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Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives

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A B S T R A C T

We introduce a new generalized Caputo-type fractional derivative which generalizes Caputo fractional derivative. Some characteristics were derived to display the new generalized derivative features. Then, we present an adaptive predictor corrector method for the numerical solution of generalized Caputo-type initial value problems. The proposed algorithm can be considered as a fractional extension of the classical Adams-Bashforth-Moulton method. Dynamic behaviors of some fractional derivative models are numerically discussed. We believe that the presented generalized Caputo-type fractional derivative and the proposed algorithm are expected to be further used to formulate and simulate many generalized Caputo type fractional models.

1. Introduction

The theory of fractional derivatives, which are natural extensions of ordinary derivatives, has become very attractive to scientists because of its applications in a variety of fields. Fractional order derivatives of a given function include the entire function history; the following state of a fractional order system is not only dependent on its current state but also on all its historical states [19,20,26,31,34,38–40]. Several fractional derivative models, where non-locality plays a very important role, in different areas, including physics, engineering, mechanics and dynamical systems, signal and image processing, control theory, biology, environmental sciences, and materials, have been introduced [4,5,8,17,19,20,22,26,27,31,33–35,38–44]. A class of fractional derivatives, such as Riemann-Liouville, Hadamard, and Caputo, are defined using fractional integrals. The Riemann-Liouville fractional integral of order $\alpha > 0$, which is one of the most studied definitions, is defined by

$$I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > a. \tag{1}$$

Based on the above fractional integral, the Riemann-Liouville fractional derivative and the Caputo fractional derivative of order $\alpha > 0$ are defined by

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$$ {}^{R}D^{\alpha}_{a+}f(t) = D^{m}I^{m-\alpha}_{a+}f(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-\alpha-1}f(s)\,ds, \quad t>a, \tag{2} $$

$$ {}^{C}D^{\alpha}_{a+}f(t) = I^{m-\alpha}_{a+}D^{m}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1}f^{(m)}(s)\,ds, \quad t>a, \tag{3} $$

respectively, where $m-1<\alpha\le m$ and $m\in\mathbb{N}$. The Caputo fractional operator is widely used to model many physical problems in fractional calculus applications because it is suitable for initial value problems and has many characteristics similar to ordinary derivatives. For example, when $m-1<\alpha\le m$, the Caputo operator satisfies the rule

$$ I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}f(t) = f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{k!}(t-a)^k, \quad t>a. \tag{4} $$

Recently, a useful generalization of fractional integral operators, by introducing fractional order integral of a given function related to another function [26,40], is introduced in the following manner [24].

Definition 1. The generalized fractional integral of the function f, $I^{\alpha,\rho}_{a+}f(t)$, of order $\alpha>0$, where $\rho>0$, is defined (provided it exists) by

$$ I^{\alpha,\rho}_{a+}f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}f(s)\,ds, \qquad \alpha>0, \quad t>a. \tag{5} $$

Accordingly, for $m-1<\alpha\le m$ where $m\in\mathbb{N}$, the corresponding generalized Riemann-type and generalized Caputo-type fractional derivatives are introduced as [25]

Definition 2. The generalized Riemann-type fractional derivative of the function f, ${}^{R}D^{\alpha,\rho}_{a+}f(t)$, of order $\alpha>0$ is defined by

$$ {}^{R}D^{\alpha,\rho}_{a+}f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}f(s)\,ds, \qquad t>a\ge 0. \tag{6} $$

Definition 3. The generalized Caputo-type fractional derivative of the function f, ${}^{C}D^{\alpha,\rho}_{a+}f(t)$, of order $\alpha>0$ is defined by

$$ {}^{C}D^{\alpha,\rho}_{a+}f(t) = \left({}^{R}D^{\alpha,\rho}_{a+}\left[f(x) - \sum_{n=0}^{m-1}\frac{f^{(n)}(a)}{n!}(x-a)^n\right]\right)(t), \qquad t>a\ge 0, \tag{7} $$

where $m=\lceil\alpha\rceil$ and $\rho>0$. In case of $0<\alpha\le 1$ and $f:[a,b]\longrightarrow\mathbb{R}$, where $f\in C^1([a,b])$, the generalized Caputo-type fractional derivative of the function f reduces to [2]

$$ {}^{C}D^{\alpha,\rho}_{a+}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t (t^\rho - s^\rho)^{-\alpha}f'(s)\,ds, \qquad 0<\alpha\le 1, \quad t>a\ge 0. \tag{8} $$

On the other hand, for the numerical simulation purposes of fractional order models, the predictor-corrector (P-C) techniques are one of the most efficient, stable and accurate methods that was implemented and modified to numerically solve fractional differential equations, where the fractional derivative is considered in the Caputo sense. In [12], an Adams-type P-C approach has been introduced for the numerical solution of the IVP with Caputo fractional differential equation

$$ {}^{C}D^{\alpha}_{a+}y(t) = f(t, y(t)), \qquad a<t\le T, \tag{9} $$

where ${}^{C}D^{\alpha}_{a+}$ is the Caputo differential operator of order $\alpha>0$, given in Eq. (3). This approach has been widely used to simulate many fractional derivative models in the literature, such as the work presented in [3,9,11,13,16,18,21,28,29,36]. In recent years, some applications including physical models with the generalized fractional operators, introduced in Eqs. (5), (6) and (7), have been studied, see for example [1,6,7,14,15,23,45]. The generalized fractional integral operator, given in Definition 1, is greatly influenced by the value of the parameters α and ρ, thus it provides a valuable tool to control and build mathematical models in fractional calculus applications. However, the presented generalized fractional derivative operators, such as those mentioned in (6) and (7), do not satisfy a generalization to the rule (4) which is very useful in solving fractional differential equations. Therefore, the main aim of the preset work is to introduce a new generalized Caputo-type fractional derivative operator whose properties are somewhat similar to the Caputo derivative operator properties. Then, to extend the P-C methods to numerically simulate IVP with the suggested generalized Caputo-type fractional derivatives. Some comments regarding the convergence and the stability of the proposed algorithm are discussed. Furthermore, test problems are examined to show the performance, efficiency and features of the proposed algorithm.

2. New generalized Caputo-type fractional derivative

This section introduces a new generalized Caputo-type fractional derivative which can be considered as a modification of the generalized Caputo-type fractional derivative given in (7). We briefly discuss the mathematical formulation approach then we present some characteristics and properties of the new generalized fractional derivative. Firstly, if we perform the change of variables $u = s^\rho/\rho$ in the generalized integral formula (5) we obtain the following result.

Theorem 1. *Let $\alpha > 0$ and $\rho > 0$. Then, for $t > a \geq 0$,*

$$I_{a+}^{\alpha,\rho} f(t) = \left[I_{a^\rho/\rho}^{\alpha}(f \circ g)\right](t^\rho/\rho), \tag{10}$$

where $g(t) = (\rho t)^{1/\rho}$ and I_a^α is the Riemann-Liouville fractional integral operator of order $\alpha > 0$ given in (1).

Now, it is important to note that the generalized Riemann-type fractional derivative, given in (6), can be obtained by replacing the operator $I_{a^\rho/\rho}^{\alpha}$ in formula (10) by the composite operator $D^m\, I_{a^\rho/\rho}^{m-\alpha}$. To derive our generalized fractional derivative operator, $\mathcal{D}_{a+}^{\alpha,\rho}$, replace the operator $I_{a^\rho/\rho}^{\alpha}$ in formula (10) by the composite operator $I_{a^\rho/\rho}^{m-\alpha}\, D^m$. In this case, we get

$$\begin{aligned}\left(\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) &= \left[I_{a^\rho/\rho}^{m-\alpha}\, D^m (f \circ g)\right](t^\rho/\rho), \\ &= \tfrac{1}{\Gamma(m-\alpha)}\left[\int_{a^\rho/\rho}^{t} (t-s)^{m-\alpha-1}\, D^m\, f(g(s))\, ds\right](t^\rho/\rho),\end{aligned} \tag{11}$$

which, using simple calculations, suggests the following definition

Definition 4. The new generalized Caputo-type fractional derivative operator, $\mathcal{D}_{a+}^{\alpha,\rho}$, of order $\alpha > 0$ is defined (provided it exists) by

$$\left(\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{m-\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right)^m f(s)\, ds, \quad t > a, \tag{12}$$

where $\rho > 0$, $a \geq 0$, and $m-1 < \alpha < m$.

If $f(t) \in C^1[a,b]$ and $0 < \alpha \leq 1$, our generalized Caputo-type fractional derivative definition is consistent over $[a,b]$ with that of generalized Caputo-type fractional derivative given in (7). Consequently, if $f(t) \in C^1[a,b]$ and $0 < \alpha < 1$, then our generalized Caputo-type fractional derivative is reduced to formula (8). In particular, if $m-1 < \alpha \leq m$, $\upsilon > m-1$ and $\upsilon \notin \mathbb{N}$, we can show that

$$\mathcal{D}_{a+}^{\alpha,\rho}\left(t^\rho - a^\rho\right)^\upsilon = \rho^\alpha \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-\alpha+1)}\left(t^\rho - a^\rho\right)^{\upsilon-\alpha}. \tag{13}$$

Now, we turn our concern to provide an alternative definition of our generalized fractional derivative operator, given in Definition 4, under the condition that the first m derivatives of the function f are assumed to be all exist and are continuous on $[a,b]$ (i.e. $f \in C^m[a,b]$).

Theorem 2. *Let $m-1 < \alpha \leq m$, $a \geq 0$, $\rho > 0$ and $f \in C^m[a,b]$. Then, for $a < t \leq b$,*

$$\left(\mathcal{D}_{a+}^{\alpha,\rho} f\right)(t) = {}^R D_{a+}^{\alpha,\rho}\left\{f(t) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(t^\rho - a^\rho\right)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}. \tag{14}$$

Proof. Using Definition 2 and integration by parts, we get

$$
\begin{aligned}
&{}^{R}D_{a+}^{\alpha,\rho}\left\{f(t)-\sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(t^\rho-a^\rho\right)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}\\
&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{m-\alpha-1}\left\{f(s)-\sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(s^\rho-a^\rho\right)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}ds,\\
&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m\int_a^t \frac{(t^\rho-s^\rho)^{m-\alpha}}{m-\alpha}\left\{f'(s)-s^{\rho-1}\sum_{n=1}^{m-1}\frac{1}{\rho^{n-1}(n-1)!}\left(s^\rho-a^\rho\right)^{n-1}\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}ds,\\
&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^{m-1}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{m-\alpha-1}\left\{\left(s^{1-\rho}\frac{d}{ds}\right)f(s)\right.\\
&\quad\left.-\sum_{n=1}^{m-1}\frac{1}{\rho^{n-1}(n-1)!}\left(s^\rho-a^\rho\right)^{n-1}\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}ds.
\end{aligned}
\tag{15}
$$

Repeating the above iteration $m-1$ times yields

$$
\begin{aligned}
&{}^{R}D_{a+}^{\alpha,\rho}\left\{f(t)-\sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(t^\rho-a^\rho\right)^n\left[\left(x^{\rho-1}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}\right\}\\
&=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{m-\alpha-1}\left\{\left(s^{1-\rho}\frac{d}{ds}\right)^m f(s)\right\}ds,
\end{aligned}
\tag{16}
$$

and so, the equality given in (14) is proved. This completes the proof.

Next, the following result introduces the relation between the generalized fractional integral given in (5) and our generalized Caputo-type fractional derivative

Theorem 3. *Let $m-1<\alpha\le m$, $a\ge 0$, $\rho>0$ and $f\in C^m[a,b]$. Then, for $a<t\le b$,*

$$
I_{a+}^{\alpha,\rho}\,\mathcal{D}_{a+}^{\alpha,\rho}f(t)=f(t)-\sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(t^\rho-a^\rho\right)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}.
\tag{17}
$$

Proof. Following Fubini's theorem, from Definitions 1 and 4, we obtain

$$
\begin{aligned}
I_{a+}^{\alpha,\rho}\,\mathcal{D}_{a+}^{\alpha,\rho}f(t)&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}\mathcal{D}_{a+}^{\alpha,\rho}f(s)\,ds\\
&=\frac{\rho^{2-m}}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}\int_a^s z^{\rho-1}\left(s^\rho-z^\rho\right)^{m-\alpha-1}\left(z^{1-\rho}\frac{d}{dz}\right)^m f(z)\,dz\,ds,\\
&=\frac{\rho^{2-m}}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_a^t z^{\rho-1}\left(z^{1-\rho}\frac{d}{dz}\right)^m f(z)\int_z^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}\left(s^\rho-z^\rho\right)^{m-\alpha-1}ds\,dz.
\end{aligned}
\tag{18}
$$

Making the substitution $s^\rho=z^\rho+\left(t^\rho-z^\rho\right)r$, using the beta function definition, the inner integral can be evaluated as

$$
\begin{aligned}
\int_z^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}\left(s^\rho-z^\rho\right)^{m-\alpha-1}ds&=\frac{1}{\rho}\left(t^\rho-z^\rho\right)^{m-1}\int_0^1(1-r)^{\alpha-1}r^{m-\alpha-1}\,dr\\
&=\frac{1}{\rho}\left(t^\rho-z^\rho\right)^{m-1}\frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}.
\end{aligned}
\tag{19}
$$

Substituting (19) into (18), we obtain

$$
I_{a+}^{\alpha,\rho}\,\mathcal{D}_{a+}^{\alpha,\rho}f(t)=\frac{\rho^{1-m}}{(m-1)!}\int_a^t z^{\rho-1}\left(t^\rho-z^\rho\right)^{m-1}\left(z^{1-\rho}\frac{d}{dz}\right)^m f(z)\,dz.
\tag{20}
$$

Integrating (20) by parts, $m-1$ times, yields

$$
\begin{aligned}
I_{a+}^{\alpha,\rho}\,\mathcal{D}_{a+}^{\alpha,\rho}f(t)&=\frac{\rho^{2-m}}{(m-2)!}\int_a^t z^{\rho-1}\left(t^\rho-z^\rho\right)^{m-2}\left(z^{1-\rho}\frac{d}{dz}\right)^{m-1}f(z)\,dz\\
&\quad+\frac{\rho^{1-m}}{(m-1)!}\left(t^\rho-a^\rho\right)^{m-1}\left[\left(x^{1-\rho}\frac{d}{dx}\right)^{m-1}f(x)\right]\Big|_{x=a},\\
&=\cdots\\
&=\int_a^t f'(z)\,dz-\sum_{n=1}^{m-1}\frac{1}{\rho^n n!}\left(t^\rho-a^\rho\right)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\right]\Big|_{x=a}.
\end{aligned}
\tag{21}
$$

Implementing the fact that $\int_a^t f'(z)\,dz = f(t) - f(a)$, formula (17) is, therefore, obtained. This completes the proof.

The final problem that we consider here is to investigate that our generalized Caputo-type fractional derivative operator $\mathcal{D}_{a+}^{\alpha,\rho}$ is a left inverse to the generalized fractional integral operator $I_{a+}^{\alpha,\rho}$.

Theorem 4. *Let $m-1<\alpha\le m$, $a\ge 0$, $\rho>0$ and $f\in C[a,b]$. Then, for $a<t\le b$,*

$$\mathcal{D}_{a+}^{\alpha,\rho}\, I_{a+}^{\alpha,\rho} f(t) = f(t). \tag{22}$$

Proof. Following the conceptual relation (11), we get

$$\left(\mathcal{D}_{a+}^{\alpha,\rho}\, I_{a+}^{\alpha,\rho} f\right)(t) = \left[I_{a^\rho/\rho}^{m-\alpha} D^m\left(\left(I_{a+}^{\alpha,\rho} f\right)\circ g\right)\right](t^\rho/\rho), \tag{23}$$

where $g(t) = (\rho t)^{1/\rho}$. Note that, using the relation (10), we have $\left(I_{a+}^{\alpha,\rho} f\right)\circ g = I_{a^\rho/\rho}^{\alpha}\left(f\circ g\right)$. Hence

$$\begin{aligned}\left(\mathcal{D}_{a+}^{\alpha,\rho}\, I_{a+}^{\alpha,\rho} f\right)(t) &= \left[I_{a^\rho/\rho}^{m-\alpha} D^m I_{a^\rho/\rho}^{\alpha}\left(f\circ g\right)\right](t^\rho/\rho),\\ &= \left[I_{a^\rho/\rho}^{m-\alpha} D\, I_{a^\rho/\rho}^{\alpha-m+1}\left(f\circ g\right)\right](t^\rho/\rho),\\ &= \left[{}^C D_{a^\rho/\rho}^{\alpha-m+1}\, I_{a^\rho/\rho}^{\alpha-m+1}\left(f\circ g\right)\right](t^\rho/\rho).\end{aligned} \tag{24}$$

Since $f\in C[a,b]$, we obtain ${}^C D_{a^\rho/\rho}^{\alpha-m+1}\, I_{a^\rho/\rho}^{\alpha-m+1}\left(f\circ g\right) = f\circ g$ (see [26]), and so $\left[{}^C D_{a^\rho/\rho}^{\alpha-m+1}\, I_{a^\rho/\rho}^{\alpha-m+1}\left(f\circ g\right)\right](t^\rho/\rho) = f(t)$. This completes the proof.

In view of the results given in Theorems 2, 3 and 4, we can see that our generalized fractional derivative has the same properties as the Caputo derivative given in (3). Thus, we can conclude that our generalized Caputo-type fractional derivative identification seems closer to ordinary derivatives than the generalized Riemann-type fractional derivative and the generalized Caputo-type fractional derivative representations given in (6) and (7), respectively. Therefore, in the next section, we consider IVPs with the proposed type of generalized fractional derivatives.

3. The adaptive predictor-corrector algorithm

In this section, we propose an adaptive algorithm of the P-C methods, namely the adaptive P-C algorithm, to effectively provide numerical solutions for IVPs with the proposed generalized Caputo-type fractional derivatives. The developed algorithm is an extension of the P-C approach, introduced in [12], that is used to solve IVPs with Caputo derivatives. For this purpose, consider the IVP governed by

$$\begin{cases} \mathcal{D}_{a+}^{\alpha,\rho} y(t) &= f(t, y(t)), \qquad t\in[0,T],\\ y^{(k)}(a) &= y_0^k, \qquad k = 0,1,\cdots,\lceil\alpha\rceil,\end{cases} \tag{25}$$

where $\mathcal{D}_{a+}^{\alpha,\rho}$ is the proposed generalized Caputo-type fractional derivative operator given in Definition 4. Initially, for $m-1<\alpha\le m$, $a\ge 0$, $\rho>0$ and $y\in C^m([a,T])$, the IVP (25) is equivalent, using Theorem 3, to the Volterra integral equation

$$y(t) = u(t) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} f(s, y(s))\,ds, \tag{26}$$

where

$$u(t) = \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)^n\left[\left(x^{1-\rho}\frac{d}{dx}\right)^n y(x)\right]\Big|_{x=a}. \tag{27}$$

The first step of our algorithm, under the assumption that the function f to be such that a unique solution exists on some interval $[a,T]$, consists of dividing the interval $[a,T]$ into N unequal subintervals $\{[t_k, t_{k+1}],\ k=0,1,\cdots N-1\}$ using the mesh points

$$\begin{cases} t_0 &= a,\\ t_{k+1} &= (t_k^\rho + h)^{1/\rho}, \qquad k = 0,1,\cdots,N-1,\end{cases} \tag{28}$$

where $h=\frac{T^{\rho}-a^{\rho}}{N}$. Now, we are going to generate the approximations y_k, $k=0,1,\cdots,N$, to solve numerically the IVP (25). The basic step, assuming that we have already evaluated the approximations $y_j\approx y(t_j)$ $(j=1,2,\cdots,k)$, is that we want to get the approximation $y_{k+1}\approx y(t_{k+1})$ by means of the integral equation

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{t_{k+1}} s^{\rho-1}(t_{k+1}^{\rho}-s^{\rho})^{\alpha-1}f(s,y(s))\,ds. \tag{29}$$

Making the substitution

$$z=s^{\rho}, \tag{30}$$

we get

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{-\alpha}}{\Gamma(\alpha)}\int_{a^{\rho}}^{t_{k+1}^{\rho}} (t_{k+1}^{\rho}-z)^{\alpha-1}f(z^{1/\rho},y(z^{1/\rho}))\,dz. \tag{31}$$

That is

$$y(t_{k+1})=u(t_{k+1})+\frac{\rho^{-\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j^{\rho}}^{t_{j+1}^{\rho}} (t_{k+1}^{\rho}-z)^{\alpha-1}f(z^{1/\rho},y(z^{1/\rho}))\,dz. \tag{32}$$

Next, if we use the trapezoidal quadrature rule with respect to the weight function $(t_{k+1}^{\rho}-.)^{\alpha-1}$ to approximate the integrals appear in the right-hand side of Eq. (32), replacing the function $f(z^{1/\rho},y(z^{1/\rho}))$ by its piecewise linear interpolant with nodes chosen at the t_j^{ρ} $(j=0,1,\cdots k+1)$, then we obtain

$$\begin{aligned}\textstyle\int_{t_j^{\rho}}^{t_{j+1}^{\rho}}(t_{k+1}^{\rho}-z)^{\alpha-1}f(z^{1/\rho},y(z^{1/\rho}))\,dz \;\approx\; &\tfrac{h^{\alpha}}{\alpha(\alpha+1)}\Big\{\Big((k-j)^{\alpha+1}-(k-j-\alpha)(k-j+1)^{\alpha}\Big)f(t_j,y(t_j))\\ &+\Big((k-j+1)^{\alpha+1}-(k-j+\alpha+1)(k-j)^{\alpha}\Big)f(t_{j+1},y(t_{j+1}))\Big\}.\end{aligned} \tag{33}$$

Thus, substituting the above approximations in to Eq. (32), we obtain the corrector formula for $y(t_{k+1})$, $k=0,1,\cdots,N-1$,

$$y(t_{k+1})\approx u(t_{k+1})+\frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k}a_{j,k+1}f(t_j,y(t_j))+\frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}f(t_{k+1},y(t_{k+1})), \tag{34}$$

where

$$a_{j,k+1}=\begin{cases}k^{\alpha+1}-(k-\alpha)(k+1)^{\alpha} & if\;\; j=0,\\ (k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}-2(k-j+1)^{\alpha+1} & if\;\; 1\le j<k.\end{cases} \tag{35}$$

The last step of our algorithm is to replace the quantity $y(t_{k+1})$ shown on the right hand side of the formula (34) with the predictor value $y^P(t_{k+1})$ that can be obtained by applying the one-step Adams-Bashforth method to the integral equation (31). In this case, by replacing the function $f(z^{1/\rho},y(z^{1/\rho}))$ by the quantity $f(t_j,y(t_j))$ at each integral in Eq. (32), we get

$$\begin{aligned} y^P(t_{k+1}) \;&\approx\; u(t_{k+1})+\tfrac{\rho^{-\alpha}}{\Gamma(\alpha)}\textstyle\sum_{j=0}^{k}\int_{t_j^{\rho}}^{t_{j+1}^{\rho}}(t_{k+1}^{\rho}-z)^{\alpha-1}f(t_j,y(t_j))\,dz\\ &=\; u(t_{k+1})+\tfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\textstyle\sum_{j=0}^{k}\big[(k+1-j)^{\alpha}-(k-j)^{\alpha}\big]f(t_j,y(t_j)).\end{aligned} \tag{36}$$

Therefore, our adaptive P-C algorithm, for evaluating the approximation $y_{k+1}\approx y(t_{k+1})$, is completely described by the formula

$$y_{k+1}\approx u(t_{k+1})+\frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{k}a_{j,k+1}f(t_j,y_j)+\frac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}f(t_{k+1},y_{k+1}^{p}), \tag{37}$$

where $y_j\approx y(t_j)$, $j=0,1,\cdots k$, and the predicted value $y_{k+1}^{P}\approx y^P(t_{k+1})$ can be determined as described in Eq. (36) with the weights $a_{j,k+1}$ being defined according to (35). It is easily observed that when $\rho=1$, our adaptive P-C algorithm will be reduced to the P-C approach presented in [12].

Remark 1. The proposed adaptive P-C algorithm uses a non uniform grid $\{t_{j+1} = (t_j^\rho + h)^\rho : j = 0, 1, \cdots, N-1\}$ with $t_0 = a$ and $h = \frac{T^\rho - a^\rho}{N}$, where N is some positive integer. We believe that the application of P-C methods to IVPs with the generalized Caputo-type derivatives becomes impossible if we use a uniform grid such as the case of [12].

Remark 2. For the purposes of implementation, the behavior of the adaptive P-C algorithm is independent of the parameters α and ρ, and it follows the same procedure as the classic Adams-Bashforth-Moulton method. However, we can see that the approximation y_{k+1} depends on the whole record $(y_0, y_1, \cdots, y_k)$ and this confirms the result that the non-local property of fractional derivatives leads to a much higher arithmetic effort.

Remark 3. From the results given in [32], we can observe that the stability properties of the adaptive P-C algorithm are closely related to those of the classic Adams-Bashforth-Moulton method and the P-C approach of [12]. So, our algorithm behaves in an acceptable manner with respect to its numerical stability.

Remark 4. By comparing our adaptive P-C formula, given in Eq. (37), with the P-C formula of [12], using the product integration methods presented in [30], we can expect that the error behaves as

$$\max_j |y(t_j) - y_j| = O((h/\rho)^p), \tag{38}$$

where $p = \min\{2, 1+\alpha\}$.

4. Test problems

In this section we investigate the efficiency of the adaptive P-C algorithm as a numerical solution tool for IVPs with the proposed generalized Caputo-type fractional derivatives. For this purpose, numerical simulations of some test problems are performed.

Example 1. Our first problem deals with the fractional Riccati equation

$$\mathcal{D}_0^{\alpha,\rho} y(t) = 2y(t) - y^2(t) + 1, \qquad t > 0, \quad 0 < \alpha \le 1, \tag{39}$$

subject to the initial condition $y(0) = 0$, where $\mathcal{D}_0^{\alpha,\rho}$ is the generalized Caputo-type fractional derivative operator, given in Definition 4, of parameters α and ρ. The exact solution of the Riccati Eq. (39), when $\alpha = 1$ and $\rho = 1$, is

$$y(t) = 1 + \sqrt{2}\tanh\Big[\sqrt{2}t + \frac{1}{2}log\Big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big)\Big]. \tag{40}$$

According to the proposed adaptive P-C algorithm, for some $T > 0$, the approximation y_{k+1} is determined by the formula

$$y_{k+1} \approx \frac{\rho^{-\alpha}h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1}\big(2y_j - y_j^2 + 1\big) + \frac{\rho^{-\alpha}h^\alpha}{\Gamma(\alpha+2)}\big(2y_{k+1}^P - (y_{k+1}^P)^2 + 1\big), \tag{41}$$

where $y_0 = 0$, $h = \frac{T^\rho}{N}$, for some $N \in \mathbb{N}$, and

$$y_{k+1}^P = \frac{\rho^{-\alpha}h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k} \big[(k+1-j)^\alpha - (k-j)^\alpha\big]\big(2y_j - y_j^2 + 1\big). \tag{42}$$

In Table 1, we display numerical solutions obtained using our adaptive P-C algorithm to fractional Riccati Eq. (39) when $\alpha = 1$ and $\rho = 1$ at $t = 1$, $t = 2$ and $t = 5$. From the data given in Table 1, we can conclude that the numerical solutions obtained using our algorithm are in high agreement with the exact solutions (y_{Exact}). Furthermore, the accuracy becomes better when the step size h gets small. Tables 2 and 3 display numerical solutions obtained using our adaptive P-C algorithm to fractional Riccati Eq. (39) at $t = 1$ and $t = 2$, respectively, for different values of α and ρ. From the convergence of the numerical results shown in Tables 2 and 3, we can observe the numerical stability feature of the adaptive P-C algorithm.

Figs. 1 and 2 show the dynamic behaviors of fractional Riccati Eq. (39) with respect to the parameters α and ρ against the time variable t. In Figs. 1 and 2, we plot numerical solutions to Eq. (39) with $T = 2.5$ obtained using our adaptive P-C algorithm when $N = 50$ and $N = 400$, respectively, for different values of α and ρ. It can be seen, using the graphical representations shown in Fig. 1, that our algorithm simulates problem (39) on non uniform grid sets whose step size depends on the parameter ρ. From Fig. 2, for fixed α, it can be observed that the solution $y(t)$ for the three different values of ρ converges to the same limit.

Table 1
Numerical solutions to fractional Riccati Eq. (39) when $\alpha = 1$ and $\rho = 1$.

h	$t = 1$	$t = 2$	$t = 5$
1/10	1.68745117	2.35530727	2.41419835
1/20	1.68896723	2.35721255	2.41420101
1/40	1.68936339	2.35763805	2.41420152
1/80	1.68946438	2.35773897	2.41420163
1/160	1.68948986	2.35776357	2.41420166
1/320	1.68949625	2.35776964	2.41420167
1/640	1.68949786	2.35777115	2.41420167
y_{Exact}	1.68949839	2.35777165	2.41420167

Table 2
Numerical solutions to fractional Riccati Eq. (39) when $t = 1$.

h	$\alpha = 1,\ \rho = 0.9$	$\alpha = 0.95,\ \rho = 0.75$	$\alpha = 0.9,\ \rho = 1.2$
1/10	1.84281224	2.06729863	1.52944766
1/20	1.84491385	2.07202706	1.53119172
1/40	1.84546411	2.07322261	1.53167452
1/80	1.84560424	2.07352741	1.53180584
1/160	1.84563955	2.07360571	1.53184129
1/320	1.84564841	2.07362592	1.53185082
1/640	1.84565063	2.07363115	1.53185339
1/1280	1.84565119	2.07363250	1.53185407

Table 3
Numerical solutions to fractional Riccati Eq. (39) when $t = 2$.

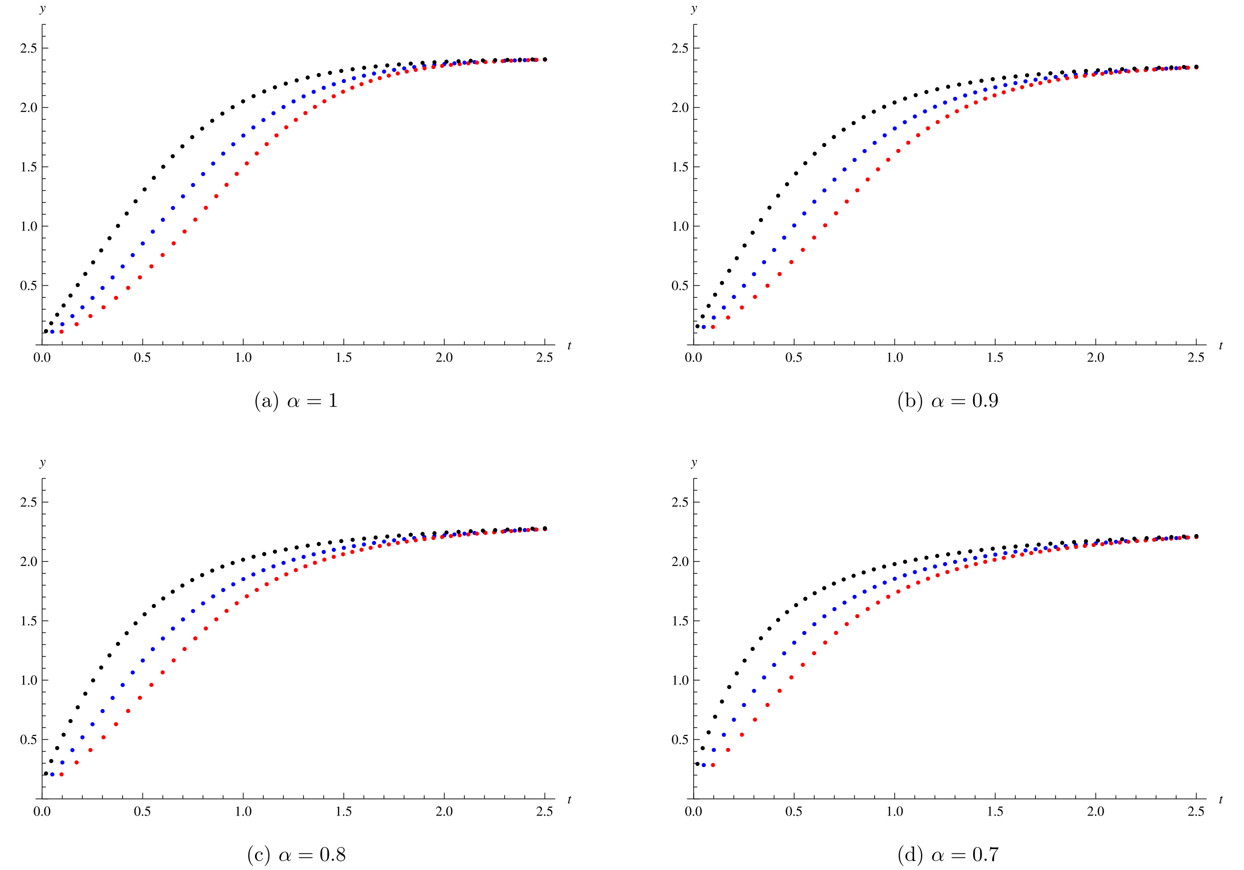
h	$\alpha = 1,\ \rho = 0.9$	$\alpha = 0.95,\ \rho = 0.75$	$\alpha = 0.9,\ \rho = 1.2$
1/10	2.36576348	2.34646084	2.26631061
1/20	2.36763874	2.34834846	2.26840179
1/40	2.36805246	2.34876916	2.26890814
1/80	2.36815011	2.34887135	2.26903810
1/160	2.36817385	2.34889710	2.26907235
1/320	2.36817971	2.34890369	2.26908148
1/640	2.36818116	2.34890540	2.26908393
1/1280	2.36818153	2.34890584	2.26908459

Example 2. Our second problem covers the fractional Chen system

$$\begin{cases} \mathcal{D}_0^{\alpha,\rho} x(t) = a(y(t) - x(t)), \\ \mathcal{D}_0^{\alpha,\rho} y(t) = (c-a)x(t) - x(t)z(t) + cy(t), \\ \mathcal{D}_0^{\alpha,\rho} z(t) = x(t)y(t) - bz(t), \end{cases} \tag{43}$$

subject to the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$, where $a, b, c \in \mathbb{R}$, $t > 0$, and $\mathcal{D}_0^{\alpha,\rho}$ is the generalized Caputo-type fractional derivative operator, given in Definition 4, of parameters α and ρ such that $0 < \alpha \leq 1$. It is found that Chen system (43), when $\alpha = 1$ and $\rho = 1$, displays chaotic attractors, for example when $(a, b, c) = (35, 3, 28)$ [10]. Also, if the generalized Caputo-type fractional derivative operator, $\mathcal{D}_0^{\alpha,\rho}$, is replaced by the Caputo fractional derivative operator, ${}^C D_0^{\alpha}$, in system (43) then it is shown in [37] that the fractional Chen system (43), when $\rho = 1$, exhibits chaotic behavior when $(a, b, c) = (35, 3, 28)$.

In view of our algorithm, following the rule (37), the approximations x_{k+1}, y_{k+1} and z_{k+1} can be simply evaluated using the iterative formulas, for $N \in \mathbb{N}$ and $T > 0$,



(a) $\alpha = 1$ (b) $\alpha = 0.9$

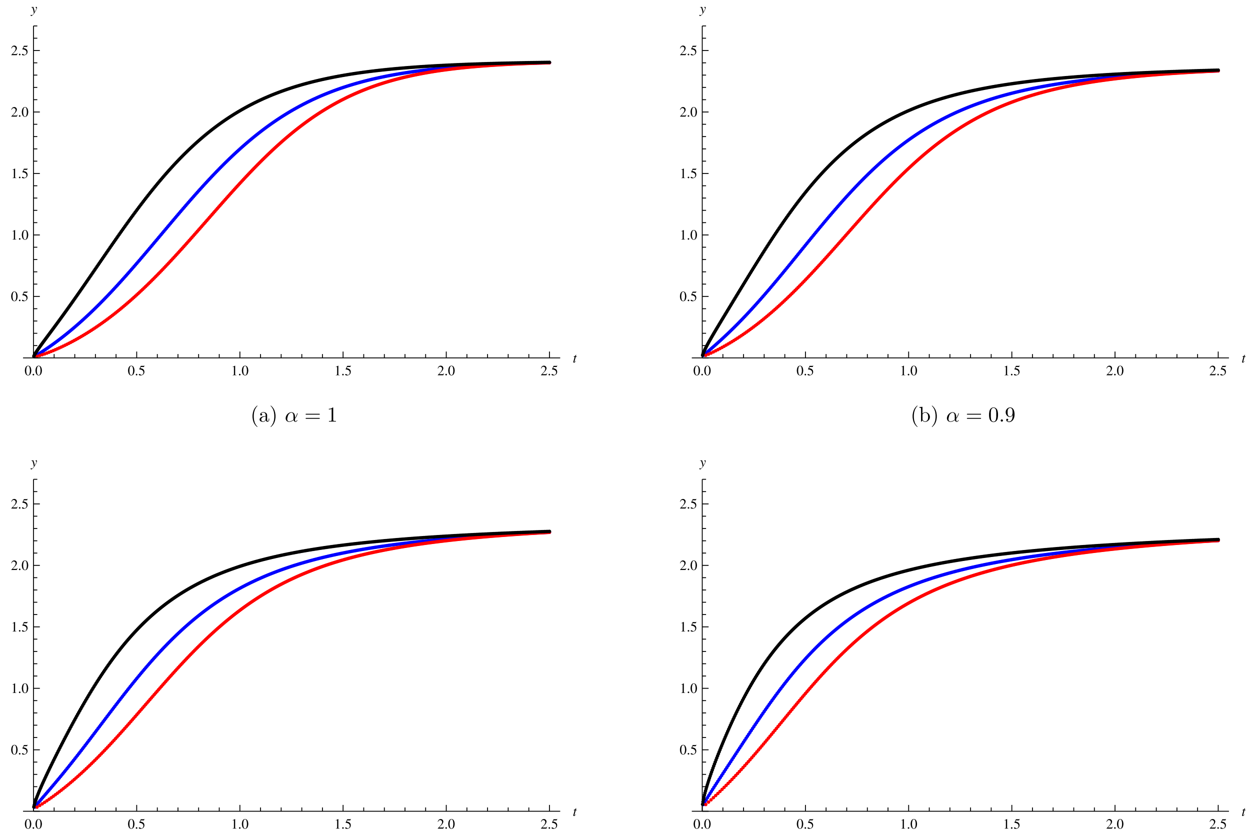
(c) $\alpha = 0.8$ (d) $\alpha = 0.7$

Fig. 1. Plots of numerical solutions for the fractional Riccati Eq. (39), when $T = 2.5$ and $N = 50$: (*Black*) $\rho = 0.8$; (*Blue*) $\rho = 1$; (*Red*) $\rho = 1.2$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\begin{cases} x_{k+1} \approx x_0 + a\dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{k} a_{j,k+1}\big(y_j - x_j\big) + a\dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\big(y^P_{k+1} - x^P_{k+1}\big), \\[2ex] y_{k+1} \approx y_0 + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{k} a_{j,k+1}\big((c-a)x_j - x_j z_j + c y_j\big) + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\big((c-a)x^P_{k+1} - x^P_{k+1}z^P_{k+1} + c y^P_{k+1}\big), \\[2ex] z_{k+1} \approx z_0 + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{k} a_{j,k+1}\big(x_j y_j - b z_j\big) + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+2)}\big(x^P_{k+1} y^P_{k+1} - b z^P_{k+1}\big), \end{cases} \tag{44}$$

where $h = \frac{T^{\rho}}{N}$ and

$$\begin{cases} x^P_{k+1} \approx x_0 + a\dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\displaystyle\sum_{j=0}^{k} \big[(k+1-j)^{\alpha} - (k-j)^{\alpha}\big]\big(y_j - x_j\big), \\[2ex] y^P_{k+1} \approx y_0 + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\displaystyle\sum_{j=0}^{k} \big[(k+1-j)^{\alpha} - (k-j)^{\alpha}\big]\big((c-a)x_j - x_j z_j + c y_j\big), \\[2ex] z^P_{k+1} \approx z_0 + \dfrac{\rho^{-\alpha}h^{\alpha}}{\Gamma(\alpha+1)}\displaystyle\sum_{j=0}^{k} \big[(k+1-j)^{\alpha} - (k-j)^{\alpha}\big]\big(x_j y_j - b z_j\big). \end{cases} \tag{45}$$



(a) $\alpha = 1$ (b) $\alpha = 0.9$ (c) $\alpha = 0.8$ (d) $\alpha = 0.7$

Fig. 2. Plots of numerical solutions for the fractional Riccati Eq. (39), when $T = 2.5$ and $N = 400$: (*Black*) $\rho = 0.8$; (*Blue*) $\rho = 1$; (*Red*) $\rho = 1.2$.

Table 4
Numerical solutions to fractional Chen system (43) when $\alpha = 1$, $\rho = 1$, $(a, b, c) = (53, 3, 28)$ and $(x_0, y_0, z_0) = (2.5, 1, 0.5)$.

	$t = 2$			$t = 5$		
h	x	y	z	x	y	z
1/10	0.76547321	0.50805837	0.24436125	1.87691249	1.90833765	1.76232887
1/20	0.76301723	0.50206750	0.24290222	1.87411505	1.93108732	1.75629505
1/40	0.76241447	0.50063345	0.24253181	1.87336221	1.93662856	1.75468966
1/80	0.76226520	0.50028279	0.24243886	1.87317222	1.93800680	1.75428450
1/160	0.76222806	0.50019611	0.24241560	1.87312481	1.93835118	1.75418327
1/320	0.76221880	0.50017456	0.24240979	1.87311299	1.93843729	1.75415800
1/640	0.76221649	0.50016919	0.24240833	1.87311004	1.93845883	1.75415169
$RK4$	0.76221572	0.50016739	0.24240783	1.87310902	1.93846581	1.75414950

Table 4 displays numerical solutions to fractional Chen system (43) when $\alpha = 1$, $\rho = 1$, $(a, b, c) = (1, 2, 1)$ and $(x_0, y_0, z_0) = (2.5, 1, 0.5)$ using the adaptive P-C algorithm and the RK4 method at $t = 2$ and $t = 5$. From the numerical results shown in Table 4, when the step size h becomes too small, we can observe that the numerical solutions obtained using our algorithm are in high agreement with those obtained using the RK4 method. Figs. 3 and 4 show the dynamic behaviors of fractional Chen system (43) when $(a, b, c) = (35, 3, 28)$ and $(x_0, y_0, z_0) = (1, 0.5, 2)$. In these figures we display the projections of fractional Chen system (43) attractors obtained using adaptive P-C algorithm when $T = 20$ and $N = 3000$ for some values of the parameters α and ρ. It can be observed, from Figs. 3 and 4, that fractional Chen system (43), where $(a, b, c) = (35, 3, 28)$, may exhibit chaotic attractor similar to its integer order counterpart when $(\alpha, \rho) = (0.85, 0.8)$ and $(\alpha, \rho) = (0.9, 1.2)$.

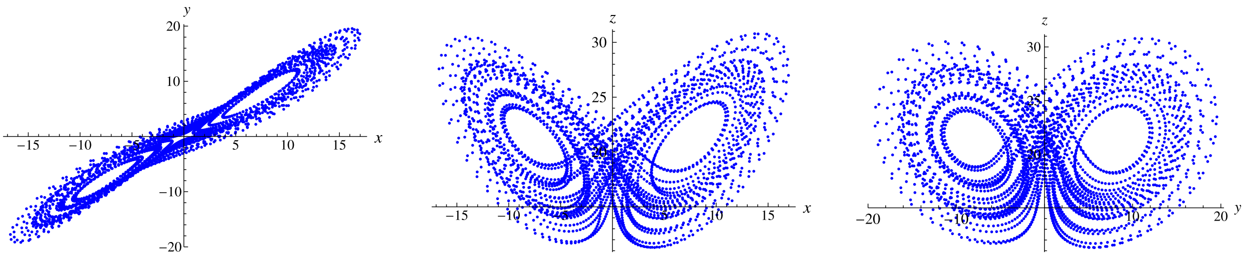


Fig. 3. Chaotic attractor of fractional Chen system (43), where $(a, b, c) = (35, 3, 28)$, when $\alpha = 0.85$ and $\rho = 0.8$.

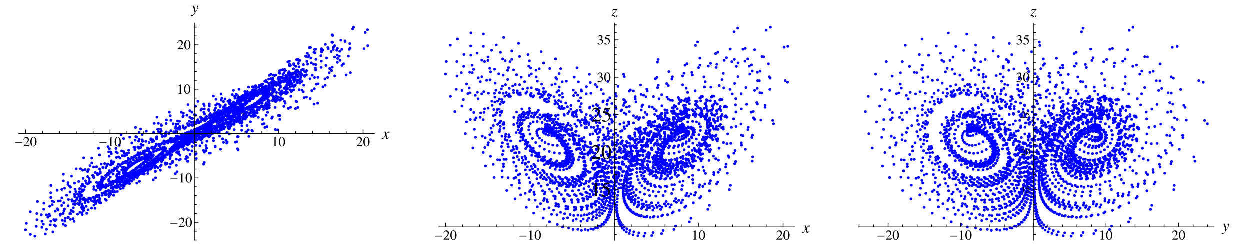


Fig. 4. Chaotic attractor of fractional Chen system (43), where $(a, b, c) = (35, 3, 28)$, when $\alpha = 0.9$ and $\rho = 1.2$.

5. Discussion and conclusions

There are two main objectives of the present work. Firstly, we introduce a generalized Caputo-type fractional derivative with properties similar to those of the Caputo derivative. Some useful features of the proposed generalized derivative operator were discussed, including the relationship between the generalized fractional integral operator and our generalized fractional derivative operator. This type of generalized derivative seems closer to ordinary derivatives than other generalized derivatives. Secondly, we present a new method, namely the adaptive P-C algorithm, for the numerical simulation of fractional order systems that include generalized Caputo-type fractional derivatives. The performed numerical simulations, in both test problems, show that the numerical results obtained using the presented algorithm are very close to the exact solutions or RK4 solutions in the integer order case when the step size h becomes too small. In the fractional case, the obtained numerical results confirm that our algorithm behaves satisfactorily regarding its numerical stability. The proposed algorithm was successfully implemented to obtain precise approximate solutions and demonstrate the dynamic behaviors of the discussed systems. Therefore, it is hoped that this study will serve as a promising tool for further implementation and investigation in the field of generalized Caputo fractional systems.

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