence of real numbers. Also, assume that a* κ*th-order differentiable mapping* $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)] \to \mathbb{R}$ *is defined on* Ω° *such that* $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1) > 0$. *If* $|\mathcal{Q}^{(\kappa)}|^{q_2}$ *is an approximately generalized* ψ*-quasiconvex on* Ω *for* $q_2 > 1$ *with* $q_1^{-1} + q_2^{-1} = 1$, *then*

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \leq 2\left(\frac{1}{3(\delta+\kappa)}\right)[\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, \\
&\quad \times |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} + \mathbb{D}(\omega_1, \omega_2)]^{\frac{1}{q_2}}. \qquad (26)
\end{aligned}$$

Proof. According to Lemma 12, the well-known power-mean inequality and utilizing the fact of approximately generalized $(\psi, \hbar)$-quasiconvex of $|\mathcal{Q}^{(\kappa)}|^{q_2}$, we obtain

$$\begin{aligned}
&|\mathcal{Y}(\kappa, n, \delta, \omega_1, \omega_2)(\mathcal{Q})| \\
&\quad \leq \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) d\xi\right)^{1-\frac{1}{q_2}} \\
&\quad \times \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)\right. \\
&\quad \times \left|\mathcal{Q}^{(\kappa)}\left(\omega_1 + \frac{1-\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right|^{q_2} \\
&\quad \left.\times d\xi\right)^{\frac{1}{q_2}} \\
&\quad + \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)\right. \\
&\quad \left.\times d\xi\right)^{1-\frac{1}{q_2}} \\
&\quad \times \left(\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right)\right|\mathcal{Q}^{(\kappa)} \\
&\quad \times \left.\left(\omega_1 + \frac{n+\xi}{n+1}\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)\right|^{q_2} d\xi\Big)^{\frac{1}{q_2}}
\end{aligned}$$

$$\le \left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}} \times \left[\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) \times (\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} + \mathbb{D}(\omega_1,\omega_2))d\xi\right]^{\frac{1}{q_2}} + \left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}} \times \left[\int_0^1 \left(\frac{2(1-\xi)^{\delta+\kappa-1} - \xi^{\delta+\kappa-1}}{3}\right) \times (\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} + \mathbb{D}(\omega_1,\omega_2))d\xi\right]^{\frac{1}{q_2}} = 2\left(\frac{1}{3(\delta+\kappa)}\right) \times [\max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} + \mathbb{D}(\omega_1,\omega_2)]^{\frac{1}{q_2}}. \tag{27}$$

This completes the proof. □

Some remarkable cases of Theorem 39 can be discussed as follows:

(I) Letting $\mathbb{D}(\omega_1,\omega_2) = -\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi) + (n+\xi)(1-\xi)^p\Big\}\left(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)\right)^p$, then we acquire HOS ψ-quasiconvex function.

Corollary 40. *For $n,\kappa \in \mathbb{N}$ with $\upsilon_1,\upsilon_2,\delta > 0$ and $\eta = \{\eta(m)\}_{m=0}^{\infty}$ a bounded sequence of real numbers. Also, assume that a κth-order differentiable mapping $\mathcal{Q} : \Omega = [\omega_1, \omega_1 + \mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1)] \to \mathbb{R}$ is defined on Ω° such that $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1) > 0$. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS generalized ψ-quasiconvex on Ω for $q_2 > 1$ with $q_1^{-1} + q_2^{-1} = 1$, then*

$$|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \le 2\left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}} \left[\frac{1}{3(\delta+\kappa)} \times \max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} - \frac{c(\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1))^p}{3(n+1)^{2p}}\{2n(\delta+\kappa+1)_2\mathfrak{F}_1 \times \left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right) - n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1 \times \left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right) + 2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right) - n^p\mathbb{B}(\delta+\kappa,2) \times{}_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\right]^{\frac{1}{q_2}}. \tag{28}$$

(II) Letting $\mathbb{D}(\omega_1,\omega_2) = -\frac{c}{(n+1)^{2p}}\Big\{(n+\xi)^p(1-\xi) + (n+\xi)(1-\xi)^p\Big\}(\omega_2-\omega_1)^p$ along with $\mathcal{F}^{\eta}_{\upsilon_1,\upsilon_2}(\omega_2-\omega_1) = \omega_2-\omega_1$, then we acquire HOS quasiconvex function.

Corollary 41. *For $n,\kappa \in \mathbb{N}$ with $\delta > 0$. Let a κth order differentiable mapping $\mathcal{Q} : \Omega \to \mathbb{R}$ be defined on Ω°. If $|\mathcal{Q}^{(\kappa)}|^{q_2}$ is a HOS quasiconvex on Ω for $q_2 > 1$ with $q_1^{-1} + q_2^{-1} = 1$, then*

$$|\mathcal{Y}(\kappa,n,\delta,\omega_1,\omega_2)(\mathcal{Q})| \le 2\left(\frac{1}{3(\delta+\kappa)}\right)^{1-\frac{1}{q_2}} \left[\frac{1}{3(\delta+\kappa)} \times \max\{|\mathcal{Q}^{(\kappa)}(\omega_1)|^{q_2}, |\mathcal{Q}^{(\kappa)}(\omega_2)|^{q_2}\} - \frac{c(\omega_2-\omega_1)^p}{3(n+1)^{2p}}\{2n(\delta+\kappa+1)_2\mathfrak{F}_1 \times \left(-1;1,\delta+\kappa+p+1,-\frac{1}{n}\right) - n\mathbb{B}(\delta+\kappa,p+1)_2\mathfrak{F}_1 \times \left(-1;\delta+\kappa,\delta+\kappa+p+1,-\frac{1}{n}\right) + 2n^p_2\mathfrak{F}_1\left(-p;1,\delta+\kappa+2,-\frac{1}{n}\right) - n^p\mathbb{B}(\delta+\kappa,2) \times{}_2\mathfrak{F}_1\left(-p;\delta+\kappa,\delta+\kappa+2,-\frac{1}{n}\right)\Big\}\right]^{\frac{1}{q_2}}. \tag{29}$$

6. CONCLUSION

Among the numerous definitions presented in convex analysis, it can be seen that the approximately generalized $(\psi, \hbar)$-convex, approximately generalized ψ-quasiconvex, HOS generalized $(\psi, \hbar)$-convex functions and HOS generalized ψ-quasiconvex functions are one of the main definitions of convex analysis that have been used successfully in optimization theory, coding theory, and machine learning. This is the most crucial reason for the need for new approaches in this field as a future direction. In the present scenario, we have established a new integral identity for κth-order differentiable functions to deal with fractional operators involving the Riemann–Liouville fractional integral and special Raina's function. New consequences and numerical comparisons demonstrate that the proposed technique is eligible to generate several existing outcomes in the relative literature by the suitable selection of parameters.[31,32] The suggested scheme has concrete application in uniformly reflex Banach spaces and the parallelogram law in L^p spaces.[26–28] For further investigation, taking into account the advanced convexity properties, in the preinvexity context, we may extend this study in inequality theory, quantum calculus, artificial intelligence, robotics and forecasting applications in different areas which are promising areas that need potential investigations. Considering these points and the demonstration of better performance with our proposed analysis, a bridge between theory and application can be established, which will eventually generate robust and optimal solutions.

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