OPEN ACCESS
Fractals, Vol. 29, No. 5 (2021) 2140018 (20 pages)
© The Author(s)
DOI: 10.1142/S0218348X21400181

World Scientific
www.worldscientific.com

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

NEW NEWTON'S TYPE ESTIMATES PERTAINING TO LOCAL FRACTIONAL INTEGRAL VIA GENERALIZED p-CONVEXITY WITH APPLICATIONS

YONG-MIN LI
Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China
ymlwww@163.com

SAIMA RASHID
Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
saimarashid@gcuf.edu.pk

ZAKIA HAMMOUCH
Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam
Department of Medical Research, China Medical University Hospital
China Medical University, Taichung 40402, Taiwan
Ecole Normale Supérieure, Moulay Ismail University of Meknes, 5000, Morocco
hammouch_zakia@tdmu.edu.vn

DUMITRU BALEANU
Department of Mathematics, Cankaya University, Ankara, Turkey
dumitru@cankaya.edu.tr

YU-MING CHU*
College of Science, Hunan City University, Yiyang 413000, P. R. China
chuyuming2005@126.com

Received July 22, 2020
Accepted November 25, 2020
Published March 10, 2021

*Corresponding author.

This is an Open Access article in the "Special Issue Section on Fractal AI-Based Analyses and Applications to Complex Systems: Part I", edited by Yeliz Karaca (University of Massachusetts Medical School, USA), Dumitru Baleanu (Cankaya University, Turkey), Majaz Moonis (University of Massachusetts Medical School, USA), Khan Muhammad (Sejong University, South Korea), Yu-Dong Zhang (University of Leicester, UK) & Osvaldo Gervasi (Perugia University, Italy) published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 (CC-BY-NC-ND) License which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

Abstract

This paper aims to investigate the notion of p-convex functions on fractal sets $\mathbb{R}^{\hat{\alpha}}$ $(0 < \hat{\alpha} \leq 1)$. Based on these novel ideas, we derived an auxiliary result depend on a three-step quadratic kernel by employing generalized p-convexity. Take into account the local fractal identity, we established novel Newton's type variants for the local differentiable functions. Several special cases are apprehended in the light of generalized convex functions and generalized harmonically convex functions. This novel strategy captures several existing results in the relative literature. Application is obtained in cumulative distribution function and generalized special weighted means to confirm the relevance and computational effectiveness of the considered method. Finally, we supposed that the consequences of this paper can stimulate those who are interested in fractal analysis.

Keywords: Generalized Convex Function; Generalized Harmonically Convex Function; Generalized p-Convex Functions; Newton's Type Inequality; Fractal Sets.

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

1. INTRODUCTION

The fractal sets in science have introduced some fascinating complex graphs and picture compressions to computer graphics. Fractal is a Latin word, derived from the word "Fractus" which signifies "Broken." The expression "fractal" was first utilized by a young mathematician, Julia[1] when he was considering Cayley's problem identified with the conduct of Newton's method in a complex plane. Fractal is frequently utilized in real-world involving: fractal antennas, fractal transistors, and fractal heat exchangers. It has applications in the music industry, the creation of photography, soil mechanics, small-angle scattering theory, and many more. It is to be emphasized that fractal theory assumes an essential job in the improvement of picturesque of fractal sets. The utilizations of fractal sets are in cryptography and other useful areas of research have increased the interest of researchers to broaden the utilization in mathematical inequalities. Fractals are the elite, arbitrary examples abandoned by the erratic developments of the disorderly world at work. The most significant utilization of fractals in software engineering is the fractal picture compression. This sort of compression utilizes the way that this present reality is very much portrayed by fractal geometry.[2–4] Interestingly, authors[5] investigated the local fractional functions on fractal space deliberately, which comprises of local fractional calculus and the monotonicity of functions. Numerous analysts contemplated the characteristics of functions on fractal space and built numerous sorts of fractional calculus by utilizing various strategies.[6–14] Mathematical inequalities[15–33] assume a significant job in engineering and applied sciences. An enormous heft of fractional differential problems and partial differential equations can be converted into problems of comprehending some estimated integral equations, see Refs. 34–38. Nevertheless, some underlying differential equations are fractal images, which are continuous and nowhere differentiable. In this phenomenon, an important tool can be utilized to cope with fractal and fractal-based problems with continuity and nondifferentiable.

The Newton's type identity is famous for its distinguished nature known as error estimation or Simpson's second formula, stated as follows:

Theorem 1. *Let $\Omega \subseteq \mathbb{R}$ be an interval and let $\chi : \Omega = [g, h] \to \mathbb{R}$ be a four times continuously differentiable function on Ω° (Ω° is the interior of Ω) and $\|\chi^{(4)}\|_\infty = \sup_{u\in[g,h]} |\chi^{(4)}| < \infty$, then*

$$\left|\frac{1}{8}\left[\chi(g) + 3\chi\left(\frac{2g+h}{3}\right) + 3\chi\left(\frac{g+2h}{3}\right) + \chi(h)\right] - \frac{1}{h-g}\int_g^h \chi(u)du\right| \leq \frac{1}{6480}\|\chi^{(4)}\|_\infty (h-g)^5. \tag{1}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

Many important results have been obtained for the investigation of the inequality (1) on fractional calculus theory and in the classical sense. Specifically, several researchers paid their interest in developing the novel versions depends on a two-step quadratic kernel and Simpson's second type results based on three-step quadratic kernel via different classes of functions, for some specific formulation, modifications, and innovative speculations, see Refs. 39–46. It is also worth pointing out that the concepts of fractal fractional calculus are also modified and further developed to adapt to the new requirements of theory and practice. Up to now, there exist several definitions of the convexity theory in the frame of local fractional integration in the literature, each being constructed to satisfy various modifications and to be consistent with physical background and applications, among which the most popular one is the p-convex functions. Inspired by the above phenomena, we intend to establish the novel version of Newton's type identity in the frame of local fractional integration by proposing generalized p-convexity property and to check the effectiveness, we provide the generalized variants in probability density functions and $\hat{\alpha}$-type weighted special means.

Additionally, integral inequalities in the context of local fractional calculus have a significant role in all fields of pure and applied mathematics. For example, Chen,[47] derived a novel version of Hölder inequality on fractals. In Ref. 14, Mo *et al.* established the fractal version of Hermite–Hadamard inequality by the use of generalized convex functions. Du *et al.* contemplated the novel generalizations for Simpson's, Hermite–Hadamard and Hermite–Hadamard-Fejér type inequalities for generalized m-convex functions concerning to local fractional calculus. In addition to these results, Luo *et al.*,[48] deduced several new Fejér–Hermite–Hadamard inequalities for a class of h-convex functions with applications. For some useful and recent studies on fractional calculus and its applications in different fields of mathematics, see Refs. 49–53.

Owing to the above phenomena, the key aim of this research is to introduce a new auxiliary result depending on the three-step quadratic kernel on local fractal sets will be given. With the aid of novel identity, we derived numerous novel generalizations of Newton's type for mappings whose powers contain local fractional derivatives in modulus are generalized p-convex. The main impetus of this study is to capture new estimates for generalized convex functions and generalized harmonically convex functions. In addition, the application of the proved results in probability density functions and the generalized weighted special mean formula is also presented. We hope that the new strategy formulated in this paper is more invigorating than the accessible one.

2. PRELIMINARIES

Now, we mention the preliminaries from the theory of local fractional calculus. These ideas and important consequences associated with the local fractional derivative and local fractional integral are mainly due to Yang.[5]

Let $\omega_1^{\hat{\alpha}}, \omega_2^{\hat{\alpha}}$ and $\omega_3^{\hat{\alpha}}$ belong to the set $\mathbb{R}^{\hat{\alpha}}$ $(0 < \hat{\alpha} \leq 1)$, then

(1) $\omega_1^{\hat{\alpha}} + \omega_2^{\hat{\alpha}}$ and $\omega_1^{\hat{\alpha}}\omega_2^{\hat{\alpha}}$ belongs to the set $\mathbb{R}^{\hat{\alpha}}$;
(2) $\omega_1^{\hat{\alpha}} + \omega_2^{\hat{\alpha}} = \omega_2^{\hat{\alpha}} + \omega_1^{\hat{\alpha}} = (\omega_1 + \omega_2)^{\hat{\alpha}} = (\omega_2 + \omega_1)^{\hat{\alpha}}$;
(3) $\omega_1^{\hat{\alpha}} + (\omega_2^{\hat{\alpha}} + \omega_3^{\hat{\alpha}}) = (\omega_1^{\hat{\alpha}} + \omega_2^{\hat{\alpha}}) + \omega_3^{\hat{\alpha}}$;
(4) $\omega_1^{\hat{\alpha}}\omega_2^{\hat{\alpha}} = \omega_2^{\hat{\alpha}}\omega_1^{\hat{\alpha}} = (\omega_1\omega_2)^{\hat{\alpha}} = (\omega_2\omega_1)^{\hat{\alpha}}$;
(5) $\omega_1^{\hat{\alpha}}(\omega_2^{\hat{\alpha}}\omega_3^{\hat{\alpha}}) = (\omega_1^{\hat{\alpha}}\omega_2^{\hat{\alpha}})\omega_3^{\hat{\alpha}}$;
(6) $\omega_1^{\hat{\alpha}}(\omega_2^{\hat{\alpha}} + \omega_3^{\hat{\alpha}}) = \omega_1^{\hat{\alpha}}\omega_2^{\hat{\alpha}} + \omega_1^{\hat{\alpha}}\omega_3^{\hat{\alpha}}$;
(7) $\omega_1^{\hat{\alpha}} + 0^{\hat{\alpha}} = 0^{\hat{\alpha}} + \omega_1^{\hat{\alpha}} = \omega_1^{\hat{\alpha}}$ and $\omega_1^{\hat{\alpha}}1^{\hat{\alpha}} = 1^{\hat{\alpha}}\omega_1^{\hat{\alpha}} = \omega_1^{\hat{\alpha}}$.

Definition 2. A nondifferentiable mapping $\chi : \mathbb{R} \to \mathbb{R}^{\hat{\alpha}}, \theta \to \chi(\epsilon)$ is said to be local fractional continuous at $\epsilon_\circ$, if for any $\epsilon > 0$, there exists $\kappa > 0$, satisfying that

$$|\chi(\epsilon) - \chi(\epsilon_\circ)| < \epsilon^{\hat{\alpha}},$$

holds for $|\epsilon - \epsilon_\circ| < \kappa$. If $\chi(\epsilon)$ is local continuous on (g, h), then we denote it by $\chi(\epsilon) \in \mathbb{C}_{\hat{\alpha}}(g, h)$.

Definition 3. The local fractional derivative of $\chi(\epsilon)$ of order $\hat{\alpha}$ at $\epsilon = \epsilon_\circ$ is defined by the expression

$$\chi^{(\hat{\alpha})}(\epsilon_\circ) = {}_{\epsilon_\circ}\mathcal{D}_\epsilon^{\hat{\alpha}}\chi(\epsilon) = \left.\frac{d^{\hat{\alpha}}\chi(\epsilon)}{d\epsilon^{\hat{\alpha}}}\right|_{\epsilon=\epsilon_\circ}$$
$$= \lim_{\epsilon\to\epsilon_\circ} \frac{\Delta^{\hat{\alpha}}(\chi(\epsilon) - \chi(\epsilon_\circ))}{(\epsilon - \epsilon_\circ)^{\hat{\alpha}}},$$

where $\Delta^{\hat{\alpha}}(\chi(\epsilon) - \chi(\epsilon_\circ)) = \Gamma(\hat{\alpha} + 1)(\chi(\epsilon) - \chi(\epsilon_\circ))$. Let $\chi^{(\hat{\alpha})}(\epsilon) = \mathcal{D}_\epsilon^{\hat{\alpha}}\chi(\epsilon)$. If there exists $\chi^{(k+1)\hat{\alpha}}(\epsilon) = \overbrace{\mathcal{D}_\epsilon^{\hat{\alpha}} \ldots \mathcal{D}_\epsilon^{\hat{\alpha}}}^{(k+1)\text{ times}}\chi(\epsilon)$ for any $\epsilon \in \Omega \subseteq \mathbb{R}$, then it is denoted by $\chi \in \mathcal{D}_{(k+1)\hat{\alpha}}(\mathcal{I})$, where $k = 0, 1, 2, \ldots$.

Definition 4. Let $\chi(\epsilon) \in \mathbb{C}_{\hat{\alpha}}[g, h]$, and let $\Delta = \{\eta_0, \eta_1, \ldots, \eta_N\}, (N \in \mathbb{N})$ be a partition of $[g, h]$ which satisfies $g = \eta_0 < \eta_1 < \cdots < \eta_N = h$. Then the local fractional integral of χ on $[g, h]$ of order $\hat{\alpha}$

is defined as follows:

$$ {}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(\epsilon) = \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h \chi(\eta)(d\eta)^{\hat{\alpha}} := \frac{1}{\Gamma(1+\hat{\alpha})}\lim_{\delta\eta\to 0}\sum_{j=0}^{N-1} \times \chi(\eta_j)(\Delta\eta_j), $$

where $\delta\eta := \max\{\Delta\eta_1, \Delta\eta_2, \ldots, \Delta\eta_{N-1}\}$ and $\Delta\eta_j := \eta_{j+1} - \eta_j, j = 0, \ldots, N-1$.

Here, it follows that ${}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(\epsilon) = 0$ if $g = h$ and ${}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(\epsilon) = -{}_h\mathcal{I}_g^{(\hat{\alpha})}\chi(\epsilon)$ if $g < h$. For any $\epsilon \in [g,h]$, if there exists ${}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(\epsilon)$, then it is denoted by $\chi(\epsilon) \in \mathcal{I}_\epsilon^{\hat{\alpha}}[g,h]$.

Lemma 5 (Ref. 5). (1) *Suppose that* $\chi(u) = \mathcal{G}^{(\hat{\alpha})}(u) \in \mathbb{C}_{\hat{\alpha}}[g,h]$, *then*

$$ {}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(u) = \mathcal{G}(h) - \mathcal{G}(g). $$

(2) *Suppose that* $\chi(u), \mathcal{G}(u) \in \mathcal{D}_{\hat{\alpha}}[g,h]$, *and* $\chi^{(\hat{\alpha})}(u)$, $\mathcal{G}^{(\hat{\alpha})}(u) \in \mathbb{C}_{\hat{\alpha}}[g,h]$, *then*

$$ {}_g\mathcal{I}_h^{(\hat{\alpha})}\chi(u)\mathcal{G}^{(\hat{\alpha})}(u) = \chi(u)\mathcal{G}(u)|_g^h -_g \mathcal{I}_h^{(\hat{\alpha})}\chi^{(\hat{\alpha})}(u)\mathcal{G}(u). $$

Lemma 6 (Ref. 5). *For* $k > 0$, *the following results hold true*:

$$ \frac{d^{\hat{\alpha}}u^{k\hat{\alpha}}}{du^{\hat{\alpha}}} = \frac{\Gamma(1+k\hat{\alpha})}{\Gamma(1+(k-1)\hat{\alpha})}u^{(k-1)\hat{\alpha}}, $$

$$ \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h u^{k\hat{\alpha}}(du)^{\hat{\alpha}} = \frac{\Gamma(1+k\hat{\alpha})}{\Gamma(1+(k+1)\hat{\alpha})} \times \left(h^{(k+1)\hat{\alpha}} - g^{(k+1)\hat{\alpha}}\right). $$

Lemma 7 (Ref. 47, Generalized Hölder's inequality). *For* $s, q > 1$ *with* $s^{-1} + q^{-1} = 1$, *and let* $\chi, \mathcal{G} \in \mathbb{C}_{\hat{\alpha}}[g,h]$, *then*

$$ \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h |\chi(u)\mathcal{G}(u)|(du)^{\hat{\alpha}} \le \left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h |\chi(u)|^s(du)^{\hat{\alpha}}\right)^{\frac{1}{s}} \times \left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h |\mathcal{G}(u)|^q(du)^{\hat{\alpha}}\right)^{\frac{1}{q}}. $$

3. MAIN RESULTS

This section dedicated to an auxiliary result associated with the local fractional integral and then by employing the said result we derive a novel version of Newton's type inequalities considering several new and existing generalized classes of convex functions.

We now present the concept of generalized p-convex functions on fractal space as follows.

Definition 8. Let $p \in \mathbb{R}\backslash\{0\}$. Then $\chi : \Omega = [g,h] \to \mathbb{R}^{\hat{\alpha}}$ is said to be generalized p-convex, if

$$ \chi\left(\left[\zeta x^p + (1-\zeta)y^p\right]^{\frac{1}{p}}\right) \le \zeta^{\hat{\alpha}}\chi(x) + (1-\zeta)^{\hat{\alpha}}\chi(y), \tag{2} $$

holds for $x, y \in \Omega$, and $\zeta \in [0,1]$.

Remark 9. In Definition 8:

(1) If we take $\alpha = 1$, then we get a definition in Ref. 54.
(2) If we take $p = 1$, then we get a definition in Ref. 14.
(3) If we take $p = -1$, then we get definition in Ref. 52.
(4) If we take $p = -1$ and $\hat{\alpha} = 1$, then we get definition in Ref. 55.
(5) If we take $p = 1$ with $\hat{\alpha} = 1$, then we get classical convex functions.

It is worth mentioning that generalized p-convex functions collapses to generalized convex, generalized harmonically convex functions, harmonically convex functions and classical convex functions as special cases. This shows that outcomes derived in this paper continue to hold for these classes of convex functions and their variant forms.

Example 10. Let $\chi : \Omega = [g,h] \to \mathbb{R}^{\hat{\alpha}}$, $\chi_1(u) = u^{\hat{\alpha}p}, p \ne 0$ and $\chi_2 : \Omega \to \mathbb{R}^{\hat{\alpha}}$, $\chi_2(u) = c^{\hat{\alpha}}$, $c^{\hat{\alpha}} \in \mathbb{R}^{\hat{\alpha}}$, then χ_1 and χ_2 are both generalized p-convex and generalized p-concave functions, respectively.

Lemma 11. *For* $p \in \mathbb{R}\backslash\{0\}$ *and let there is a function* $\chi : \Omega^\circ \subset \mathbb{R} \to \mathbb{R}^{\hat{\alpha}}$ *such that* $\chi \in \mathcal{D}_{\hat{\alpha}}(\Omega^\circ)$ (Ω° *is the interior of* Ω) *and* $\chi^{(\hat{\alpha})} \in \mathcal{C}_{\hat{\alpha}}[g,h]$. *Then, for all* $\mu \in [g,h]$, *we have*

$$ \left(\frac{1}{8}\right)^{\hat{\alpha}}\left[\chi(g) + 3^{\hat{\alpha}}\chi\left(\left[\frac{g^p + 2h^p}{3}\right]^{\frac{1}{p}}\right) + 3^{\hat{\alpha}}\chi\left(\left[\frac{2g^p + h^p}{3}\right]^{\frac{1}{p}}\right) + \chi(h)\right] $$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$-\left(\frac{p}{h^p-g^p}\right)^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\,{}_g\mathcal{I}_h^{(\hat{\alpha})}\times\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}$$
$$=\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_0^1\frac{\mu(\eta)}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}},\tag{3}$$

where

$$\mu(\eta)=\begin{cases}\left(\eta-\dfrac{1}{8}\right)^{\hat{\alpha}}, & \eta\in\left[0,\dfrac{1}{3}\right),\\ \left(\eta-\dfrac{1}{2}\right)^{\hat{\alpha}}, & \eta\in\left[\dfrac{1}{3},\dfrac{2}{3}\right),\\ \left(\eta-\dfrac{7}{8}\right)^{\hat{\alpha}}, & \eta\in\left[\dfrac{2}{3},1\right].\end{cases}$$

Proof. Utilizing the given assumption and it suffices that

$$I=\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_0^1\frac{\mu(\eta)}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$=\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_0^{\frac{1}{3}}\frac{\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\left(\eta-\frac{1}{2}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}$$
$$\times\int_{\frac{2}{3}}^{1}\frac{\left(\eta-\frac{7}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$=I_1+I_2+I_3.\tag{4}$$

Considering I_1 and applying the local fractional integration by parts, one obtains

$$I_1=\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_0^{\frac{1}{3}}\frac{\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$=\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)\Big|_0^{\frac{1}{3}}-\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\chi^{(\hat{\alpha})}\times\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$=\left(\frac{5}{24}\right)^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right)-\left(-\frac{1}{8}\right)^{\hat{\alpha}}\chi(h)-\frac{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}{(h^p-g^p)^{\hat{\alpha}}}\,{}_{\left([\frac{g^p+2h^p}{3}]^{\frac{1}{p}}\right)}\mathcal{I}_h^{(\hat{\alpha})}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}.\tag{5}$$

Analogously, we have

$$I_2=\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\times\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\left(\eta-\frac{1}{2}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$
$$=\left(\frac{1}{6}\right)^{\hat{\alpha}}\chi\left(\left[\frac{2g^p+h^p}{3}\right]\right)-\left(-\frac{1}{6}\right)^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]\right)-\frac{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}{(h^p-g^p)^{\hat{\alpha}}}\,{}_{\left([\frac{2g^p+h^p}{3}]^{\frac{1}{p}}\right)}\mathcal{I}^{(\hat{\alpha})}_{\left([\frac{g^p+2h^p}{3}]^{\frac{1}{p}}\right)}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}\tag{6}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

and

$$I_3 = \frac{(h^p - g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times \int_{\frac{2}{3}}^{1} \frac{\left(\eta - \frac{7}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p + (1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}} \times \chi^{(\hat{\alpha})}\left(\left[\eta g^p + (1-\eta)h^p\right]^{\frac{1}{p}}\right)(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{1}{8}\right)^{\hat{\alpha}} \chi(g) - \left(\frac{-5}{24}\right)^{\hat{\alpha}} \times \chi\left(\left[\frac{2g^p + h^p}{3}\right]\right) - \frac{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}{(h^p - g^p)^{\hat{\alpha}}} {}_g\mathcal{I}^{(\hat{\alpha})}_{([\frac{2g^p+h^p}{3}]^{\frac{1}{p}})} \frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}. \quad (7)$$

Combining (4)–(6) gives the desired identity (3). This completes the proof. □

Remark 12. In Lemma 11:

(1) If one takes $p = -1$, then we get result in Ref. 41.
(2) If one takes $p = -1$ and $\hat{\alpha} = 1$, then we get a result in Ref. 43.
(3) If one takes $p = 1$, then we get a results for generalized convexity and classical convex functions in the literature, respectively.

Theorem 13. *For $\hat{\alpha} \in (0,1]$) with $s^{-1} + q^{-1} = 1$, $q \geq 1$ let there is a function $\chi : \Omega^\circ \subset \mathbb{R} \to \mathbb{R}^{\hat{\alpha}}$ such that $\chi \in \mathcal{D}_{\hat{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω) and $\chi^{(\hat{\alpha})} \in \mathcal{C}_{\hat{\alpha}}[g,h]$. If $|\chi^{(\hat{\alpha})}|^q$ is generalized p-convex function on Ω, then*

$$\Bigg|\left(\frac{1}{8}\right)^{\hat{\alpha}}\left[\chi(g) + 3^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right) + 3^{\hat{\alpha}}\chi\left(\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right) + \chi(h)\right] - \left(\frac{p}{h^p-g^p}\right)^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\, {}_g\mathcal{I}_h^{(\hat{\alpha})}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}\Bigg|$$

$$\leq \left(\frac{h^p-g^p}{p}\right)^{\hat{\alpha}}\Bigg[\left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times \left[\Upsilon_1^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(g)\right|^q + \Upsilon_2^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(h)\right|^q\right]^{\frac{1}{q}}$$

$$+ \left(\left(\frac{1}{18}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times \left[\Upsilon_3^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(g)\right|^q + \Upsilon_4^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(h)\right|^q\right]^{\frac{1}{q}}$$

$$+ \left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times \left[\Upsilon_5^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(g)\right|^q + \Upsilon_6^{(\hat{\alpha})}(p,q;g,h)\left|\chi^{(\hat{\alpha})}(h)\right|^q\right]^{\frac{1}{q}}\Bigg],$$

$$\Upsilon_1^{(\hat{\alpha})}(p,q;g,h) := \frac{p^{\hat{\alpha}}(8h^p)^{\hat{\alpha}} + (g^p+7h^p)^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}} \times (2-q(p+1))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times \left[\left(\frac{1}{h^{p(q-1)-2}} - \left(\frac{8}{g^p+7h^p}\right)^{p(q-1)-2/p}\right)^{\hat{\alpha}} + \left(\left(\frac{8}{g^p+7h^p}\right)^{p(q-1)-2/p} - \left(\frac{3}{g^p+2h^p}\right)^{\hat{\alpha}}\right)\right]$$

$$- \frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}(2+p+q(1-p))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times \left[\left(\frac{1}{h^{p(q-1)-q-2}} - \left(\frac{8}{g^p+7h^p}\right)^{(q-1)-\frac{q+2}{p}}\right)^{\hat{\alpha}} + \left(\left(\frac{8}{g^p+7h^p}\right)^{(q-1)-\frac{q+2}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{(q-1)-\frac{q+2}{p}}\right)^{\hat{\alpha}}\right]$$

$$- \frac{p^{\hat{\alpha}}(h^p(g^p+7h^p))^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}(1-p+q(1-p))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}$$

Fractals 2021.29. Downloaded from www.worldscientific.com by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$\times\Bigg[\Bigg(\frac{1}{h^{q(p-1)+p-2}} - \left(\frac{8}{g^p+7h^p}\right)^{(q+1)-\frac{q+2}{p}}\Bigg)^{\hat{\alpha}} + \frac{(h^p(g^p+7h^p))^{\hat{\alpha}}}{(1-p+q(1-p))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times\Bigg(\left(\frac{8}{g^p+7h^p}\right)^{(q+1)-\frac{q+2}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{(q+1)-\frac{q+2}{p}}\Bigg)^{\hat{\alpha}}\Bigg], \quad (8)$$

$$\Upsilon_2^{(\hat{\alpha})}(p,q;g,h) := \frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}\Gamma(1+\hat{\alpha}) \times(2+q+p(3-q))^{\hat{\alpha}}} \times\Bigg[\Bigg(\frac{1}{h^{p(q-3)-(q+2)}} - \left(\frac{3}{g^p+2h^p}\right)^{(q-3)-\frac{q+2}{p}}\Bigg)^{\hat{\alpha}} + \Bigg(\left(\frac{3}{g^p+2h^p}\right)^{(q-3)-\frac{q+2}{p}} - \left(\frac{8}{g^p+7h^p}\right)^{(q-3)-\frac{q+2}{p}}\Bigg)^{\hat{\alpha}}\Bigg]$$

$$- \frac{p^{\hat{\alpha}}\big((8g^p)^{\hat{\alpha}}+(g^p+7h^p)^{\hat{\alpha}}\big)^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}\Gamma \times(1+\hat{\alpha})(q+p(2-q))^{\hat{\alpha}}} \times\Bigg[\Bigg(\frac{1}{h^{p(q-2)-q}} - \left(\frac{3}{g^p+2h^p}\right)^{(2-q)-\frac{q}{p}}\Bigg)^{\hat{\alpha}} + \Bigg(\left(\frac{3}{g^p+2h^p}\right)^{(2-q)-\frac{q}{p}} - \left(\frac{8}{g^p+7h^p}\right)^{(2-q)-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg]$$

$$+ \frac{p^{\hat{\alpha}}(g^p(g^p+7h^p))}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}\Gamma \times(1+\hat{\alpha})(q+p(1-q))^{\hat{\alpha}}} \times\Bigg[\Bigg(\frac{1}{h^{p(1-q)+q}} - \left(\frac{3}{g^p+2h^p}\right)^{(q-1)-\frac{q}{p}}\Bigg)^{\hat{\alpha}} + \Bigg(\left(\frac{3}{g^p+2h^p}\right)^{(q-1)-\frac{q}{p}} - \left(\frac{8}{g^p+7h^p}\right)^{(q-1)-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg], \quad (9)$$

$$\Upsilon_3^{(\hat{\alpha})}(p,q;g,h) := \frac{p^{\hat{\alpha}}\big((2h^p)^{\hat{\alpha}}+(g^p+h^p)^{\hat{\alpha}}\big)}{2^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}} \times(q+p(2-q))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}\Bigg)^{\hat{\alpha}} - \Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{2}{g^p+h^p}\right)^{q-2-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg]$$

$$- \frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}(q+p(3-q))^{\hat{\alpha}} \times\Gamma(1+\hat{\alpha})} \times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-3-\frac{q}{p}} - \left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}}\Bigg)^{\hat{\alpha}} - \Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}} - \left(\frac{2}{g^p+h^p}\right)^{q-3-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg]$$

$$- \frac{p^{\hat{\alpha}}(h^p(g^p+h^p))^{\hat{\alpha}}}{2^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}(q+p(1-q))^{\hat{\alpha}} \times\Gamma(1+\hat{\alpha})} \times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-1-\frac{q}{p}} - \left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}} - \Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$-\left(\frac{2}{g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg], \tag{10}$$

$$\begin{aligned}\Upsilon_4^{(\hat{\alpha})}&(p,q;g,h)\\ :=& \frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}(q+p(3-q))^{\hat{\alpha}}}\times\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-3-\frac{q}{p}}-\left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}}-\left(\frac{2}{g^p+h^p}\right)^{q-3-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg]\\ &-\frac{p^{\hat{\alpha}}\left((2g^p)^{\hat{\alpha}}+(g^p+h^p)^{\hat{\alpha}}\right)}{2^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}(q+p(2-q))^{\hat{\alpha}}}\times\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-2-\frac{q}{p}}-\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}-\left(\frac{2}{g^p+h^p}\right)^{q-2-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg]\\ &+\frac{p^{\hat{\alpha}}(g^p(g^p+h^p))^{\hat{\alpha}}}{2^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}(q+p(1-q))^{\hat{\alpha}}}\times\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{g^p+2h^p}\right)^{q-1-\frac{q}{p}}-\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}-\left(\frac{2}{g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\Bigg],\end{aligned} \tag{11}$$

$$\begin{aligned}\Upsilon_5^{(\hat{\alpha})}&(p,q;g,h)\\ :=& \frac{p^{\hat{\alpha}}\left((8h^p)^{\hat{\alpha}}+(7g^p+h^p)^{\hat{\alpha}}\right)}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}}\times(q+p(2-q))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}}-\frac{1}{g^{q-2-\frac{q}{p}}}\Bigg)^{\hat{\alpha}}\Bigg]\\ &-\frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}(q+p(3-q))^{\hat{\alpha}}}\times\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-3-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{8}{7g^p+h^p}\right)^{q-3-\frac{q}{p}}-\frac{1}{g^{q-3-\frac{q}{p}}}\Bigg)^{\hat{\alpha}}\Bigg]\\ &-\frac{p^{\hat{\alpha}}(h^p(g^p+h^p))^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}(q+p(1-q))^{\hat{\alpha}}}\times\Gamma(1+\hat{\alpha})\\ &\times\Bigg[\Bigg(\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}}\\ &-\Bigg(\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}-\frac{1}{g^{q-1-\frac{q}{p}}}\Bigg)^{\hat{\alpha}}\Bigg]\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Upsilon_6^{(\hat{\alpha})}&(p,q;g,h)\\ :=& -\frac{p^{\hat{\alpha}}\left((8g^p)^{\hat{\alpha}}+(7g^p+h^p)^{\hat{\alpha}}\right)}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}}\times(q+p(2-q))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\end{aligned}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$\times\left[\left(\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}}-\left(\left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}}-\frac{1}{g^{q-2-\frac{q}{p}}}\right)^{\hat{\alpha}}\right]$$

$$+\frac{p^{\hat{\alpha}}}{(h^p-g^p)^{3\hat{\alpha}}(q+p(3-q))^{\hat{\alpha}}\times\Gamma(1+\hat{\alpha})}$$

$$\times\left[\left(\left(\frac{3}{2g^p+h^p}\right)^{q-3-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-3-\frac{q}{p}}\right)^{\hat{\alpha}}-\left(\left(\frac{8}{7g^p+h^p}\right)^{q-3-\frac{q}{p}}-\frac{1}{g^{q-3-\frac{q}{p}}}\right)^{\hat{\alpha}}\right]$$

$$+\frac{p^{\hat{\alpha}}(g^p(g^p+h^p))^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}\times(q+p(1-q))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}$$

$$\times\left[\left(\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}-\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}\right)^{\hat{\alpha}}-\left(\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}-\frac{1}{g^{q-1-\frac{q}{p}}}\right)^{\hat{\alpha}}\right]. \quad (13)$$

Proof. Considering Lemma 11, the generalized power mean inequality and the generalized p-convexity of $|\chi^{(\hat{\alpha})}|^q$, one obtains

$$\left|\left(\frac{1}{8}\right)^{\hat{\alpha}}\left[\chi(g)+3^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right)+3^{\hat{\alpha}}\chi\left(\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right)+\chi(h)\right]-\left(\frac{p}{h^p-g^p}\right)^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\,{}_g\mathcal{I}_h^{(\hat{\alpha})}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}\right|$$

$$\leq\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\times\frac{\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\left|\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)\right|$$

$$\times(d\eta)^{\hat{\alpha}}+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\times\frac{\left|\eta-\frac{1}{2}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\left|\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)\right|$$

$$\times(d\eta)^{\hat{\alpha}}+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1}\times\frac{\left|\eta-\frac{7}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}\times\left|\chi^{(\hat{\alpha})}\left(\left[\eta g^p+(1-\eta)h^p\right]^{\frac{1}{p}}\right)\right|(d\eta)^{\hat{\alpha}}$$

$$\leq\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\times\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\times\frac{\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}\times\left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q+(1-\eta)^{\hat{\alpha}}\right|\times\left|\chi^{(\hat{\alpha})}(h)\right|^q\right](d\eta)^{\hat{\alpha}}\right)^{\frac{1}{q}}$$

$$+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\times\left|\eta-\frac{1}{2}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\times\frac{\left|\eta-\frac{1}{2}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}\right.$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$\times \left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q + (1-\eta)^{\hat{\alpha}}\right| \times |\chi^{(\hat{\alpha})}(h)|^q\Big](d\eta)^{\hat{\alpha}}\Bigg)^{\frac{1}{q}} + \frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1} \times \left|\eta-\frac{7}{8}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\right)^{1-\frac{1}{q}} \times\Bigg(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1} \times \frac{\left|\eta-\frac{7}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}} \times \left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q + (1-\eta)^{\hat{\alpha}}\right| \times |\chi^{(\hat{\alpha})}(h)|^q\Big](d\eta)^{\hat{\alpha}}\Bigg)^{\frac{1}{q}}.$$

Utilizing Lemma 6 and using the change of the variable, yields

$$\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}} = \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{-1}{8}}^{\frac{5}{24}}\nu^{\hat{\alpha}}(d\nu)^{\hat{\alpha}} = \left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},$$

$$\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\eta-\frac{1}{2}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}} = \left(\frac{1}{18}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},$$

$$\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1}\left|\eta-\frac{7}{8}\right|^{\hat{\alpha}}(d\eta)^{\hat{\alpha}} = \left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}. \tag{14}$$

Now, using Lemma 6 and the change of the variable technique, it follows that

$$\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\frac{\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}\eta^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}$$

$$= \frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{8}} \times \frac{\left(\frac{1}{8}-\eta\right)^{\hat{\alpha}}\eta^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}} + \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{8}}^{\frac{1}{3}} \times \frac{\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}\eta^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}$$

$$= \frac{p^{\hat{\alpha}}}{8^{\hat{\alpha}}(h^p-g^p)^{3\hat{\alpha}}}\frac{1}{\Gamma(1+\hat{\alpha})} \times\Bigg[\int_{(\frac{g^p+7h^p}{8})^{1/p}}^{h}\Bigg(\frac{(8h^p)^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-1)}} - \frac{8^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-p-1)}} - \frac{(h^p(g^p+7h^p))^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q+p-1)}} + \frac{(g^p+7h^p)^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-1)}}\Bigg)(d\nu)^{\hat{\alpha}} + \int_{(\frac{g^p+2h^p}{3})^{\frac{1}{p}}}^{(\frac{g^p+7h^p}{8})^{1/p}}\Bigg(\frac{(8h^p)^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-1)}} - \frac{8^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-p-1)}} - \frac{(h^p(g^p+7h^p))^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q+p-1)}} + \frac{(g^p+7h^p)^{\hat{\alpha}}}{\nu^{\hat{\alpha}(qp-q-1)}}\Bigg)(d\nu)^{\hat{\alpha}}\Bigg]. \tag{15}$$

Now, applying the change of variable $\frac{1}{\nu^{qp-p-2}}=\tau$ and from $\frac{1}{\nu^{\hat{\alpha}(qp-p-1)}}(du)^{\hat{\alpha}} = \frac{(d\tau)^{\hat{\alpha}}}{(2-q(p+1))^{\hat{\alpha}}}$, we get

$$\frac{(8h^p)^{\hat{\alpha}}+(g^p+7h^p)^{\hat{\alpha}}}{\Gamma(1+\hat{\alpha})}\int_{(\frac{g^p+7h^p}{8})^{1/p}}^{h}\frac{(d\nu)^{\hat{\alpha}}}{\nu^{qp-p-1}} = \frac{(8h^p)^{\hat{\alpha}}+(g^p+7h^p)^{\hat{\alpha}}}{(2-q(p+1))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times\int_{(\frac{8}{g^p+7h^p})^{\frac{p(q-1)-2}{p}}}^{\frac{1}{h^{p(q-1)-2}}}(d\tau)^{\hat{\alpha}}$$

$$= \frac{(8h^p)^{\hat{\alpha}}+(g^p+7h^p)^{\hat{\alpha}}}{(2-q(p+1))^{\hat{\alpha}}\Gamma(1+\hat{\alpha})} \times\left(\frac{1}{h^{p(q-1)-2}} - \left(\frac{8}{g^p+7h^p}\right)^{\frac{p(q-1)-2}{p}}\right)^{\hat{\alpha}} \tag{16}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

and

$$\frac{(8h^p)^{\hat\alpha}+(g^p+7h^p)^{\hat\alpha}}{\Gamma(1+\hat\alpha)}\int_{(\frac{g^p+2h^p}{3})^{1/p}}^{(\frac{g^p+7h^p}{8})^{1/p}} \times \frac{(d\nu)^{\hat\alpha}}{\nu^{qp-p-1}}$$
$$= \frac{(8h^p)^{\hat\alpha}+(g^p+7h^p)^{\hat\alpha}}{(2-q(p+1))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times \int_{(\frac{3}{g^p+2h^p})^{\frac{p(q-1)-2}{p}}}^{(\frac{8}{g^p+7h^p})^{\frac{p(q-1)-2}{p}}} (d\tau)^{\hat\alpha}$$
$$= \frac{(8h^p)^{\hat\alpha}+(g^p+7h^p)^{\hat\alpha}}{(2-q(p+1))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times\left(\left(\frac{8}{g^p+7h^p}\right)^{\frac{p(q-1)-2}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{\frac{p(q-1)-2}{p}}\right)^{\hat\alpha}. \tag{17}$$

Again, applying the change of variable $\frac{1}{\nu^{qp-q-p-2}} = \tau$ and from $\frac{1}{\nu^{\hat\alpha(qp-q-p-1)}}(du)^{\hat\alpha} = (d\tau)^{\hat\alpha}$, we get

$$\frac{8^{\hat\alpha}}{\Gamma(1+\hat\alpha)}\int_{(\frac{g^p+7h^p}{8})^{1/p}}^{h} \frac{(d\nu)^{\hat\alpha}}{\nu^{\hat\alpha(qp-q-p-1)}}$$
$$= \frac{8^{\hat\alpha}}{(2+p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times \int_{(\frac{8}{g^p+7h^p})^{(q-1)-\frac{q+2}{p}}}^{\frac{1}{h^{p(q-1)-q-2}}} (d\tau)^{\hat\alpha}$$
$$= \frac{8^{\hat\alpha}}{(2+p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times\left(\frac{1}{h^{p(q-1)-q-2}} - \left(\frac{8}{g^p+7h^p}\right)^{(q-1)-\frac{q+2}{p}}\right)^{\hat\alpha} \tag{18}$$

and

$$\frac{8^{\hat\alpha}}{\Gamma(1+\hat\alpha)}\int_{(\frac{g^p+2h^p}{3})^{1/p}}^{(\frac{g^p+7h^p}{8})^{1/p}} \frac{(d\nu)^{\hat\alpha}}{\nu^{\hat\alpha(qp-q-p-1)}}$$
$$= \frac{8^{\hat\alpha}}{(2+p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times \int_{(\frac{3}{g^p+2h^p})^{(q-1)-\frac{q+2}{p}}}^{(\frac{8}{g^p+7h^p})^{(q-1)-\frac{q+2}{p}}} (d\tau)^{\hat\alpha}$$
$$= \frac{8^{\hat\alpha}}{(2+p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times\left(\left(\frac{8}{g^p+7h^p}\right)^{(q-1)-\frac{q+2}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{(q-1)-\frac{q+2}{p}}\right)^{\hat\alpha}. \tag{19}$$

Further, applying the change of variable $\frac{1}{\nu^{qp-q+p-2}} = \tau$ and from $\frac{1}{\nu^{\hat\alpha(qp-q-p-1)}}(du)^{\hat\alpha} = (d\tau)^{\hat\alpha}$, we get

$$\frac{(h^p(g^p+7h^p))^{\hat\alpha}}{\Gamma(1+\hat\alpha)}\int_{(\frac{g^p+7h^p}{8})^{1/p}}^{h} \frac{(d\nu)^{\hat\alpha}}{\nu^{\hat\alpha(qp-q+p-1)}}$$
$$= \frac{(h^p(g^p+7h^p))^{\hat\alpha}}{(2-p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times \int_{(\frac{8}{g^p+7h^p})^{(q+1)-\frac{q+2}{p}}}^{\frac{1}{h^{q(p-1)+p-2}}} (d\tau)^{\hat\alpha}$$
$$= \frac{(h^p(g^p+7h^p))^{\hat\alpha}}{(2-p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times\left(\frac{1}{h^{q(p-1)+p-2}} - \left(\frac{8}{g^p+7h^p}\right)^{(q+1)-\frac{q+2}{p}}\right)^{\hat\alpha} \tag{20}$$

and

$$\frac{(h^p(g^p+7h^p))^{\hat\alpha}}{\Gamma(1+\hat\alpha)}\int_{(\frac{g^p+2h^p}{3})^{1/p}}^{(\frac{g^p+7h^p}{8})^{1/p}} \frac{(d\nu)^{\hat\alpha}}{\nu^{\hat\alpha(qp-q-p-1)}}$$
$$= \frac{(h^p(g^p+7h^p))^{\hat\alpha}}{(1-p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times \int_{(\frac{3}{g^p+2h^p})^{(q+1)-\frac{q+2}{p}}}^{(\frac{8}{g^p+7h^p})^{(q+1)-\frac{q+2}{p}}} (d\tau)^{\hat\alpha}$$
$$= \frac{(h^p(g^p+7h^p))^{\hat\alpha}}{(1-p+q(1-p))^{\hat\alpha}\Gamma(1+\hat\alpha)} \times\left(\left(\frac{8}{g^p+7h^p}\right)^{(q+1)-\frac{q+2}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{(q+1)-\frac{q+2}{p}}\right)^{\hat\alpha}. \tag{21}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

A combination of (16)–(21) yields

$$\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}} \frac{|\eta-\frac{1}{8}|^{\hat{\alpha}}\eta^{\hat{\alpha}}}{[\eta g^p+(1-\eta)h^p]^{q\hat{\alpha}(1-\frac{1}{p})}} \times (d\eta)^{\hat{\alpha}} := \Upsilon_1^{(\hat{\alpha})}(p,q;g,h).$$

Adopting the aforementioned procedure, and after substituting the computed identities (14) in (14), we get immediate consequence (8). This completes the proof. □

Remark 14. In Theorem 13:

(1) If one takes $p=-1$, then we get result in Ref. 41.
(2) If one takes $p=-1$ and $\hat{\alpha}=1$, then we get a result in Ref. 43.
(3) If one takes $p=1$, then we get results for generalized convexity and classical convex functions in the literature, respectively.

Theorem 15. *For $p\in\mathbb{R}\backslash\{0\}$ with $s^{-1}+q^{-1}=1$, $s>1$ let there is a function $\chi:\Omega^\circ\subset\mathbb{R}\to\mathbb{R}^{\hat{\alpha}}$ such that $\chi\in\mathcal{D}_{\hat{\alpha}}(\Omega^\circ)$ (Ω° is the interior of Ω) and $\chi^{(\hat{\alpha})}\in\mathcal{C}_{\hat{\alpha}}[g,h]$. If $|\chi^{(\hat{\alpha})}|^q$ is generalized p-convex function on Ω, then*

$$\begin{aligned}&\left|\left(\frac{1}{8}\right)^{\hat{\alpha}}\left[\chi(g)+3^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right)+3^{\hat{\alpha}}\chi\left(\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right)+\chi(h)\right]-\left(\frac{p}{h^p-g^p}\right)^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\,{}_g\mathcal{I}_h^{(\hat{\alpha})}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}\right|\\
&\le\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\Bigg[\left(\Upsilon_8^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}\times\left[|\chi^{(\hat{\alpha})}(g)|^q+|\chi^{(\hat{\alpha})}(h)|^q\right]\\
&\quad+\left(\Upsilon_7^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}\times\left[|\chi^{(\hat{\alpha})}(g)|^q+|\chi^{(\hat{\alpha})}(h)|^q\right]\\
&\quad+\left(\Upsilon_9^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}\times\left[|\chi^{(\hat{\alpha})}(g)|^q+|\chi^{(\hat{\alpha})}(h)|^q\right]\Bigg].\end{aligned}\tag{22}$$

Proof. Considering Lemma 11, the generalized power mean inequality and the generalized p-convexity of $|\chi^{(\hat{\alpha})}|^q$, one obtains

$$\begin{aligned}&\left|\left(\frac{1}{8}\right)^{\hat{\alpha}}\left[\chi(g)+3^{\hat{\alpha}}\chi\left(\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right)+3^{\hat{\alpha}}\chi\left(\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right)+\chi(h)\right]-\left(\frac{p}{h^p-g^p}\right)^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\,{}_g\mathcal{I}_h^{(\hat{\alpha})}\frac{\chi(u)}{u^{\hat{\alpha}(1-p)}}\right|\\
&\le\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\frac{|\eta-\frac{1}{8}|^{\hat{\alpha}}}{[\eta g^p+(1-\eta)h^p]^{\hat{\alpha}(1-\frac{1}{p})}}\times\left|\chi^{(\hat{\alpha})}\left([\eta g^p+(1-\eta)h^p]^{\frac{1}{p}}\right)\right|\times(d\eta)^{\hat{\alpha}}\\
&\quad+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\eta-\frac{1}{2}|^{\hat{\alpha}}}{[\eta g^p+(1-\eta)h^p]^{\hat{\alpha}(1-\frac{1}{p})}}\times\left|\chi^{(\hat{\alpha})}\left([\eta g^p+(1-\eta)h^p]^{\frac{1}{p}}\right)\right|\\
&\quad\times(d\eta)^{\hat{\alpha}}+\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1}\end{aligned}$$

Fractals 2021.29. Downloaded from www.worldscientific.com by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$\times \frac{\left|\eta - \frac{7}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p + (1-\eta)h^p\right]^{\hat{\alpha}(1-\frac{1}{p})}}$$

$$\times \left|\chi^{(\hat{\alpha})}\left(\left[\eta g^p + (1-\eta)h^p\right]^{\frac{1}{p}}\right)\right|(d\eta)^{\hat{\alpha}}$$

$$\leq \frac{(h^p - g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\right.$$

$$\times \frac{\left|\eta - \frac{1}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p + (1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}$$

$$\left.\times (d\eta)^{\hat{\alpha}}\right)^{\frac{1}{s}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\right.$$

$$\times \left|\eta - \frac{1}{8}\right|^{\hat{\alpha}}\left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q\right.$$

$$\left.+ (1-\eta)^{\hat{\alpha}}\left|\left|\chi^{(\hat{\alpha})}(h)\right|^q\right](d\eta)^{\hat{\alpha}}\right)^{\frac{1}{q}}$$

$$+ \frac{(h^p - g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\right.$$

$$\times \frac{\left|\eta - \frac{1}{2}\right|^{\hat{\alpha}}}{\left[\eta g^p + (1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}$$

$$\left.\times (d\eta)^{\hat{\alpha}}\right)^{\frac{1}{s}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}}\right.$$

$$\times \left|\eta - \frac{1}{2}\right|^{\hat{\alpha}}\left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q\right.$$

$$\left.+ (1-\eta)^{\hat{\alpha}}\left|\left|\chi^{(\hat{\alpha})}(h)\right|^q\right](d\eta)^{\hat{\alpha}}\right)^{\frac{1}{q}}$$

$$+ \frac{(h^p - g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1}\right.$$

$$\times \frac{\left|\eta - \frac{7}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p + (1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}$$

$$\left.\times (d\eta)^{\hat{\alpha}}\right)^{\frac{1}{s}}\left(\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1}\right.$$

$$\times \left|\eta - \frac{7}{8}\right|^{\hat{\alpha}}\left[\eta^{\hat{\alpha}}\left|\chi^{(\hat{\alpha})}(g)\right|^q\right.$$

$$\left.+ (1-\eta)^{\hat{\alpha}}\left|\left|\chi^{(\hat{\alpha})}(h)\right|^q\right](d\eta)^{\hat{\alpha}}\right)^{\frac{1}{q}}. \tag{23}$$

From Lemma 6 and simple computations, yields

$$\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{8}}\eta^{\hat{\alpha}}\left(\frac{1}{8}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{1}{512}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})} - \left(\frac{1}{512}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})},$$

$$\times \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{8}}^{\frac{1}{3}}\eta^{\hat{\alpha}}\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{485}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}$$

$$- \left(\frac{165}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},$$

$$\times \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{1}{2}}\eta^{\hat{\alpha}}\left(\frac{1}{2}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{15}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}$$

$$- \left(\frac{19}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})},$$

$$\times \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{\frac{7}{8}}\eta^{\hat{\alpha}}\left(\frac{7}{8}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{3885}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}$$

$$- \left(\frac{5165}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}, \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{7}{8}}^{1}\eta^{\hat{\alpha}}$$

$$\times \left(\eta-\frac{7}{8}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{4563}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}$$

$$- \left(\frac{2835}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},$$

$$\times \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{2}}^{\frac{2}{3}}\eta^{\hat{\alpha}}\left(\eta-\frac{1}{2}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}$$

$$= \left(\frac{37}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}$$

$$- \left(\frac{21}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}. \tag{24}$$

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

Analogously, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{8}}(1-\eta)^{\hat{\alpha}}\left(\frac{1}{8}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{4563}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\\
&\qquad-\left(\frac{2835}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},\\
&\qquad\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{8}}^{\frac{1}{3}}(1-\eta)^{\hat{\alpha}}\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{3885}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\\
&\qquad-\left(\frac{5165}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})},\\
&\qquad\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{1}{2}}(1-\eta)^{\hat{\alpha}}\\
&\qquad\times\left(\frac{1}{2}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{37}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\\
&\qquad-\left(\frac{21}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},\\
&\qquad\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{2}}^{\frac{2}{3}}(1-\eta)^{\hat{\alpha}}\\
&\qquad\times\left(\eta-\frac{1}{2}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{15}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\\
&\qquad-\left(\frac{19}{216}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})},\\
&\qquad\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{\frac{7}{8}}(1-\eta)^{\hat{\alpha}}\\
&\qquad\times\left(\frac{7}{8}-\eta\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{485}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\\
&\qquad-\left(\frac{165}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})},\\
&\qquad\times\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{7}{8}}^{1}(1-\eta)^{\hat{\alpha}}\left(\eta-\frac{7}{8}\right)^{\hat{\alpha}}(d\eta)^{\hat{\alpha}}\\
&\quad=\left(\frac{1}{512}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}-\left(\frac{1}{512}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}.
\end{aligned}\tag{25}$$

Again, utilizing Lemma 6 and the change of variable technique, we have

$$\begin{aligned}
&\Upsilon_7(q,p;g,h)\\
&\quad=\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{3}}\\
&\qquad\times\frac{\left|\eta-\frac{1}{8}\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}\\
&\quad=\frac{1}{\Gamma(1+\hat{\alpha})}\int_0^{\frac{1}{8}}\frac{\left(\frac{1}{8}-\eta\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}\\
&\qquad\times(d\eta)^{\hat{\alpha}}+\frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{8}}^{\frac{1}{3}}\\
&\qquad\times\frac{\left(\eta-\frac{1}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}\\
&\quad=\frac{1}{\Gamma(1+\hat{\alpha})}\left[\left(\frac{p}{(h^p-g^p)^2(q+p(2-q))}\right)^{\hat{\alpha}}\right.\\
&\qquad\times\left(\frac{1}{g^{q-2-\frac{q}{p}}}-\left(\frac{8}{g^p+7h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}}\\
&\qquad-\left(\frac{p(g^p+7h^p)}{8(h^p-g^p)^2(q+p(1-q))}\right)^{\hat{\alpha}}\\
&\qquad\times\left(\frac{1}{g^{q-1-\frac{q}{p}}}-\left(\frac{8}{g^p+7h^p}\right)^{q-1-\frac{q}{p}}\right)^{\hat{\alpha}}\\
&\qquad+\left(\frac{p(g^p+7h^p)}{8(h^p-g^p)^2(q+p(1-q))}\right)^{\hat{\alpha}}\\
&\qquad\times\left(\left(\frac{8}{g^p+7h^p}\right)^{q-1-\frac{q}{p}}\right.\\
&\qquad-\left.\left(\frac{3}{g^p+2h^p}\right)^{q-1-\frac{q}{p}}\right)^{\hat{\alpha}}\\
&\qquad-\left(\frac{p}{(h^p-g^p)^2(q+p(2-q))}\right)^{\hat{\alpha}}
\end{aligned}$$

$$\times\left(\left(\frac{8}{g^p+7h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{3}{g^p+2h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}}\Bigg], \tag{26}$$

$$\Upsilon_8(q,p;g,h) = \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{2}{3}} \times \frac{\left|\frac{1}{2}-\eta\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}$$

$$= \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{3}}^{\frac{1}{2}} \times \frac{\left(\frac{1}{2}-\eta\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}} + \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{1}{2}}^{\frac{2}{3}} \times \frac{\left(\eta-\frac{1}{2}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}$$

$$= \frac{1}{\Gamma(1+\hat{\alpha})}\left[\left(\frac{p^{\hat{\alpha}}}{(h^p-g^p)^{2\hat{\alpha}} \times (q+p(2-p))^{\hat{\alpha}}}\right)^{\hat{\alpha}} \times\left\{\left(\left(\frac{3}{g^p+2h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{2}{g^p+h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}} - \left(\left(\frac{2}{g^p+h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}}\right\}\right.$$

$$- \left(\frac{p^{\hat{\alpha}}(g^p+h^p)^{\hat{\alpha}}}{2^{\hat{\alpha}}(h^p-g^p)^{2\hat{\alpha}} \times (q+p(1-q))^{\hat{\alpha}}}\right)^{\hat{\alpha}} \times\left\{\left(\left(\frac{3}{g^p+2h^p}\right)^{q-1-\frac{q}{p}} - \left(\frac{2}{g^p+h^p}\right)^{q-1-\frac{q}{p}}\right)^{\hat{\alpha}} - \left(\left(\frac{2}{g^p+h^p}\right)^{q-1-\frac{q}{p}} - \left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}\right)^{\hat{\alpha}}\right\}\Bigg] \tag{27}$$

and

$$\Upsilon_9(q,p;g,h) = \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{1} \times \frac{\left|\frac{7}{8}-\eta\right|^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}$$

$$= \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{2}{3}}^{\frac{7}{8}} \frac{\left(\frac{7}{8}-\eta\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}} \times (d\eta)^{\hat{\alpha}} + \frac{1}{\Gamma(1+\hat{\alpha})}\int_{\frac{7}{8}}^{1} \times \frac{\left(\eta-\frac{7}{8}\right)^{\hat{\alpha}}}{\left[\eta g^p+(1-\eta)h^p\right]^{q\hat{\alpha}(1-\frac{1}{p})}}(d\eta)^{\hat{\alpha}}$$

$$= \frac{1}{\Gamma(1+\hat{\alpha})}\left[\left(\frac{p}{(h^p-g^p)(q+p(2-q))}\right)^{\hat{\alpha}} \times\left\{\left(\left(\frac{3}{2g^p+h^p}\right)^{q-2-\frac{q}{p}} - \left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}}\right)^{\hat{\alpha}} - \left(\left(\frac{8}{7g^p+h^p}\right)^{q-2-\frac{q}{p}} - \frac{1}{g^{q-2-\frac{q}{p}}}\right)^{\hat{\alpha}}\right\}\right.$$

$$-\left(\frac{p(h^p+7g^p)}{(h^p-g^p)(q+p(1-q))}\right)^{\hat{\alpha}} \times\left\{\left(\left(\frac{3}{2g^p+h^p}\right)^{q-1-\frac{q}{p}}\right.\right.$$

Fractals 2021.29. Downloaded from www.worldscientific.com by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$-\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}\Bigg)^{\hat{\alpha}}$$

$$-\left(\left(\frac{8}{7g^p+h^p}\right)^{q-1-\frac{q}{p}}-\frac{1}{g^{q-1-\frac{q}{p}}}\right)^{\hat{\alpha}}\Bigg\}\Bigg]. \tag{28}$$

Substituting (24)–(28) in (23) gives the desired inequality (22). This completes the proof. □

Remark 16. In Theorem 15:

(1) If one takes $p = -1$, then we get result in Ref. 41.
(2) If one takes $p = -1$ and $\hat{\alpha} = 1$, then we get a result in Ref. 43.
(3) If one takes $p = 1$, then we get results for generalized convexity and classical convex functions in the literature, respectively.

4. APPLICATIONS

4.1. Probability Density Functions

Consider a random variable $\mathcal{X}$ whose generalized probability density function is $\mathfrak{p} : [g,h] \to [0^{\hat{\alpha}}, 1^{\hat{\alpha}}]$, which is generalized convex having cumulative distribution function $\mathcal{F}_{\hat{\alpha}}$ is stated as

$$P_{\hat{\alpha}}(\mathcal{X} \le x) = \mathcal{F}_{\hat{\alpha}}(u) := \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^u \mathfrak{p}(\zeta)(d\zeta)^{\hat{\alpha}}.$$

Moreover, the generalized expectation can be expressed as

$$E_{\hat{\alpha}} = \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h \zeta^{\hat{\alpha}}\mathfrak{p}(\zeta)(d\zeta)^{\hat{\alpha}}.$$

For more information related to probability density function, see Ref. 50.

Clearly, we see that

$$E_{\hat{\alpha}}(u) = \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^x \zeta^{\hat{\alpha}} d\mathcal{F}_{\hat{\alpha}}\zeta = h^{\hat{\alpha}} - \frac{1}{\Gamma(1+\hat{\alpha})}\int_g^h \mathcal{F}_{\hat{\alpha}}(\zeta)(d\zeta)^{\hat{\alpha}}.$$

The following results are associated with Sec. 4 and obtained as follows.

Proposition 17. *In Theorem* 13, *and* $p \in \mathbb{R}\backslash\{0\}$, *we have*

$$\Bigg|\left(\frac{1}{8}\right)^{\hat{\alpha}}\Bigg[P_{\hat{\alpha}}(\mathcal{X}\le g) + 3^{\hat{\alpha}}P_{\hat{\alpha}}\left(\mathcal{X}\le\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right) + 3^{\hat{\alpha}}P_{\hat{\alpha}}\left(\mathcal{X}\le\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right) + P_{\hat{\alpha}}(\mathcal{X}\le h)\Bigg)\Bigg] - \frac{h^{\hat{\alpha}}-E_{\hat{\alpha}}(\mathcal{X})}{(h-g)^{\hat{\alpha}}}\Bigg|$$
$$\le \left(\frac{h^p-g^p}{p}\right)^{\hat{\alpha}} \times\Bigg[\left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times\left[\Upsilon_1^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(g)|^q + \Upsilon_2^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(h)|^q\right]^{\frac{1}{q}}$$
$$+\left(\left(\frac{1}{18}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times\left[\Upsilon_3^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(g)|^q + \Upsilon_4^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(h)|^q\right]^{\frac{1}{q}}$$
$$+\left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}} \times\left[\Upsilon_5^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(g)|^q + \Upsilon_6^{(\hat{\alpha})}(p,q;g,h)|\mathfrak{p}(h)|^q\right]^{\frac{1}{q}}\Bigg],$$

where $\Upsilon_1^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_2^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_3^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_4^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_5^{(\hat{\alpha})}(p,q;g,h)$, *and* $\Upsilon_6^{(\hat{\alpha})}(p,q;g,h)$ *are given in* (8)–(13), *respectively.*

Proposition 18. *In Theorem* 15, *and* $p \in \mathbb{R}\backslash\{0\}$, *then we have*

$$\Bigg|\left(\frac{1}{8}\right)^{\hat{\alpha}}\Bigg[P_{\hat{\alpha}}(\mathcal{X}\le g) + 3^{\hat{\alpha}}P_{\hat{\alpha}} \times\left(\mathcal{X}\le\left[\frac{g^p+2h^p}{3}\right]^{\frac{1}{p}}\right)$$

Fractals 2021.29. Downloaded from www.worldscientific.com by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

$$+3^{\hat{\alpha}}P_{\hat{\alpha}}\left(\mathcal{X}\leq\left[\frac{2g^p+h^p}{3}\right]^{\frac{1}{p}}\right)$$
$$+P_{\hat{\alpha}}(\mathcal{X}\leq h)\bigg)\bigg]-\frac{h^{\hat{\alpha}}-E_{\hat{\alpha}}(\mathcal{X})}{(h-g)^{\hat{\alpha}}}\bigg|$$
$$\leq\frac{(h^p-g^p)^{\hat{\alpha}}}{p^{\hat{\alpha}}}\bigg[\left(\Upsilon_7^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}$$
$$\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right.\right.$$
$$\left.\left.+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}$$
$$\times\left[|\mathfrak{p}(g)|^q+|\mathfrak{p}(h)|^q\right]$$
$$+\left(\Upsilon_8^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}$$
$$\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right.\right.$$
$$\left.\left.+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}$$
$$\times\left[|\mathfrak{p}(g)|^q+|\mathfrak{p}(h)|^q\right]$$
$$+\left(\Upsilon_9^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}}$$
$$\times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right.\right.$$
$$\left.\left.+\left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}}$$
$$\times\left[|\mathfrak{p}(g)|^q+|\mathfrak{p}(h)|^q\right]\bigg],$$

where $\Upsilon_7^{(\hat{\alpha})}(p;q,g,h)$, $\Upsilon_8^{(\hat{\alpha})}(p;q,g,h)$ *and* $\Upsilon_9^{(\hat{\alpha})}(p;q,g,h)$ *are given in* (26)–(28), *respectively.*

4.2. Generalized Special Means

Considering the following $\hat{\alpha}$-type special means.[56] For $g<h$ and $g,h\in\mathbb{R}^{\hat{\alpha}}$, we have:

(I) The generalized arithmetic mean:

$$\mathcal{A}_{\hat{\alpha}}(g,h):=\left(\frac{g+h}{2}\right)^{\hat{\alpha}}=\frac{g^{\hat{\alpha}}+h^{\hat{\alpha}}}{2^{\hat{\alpha}}}.$$

(II) The generalized weighted arithmetic mean:

$$\mathcal{A}_{\hat{\alpha}}(g,h;w_1,w_2):=\frac{w_1^{\hat{\alpha}}g^{\hat{\alpha}}+w_2^{\hat{\alpha}}h^{\hat{\alpha}}}{w_1^{\hat{\alpha}}+w_2^{\hat{\alpha}}},w_1^{\hat{\alpha}},\quad w_2^{\hat{\alpha}}\in\mathbb{R}^{\alpha}.$$

(III) The generalized logarithmic mean:

$$\mathcal{L}_{\hat{\alpha}}(g,h):=\left[\frac{\Gamma(1+n\hat{\alpha})}{\Gamma(1+(n+1)\hat{\alpha})}\frac{h^{\hat{\alpha}+1}-g^{\hat{\alpha}+1}}{(h-g)^{\hat{\alpha}}}\right],$$
$$n\in\mathbb{Z}\backslash\{-1,0\};g,h\in\mathbb{R}\quad\text{with } g\neq h.$$

Consider $\chi(u)=u^{n\hat{\alpha}}(u\in\mathbb{R}:n\in\mathbb{Z},|n|\geq 2)$ in Theorems 13 and 15, we obtain the following inequalities asserted by the Propositions 19 and 20, respectively.

Proposition 19. *Let* $g,h\in\mathbb{R}$ *with* $g<h$, $0\notin[g,h]$, *and* $p\in\mathbb{R}\backslash\{0\}$, *and* $n\in\mathbb{N}\backslash\{1\}$. *Then*

$$\bigg|\bigg[\mathcal{A}_{\hat{\alpha}}^n\bigg(g,\left(\frac{g^p+2h^p}{3}\right)^{\frac{1}{p}},$$
$$\times\left(\frac{2g^p+h^p}{3}\right)^{\frac{1}{p}},h;1^{\hat{\alpha}},3^{\hat{\alpha}},3^{\hat{\alpha}},1^{\hat{\alpha}}\bigg)\bigg]$$
$$-p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\mathcal{L}_{\hat{\alpha}(p+(n-1))}\bigg|$$
$$\leq\left(\frac{h^p-g^p}{p}\right)^{\hat{\alpha}}$$
$$\times\left(\frac{\Gamma(1+n\hat{\alpha})}{\Gamma(1+(n-1)\hat{\alpha})}\right)^{\frac{1}{q}}$$
$$\times\bigg[\left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}}$$
$$\times\left[\Upsilon_1^{(\hat{\alpha})}(p,q;g,h)\left|g^{\hat{\alpha}(n-1)}\right|^q\right.$$
$$\left.+\Upsilon_2^{(\hat{\alpha})}(p,q;g,h)\left|h^{\hat{\alpha}(n-1)}\right|^q\right]^{\frac{1}{q}}$$
$$+\left(\left(\frac{1}{18}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}}$$
$$\times\left[\Upsilon_3^{(\hat{\alpha})}(p,q;g,h)\left|g^{\hat{\alpha}(n-1)}\right|^q\right.$$
$$\left.+\Upsilon_4^{(\hat{\alpha})}(p,q;g,h)\left|h^{\hat{\alpha}(n-1)}\right|^q\right]^{\frac{1}{q}}$$
$$+\left(\left(\frac{17}{288}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})}\right)^{1-\frac{1}{q}}$$
$$\times\left[\Upsilon_5^{(\hat{\alpha})}(p,q;g,h)\left|g^{\hat{\alpha}(n-1)}\right|^q\right.$$
$$\left.+\Upsilon_6^{(\hat{\alpha})}(p,q;g,h)\left|h^{\hat{\alpha}(n-1)}\right|^q\right]^{\frac{1}{q}}\bigg],$$

where $\Upsilon_1^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_2^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_3^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_4^{(\hat{\alpha})}(p,q;g,h)$, $\Upsilon_5^{(\hat{\alpha})}(p,q;g,h)$ *and* $\Upsilon_6^{(\hat{\alpha})}(p,q;g,h)$ *are given in* (8)–(13), *respectively.*

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

Proposition 20. *Let* $g, \lambda_2 \in \mathbb{R}$ *with* $g < h$, $0 \notin [g,h]$, $p \in \mathbb{R}\backslash\{0\}$, *and* $n \in \mathbb{N}\backslash\{1\}$. *Then*

$$\Bigg|\Bigg[\mathcal{A}^n_{\hat{\alpha}}\Bigg(g, \left(\frac{g^p+2h^p}{3}\right)^{\frac{1}{p}}, \left(\frac{2g^p+h^p}{3}\right)^{\frac{1}{p}}, h; 1^{\hat{\alpha}}, 3^{\hat{\alpha}}, 3^{\hat{\alpha}}, 1^{\hat{\alpha}}\Bigg)\Bigg] - p^{\hat{\alpha}}\Gamma(1+\hat{\alpha})\mathcal{L}_{\hat{\alpha}(p+(n-1))}\Bigg|$$

$$\leq \left(\frac{h^p-g^p}{p}\right)^{\hat{\alpha}}\left(\frac{\Gamma(1+n\hat{\alpha})}{\Gamma(1+(n-1)\hat{\alpha})}\right)^{\frac{1}{q}} \times\Bigg[\left(\Upsilon_8^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}} \times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})} + \left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}} \times\left[|g^{\hat{\alpha}(n-1)}|^q + |h^{\hat{\alpha}(n-1)}|^q\right]$$

$$+\left(\Upsilon_7^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}} \times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})} + \left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}} \times\left[|g^{\hat{\alpha}(n-1)}|^q + |h^{\hat{\alpha}(n-1)}|^q\right]$$

$$+\left(\Upsilon_9^{(\hat{\alpha})}(q,p;g,h)\right)^{\frac{1}{s}} \times\left(\left[\left(\frac{528}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+\hat{\alpha})}{\Gamma(1+2\hat{\alpha})} + \left(\frac{1008}{13{,}824}\right)^{\hat{\alpha}}\frac{\Gamma(1+2\hat{\alpha})}{\Gamma(1+3\hat{\alpha})}\right]\right)^{\frac{1}{q}} \times\left[|g^{\hat{\alpha}(n-1)}|^q + |h^{\hat{\alpha}(n-1)}|^q\right]\Bigg],$$

where $\Upsilon_7^{(\hat{\alpha})}(p;q,g,h)$, $\Upsilon_8^{(\hat{\alpha})}(p;q,g,h)$ *and* $\Upsilon_9^{(\hat{\alpha})}(p;q,g,h)$ *are given in* (26)–(28), *respectively.*

5. CONCLUSIONS

Newton's type inequalities are obtained for generalized p-convex functions within the local fractional calculus. They include a large number of particular cases of generalized convexity of segmental type and generalized harmonically convex functions. The techniques used to prove the results in this paper may generate new results within other classes of generalized convexity, which are not included in generalized p-convexity type. These types of results are useful in all the pure and applied domains of science and technique when approximation schemes are involved. Also, the need for error estimation may often lead to using inequalities of this kind.

REFERENCES

1. G. Julia, Memoire sur l'iteration des functions rationnelles, *J. Math. Pures Appl.* **8** (1918) 737–747.
2. Y. C. Kwun, A. A. Shahid, W. Nazeer, M. Abbas and S. M. Kang, Fractel generation via CR iteration scheme with s-convexity, *IEEE Access* **7** (2019) 69986–69997.
3. S. Kumari, M. Kumari and R. Chugh, Generation of new fractals via SP orbit with s-convexity, *Int. J. Eng. Tech.* **9**(3) (2017) 2491–2504.
4. J. Yang, D. Baleanu and X. J. Yang, Analysis of fractal wave equations by local fractional Fourier series method, *Adv. Math. Phys.* **2013** (2013) 632309.
5. X. J. Yang, *Advanced Local Fractional Calculus and its Applications* (World Science Publisher, New York, 2012).
6. S. Alizadeh, D. Baleanu and S. Rezapour, Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative, *Adv. Differ. Eqs.* **2020** (2020) 55.
7. D. Baleanu, K. Ghafarnezhad and S. Rezapour, On a three step crisis integro-differential equation, *Adv. Diff. Eqs.* **2019** (2019) 153.
8. D. Baleanu, H. Mohammadi and S. Rezapour, Analysis of the model of HIV-1 infection of $CD4^+$ T-cell with a new approach of fractional derivative, *Adv. Diff. Eqs.* **2020** (2020) 71.
9. D. Baleanu, H. Mohammadi and S. Rezapour, A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model, *Adv. Diff. Eqs.* **2020** (2020) 184.
10. D. Baleanu, H. Mohammadi and S. Rezapour, A fractional differential equation model for the COVID-19 transmission by using the Caputo–Fabrizio derivative, *Adv. Diff. Eqs.* **2020** (2020) 229.
11. D. Baleanu, S. Rezapour and Z. Saberpour, On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation, *Boundary Value Prob.* **2019** (2019) 79.
12. H. Mo, Generalized Herrmite–Hadamard type inequalities involving local fractional integrals, preprint (2014), arXiv: 1410.1062 [math.CA].

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

13. H. X. Mo and X. Sui, Hermite–Hadamard-type inequalities for generalized s-convex functions on real linear fractal set $\mathbb{R}^{\hat{\alpha}}(0 \leq \hat{\alpha} < 1)$, *Math. Sci.* **11** (2017) 241–246.
14. H. X. Mo, X. Sui and D. Y. Yu, Generalized convex functions on fractal sets and two related inequalities, *Abs. Appl. Anal.* **2014** (2014) 636–751.
15. T. Abdeljawad, S. Rashid, Z. Hammouch and Y.-M. Chu, Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications, *Adv. Differ. Eqs.* **2020** (2020) 406.
16. T. Abdeljawad, S. Rashid, A. A. El Deeb, Z. Hammouch and Y.-M. Chu, Certain new weighted estimates proposing generalized proportional fractional operator in another sense, *Adv. Differ. Eqs.* **2020** (2020) 463.
17. T. Abdeljawad, S. Rashid, H. Khan and Y.-M. Chu, On new fractional integral inequalities for p-convexity within interval-valued functions, *Adv. Differ. Eqs.* **2020** (2020) 330.
18. S.-B. Chen, S. Rashid, M. A. Noor, R. Ashraf and Y.-M. Chu, A new approach on fractional calculus and probability density function, *AIMS Math.* **5**(6) (2020) 7041–7054.
19. H.-H. Chu, S. Rashid, Z. Hammouch and Y.-M. Chu, New fractional estimates for Hermite–Hadamard-Mercer's type inequalities, *Alexandria Eng. J.* (2020), doi:10.1016/j.aej.2020.06.040.
20. J.-F. Li, S. Rashid, J.-B. Liu, A. O. Akdemir and F. Safdar, Inequalities involving conformable approach for exponentially convex functions and their applications, *J. Fun. Spaces* **2020** (2020) 6517068.
21. D. Nie, S. Rashid, A. O. Akdemir, D. Baleanu and J.-B. Liu, On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications, *Mathematics* **7** (2019) 727.
22. S. Rashid, T. Abdeljawad, F. Jarad and M. A. Noor, Some estimates for generalized Riemann–Liouville fractional integrals of exponentially convex functions and their applications, *Mathematics* **7** (2019) 807.
23. S. Rashid, R. Ashraf, M. A. Noor, K. I. Noor and Y.-M. Chu, New weighted generalizations for differentiable exponentially convex mapping with application, *AIMS Math.* **5**(4) (2020) 3525–3546.
24. S. Rashid, D. Baleanu and Y.-M. Chu, Some new extensions for fractional integral operator having exponential in the kernel and their applications in physical systems, *Open. Phys.* **18** (2020) 478–491.
25. S. Rashid, Z. Hammouch, D. Baleanu and Y.-M. Chu, New generalizations in the sense of the weighted non-singular fractional integral operator, *Fractals*, doi:10.1142/S0218348X20400034.
26. S. Rashid, Z. Hammouch, H. Kalsoom, R. Ashraf and Y.-M. Chu, New investigation on the generalized K-fractional integral operators, *Front. Phys.* **8** (2020) 25.
27. S. Rashid, H. Kalsoom, Z. Hammouch, R. Ashraf, D. Baleanu and Y.-M. Chu, New multi-parametrized estimates having pth-order differentiability in fractional calculus for predominating h-convex functions in Hilbert space, *Symmetry* **12** (2020) 222.
28. S. Rashid, A. Khalid, G. Rahman, K. S. Nisar and Y.-M. Chu, On new modifications governed by quantum Hahn's integral operator pertaining to fractional calculus, *J. Fun. Spaces* **2020** (2020) 8262860.
29. S. Rashid, M. A. Latif, Z. Hammouch and Y.-M. Chu, Fractional integral inequalities for strongly h-preinvex functions for a kth order differentiable functions, *Symmetry* **11** (2019) 1448.
30. S. Rashid, M. A. Noor, K. I. Noor, F. Safdar and Y.-M. Chu, Hermite–Hadamard type inequalities for the class of convex functions on time scale, *Mathematics* **7** (2019) 956.
31. S. Rashid, M. A. Noor, K. I. Noor and Y.-M. Chu, Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions, *AIMS Math.* **5**(3) (2020) 2629–2645.
32. J.-M. Shen, S. Rashid, M. A. Noor, R. Ashraf and Y.-M. Chu, Certain novel estimates within fractional calculus theory on time scales, *AIMS. Math.* **5**(6) (2020) 6073–6086.
33. L. Xu, Y.-M. Chu, S. Rashid, A. A. El-Deeb and K. S. Nisar, On new unified bounds for a family of functions via fractional q-calculus theory, *J. Fun. Spaces* **2020** (2020) 4984612.
34. S. Rashid, İ. İşcan, D. Baleanu and Y.-M. Chu, Generation of new fractional inequalities via n-polynomials s-type convexity with applications, *Adv. Differ. Eqs.* **2020** (2020) 264.
35. S. Rashid, F. Jarad, H. Kalsoom and Y.-M. Chu, On Pólya-Szegö and Čebyšev type inequalities via generalized K-fractional integrals, *Adv. Diff. Eqs.* **2020** (2020) 125.
36. S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom and Y.-M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics* **7** (2019) 1225.
37. S.-S. Zhou, S. Rashid, F. Jarad, H. Kalsoom and Y.-M. Chu, New estimates considering the generalized proportional Hadamard fractional integral operators, *Adv. Differ. Eqs.* **2020** (2020) 275.
38. S.-S. Zhou, S. Rashid, M. A. Noor, K. I. Noor, F. Safdar and Y.-M. Chu, New Hermite–Hadamard type inequalities for exponentially convex functions and applications, *AIMS Math.* **5**(6) (2020) 6874–6901.

Fractals 2021.29. Downloaded from www.worldscientific.com
by CANKAYA UNIVERSITY on 07/07/22. Re-use and distribution is strictly not permitted, except for Open Access articles.

39. M. W. Alomari, M. Darus and S. S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, *RGMIA Res. Rep. Coll.* **12**(4) (2009) 9.
40. S. S. Dragomir, R. P. Agarwal and P. Cerone, On Simpson's inequality and applications, *J. Inequal. Appl.* **5** (2000) 533–579.
41. S. Iftikhar, S. Erden, P. Kumam and M. U. Awan, Local fractional newton's inequalities involving generalized harmonic convex functions, *Adv. Diff. Eqs.* **2020** (2020) 185, doi:10.1186/s13662-020-02637-6.
42. B. Z. Liu, An inequality of Simpson type, *Proc. R. Soc. A* **261** (2005) 2155–2158.
43. M. A. Noor, K. I. Noor and S. Iftikhar, Some newton's type inequalities for harmonic convex functions, *J. Adv. Math. Stud.* **9**(1) (2016) 7–16.
44. J. Park, Generalization of some Simpson-like type inequalities via differentiable s-convex mappings in the second sense, *Int. J. Math. Sci.* **2011** (2011) 493531.
45. S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor and K. I. Noor, Simpson's type integral inequalities for K-fractional integrals and their applications, *AIMS Math.* **4**(4) (2019) 1087–1100, doi:10.3934/math.2019.4.1087.
46. M. Z. Sarikaya, H. Budak and S. Erden, On new inequalities of Simpson's type for generalized convex functions, *Korean J. Math.* **27**(2) (2019) 279–295, doi:10.11568/kjm.2019.27.2.279.
47. G.-S. Chen, Generalizations of Hölder's and some related integral inequalities on fractal space, *J. Funct. Spaces Appl.* **2013** (2013) 198405.
48. C. Luo, H. Wang and T. Dua, Fejér–Hermite–Hadamard type inequalities involving generalized h-convexity on fractal sets and their applications, *Chaos Solitons Fractals* **131** (2020) 109547.
49. H. Budak, M. Z. Sarikaya and H. Yildirim, New inequalities for local fractional integrals, *Iran. J. Sci. Tech. Trans. Sci.* **41**(4) (2017) 1039–3104.
50. S. Erden, M. Z. Sarikaya and N. Celik, Some generalized inequalities involving local fractional integrals and their applications for random variables and numerical integration, *J. Appl. Math. Stat. Inform.* **12**(2) (2016) 49–65.
51. M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals, *Proc. Amer. Math. Soc.* **145**(4) (2017) 1527–1538.
52. W. Sun, On generalization of some inequalities for generalized harmonically convex functions via local fractional integrals, *Quaest. Math.* **42**(9) (2019) 1159–1183.
53. M. Vivas, J. Hernandez and N. Merentes, New Hermite–Hadamard and Jensen type inequalities for h-convex functions on fractal sets, *Rev. Colombiana Mat.* **50**(2) (2016) 145–164.
54. K. S. Zhang and J. P. Wan, p-convex functions and their properties, *Pure Appl. Math.* **23**(1) (2007) 130–133.
55. İ. İşcan, Hermite–Hadamard type inequalities for harmonically convex functions, *Hacet. J. Math. Stat.* **43**(6) (2014) 935–942.
56. C. E. M. Pearce and J. Pecaric, Inequalities for differentiabel mappins with application to special means and quadrature formulae, *Appl. Math. Lett.* **13**(2) (2000) 51–55.