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Research article

Further characterizations of the m-weak group inverse on complex matrices

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Abstract: A new definition of the m-weak group inverse on complex matrices is introduced. The m-weak group inverse extends the notions of the weak group inverse and Drazin inverse on complex matrices. The characterizations, properties and representations of the m-weak group inverse are presented. The relationship between the m-weak group inverse and nonsingular bordered matrix is established. Then the Cramer's rule for the solution of the restricted matrix equation is followed by it.

Keywords: weak group inverse; m-weak group inverse; core-EP inverse; core-EP decomposition
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1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. The symbol $r(A)$ represents the rank of $A \in \mathbb{C}^{m\times n}$. The symbol $\mathbb{Z}^+$ denotes the set of all positive integers. The index of $A \in \mathbb{C}^{n\times n}$, denoted by $\mathrm{Ind}(A)$, is the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$. Let $A \in \mathbb{C}^{n\times n}_k$ be the set of all $n \times n$ complex matrices with index k. The symbol $\mathbb{C}^{\mathrm{CM}}_n$ stands for the set of all core matrices (or group inverse matrices), i.e.,

$$\mathbb{C}^{\mathrm{CM}}_n = \{A | A \in \mathbb{C}^{n\times n}, r(A) = r(A^2)\}.$$

The Drazin inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by A^D [1], is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying: $XA^{k+1} = A^k, XAX = X$ and $AX = XA$. Especially, when $A \in \mathbb{C}^{\mathrm{CM}}_n$, then X is called the group inverse of A and denoted by $A^\#$. In 2018, Wang [2] introduced the weak group inverse on complex square matrices by core-EP decomposition [3] and gave some characterizations of it.

Recently, there has been a growing interest in the weak group inverse and related results. Here, we mention part of the works. Wang et al. [4] introduced the definition of the weak group inverse matrices and proved that the set of the weak group inverse matrices was more inclusive than that of the group inverse matrices. The weak group inverse was also generalized to proper *-rings and characterized by three equations in [5]. For more details of the weak group inverse in proper *-rings, it can be seen from [6,7]. The authors of the paper [8] extended the notion of the weak group inverse to rectangular matrices.

In [6], Zhou and Chen proposed the m-weak group in ring and gave some characterizations of it. Let R be a unitary ring with involution, $a \in R$ and $m \in \mathbb{Z}^+$. If there exist $x \in R$ and $k \in \mathbb{Z}^+$ such that

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^k)^* a^{m+1} x = (a^m)^* a^k,$$

then x is called the m-weak group inverse of a. When a is m-weak group invertible, the m-weak group inverse of a may not be unique. If the m-weak group inverse of a is unique, then it is denoted by $a^{\textcircled{w}_m}$. Then the authors [9] investigated the relationship between the weak core inverse and the m-weak group inverse. Also they provided a necessary and sufficient condition that the Drazin inverse coincides with the m-weak group inverse of a complex matrix by core-EP decomposition.

Let $A \in \mathbb{C}_k^{n\times n}$. From [2], it is known that X is the weak group inverse of A if $X \in \mathbb{C}^{n\times n}$ is the unique solution of the system of equations below

$$AX^2 = X, \quad AX = A^{\textcircled{\dagger}} A,$$

in which case X is denoted by $A^{\textcircled{w}}$. Now, we consider the system of equations

$$AX^2 = X, \quad AX = (A^{\textcircled{\dagger}})^m A^m. \tag{1.1}$$

Interestingly, the X satisfying (1.1) coincides with the m-weak group inverse on complex matrices, in which case X exists for every $A \in \mathbb{C}^{n\times n}$ and is unique.

The Drazin inverse has been widely applied in different fields and has huge literatures. Here we only mention the part. The perturbation theory, additive results for the Drazin inverse were investigated in [10–12]. In [13], the proposed algorithms based on the discrete Fourier transform were shown to be more efficient by computing the Drazin inverse of a polynomial matrix in the case where the degree and the size of the polynomial matrix got bigger. Karampetakis and Stanimirović [14] presented two algorithms for symbolic computation of the Drazin inverse of a given square one-variable polynomial matrix, which were effective with respect to CPU time and the elimination of redundant computations. Some computable representations of the W-weighted Drazin inverse were investigated and the computational complexities of the representations were also estimated in [15]. Kyrchei [16] generalized the weighted Drazin inverse, the weighted DMP-inverse and the weighted dual DMP-inverse [17–19] to matrices over the quaternion skew field and provided their determinantal representations by using noncommutative column and row determinants. In [20], the authors considered the quaternion two-sided restricted matrix equations and gave their unique solutions by the DMP-inverse and dual DMP-inverse.

Motivated by the above discussion, we redefine the m-weak group inverse on complex matrices by (1.1) and prove the existence and uniqueness of it for every $A \in \mathbb{C}^{n\times n}$. Some new characterizations of the m-weak group inverse are derived in terms of the range space, null space, rank equalities and projectors. We present some representations of the m-weak group inverse involving some known generalized inverses and limit expressions. Also we investigate the relationships between the m-weak group inverse and other generalized inverses. Finally, we consider the relationship between the m-weak group inverse and the nonsingular bordered matrix, which is applied to the Cramer's rule for the solution of the restricted matrix equation.

This paper is organized as follows. In Section 2, we present some necessary definitions and lemmas. In Section 3, we provide a new definition, a representation and some basic properties of the m-weak group inverse on the complex matrices. In Section 4, we give the characterizations of the m-weak group inverse. In Section 5, we provide several expressions of the m-weak group inverse which are good in computational accuracy. In Section 6, we investigate some properties of m-weak group inverse as well as the relationships between the m-weak group inverse and other generalized inverses by core-EP decomposition. In Section 7, we show the applications of the m-weak group inverse concerned with the bordered matrices and the Cramer's rule for the solution of the restricted matrix equation.

2. Preliminaries

The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* denote the range space, null space and rank of $A \in \mathbb{C}^{m\times n}$, respectively. The symbol I_n denotes the identity matrix of order n. Let $P_{\mathcal{L},\mathcal{M}}$ be the projector on the space $\mathcal{L}$ along the $\mathcal{M}$, where $\mathcal{L}, \mathcal{M} \leq \mathbb{C}^n$ and $\mathcal{L} \oplus \mathcal{M} = \mathbb{C}^n$. For $A \in \mathbb{C}^{m\times n}$, P_A represents the orthogonal projection onto $\mathcal{R}(A)$, i.e., $P_A = P_{\mathcal{R}(A)} = AA^\dagger$. The symbols $\mathbb{C}_n^{\mathrm{P}}$ and $\mathbb{C}_n^{\mathrm{H}}$ represent the sets of $\mathbb{C}^{n\times n}$ consisting of idempotent matrices and Hermitian matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathrm{P}} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A\},$$
$$\mathbb{C}_n^{\mathrm{H}} = \{A|A \in \mathbb{C}^{n\times n}, A = A^*\}.$$

Next, we first recall the definitions of some generalized inverses. Let $A \in \mathbb{C}^{m\times n}$, the MP-inverse $A^\dagger$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [21–23]:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (1) above is called an inner inverse of A and the set of all inner inverses of A is denoted by $A\{1\}$. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies condition (2) above is called an outer inverse of A. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies both conditions (1) and (2) above is called a reflexive g-inverse of A. For $A \in \mathbb{C}^{m\times n}$, if a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $X = XAX$, $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$, where $\mathcal{T}$ and $\mathcal{S}$ are the subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, then it is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. If $A^{(2)}_{\mathcal{T},\mathcal{S}}$ exists, then it is unique. The notion of the core inverse on the $\mathbb{C}_n^{\mathrm{CM}}$ was proposed and was denoted by $A^{\#\!\!\!\bigcirc}$ [24–26]. In addition, it was proved that $A^{\#\!\!\!\bigcirc} = A^\# AA^\dagger$. The core-EP inverse of $A \in \mathbb{C}_k^{n\times n}$, written as $A^{\dagger\!\!\!\bigcirc}$ [27–29], was presented. Moreover, it was seen that $A^{\dagger\!\!\!\bigcirc} = (A^{k+1}(A^k)^\dagger)^\dagger$. The DMP-inverse of $A \in \mathbb{C}_k^{n\times n}$, denoted by $A^{D,\dagger}$ [17,18], was introduced. Moreover, it was known that $A^{D,\dagger} = A^D AA^\dagger$. Also, the dual DMP-inverse of A was proposed in [17], namely $A^{\dagger,D} = A^\dagger AA^D$. The (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [30,31], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying: $XAB = B, CAX = C, \mathcal{R}(X) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(C)$, where B and $C \in \mathbb{C}^{n\times m}$.

In order to discuss the m-weak group inverse, some lemmas are given. First, the lemma below gives the core-EP decomposition as an important tool in this paper.

Lemma 2.1. [3] *Let $A \in \mathbb{C}_k^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = A_1 + A_2 = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \tag{2.1}$$

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{2.2}$$

where $T \in \mathbb{C}^{t\times t}$ is nonsingular with $t = r(T) = r(A^k)$ and N is nilpotent of index k. The representation (2.1) is called the core-EP decomposition of A, A_1 and A_2 are termed the core part and nilpotent part of A, respectively.

From [2, 3, 32], it is known that

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.3}$$

$$A^{\textcircled{W}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*, \tag{2.4}$$

$$A^D = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k \\ 0 & 0 \end{bmatrix}U^*, \tag{2.5}$$

where $T_k = \sum\limits_{j=0}^{k-1} T^jSN^{k-1-j}$.

By direct computations, we get that $A \in \mathbb{C}_n^{\mathrm{CM}}$ is equivalent to $N = 0$, in which case

$$A^\# = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*, \tag{2.6}$$

$$A^{\textcircled{\#}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{2.7}$$

Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1), $m \in \mathbb{Z}^+$. The notations below will be frequently used in this paper:

$$M = S(I_{n-t} - N^\dagger N),$$
$$\triangle = (TT^* + MS^*)^{-1},$$
$$T_m = \sum_{j=0}^{m-1} T^jSN^{m-1-j}.$$

Lemma 2.2. [*33, Lemma 6*] *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form* (2.1). *Then*

$$A^\dagger = U\begin{bmatrix} T^*\triangle & T^*\triangle SN^\dagger \\ M^*\triangle & N^\dagger - M^*\triangle SN^\dagger \end{bmatrix}U^*. \tag{2.8}$$

From (2.8) and [10, Theorem 2.2], we get that

$$AA^\dagger = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^*, \tag{2.9}$$

$$A^\dagger A = U\begin{bmatrix} T^*\triangle T & -T^*\triangle M \\ M^*\triangle T & N^\dagger N + M^*\triangle M \end{bmatrix}U^*, \tag{2.10}$$

$$A^k = U\begin{bmatrix} T^k & T_k \\ 0 & 0 \end{bmatrix}U^*, \tag{2.11}$$

$$A^m = U\begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix}U^*, \tag{2.12}$$

$$P_{A^k} = A^k(A^k)^\dagger = U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.13}$$

where $t = r(A^k)$.

Lemma 2.3. [28, 34, 35] *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. Then*
(*a*) $AA^{\textcircled{\dagger}} = P_{A^k}$;
(*b*) $A^{\textcircled{\dagger}}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^{k+1})^*A)}$;
(*c*) $A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$;
(*d*) $(A^{\textcircled{\dagger}})^m P_{A^k} = (A^{\textcircled{\dagger}})^m$.

Lemma 2.4. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. Then $A^m(A^{\textcircled{\dagger}})^m = P_{A^k}$.*

Proof. Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1). By (2.3), (2.12) and (2.13), it follows that

$$A^m(A^{\textcircled{\dagger}})^m = U\begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix}\begin{bmatrix} T^{-m} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}U^* = P_{A^k}.$$

□

3. *m*-weak group inverse on complex matrices

In this paper, we stipulate that $A^0 = I_n$ for any $A \in \mathbb{C}_k^{n\times n}$. Then we apply the core-EP decomposition to give another definition of the m-weak group inverse on complex matrices. Moreover, some properties of it are derived.

Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}^+$. Consider the system of equations below

$$(1)\ AX^2 = X, \quad (2)\ AX = (A^{\textcircled{\dagger}})^m A^m. \tag{3.1}$$

Theorem 3.1. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1)*, $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}^+$. Then the system of Eq* (3.1) *is consistent and has a unique solution X:*

$$X = (A^{\textcircled{\dagger}})^{m+1}A^m = U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^*. \tag{3.2}$$

Proof. If $m = 1$, then X coincides with $A^{\textcircled{w}}$. Clearly, X is the unique solution of (3.1) according to the definition of the weak group inverse. If $m \neq 1$, by (3.1), Lemmas 2.3 (*d*) and 2.4, then it follows that

$$X = (AX)X = (A^{\textcircled{\dagger}})^m A^m X = (A^{\textcircled{\dagger}})^m A^{m-1}(A^{\textcircled{\dagger}})^m A^m = (A^{\textcircled{\dagger}})^m P_{A^k} A^{\textcircled{\dagger}} A^m = (A^{\textcircled{\dagger}})^m A^{\textcircled{\dagger}} A^m = (A^{\textcircled{\dagger}})^{m+1} A^m.$$

Thus, by (2.3) and (2.12), we have that

$$X = (A^{\textcircled{\dagger}})^{m+1}A^m = U\begin{bmatrix} T^{-(m+1)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix}U^*.$$

□

Definition 3.2. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. The m-weak group inverse of A, denoted as $A^{\textcircled{w}_m}$, is defined as the unique solution of the system* (3.1).

Remark 3.3. *The m-weak group inverse is a generation of the weak group inverse and Drazin inverse. More precisely, we have the following statements:*
Let $A \in \mathbb{C}_k^{n\times n}$, $m \in \mathbb{Z}^+$.
(*a*) *If $m = 1$, then* 1*-weak group inverse of A coincides with the weak group inverse of A*;
(*b*) *If $m \geq k$, then m-weak group inverse of A coincides with the Drazin inverse of A.*

In the following example, we show that the m-weak group inverse is different from some known generalized inverses.

Example 3.4. *Let $A = \begin{bmatrix} I_3 & I_3 \\ 0 & N \end{bmatrix}$, where $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It can be verified that* $\mathrm{Ind}(A) = 3$. *By direct computations, some generalized inverses are derived below:*

$$A^\dagger = \begin{bmatrix} H_1 & -N^\dagger \\ I_3 - H_1 & N^\dagger \end{bmatrix}, \quad A^D = \begin{bmatrix} I_3 & H_2 \\ 0 & 0 \end{bmatrix}, A^{\textcircled{\dagger}} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^{D,\dagger} = \begin{bmatrix} I_3 & H_3 \\ 0 & 0 \end{bmatrix}, \quad A^{\dagger,D} = \begin{bmatrix} H_1 & H_4 \\ I_3 - H_1 & H_2 - H_4 \end{bmatrix}, A^{\textcircled{w}} = \begin{bmatrix} I_3 & I_3 \\ 0 & 0 \end{bmatrix},$$

where $H_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $H_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $N^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

It is clear that $A^{\textcircled{w}_2} = (A^{\textcircled{\dagger}})^3A^2 = \begin{bmatrix} I_3 & I_3 + N \\ 0 & 0 \end{bmatrix}$.

According to Example 3.4, the m-weak group inverse is indeed a new generalized inverse. Next, we consider some basic properties of the m-weak group inverse including the range space, null space, rank and projectors in the following results.

Theorem 3.5. *Let $A \in \mathbb{C}_k^{n\times n}$ be decomposed by $A = A_1 + A_2$ in* (2.1) *and $m \in \mathbb{Z}^+$. Then*
(*a*) *$A^{\textcircled{w}_m}$ is an outer inverse of A*;
(*b*) *$A^{\textcircled{w}_m}$ is a reflexive g-inverse of A_1.*

Proof. (a). By Lemmas 2.3 (d), 2.4 and the definition of $A^{\textcircled{W}_m}$, it follows that

$$A^{\textcircled{W}_m}AA^{\textcircled{W}_m} = (A^{\textcircled{\dagger}})^{m+1}A^mA(A^{\textcircled{\dagger}})^{m+1}A^m = (A^{\textcircled{\dagger}})^{m+1}P_{A^k}A^m = A^{\textcircled{W}_m}.$$

(b).

$$A_1A^{\textcircled{W}_m}A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* = A_1.$$

From [3, Theorem 3.4], we get $A_1 = AA^{\textcircled{\dagger}}A$. By the fact that $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$ and the statement (a) above, it follows that

$$A^{\textcircled{W}_m}A_1A^{\textcircled{W}_m} = A^{\textcircled{W}_m}AA^{\textcircled{\dagger}}A(A^{\textcircled{\dagger}})^{m+1}A^m = A^{\textcircled{W}_m}A(A^{\textcircled{\dagger}})^{m+1}A^m = A^{\textcircled{W}_m}AA^{\textcircled{W}_m} = A^{\textcircled{W}_m}.$$

Hence $A^{\textcircled{W}_m}$ is a reflexive g-inverse of A_1. □

Theorem 3.6. *Let* $A \in \mathbb{C}_k^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then*
(a) $r(A^{\textcircled{W}_m}) = r(A^k)$;
(b) $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$, $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^k)^*A^m)$;
(c) $A^{\textcircled{W}_m} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$.

Proof. (a). Assume that A be of the form (2.1). From (2.11) and (3.2), it is clear that $r(A^{\textcircled{W}_m}) = t = r(A^k)$.

(b). Since $A^{\textcircled{W}_m} = (A^{\textcircled{\dagger}})^{m+1}A^m$ implies that $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}((A^{\textcircled{\dagger}})^{m+1}A^m) \subseteq \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}(A^k)$ and since $r(A^{\textcircled{W}_m}) = r(A^k)$, we get $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$. From $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^{\textcircled{\dagger}})^{m+1}A^m) \supseteq \mathcal{N}(A^{\textcircled{\dagger}}A^m)$ and $r(A^{\textcircled{W}_m}) = t = r(A^{\textcircled{\dagger}}A^m)$, we get $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}}A^m)$. If $x \in \mathcal{N}(A^{\textcircled{\dagger}}A^m)$, we get that $A^mx \in \mathcal{N}(A^{\textcircled{\dagger}}) = \mathcal{N}((A^k)^*)$. Then $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}}A^m) \subseteq \mathcal{N}((A^k)^*A^m)$, and by $r(A^{\textcircled{W}_m}) = r((A^k)^*A^m)$, it follows that $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{\dagger}}A^m) = \mathcal{N}((A^k)^*A^m)$.

(c). It is a direct consequence from Theorems 3.5 (a) and 3.6 (b). □

Theorem 3.7. *Let* $A \in \mathbb{C}_k^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then*
(a) $AA^{\textcircled{W}_m} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$;
(b) $A^{\textcircled{W}_m}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$.

Proof. (a). From Theorem 3.5 (a), we gain that $AA^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{P}}$. By the definition of $A^{\textcircled{W}_m}$ and (3.2), it can be proved that $\mathcal{R}(AA^{\textcircled{W}_m}) = \mathcal{R}((A^{\textcircled{\dagger}})^mA^m) \subseteq \mathcal{R}(A^{\textcircled{\dagger}}) = \mathcal{R}(A^k)$ and $r(AA^{\textcircled{W}_m}) = r(A^{\textcircled{W}_m}) = r(A^k) = t$. Hence $\mathcal{R}(AA^{\textcircled{W}_m}) = \mathcal{R}(A^k)$. Similarly, we get that $\mathcal{N}(AA^{\textcircled{W}_m}) = \mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^k)^*A^m)$. Therefore, $AA^{\textcircled{W}_m} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$.

(b). The proof follows in a similar manner above. □

4. Some characterizations of the *m*-weak group inverse

In this part, we present some characterizations of the m-weak group inverse in terms of the range space, null space, rank equalities and projectors. The next theorem gives several characterizations of $A^{\textcircled{W}_m}$ mainly by the fact that $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$ in Theorem 3.6.

Theorem 4.1. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}^+$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{w}_m}$;
(b) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $AX = (A^{\textcircled{\dagger}})^m A^m$;
(c) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $A^{m+1}X = P_{A^k}A^m$;
(d) $\mathcal{R}(X) = \mathcal{R}(A^k)$, $(A^k)^*A^{m+1}X = (A^k)^*A^m$.

Proof. (a) ⇒ (b). This follows directly by Theorem 3.6 (b) and the definition of $A^{\textcircled{w}_m}$.
(b) ⇒ (c). Premultiplying $AX = (A^{\textcircled{\dagger}})^m A^m$ by A^m, and by Lemma 2.4, it follows that

$$A^{m+1}X = (A)^m(A^{\textcircled{\dagger}})^m A^m = P_{A^k}A^m.$$

(c) ⇒ (d). Premultiplying $A^{m+1}X = P_{A^k}A^m$ by $(A^k)^*$, it follows that

$$(A^k)^*A^{m+1}X = (A^k)^*P_{A^k}A^m = (A^k)^*A^m.$$

(d) ⇒ (a). Let A be of the form (2.1). By (2.11) and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we obtain that

$$X = U\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}U^*,$$

where $X_1 \in \mathbb{C}^{t\times t}$ and $X_2 \in \mathbb{C}^{t\times(n-t)}$. Thus

$$\begin{aligned}(A^k)^*A^{m+1}X = (A^k)^*A^m \quad &\Longrightarrow \quad U\begin{bmatrix} (T^k)^*T^{m+1}X_1 & (T^k)^*T^{m+1}X_2 \\ (\tilde{T})^*T^{m+1}X_1 & (\tilde{T})^*T^{m+1}X_2 \end{bmatrix}U^* = U\begin{bmatrix} (T^k)^*T^m & (T^k)^*T_m \\ (\tilde{T})^*T^m & (\tilde{T})^*T_m \end{bmatrix}U^* \\ &\Longrightarrow \quad X_1 = T^{-1} \textit{ and } X_2 = (T^{m+1})^{-1}T_m,\end{aligned}$$

which imply $X = U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{w}_m}$. □

By Theorem 3.5, it is known that $A^{\textcircled{w}_m}$ is an outer inverse of $A \in \mathbb{C}_k^{n\times n}$, i.e., $A^{\textcircled{w}_m}AA^{\textcircled{w}_m} = A^{\textcircled{w}_m}$. Using this result, we obtain some characterizations of $A^{\textcircled{w}_m}$.

Theorem 4.2. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}^+$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{w}_m}$;
(b) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^m)$;
(c) $XAX = X$, $XA^{k+1} = A^k$, $AX = (A^{\textcircled{\dagger}})^m A^m$;
(d) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$, $(A^m)^*A^{m+1}X \in \mathbb{C}_n^{\mathrm{H}}$.

Proof. (a) ⇒ (b). It is a direct consequence from Theorem 3.6 (c).
(b) ⇒ (c). By $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, it follows that

$$\mathcal{R}(AX) = A\mathcal{R}(X) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k) = \mathcal{R}((A^{\textcircled{\dagger}})^m A^m)$$

and

$$\mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}((A^k)^*A^m) = \mathcal{N}((A^{\textcircled{\dagger}})^m A^m).$$

For AX, $(A^{\textcircled{\dagger}})^m A^m \in \mathbb{C}_n^{\mathrm{P}}$, we have $AX = (A^{\textcircled{\dagger}})^m A^m$. By $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XAX = X$, we obtain that $XA^{k+1} = A^k$.

$(c) \Rightarrow (d)$. By the conditions, we obtain that

$$r(X) = r(AX) = r((A^{\textcircled{\dagger}})^m A^m) = r(A^k),$$

and by $\mathcal{R}(A^k) = \mathcal{R}(XA^{k+1}) \subseteq \mathcal{R}(X)$, we get $\mathcal{R}(XA) = \mathcal{R}(A^k)$. Since $A^m(A^{\textcircled{\dagger}})^m = P_{A^k} \in \mathbb{C}_n^{\mathrm{H}}$, it follows that

$$(A^m)^* A^{m+1} X = (A^m)^*(A^m(A^{\textcircled{\dagger}})^m)A^m \in \mathbb{C}_n^{\mathrm{H}}.$$

$(d) \Rightarrow (a)$. Let A be of the form (2.1). From $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we get that $XA^{k+1} = A^k$, it is easy to derive that

$$X = U \begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix} U^*,$$

where $X_2 \in \mathbb{C}^{t\times(n-t)}$.

For

$$\begin{aligned} (A^m)^* A^{m+1} X &= U \begin{bmatrix} (T^m)^* & 0 \\ (T_m)^* & (N^m)^* \end{bmatrix} \begin{bmatrix} T^{m+1} & TT_m + SN^m \\ 0 & N^{m+1} \end{bmatrix} \begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (T^m)^* T^m & (T^m)^* T^{m+1} X_2 \\ (T_m)^* T^m & (T_m)^* T^{m+1} X_2 \end{bmatrix} U^* \in \mathbb{C}_n^{\mathrm{H}}, \end{aligned}$$

we obtain that $X_2 = T^{-(m+1)} T_m$. Hence $X = U \begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix} U^* = A^{\textcircled{W}_m}$. □

From [1], it is known that the definition of the Drazin of a given matrix $A \in \mathbb{C}_k^{n\times n}$ is defined by three matrix equations $XA^{k+1} = A^k, XAX = X$ and $AX = XA$. Motivated by the first two matrix equations, we provide some similar characterizations of $A^{\textcircled{W}_m}$.

Theorem 4.3. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}^+$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{W}_m}$;
(b) $XA^{k+1} = A^k$, $AX^2 = X$, $(A^m)^* A^{m+1} X \in \mathbb{C}_n^{\mathrm{H}}$;
(c) $XA^{k+1} = A^k$, $AX^2 = X$, $A^{m+1}X = P_{A^k}A^m$;
(d) $XA^{k+1} = A^k$, $AX = (A^{\textcircled{\dagger}})^m A^m$, $r(X) = r(A^k)$.

Proof. $(a) \Leftrightarrow (b)$. It is the result of Proposition 4.2 in [6].

$(a) \Rightarrow (c)$. It is a direct consequence from the definition of $A^{\textcircled{W}_m}$, Theorems 4.1 (c) and 4.2 (c).

$(c) \Rightarrow (d)$. Let A be of the form (2.1). By $XA^{k+1} = A^k$, we get that

$$X = U \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} U^*,$$

where $X_2 \in \mathbb{C}^{t\times(n-t)}$ and $X_4 \in \mathbb{C}^{(n-t)\times(n-t)}$.

By $AX^2 = X$, we have that $X_4 = NX_4{}^2$, which implies that

$$X_4 = NX_4{}^2 = N^2 X_4{}^3 = \cdots = N^k X_4{}^{k+1} = 0.$$

Using (2.12) and (2.13) to $A^{m+1}X = P_{A^k}A^m$, we get

$$X = U \begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix} U^*.$$

Then the rest proof is trivial.

(d) $\Rightarrow$ (a). Since $XA^{k+1} = A^k$, it follows that $\mathcal{R}(A^k) = \mathcal{R}(XA^{k+1}) \subseteq \mathcal{R}(X)$ and by $r(X) = r(A^k)$, we get $\mathcal{R}(A^k) = \mathcal{R}(X)$. Hence, according to Theorem 4.1 (b), we get $X = A^{Ⓦ_m}$. □

According to Theorem 3.7, it holds that $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ when $X = A^{Ⓦ_m}$. Conversely, the conclusion might not hold. Here's an example below.

Example 4.4. *Let* $A = \begin{bmatrix} I_3 & L \\ 0 & N \end{bmatrix}$, $X = \begin{bmatrix} I_3 & L \\ 0 & L \end{bmatrix}$, *where* $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Then it is clear that* $k = \mathrm{Ind}(A) = 3$ *and* $A^{Ⓦ_2} = \begin{bmatrix} I_3 & L \\ 0 & 0 \end{bmatrix}$. *It can be directly verified that* $AX = P_{\mathcal{R}(A^3),\mathcal{N}((A^3)^*A^2)}$, $XA = P_{\mathcal{R}(A^3),\mathcal{N}((A^3)^*A^3)}$. *However,* $X \neq A^{Ⓦ_2}$.

Based on the example above, the next theorem, we consider other characterizations of $A^{Ⓦ_m}$ by using $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$.

Theorem 4.5. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be of the form (2.1),* $X \in \mathbb{C}^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then the following statements are equivalent:*

(a) $X = A^{Ⓦ_m}$;

(b) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$;

(c) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $XAX = X$;

(d) $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $AX^2 = X$.

Proof. (a) $\Rightarrow$ (b). It is a direct consequence from Theorems 3.6 (a) and 3.7.

(b) $\Rightarrow$ (c). Since $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$, we get that $\mathcal{R}(X) = \mathcal{R}(A^k)$ and by $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$, we obtain $XAX = X$.

(c) $\Rightarrow$ (d). From $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $r(X) = r(A^k)$, we have that $\mathcal{R}(X) = \mathcal{R}(A^k)$ and by $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, it follows that $AX^2 = X$.

(d) $\Rightarrow$ (a). By $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ and $AX^2 = X$, it follows that

$$\mathcal{R}(A^k) = \mathcal{R}(XA) \subseteq \mathcal{R}(X) = \mathcal{R}(AX^2) = \cdots = \mathcal{R}(A^kX^{k+1}) \subseteq \mathcal{R}(A^k),$$

which implies $\mathcal{R}(X) = \mathcal{R}(A^k)$. By $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$, we get that

$$(A^k)^*A^{m+1}X = (A^k)^*A^m.$$

According to Theorem 4.1 (d), we derive $X = A^{Ⓦ_m}$. □

Analogously, we characterize the $A^{Ⓦ_m}$ by using $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}$ or $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})}$ as follows:

Theorem 4.6. *Let* $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then*

(a) $X = A^{Ⓦ_m}$ *is the unique solution of the system of equations below*:

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)},\ \mathcal{R}(X) = \mathcal{R}(A^k); \tag{4.1}$$

(b) $X = A^{Ⓦ_m}$ *is the unique solution of the system of equations below*:

$$XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^{m+1})},\ \mathcal{N}(X) = \mathcal{N}((A^k)^*A^m). \tag{4.2}$$

Proof. (*a*). If $X = A^{\textcircled{w}_m}$, by Theorems 3.6 (*b*) and 3.7 (*a*), it is evident that the system of Eq (4.1) holds. Conversely, if the system of Eq (4.1) holds, it follows that $(A^k)^*A^mAX = (A^k)^*A^m$. Hence $X = A^{\textcircled{w}_m}$ from Theorem 4.1 (*d*).

(*b*). For $X = A^{\textcircled{w}_m}$, by Theorems 3.6 (*b*) and 3.7 (*b*), it is evident that the system of Eq (4.2) holds. Next, we prove the uniqueness of X.

Assume that X_1, X_2 satisfy the system of Eq (4.2). Then we obtain that $X_1A = X_2A$ and $\mathcal{N}(X_1) = \mathcal{N}(X_2) = \mathcal{N}((A^k)^*A^m)$. Thus, we get that $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{N}(A^*) \subseteq \mathcal{N}((A^k)^*)$ and $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{R}((A^m)^*A^k)$. For any $\eta \in \mathcal{N}((A^k)^*) \cap \mathcal{R}((A^m)^*A^k)$, we obtain that $(A^k)^*\eta = 0$, $\eta = (A^m)^*A^k\xi$ for some $\xi \in \mathbb{C}^n$. Since $\mathrm{Ind}(A) = k$, we derive that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+m})$, and it follows that $A^k\xi = A^{k+m}\xi_0$ for some $\xi_0 \in \mathbb{C}^{n\times n}$. Then we have that

$$0 = (A^k)^*\eta = (A^{k+m})^*A^{k+m}\xi_0.$$

Premultiplying the equation above by ξ_0^*, we derive that $(A^{k+m}\xi_0)^*A^{k+m}\xi_0 = 0$, which implies $A^{k+m}\xi_0 = 0$. Hence $\eta = 0$, i.e., $\mathcal{R}(X_1^* - X_2^*) = \{0\}$, which implies $X_1 = X_2$. □

Remark 4.7. *Notice that the condition* $\mathcal{R}(X) = \mathcal{R}(A^k)$ *in Theorem 4.6* (*a*) *can be replaced by* $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. *Also the condition* $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^m)$ *in Theorem 4.6* (*b*) *can be replaced by* $\mathcal{N}(X) \supseteq \mathcal{N}((A^k)^*A^m)$.

5. Representations of the *m*-weak group inverse

From Theorems 3.1 and 3.2, we have derived an expression of $A^{\textcircled{w}_m}$ by $A^{\textcircled{\dagger}}$. In the next results, we present several expressions of $A^{\textcircled{w}_m}$ involving other known generalized inverses and limitations.

Theorem 5.1. *Let* $A \in \mathbb{C}_k^{n\times n}$ *and* $m \in \mathbb{Z}^+$. *Then the following statements hold:*
(*a*) $A^{\textcircled{w}_m} = (A^D)^{m+1}P_{A^k}A^m$;
(*b*) $A^{\textcircled{w}_m} = A^{k-m}(A^{k+1})^{\textcircled{\#}}A^m$ $(k \geq m)$;
(*c*) $A^{\textcircled{w}_m} = (A^k)^{\#}A^{k-m-1}P_{A^k}A^m$ $(k \geq m+1)$;
(*d*) $A^{\textcircled{w}_m} = (A^{m+1}P_{A^k})^{\dagger}A^m$;
(*e*) $A^{\textcircled{w}_m} = A^{m-1}P_{A^k}(A^m)^{\textcircled{w}}$.

Proof. Let A be of the form (2.1). By (2.3)–(2.7) and (2.11)–(2.13), we get that

$$A^{m+1}P_{A^k} = U\begin{bmatrix} T^{m+1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{5.1}$$

$$(A^{m+1}P_{A^k})^{\dagger} = U\begin{bmatrix} T^{-m-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{5.2}$$

$$(A^D)^{m+1} = U\begin{bmatrix} T^{-m-1} & T^{-2-m-k}T_k \\ 0 & 0 \end{bmatrix}U^*, \tag{5.3}$$

$$(A^k)^{\#} = U\begin{bmatrix} T^{-k} & T^{-2k}T_k \\ 0 & 0 \end{bmatrix}U^*, \tag{5.4}$$

$$(A^{k+1})^{\textcircled{\#}} = U\begin{bmatrix} T^{-k-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{5.5}$$

$$(A^m)^{\textcircled{w}} = U \begin{bmatrix} T^{-m} & T^{-2m}T_m \\ 0 & 0 \end{bmatrix} U^*. \tag{5.6}$$

(a). By (2.12), (2.13) and (4.5), it follows that

$$\begin{aligned} (A^D)^{m+1}P_{A^k}A^m &= U \begin{bmatrix} T^{-1} & T^{-k-1}T_k \\ 0 & 0 \end{bmatrix}^{m+1} \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^m & T_m \\ 0 & N^m \end{bmatrix} U^* \\ &= U \begin{bmatrix} T^{-1} & T^{-(m+1)}T_m \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Hence $A^{\textcircled{w}_m} = (A^D)^{m+1}P_{A^k}A^m$.

The proofs of (b)–(e) are analogous to that of (a). □

Next, we consider the the accuracy of the expression in Theorem 5.1 (a) for computing the m-weak group inverse.

Example 5.2. *Let*

$$A = \begin{bmatrix} 0.8485 + 0.1676i & 0.2540 + 0.5983i & 0.6425 + 0.9363i & 0.8275 + 0.4257i & 0.6969 + 0.8590i & 0.5510 + 0.6352i & 0.2347 + 0.9504i \\ 0.1680 + 0.6196i & 0.3756 + 0.1390i & 0.2441 + 0.4382i & 0.7763 + 0.2670i & 0.2739 + 0.3483i & 0.4470 + 0.4406i & 0.7125 + 0.1951i \\ 0.4884 + 0.8135i & 0.1611 + 0.8553i & 0.3944 + 0.4832i & 0.7271 + 0.4400i & 0.1657 + 0.8773i & 0.6828 + 0.6348i & 0.1984 + 0.7051i \\ 0.1033 + 0.3637i & 0.5945 + 0.2874i & 0.1809 + 0.4247i & 0.5422 + 0.2813i & 0.2855 + 0.1739i & 0.5710 + 0.6704i & 0.6415 + 0.3145i \\ 0.4750 + 0.7706i & 0.5137 + 0.4274i & 0.4025 + 0.1004i & 0.4356 + 0.3288i & 0.1589 + 0.1206i & 1.0058 + 0.7174i & 0.6422 + 0.1015i \\ 0.3229 + 0.7518i & 0.5552 + 0.4735i & 0.4742 + 0.2084i & 0.2175 + 0.6228i & 0.2705 + 0.1671i & 0.7580 + 0.5195i & 0.1824 + 0.6410i \\ 0.2069 + 0.0437i & 0.6633 + 0.5112i & 0.3382 + 0.8101i & 0.6209 + 0.2514i & 0.5148 + 0.5723i & 0.9051 + 0.5467i & 0.3012 + 0.3692i \end{bmatrix}.$$

Assume that A is of the form (2.1), we obtain that

$$U = \begin{bmatrix} 0.4825 - 0.0849i & 0.0469 - 0.1756i & 0.5890 + 0.3816i & -0.2856 + 0.1754i & -0.2172 - 0.1097i & -0.1298 - 0.0671i & 0.0533 + 0.1967i \\ 0.2893 - 0.0752i & -0.1794 - 0.3491i & -0.5178 + 0.1008i & -0.0098 + 0.1153i & -0.1657 + 0.2133i & -0.5416 + 0.2449i & 0.1310 - 0.1466i \\ 0.4536 + 0.0132i & 0.2822 + 0.1450i & -0.0509 - 0.0460i & 0.0355 - 0.1626i & 0.0535 + 0.7433i & 0.2983 - 0.1107i & 0.0663 + 0.0081i \\ 0.2796 - 0.0694i & -0.2498 - 0.2853i & -0.2797 - 0.1038i & 0.0927 + 0.0482i & 0.0299 - 0.2892i & 0.5633 + 0.0729i & 0.3269 + 0.3991i \\ 0.3355 - 0.1257i & -0.1231 + 0.3086i & -0.0681 + 0.0045i & 0.6071 + 0.0994i & 0.0793 - 0.1390i & -0.2447 - 0.2386i & -0.3902 + 0.2889i \\ 0.3535 - 0.0406i & 0.3291 + 0.4917i & -0.1044 - 0.1231i & -0.0927 + 0.2237i & 0.1346 - 0.3811i & -0.0086 + 0.2839i & 0.2660 - 0.3518i \\ 0.3446 - 0.0896i & -0.3255 - 0.0457i & 0.0798 - 0.3137i & -0.2170 - 0.5928i & -0.0771 - 0.1818i & 0.0470 + 0.0421i & -0.3504 - 0.3101i \end{bmatrix},$$

$$T = \begin{bmatrix} 3.1562 + 3.3883i & -0.4572 + 0.0272i & -0.3260 - 0.7290i & 0.1393 + 0.2261i \\ 0.0000 + 0.0000i & 0.4103 - 0.7005i & 0.3751 + 0.2669i & -0.0078 + 0.2713i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.6433 - 0.3382i & 0.4614 + 0.3799i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & -0.8309 - 0.2692i \end{bmatrix},$$

$$S = \begin{bmatrix} 0.2397 - 0.7281i & -0.1778 + 0.2997i & 0.6513 + 0.2162i \\ 0.1628 - 0.2162i & 0.0120 + 0.0676i & -0.0751 - 0.0869i \\ -0.1639 + 0.2880i & -0.0551 - 0.4207i & 0.2045 + 0.1170i \\ 0.3038 - 0.3764i & -0.1479 + 0.0719i & 0.0834 + 0.0878i \end{bmatrix}, N = \begin{bmatrix} 0.0000 + 0.0000i & -0.4926 - 0.4478i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.1329 - 0.0269i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}.$$

Then it is clear that $k = \mathrm{Ind}(A) = 3$. *According to (2.12), (2.13), (3.2) and (5.3), a straightforward computation shows that*

$$A^{\textcircled{w}_2} = \begin{bmatrix} 0.5500 + 0.0731i & -0.3216 - 0.5308i & 0.3542 + 0.2685i & 0.0240 - 0.0992i & 0.1347 + 0.1272i & -0.2551 - 0.5635i & -0.5893 + 0.1896i \\ -0.5670 + 0.5317i & 0.8843 + 0.3216i & -0.4879 - 0.7221i & -0.2015 + 0.4591i & -0.1916 - 0.3123i & 0.4252 - 0.0781i & 0.5168 - 0.0711i \\ 0.0613 - 0.3227i & -0.3442 + 0.1506i & 0.3922 + 0.1385i & -0.0511 - 0.3461i & 0.1347 + 0.1018i & 0.1026 + 0.0907i & -0.2209 - 0.0521i \\ -0.3141 + 0.2405i & 0.6100 + 0.3344i & -0.4033 - 0.5827i & -0.1523 + 0.3461i & -0.0966 - 0.1740i & 0.1994 - 0.0641i & 0.3864 + 0.0751i \\ 0.4276 + 0.1334i & -0.2170 - 0.1890i & -0.3404 + 0.1071i & 0.3855 - 0.1231i & -0.3630 + 0.1546i & 0.0708 + 0.1494i & 0.1807 - 0.5088i \\ -0.0347 - 0.3614i & -0.7335 - 0.1810i & 0.6827 + 0.3219i & 0.1600 - 0.4144i & 0.1112 - 0.1606i & -0.1645 + 0.4531i & 0.0694 - 0.0409i \\ 0.1126 - 0.5274i & 0.3706 + 0.0871i & -0.1810 + 0.2321i & -0.1211 + 0.0487i & 0.4600 + 0.2349i & -0.2612 - 0.3429i & -0.2482 + 0.4110i \end{bmatrix},$$

$$A^D = \begin{bmatrix} 0.5519 + 0.0637i & -0.3151 - 0.5243i & 0.3573 + 0.2679i & 0.0382 - 0.1188i & 0.1155 + 0.1148i & -0.2416 - 0.5478i & -0.6048 + 0.2051i \\ -0.5868 + 0.5393i & 0.8886 + 0.3016i & -0.4926 - 0.7271i & -0.2550 + 0.4561i & -0.1919 - 0.2619i & 0.4384 - 0.1220i & 0.5643 - 0.0607i \\ 0.0712 - 0.3336i & -0.3406 + 0.1643i & 0.3968 + 0.1401i & -0.0170 - 0.3611i & 0.1191 + 0.0704i & 0.1082 + 0.1221i & -0.2536 - 0.0436i \\ -0.3314 + 0.2454i & 0.6152 + 0.3178i & -0.4069 - 0.5873i & -0.1973 + 0.3394i & -0.1009 - 0.1314i & 0.2140 - 0.1002i & 0.4257 + 0.0876i \\ 0.4426 + 0.1457i & -0.2348 - 0.1834i & -0.3423 + 0.1131i & 0.4074 - 0.0792i & -0.3228 + 0.1320i & 0.0300 + 0.1588i & 0.1674 - 0.5512i \\ -0.0182 - 0.3618i & -0.7419 - 0.1674i & 0.6848 + 0.3269i & 0.1986 - 0.3982i & 0.1247 - 0.1976i & -0.1858 + 0.4820i & 0.0371 - 0.0610i \\ 0.1003 - 0.5340i & 0.3824 + 0.0806i & -0.1805 + 0.2276i & -0.1428 + 0.0207i & 0.4347 + 0.2565i & -0.2336 - 0.3553i & -0.2329 + 0.4392i \end{bmatrix},$$

$$P_{A^3} = \begin{bmatrix} 0.8779 + 0.0000i & -0.0446 - 0.1662i & 0.1193 - 0.1338i & -0.0432 + 0.0514i & -0.0818 + 0.1456i & 0.0604 - 0.0110i & 0.0519 + 0.0811i \\ -0.0446 + 0.1662i & 0.5351 + 0.0000i & 0.0316 - 0.1369i & 0.3696 - 0.0357i & 0.0621 + 0.1759i & -0.0570 - 0.1243i & 0.0416 - 0.0797i \\ 0.1193 + 0.1338i & 0.0316 + 0.1369i & 0.3389 + 0.0000i & 0.0285 + 0.0703i & 0.1692 - 0.1423i & 0.2953 - 0.0623i & 0.1557 + 0.0476i \\ -0.0432 - 0.0514i & 0.3696 + 0.0357i & 0.0285 - 0.0703i & 0.3267 + 0.0000i & 0.1249 + 0.1524i & -0.0766 - 0.0331i & 0.1585 + 0.0310i \\ -0.0818 - 0.1456i & 0.0621 - 0.1759i & 0.1692 + 0.1423i & 0.1249 - 0.1524i & 0.6219 + 0.0000i & 0.2074 - 0.0227i & -0.0447 + 0.1980i \\ 0.0604 + 0.0110i & -0.0570 + 0.1243i & 0.2953 + 0.0623i & -0.0766 + 0.0331i & 0.2074 + 0.0227i & 0.5614 + 0.0000i & -0.0864 - 0.2733i \\ 0.0519 - 0.0811i & 0.0416 + 0.0797i & 0.1557 - 0.0476i & 0.1585 - 0.0310i & -0.0447 - 0.1980i & -0.0864 + 0.2733i & 0.7381 + 0.0000i \end{bmatrix},$$

$$A^2 = \begin{bmatrix} -0.7769+3.6386i & -0.5344+4.1046i & -0.4095+3.6111i & 0.7045+4.1930i & -0.6174+3.1904i & 0.1077+5.9761i & -0.2581+3.9581i \\ -0.4248+2.4587i & 0.2472+2.2693i & -0.2516+2.5356i & 0.6811+2.4977i & -0.2256+2.0334i & 0.6321+3.4819i & -0.0733+2.0920i \\ -1.3540+3.2996i & -0.9052+3.4946i & -1.2506+3.2411i & -0.2180+4.0436i & -1.1420+2.8767i & -0.9506+5.1147i & -1.1106+3.6001i \\ -0.5540+2.3649i & 0.2360+2.2815i & -0.2197+2.3588i & 0.5680+2.4478i & -0.1974+1.8996i & 0.5955+3.4055i & -0.0409+2.0276i \\ 0.0321+2.9089i & 0.5594+2.7931i & 0.0854+2.9168i & 0.9166+2.9714i & 0.1421+2.6006i & 1.0418+3.6923i & -0.1839+2.8256i \\ -0.3304+2.9125i & -0.3410+2.8783i & -0.8216+2.5238i & 0.3039+3.2251i & -0.7160+2.5512i & -0.0904+3.8501i & -0.6645+2.7485i \\ -0.8272+3.1061i & 0.1375+2.8116i & 0.0451+2.5280i & 0.6730+3.1529i & -0.3009+2.0044i & 0.6401+4.3322i & 0.2366+2.7781i \end{bmatrix}.$$

Let $K = (A^D)^3 P_{A^3} A^2$. Then it follows that

$$K = \begin{bmatrix} 0.5500+0.0731i & -0.3216-0.5308i & 0.3542+0.2685i & 0.0240-0.0992i & 0.1347+0.1272i & -0.2551-0.5635i & -0.5893+0.1896i \\ -0.5670+0.5317i & 0.8843+0.3216i & -0.4879-0.7221i & -0.2015+0.4591i & -0.1916-0.3123i & 0.4252-0.0781i & 0.5168-0.0711i \\ 0.0613-0.3227i & -0.3442+0.1506i & 0.3922+0.1385i & -0.0511-0.3461i & 0.1347+0.1018i & 0.1026+0.0907i & -0.2209-0.0521i \\ -0.3141+0.2405i & 0.6100+0.3344i & -0.4033-0.5827i & -0.1523+0.3461i & -0.0966-0.1740i & 0.1994-0.0641i & 0.3864+0.0751i \\ 0.4276+0.1334i & -0.2170-0.1890i & -0.3404+0.1071i & 0.3855-0.1231i & -0.3630+0.1546i & 0.0708+0.1494i & 0.1807-0.5088i \\ -0.0347-0.3614i & -0.7335-0.1810i & 0.6827+0.3219i & 0.1600-0.4144i & 0.1112-0.1606i & -0.1645+0.4531i & 0.0694-0.0409i \\ 0.1126-0.5274i & 0.3706+0.0871i & -0.1810+0.2321i & -0.1211+0.0487i & 0.4600+0.2349i & -0.2612-0.3429i & -0.2482+0.4110i \end{bmatrix}$$

and

$$r_1 = \| A^{\textcircled{W}_2} - K \| = 6.6885 \times 10^{-14},$$

where $\| \cdot \|$ is the Frobenius norm.

Hence, the representation in Theorem 5.1 (a) get a good result in terms of computational accuracy.

In the following theorem, we present a connection between the (B, C)-inverse and the m-weak group inverse showing that the m-weak group inverse of $A \in \mathbb{C}_k^{n\times n}$ is its $(A^k, (A^k)^* A^m)$-inverse.

Theorem 5.3. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. Then $A^{\textcircled{W}_m} = A^{(A^k,(A^k)^* A^m)}$.*

Proof. By Theorem 3.7, it follows that $A^{\textcircled{W}_m} A A^k = A^k$ and $((A^k)^* A^m) A A^{\textcircled{W}_m} = (A^k)^* A^m$. From Theorem 3.6 (b), we derive that $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^{\textcircled{W}_m}) = \mathcal{N}((A^k)^* A^m)$. According to the definition of the (B, C)-inverse, it is clear that $A^{\textcircled{W}_m} = A^{(A^k,(A^k)^* A^m)}$. □

Next, we give some limit expressions of $A^{\textcircled{W}_m}$. Before the theorem, we need the lemma below:

Lemma 5.4. [36] *Let $A \in \mathbb{C}^{m\times n}, X \in \mathbb{C}^{n\times p}$ and $Y \in \mathbb{C}^{p\times m}$. Then the following conditions are equivalent:*
(a) $\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1} Y$ *exists;*
(b) $r(XYAXY) = r(XY)$;
(c) $A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}$ *exists,*
in which case,

$$\lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1} Y = A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}.$$

Theorem 5.5. *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1) and $m \in \mathbb{N}^+$. Then the following statements hold:*
(a) $A^{\textcircled{W}_m} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^* A^{k+m+1})^{-1} (A^k)^* A^m$;
(b) $A^{\textcircled{W}_m} = \lim_{\lambda\to 0} A^k (A^k)^* (\lambda I_n + A^{k+m+1} (A^k)^*)^{-1} A^m$;
(c) $A^{\textcircled{W}_m} = \lim_{\lambda\to 0} A^k (A^k)^* A^m (\lambda I_n + A^{k+1} (A^k)^* A^m)^{-1}$;
(d) $A^{\textcircled{W}_m} = \lim_{\lambda\to 0} (\lambda I_n + A^k (A^k)^* A^{m+1})^{-1} A^k (A^k)^* A^m$.

Proof. (a) It is easy to check that $r(A^k (A^k)^* A^m) = r((A^k)^* A^m) = r(A^k) = t$. By Theorem 3.6, we get that $\mathcal{R}(A^k) = \mathcal{R}(A^k (A^k)^* A^m)$, $\mathcal{N}((A^k)^* A^m) = \mathcal{N}(A^k (A^k)^* A^m)$. From Theorem 3.6, then $A^{\textcircled{W}_m} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^* A^m)} = A^{(2)}_{\mathcal{R}(A^k (A^k)^* A^m),\mathcal{N}(A^k (A^k)^* A^m)}$. Let $X = A^k, Y = (A^k)^* A^m$, according to Lemma 5.4, we get that

$$A^{\textcircled{W}_m} = \lim_{\lambda\to 0} A^k(\lambda I_n + (A^k)^* A^{k+m+1})^{-1} (A^k)^* A^m.$$

The statements (b)–(d) can be similarly proved. □

The following example will test the accuracy of expression in Theorem 5.5 (a) for computing the m-weak group inverse.

Example 5.6. *Let*

$$A = \begin{bmatrix} 4.8990 + 7.3786i & 6.8197 + 3.0145i & 7.2244 + 1.2801i & 4.5380 + 1.9043i & 8.3138 + 3.7627i & 6.2797 + 3.8462i & 3.7241 + 9.8266i \\ 1.6793 + 2.6912i & 0.4243 + 7.0110i & 1.4987 + 9.9908i & 4.3239 + 3.6892i & 8.0336 + 1.9092i & 2.9198 + 5.8299i & 1.9812 + 7.3025i \\ 9.7868 + 4.2284i & 0.7145 + 6.6634i & 6.5961 + 1.7112i & 8.2531 + 4.6073i & 0.6047 + 4.2825i & 4.3165 + 2.5181i & 4.8969 + 3.4388i \\ 7.1269 + 5.4787i & 5.2165 + 5.3913i & 5.1859 + 0.3260i & 0.8347 + 9.8164i & 3.9926 + 4.8202i & 0.1549 + 2.9044i & 3.3949 + 5.8407i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 9.8406 + 6.1709i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 9.2033 + 9.0631i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}$$

with $k = \mathrm{Ind}(A) = 3$. *By* $A^{\textcircled{\dagger}} = (A^{k+1}(A^k)^\dagger)^\dagger$, *we get*

$$\begin{aligned} A^{\textcircled{\dagger}} &= (A^4(A^3)^\dagger)^\dagger \\ &= \begin{bmatrix} -0.0032 - 0.1411i & 0.0709 + 0.0429i & 0.0106 - 0.0097i & 0.0187 + 0.0560i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0878 + 0.0001i & 0.0438 + 0.0326i & -0.1096 - 0.0524i & 0.0490 - 0.0158i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ -0.0896 + 0.0173i & -0.0314 - 0.1473i & 0.0445 + 0.0629i & 0.0118 + 0.0056i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0431 + 0.0788i & -0.0348 + 0.0182i & 0.0536 + 0.0096i & -0.0658 - 0.0844i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}. \end{aligned}$$

Together with (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_2} &= (A^{\textcircled{\dagger}})^3 A^2 \\ &= \begin{bmatrix} -0.0032 - 0.1411i & 0.0709 + 0.0429i & 0.0106 - 0.0097i & 0.0187 + 0.0560i & -0.0151 - 0.0310i & -0.2502 + 0.2087i & -0.1223 + 0.0627i \\ 0.0878 + 0.0001i & 0.0438 + 0.0326i & -0.1096 - 0.0524i & 0.0490 - 0.0158i & 0.1788 + 0.2332i & -0.3818 + 0.4296i & -0.1105 + 0.0067i \\ -0.0896 + 0.0173i & -0.0314 - 0.1473i & 0.0445 + 0.0629i & 0.0118 + 0.0056i & -0.0415 - 0.2864i & 0.7974 - 0.1768i & 0.2687 + 0.0926i \\ 0.0431 + 0.0788i & -0.0348 + 0.0182i & 0.0536 + 0.0096i & -0.0658 - 0.0844i & 0.0125 - 0.0262i & 0.0972 - 0.3512i & 0.0454 - 0.1317i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}. \end{aligned}$$

Let $L = \lim_{\lambda \to 0} A^3(\lambda I_n + (A^3)^* A^6)^{-1}(A^3)^* A^2$. *Then it follows that*

$$L = \begin{bmatrix} -0.003189 - 0.1411i & 0.07088 + 0.04294i & 0.01062 - 0.009663i & 0.01868 + 0.05603i & -0.01512 - 0.03101i & -0.2502 + 0.2087i & -0.1223 + 0.06266i \\ 0.08776 + 5.907e - 5i & 0.04379 + 0.03259i & -0.1096 - 0.05243i & 0.04899 - 0.01576i & 0.1788 + 0.2332i & -0.3818 + 0.4296i & -0.1105 + 0.006749i \\ -0.08964 + 0.01729i & -0.03136 - 0.1473i & 0.04449 + 0.06285i & 0.01184 + 0.005646i & -0.04154 - 0.2864i & 0.7974 - 0.1768i & 0.2687 + 0.09263i \\ 0.04314 + 0.07879i & -0.03482 + 0.01822i & 0.05364 + 0.009637i & -0.06583 - 0.08442i & 0.01248 - 0.02618i & 0.09717 - 0.3512i & 0.04544 - 0.1317i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{bmatrix}$$

and

$$r_2 = \| A^{\textcircled{W}_2} - L \| = 6.136 \times 10^{-11},$$

where $\| \cdot \|$ *is the Frobenius norm. Hence, the representation in Theorem 5.5 (a) is efficient for computing the m-weak group inverse.*

6. Relationships between the m-weak group inverse and other generalized inverses

Next, we consider some relationships between the m-weak group inverse and other generalized inverses as well as some matrix classes. First we present some classes of matrix as follows.

These symbols $\mathbb{C}_n^{\mathrm{OP}}$, $\mathbb{C}_n^{\mathrm{EP}}$, $\mathbb{C}_n^{i-\mathrm{EP}}$ and $\mathbb{C}_n^{k,\textcircled{\dagger}}$ represent the sets of $\mathbb{C}^{n\times n}$ consisting of orthogonal projectors (Hermitian idempotent matrices), EP (Range-Hermitian) matrices, i-EP matrices and k-core-EP matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathrm{OP}} = \{A | A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\} = \{A | A \in \mathbb{C}^{n\times n}, A^2 = A = A^\dagger\},$$

$$\mathbb{C}_n^{\mathrm{EP}} = \{A|A \in \mathbb{C}^{n\times n}, AA^{\dagger} = A^{\dagger}A\} = \{A|A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\},$$
$$\mathbb{C}_n^{i-\mathrm{EP}} = \{A|A \in \mathbb{C}_k^{n\times n}, A^k(A^k)^{\dagger} = (A^k)^{\dagger}A^k\},$$
$$\mathbb{C}_n^{k,\textcircled{\dagger}} = \{A|A \in \mathbb{C}_k^{n\times n}, A^kA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}A^k\}.$$

According to [4, Theorem 2.3], it is known that $T_k = 0$ is equivalent to $S = 0$. Also we have the following lemma:

Lemma 6.1. *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1). Then*

$$T_m = 0 \Longleftrightarrow S = 0.$$

Proof. Notice that $T_m = 0$ can be equivalently expressed as the equation below:

$$T^{m-1}S + T^{m-2}SN + \cdots + TSN^{m-2} + SN^{m-1} = 0. \tag{6.1}$$

Multiplying by N^{k-1} on the right of the equation above, we get $SN^{k-1} = 0$. Multiplying by N^{k-2} on the right of the equation above, for $SN^{k-1} = 0$, then we have $SN^{k-2} = 0$. In the same manner, we derive $SN^{k-3} = 0, \cdots, SN = 0$. From Eq (6.1), it follows that $T^{m-1}S = 0$, i.e, $S = 0$. □

The next theorem provides some necessary and sufficient conditions for $A^{\textcircled{W}_m}$ to be equal to various transformations of $A \in \mathbb{C}_k^{n\times n}$.

Theorem 6.2. *Let $A \in \mathbb{C}_k^{n\times n}$, $m \in \mathbb{Z}^+$. Then the following statements hold:*
(*a*) $A^{\textcircled{W}_m} \in A\{1\} \Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{CM}}$;
(*b*) $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{CM}}$;
(*c*) $A^{\textcircled{W}_m} = A \Longleftrightarrow A = A^3$;
(*d*) $A^{\textcircled{W}_m} = A^* \Longleftrightarrow AA^* \in \mathbb{C}_n^{\mathrm{OP}}$ *and* $A \in \mathbb{C}_n^{\mathrm{EP}}$;
(*e*) $A^{\textcircled{W}_m} = P_A \Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{OP}}$.

Proof. Let A be of the form (2.1).
(*a*). From (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_m} \in A\{1\} &\Longleftrightarrow AA^{\textcircled{W}_m}A = A \\ &\Longleftrightarrow U\begin{bmatrix} T & S + T^{-m}T_mN \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &\Longleftrightarrow N = 0 \\ &\Longleftrightarrow A \in \mathbb{C}_n^{\mathrm{CM}}. \end{aligned}$$

(*b*). By (3.2), it is clear that $r(A^{\textcircled{W}_m}) = r((A^{\textcircled{W}_m})^2) = t$, which implies $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{CM}}$.
(*c*). From (3.2), we get that

$$\begin{aligned} A^{\textcircled{W}_m} = A &\Longleftrightarrow U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &\Longleftrightarrow T^2 = I_t \text{ and } N = 0 \\ &\Longleftrightarrow A = A^3. \end{aligned}$$

(d). According to (3.2), we obtain that

$$\begin{aligned} A^{\textcircled{W}_m} = A^* &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}U^* \\ &\iff T^{-1} = T^*, S = 0 \text{ and } N = 0 \\ &\iff AA^* \in \mathbb{C}_n^{\mathrm{OP}} \text{and } A \in \mathbb{C}_n^{\mathrm{EP}}. \end{aligned}$$

(e). Since (2.9) and (3.2), it follows that

$$\begin{aligned} A^{\textcircled{W}_m} = P_A &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I_t & 0 \\ 0 & NN^\dagger \end{bmatrix}U^* \\ &\iff T = I_t, NN^\dagger = 0 \text{ and } T_m = 0 \\ &\iff T = I_t, S = 0 \text{ and } N = 0. \end{aligned}$$

Hence $A^{\textcircled{W}_m} = P_A$ is equivalent to $A \in \mathbb{C}_n^{\mathrm{OP}}$. □

Interestingly, we find that $A \in \mathbb{C}_n^{i-\mathrm{EP}}$ is equivalent to $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$ by using core-EP decomposition. Therefore, we consider more equivalent conditions for $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$ by using core-EP decomposition in the next results.

Lemma 6.3. [4] *Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1). Then $A \in \mathbb{C}_n^{i-\mathrm{EP}}$ if and only if $S = 0$.*

Moreover, $S = 0$ if and only if $A \in \mathbb{C}_n^{k,\textcircled{\dagger}}$.

Theorem 6.4. *Let $A \in \mathbb{C}_k^{n\times n}$ and $m \in \mathbb{Z}^+$. Then the following statements are equivalent:*
(a) $A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}}$;
(b) $A \in \mathbb{C}_n^{i-\mathrm{EP}}$;
(c) $A^{\textcircled{W}} \in \mathbb{C}_n^{\mathrm{EP}}$;
(d) $A^{\textcircled{W}_m} = A^{\textcircled{\dagger}}$;
(e) $AA^{\textcircled{W}_m} = AA^{\textcircled{\dagger}}$.

Proof. Let $A \in \mathbb{C}_k^{n\times n}$ be of the form (2.1). According to Lemma 6.3, we will prove that each of the statements (a)–(e) is equivalent to $S = 0$.

(a). According to (3.2) and Lemma 6.1, it follows that

$$\begin{aligned} A^{\textcircled{W}_m} \in \mathbb{C}_n^{\mathrm{EP}} &\iff \mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}((A^{\textcircled{W}_m})^*) \\ &\iff (T^{m+1})^{-1}T_m = 0 \\ &\iff S = 0. \end{aligned}$$

(c). By (2.4), we get that

$$\begin{aligned} A^{\textcircled{W}} \in \mathbb{C}_n^{\mathrm{EP}} &\iff \mathcal{R}(A^{\textcircled{W}}) = \mathcal{R}((A^{\textcircled{W}})^*) \\ &\iff T^{-2}S = 0 \\ &\iff S = 0. \end{aligned}$$

(d). Since (2.3), (3.2) and Lemma 6.1, it follows that

$$
\begin{aligned}
A^{\textcircled{w}_m} = A^{\textcircled{\dagger}} &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&\iff (T^{m+1})^{-1}T_m = 0 \\
&\iff S = 0.
\end{aligned}
$$

(e). From (2.3), (3.2) and Lemma 6.1, we get that

$$
\begin{aligned}
AA^{\textcircled{w}_m} = AA^{\textcircled{\dagger}} &\iff U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \\
&\iff T^{-m}T_m = 0 \\
&\iff S = 0.
\end{aligned}
$$

□

In [9], the authors have proved that $A^{\textcircled{w}_m} = A^D$ if and only if $SN^m = 0$. In the next results, we investigate the relationships between the m-weak group inverse and other generalized inverses such as the MP-inverse, group inverse, core inverse, DMP-inverse, dual DMP-inverse, weak group inverse by core-EP decomposition.

Theorem 6.5. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *and $m \in \mathbb{Z}^+$. Then the following statements hold:*
(a) $A^{\textcircled{w}_m} = A^\dagger \iff A \in \mathbb{C}_n^{\mathrm{EP}}$;
(b) $A^{\textcircled{w}_m} = A^\# \iff A \in \mathbb{C}_n^{\mathrm{CM}}$;
(c) $A^{\textcircled{w}_m} = A^{\textcircled{\#}} \iff A \in \mathbb{C}_n^{\mathrm{CM}}$;
(d) $A^{\textcircled{w}_m} = A^{D,\dagger} \iff T^{k-m}T_m = T_kNN^\dagger$;
(e) $A^{\textcircled{w}_m} = A^{\dagger,D} \iff SN^m = 0$ *and* $S = SN^\dagger N$;
(f) $A^{\textcircled{w}_m} = A^{\textcircled{w}} \iff SN = 0$ $(m > 1)$.

Proof. (a). By (2.8) and (3.2), it follows that

$$
\begin{aligned}
A^{\textcircled{w}_m} = A^\dagger &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*\triangle & -T^*\triangle SN^\dagger \\ M^*\triangle & N^\dagger - M^*\triangle SN^\dagger \end{bmatrix}U^* \\
&\iff M^* = 0, N^\dagger = 0, T^{-1} = T^*\triangle \text{ and } (T^{m+1})^{-1}T_m = -T^*\triangle SN^\dagger \\
&\iff S = 0 \text{ and } N = 0 \\
&\iff A \in \mathbb{C}_n^{\mathrm{EP}}.
\end{aligned}
$$

(b). It has been mentioned that $A^\#$ exits if and only if $A \in \mathbb{C}_n^{\mathrm{CM}}$, which is equivalent to $N = 0$. Then, by (2.6) and (3.2), it follows that

$$
\begin{aligned}
A^{\textcircled{w}_m} = A^\# &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \text{ and } N = 0 \\
&\iff (T^{m+1})^{-1}T_m = T^{-2}S \text{ and } N = 0 \\
&\iff N = 0 \\
&\iff A \in \mathbb{C}_n^{\mathrm{CM}}.
\end{aligned}
$$

(c). The proof follows in a similar manner above.

(d). Using (2.5) and (2.9) to $A^{D,\dagger} = A^D AA^\dagger$, we derive

$$A^{D,\dagger} = \begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k NN^\dagger \\ 0 & 0 \end{bmatrix},$$

and by (3.2), it follows that

$$\begin{aligned} A^{\textcircled{w}_m} = A^{D,\dagger} &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & (T^{k+1})^{-1}T_k NN^\dagger \\ 0 & 0 \end{bmatrix}U^* \\ &\iff T^{k-m}T_m = T_k NN^\dagger. \end{aligned}$$

(e). Using (2.5) and (2.10) to $A^{\dagger,D} = A^\dagger AA^D$, we obtain that

$$A^{\dagger,D} = \begin{bmatrix} T^*\triangle & -T^*\triangle T^{-k}T_k \\ M^*\triangle & M^*\triangle T^{-k}T_k \end{bmatrix},$$

together with (3.2), we derive that

$$\begin{aligned} A^{\textcircled{w}_m} = A^{\dagger,D} &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*\triangle & T^*\triangle T^{-k}T_k \\ M^*\triangle & M^*\triangle T^{-k}T_k \end{bmatrix}U^* \\ &\iff M^* = 0, T^{-1} = T^*\triangle \text{ and } (T^{m+1})^{-1}T_m = T^*\triangle T^{-k}T_k \\ &\iff S = SN^\dagger N \text{ and } T^{k-m}T_m = T_k \\ &\iff S = SN^\dagger N \text{ and } SN^m = 0. \end{aligned}$$

(f). If $m > 1$, from (2.4) and (3.2), it leads to

$$\begin{aligned} A^{\textcircled{w}_m} = A^{\textcircled{w}} &\iff U\begin{bmatrix} T^{-1} & (T^{m+1})^{-1}T_m \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \\ &\iff (T^{m+1})^{-1}T_m = T^{-2}S. \end{aligned}$$

Clearly, $(T^{m+1})^{-1}T_m = T^{-2}S$ is equivalent to $T^{-3}SN + \cdots + (T^{m+1})^{-1}SN^{m-1} = 0$, which is further equivalent to $SN = 0$. Hence $A^{\textcircled{w}_m} = A^{\textcircled{w}} \iff SN = 0$. □

7. Applications of the m-weak group inverse

In this section, we consider the relationship between the m-weak group inverse and the nonsingular bordered matrix, which is applied to the Cramer's rule for the solution of the restricted matrix equation.

Theorem 7.1. *Let $A \in \mathbb{C}_k^{n\times n}$ with $r(A^k) = t$ and $m \in \mathbb{Z}^+$. Let $B \in \mathbb{C}^{n\times(n-t)}$ and $C^* \in \mathbb{C}^{n\times(n-t)}$ be of full column rank such that $\mathcal{N}((A^k)^*A^m) = \mathcal{R}(B)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$. Then the bordered matrix*

$$K = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is invertible and

$$K^{-1} = \begin{bmatrix} A^{\textcircled{w}_m} & (I_n - A^{\textcircled{w}_m}A)C^\dagger \\ B^\dagger(I_n - AA^{\textcircled{w}_m}) & B^\dagger(AA^{\textcircled{w}_m}A - A)C^\dagger \end{bmatrix}.$$

Proof. Let $X = \begin{bmatrix} A^{\textcircled{W}_m} & (I_n - A^{\textcircled{W}_m}A)C^{\dagger} \\ B^{\dagger}(I_n - AA^{\textcircled{W}_m}) & B^{\dagger}(AA^{\textcircled{W}_m}A - A)C^{\dagger} \end{bmatrix}$. By $\mathcal{R}(A^{\textcircled{W}_m}) = \mathcal{R}(A^k) = \mathcal{N}(C)$, we get $CA^{\textcircled{W}_m} = 0$. Since C is a full row rank matrix, we get $CC^{\dagger} = I_{n-t}$. Using

$$\mathcal{R}(I_n - AA^{\textcircled{W}_m}) = \mathcal{N}(AA^{\textcircled{W}_m}) = \mathcal{N}((A^k)^*A^m) = \mathcal{R}(B) = \mathcal{R}(BB^{\dagger}),$$

we get $BB^{\dagger}(I_n - AA^{\textcircled{W}_m}) = I_n - AA^{\textcircled{W}_m}$. Then, we have

$$\begin{aligned} KX &= \begin{bmatrix} AA^{\textcircled{W}_m} + BB^{\dagger}(I_n - AA^{\textcircled{W}_m}) & A(I_n - A^{\textcircled{W}_m}A)C^{\dagger} + BB^{\dagger}(AA^{\textcircled{W}_m}A - A)C^{\dagger} \\ CA^{\textcircled{W}_m} & C(I_n - A^{\textcircled{W}_m}A)C^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} AA^{\textcircled{W}_m} + I_n - AA^{\textcircled{W}_m} & A(I_n - A^{\textcircled{W}_m}A)C^{\dagger} - (I_n - AA^{\textcircled{W}_m})AC^{\dagger} \\ 0 & CC^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ 0 & I_{n-t} \end{bmatrix}. \end{aligned}$$

Thus, $X = K^{-1}$. The proof is completed. □

The next result, we discuss the solution of the restricted matrix equation below by the m-weak group inverse.

Theorem 7.2. *Let $A \in \mathbb{C}_k^{n\times n}$, $X \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{n\times p}$. If $\mathcal{R}(D) \subseteq \mathcal{R}(A^k)$, then the restricted matrix equation*

$$AX = D, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k) \tag{7.1}$$

has a unique solution $X = A^{\textcircled{W}_m}D$.

Proof. Since $\mathcal{R}(A^k) = \mathcal{R}(AA^k)$ and $\mathcal{R}(D) \subseteq \mathcal{R}(A^k)$, we get that $\mathcal{R}(D) \subseteq A\mathcal{R}(A^k)$, which implies the restricted matrix equation (7.1) has a solution. Obviously, $X = A^{\textcircled{W}_m}D$ is a solution of (7.1). Then we prove the uniqueness of X. If X_1 also satisfies (7.1), then

$$X = A^{\textcircled{W}_m}D = A^{\textcircled{W}_m}AX_1 = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^m)}X_1 = X_1.$$

□

Based on the nonsingular bordered matrix in Theorem 7.1, we show the Cramer's rule for solving the restricted matrix Eq (7.1).

Theorem 7.3. *Let $A \in \mathbb{C}_k^{n\times n}$ with $r(A^k) = t$, $X \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{n\times p}$. Let $B \in \mathbb{C}^{n\times(n-t)}$ and $C^* \in \mathbb{C}^{n\times(n-t)}$ be of full column rank such that $\mathcal{N}((A^k)^*A^m) = \mathcal{R}(B)$ and $\mathcal{R}(A^k) = \mathcal{N}(C)$. Then the elements of the unique solution $X = [x_{ij}]$ of the restricted matrix Eq (7.1) are given by*

$$x_{ij} = \frac{det\begin{bmatrix} A(i\rightarrow d_j) & B \\ C(i\rightarrow 0) & 0 \end{bmatrix}}{det\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}}, \ i = 1, 2, ..., n, \ j = 1, 2, ..., m, \tag{7.2}$$

where d_j denotes the j-th column of D.

Proof. Since X is the solution of the restricted matrix Eq (7.1), we get that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k) = \mathcal{N}(C)$, which implies $CX = 0$. Then the restricted matrix Eq (7.1) can be rewritten as

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} AX \\ CX \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

By Theorem 7.1, we have that $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is invertible. Consequently, (7.2) follows from the Cramer's rule for the above equation. □

Example 7.4. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 10 & 14 & 24 & 28 \\ 6 & 19 & 20 & 22 \\ 4 & 10 & 14 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \\ -2 & -4 & -7 \\ 1 & 2 & 4 \\ -1 & -3 & -6 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

It can be verified that $\mathrm{Ind}(A) = 3$. *Then we get that*

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^{\textcircled{W}_2} = (A^{\textcircled{\dagger}})^3 A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that

$$X = A^{\textcircled{W}_2} D = \begin{bmatrix} 10 & 14 & 24 & 28 \\ 6 & 19 & 20 & 22 \\ 4 & 10 & 14 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies the restricted matrix equation $AX = D$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}(A^3)$. *By simple calculations, we can also get that the components of* X *can be expressed in (7.2).*

8. Conclusions

This paper gives a new definition of the m-weak group inverse on complex matrices, which extends the notions of the Drazin inverse and the weak group inverse. Some characterizations of the m-weak group inverse in terms of the range space, null space, rank and projectors are presented. Several representations of the m-weak group inverse involving some known generalized inverses as well as limitations are also derived. The representation in Theorem 5.1 gives better result in term of the computational accuracy (see Examples 5.2 and 5.6). The m-weak group inverse are concerned with the solution of a restricted matrix Eq (7.1). The solution of (7.1) can also be expressed by the Cramer's rule (see Theorem 7.3). In [37–39], there are some iterative methods and algorithms to compute the outer inverses. Motivated by these, further investigations deserve more attention as follows:

(1) The applications of the m-weak group inverse in linear equations and matrix equations;
(2) Perturbation formulae as well as perturbation bounds for the m-weak group inverse;
(3) Iterative algorithm, splitting method for computing the m-weak group inverse;
(4) Other representations of the m-weak group inverse.

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Conflict of interest

The authors declare no conflict of interest.

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