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NONLINEAR SINGULAR p -LAPLACIAN BOUNDARY VALUE PROBLEMS IN THE FRAME OF CONFORMABLE DERIVATIVE

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Abstract. This paper studies a class of fourth point singular boundary value problem of p-Laplacian operator in the setting of a specific kind of conformable derivatives. By using the upper and lower solutions method and fixed point theorems on cones., necessary and sufficient conditions for the existence of positive solutions are obtained. In addition, we investigate the dependence of the solution on the order of the conformable differential equation and on the initial conditions.

1. **Introduction.**

The usual calculus, dealing with differentiation and integration of integer orders, causes some obstacles for scientists who search for the best tools to model the real world problem they investigate. The derivatives and integrals of fractional order are reported to aid these secientists to better understand real world phenomena, For details, see [12, 18, 24, 26] and references therein.

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As it can be obviously in the literature the most well-known Riemann, Liouville, Hadamard, Caputo and Grunwald-Letnikov fractional operators with singular kernels have found plenty of applications in various fields of science and engineering because of the fact that they convey the memory and hereditary effect, see, for example [9, 10].

The fractional differential operators with non-singular kernels, introduced recently in the literature such as the Caputo-Fabrizio [7], the extended fractional Caputo-Fabrizio derivative and the Atangana-Baleanu derivatives [3] have been extensively investigated parallel to the classical fractional operators with singular kernels.

The local fractional derivatives can be candidates to be used to portray real world problems. The conformable derivative introduced in [17] and modified in [1] is considered to be a kind of local derivatives and has received the attention of many researchers since it was applied in many problems where the traditional derivative can not be used. See, for example, [13, 28].

The research on differential equations in the frame of non conventional derivatives is very important in both theory and applications. For fractional boundary value problems at resonance we refer to [33], for fractional multi-point problems with nonresonance [31, 32], [2, 4, 6, 8, 11] and to [5] for fractional problems with lower and upper solutions.

The theory of singular boundary value problems has become an important area of investigation in recent years [16, 22, 29, 30, 33]. The search for the existence of positive solutions and multiple positive solutions to nonlinear fractional boundary value problems with p-Laplacian operator by the use of techniques of nonlinear analysis, have been studied by several authors [4, 6, 11, 28, 34, 35]. For studying this type of problem, in [19], L. S. Leibenson introduced the p-Laplacian equation as follows

$$(\varphi_p (x' (t)))' = f (t, x (t), x' (t)), \tag{1}$$

where $\varphi_p (s) = |s|^{p-2} s,\ p > 1,\ s \in \mathbb{R}$.

The equations can be applied in various fields of science such as physics, electronics, mechanics, calculus of variations, control theory, etc.. In [34, 35], Zhang and Liu considered the following fourth-order four-point boundary value problem

$$(\varphi_p (x'' (t)))'' = f (t, x (t)),\ t \in (0, 1), \tag{2}$$

with the four-point boundary conditions

$$x (0) = 0,\ x (1) = b_1 x (\xi_1),\ x'' (0) = 0,\ x'' (1) = b_2 x'' (\xi_2). \tag{3}$$

As the intensive development of fractional derivative, a natural generalization of the p-Laplacian differential equation is to replace the ordinary derivative by a fractional derivative to yield fractional p-Laplacian equation, which can be considered as a particular case of the generalization of the p-Laplacian differential equation.

In recent years, some results have been obtained under different assumptions on f [4, 5, 6], as for fractional boundary value problems, in [29], J. Wang and H. Xiang have investigated the following the fractional boundary value problem

$$\left(D_t^{\beta}\right) (\varphi_p (D_t^{\alpha} x)) (t) = f (t, x (t)) \text{ in } (0, 1), \tag{4}$$

and boundary conditions

$$x (0) = 0,\ x (1) = b_1 x (\xi_1),\ D_t^{\alpha} x (0) = 0,\ D_t^{\alpha} x (1) = b_2 D_t^{\alpha} x (\xi_2), \tag{5}$$

where $D_t^{\alpha},\ D_t^{\beta}$ are Riemann-Liouville fractional operators with $1<\alpha,\beta\leq 2$, see also [25].

Motivated by the above-mentioned works, we investigate the following boundary value problems of conformable nonlinear differential equations with p-Laplacian operator and a nonlinear term dependent on the local fractional derivative of the unknown function

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)\right)(t)=f\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)\right),\ t\in(0,1),\tag{6}$$

with the four-point boundary conditions

$$x\left(0\right)=0,\ x\left(1\right)=b_1x\left(\xi_1\right),\ \mathbf{T}_{0+}^{\alpha}x\left(0\right)=0,\ \mathbf{T}_{0+}^{\alpha}x\left(1\right)=b_2\mathbf{T}_{0+}^{\alpha}x\left(\xi_2\right),\tag{7}$$

where $\mathbf{T}_{0+}^{\beta}$ and $\mathbf{T}_{0+}^{\alpha}$ are the conformable derivatives with $1<\alpha,\beta\leq 2, 1<\alpha\leq\alpha+\beta-1,\ 0\leq b_1,b_2\leq 1,\ 0<\xi_1,\xi_2<1$.

In the special case $p=\alpha=\beta=2$ and $b_1=b_2=0$, the problem (6-7) becomes the two point boundary value problems of fourth order ordinary differential equation. When f is continuous, problem is nonsingular, the existence and uniqueness of positive solutions in this case have been studied by papers [2, 27]. The theorems we present include and extend some previous results.

The remainder of the paper is organized as follows: Firstly, we present some necessary definitions and Lemmas that are needed in the subsequent sections. In Section 3, we construct the Green functions for the homogeneous conformable boundary value corresponding to (6-7) and estimate the bounds for the Green functions. One of the difficulties here is that the corresponding Green's function is singular at $s=0$. By applying the upper and lower solutions method associated with the Krasnosel'skii's fixed point theorem in a cone, the existence of at least one positive solution are established is dealt with in section 4. Furthermore, example is presented to illustrate the main results. In the final section of the paper, we look at the question as to how the solution x varies when we change the order of the conformable differential operator or the initial values and the dependence on parameters of nonlinear term f is also established. Our approach is new and the current results are totally different from the ones obtained in literature.

2. **Preliminary and some lemmas.** The left-sided conformable derivative of order $\alpha\in(0,1]$ is given by [17]

$$T_{0+}^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},\quad T_{0+}^{\alpha}f(0)=\lim_{t\to 0^+}T_{0+}^{\alpha}f(t).\tag{8}$$

The properties of $\left(T_{0+}^{\alpha}f\right)$ can be found in [1, 17].

Definition 2.1. Let $\alpha\in(n,n+1]$ and f be a n -differentiable function at $t>0$, then the left sided conformable derivative of order α at $t>0$ is given by

$$\left(\mathbf{T}_{0+}^{\alpha}f\right)(t)=\left(T^{\alpha-n}\right)f^{(n)}\left(t\right)=\lim_{\delta\to 0}[f^{(n)}(t+\delta t^{n+1-\alpha})-f^{(n)}(t)]/\left(\delta t^{n+1-\alpha}\right).\tag{9}$$

Lemma 2.2. *Let $t>0,\ \alpha\in(n,n+1]$. The function f is $(n+1)$-differentiable if and only if f is α-differentiable, moreover, $\left(\mathbf{T}_{0+}^{\alpha}f\right)(t)=t^{n+1-\alpha}f^{(n+1)}\left(t\right)$*

Remark 1. As a basic example, given $\alpha\in(n,n+1]$, we have, $\mathbf{T}_{0+}^{\alpha}\left(t^k\right)=0$ where $k=0,1,...,n$.

Definition 2.3. Let $\alpha \in (n, n+1]$. The left sided conformable integral of order α at $t > 0$ of a function $f \in C((0, +\infty), \mathbb{R})$ is given by

$$\mathbf{I}_{0+}^{\alpha} f(t) = J_{0+}^{n+1}\left(t^{\alpha-n-1} f(t)\right) = \frac{1}{n!}\int_0^t (t-s)^n s^{\alpha-n-1} f(s)\,\mathrm{d}s, \tag{10}$$

when $f^{(n)}(t)$ exists.

The following lemma plays a fundamental role in obtaining an equivalent integral representation to the boundary value problem (6-7).

Lemma 2.4. *Let $\alpha \in (n, n+1]$. If $x \in C(0,1]$ and $\mathbf{T}_{0+}^{\alpha} x \in L^1[0,1]$, then*

$$\mathbf{I}_{0+}^{\alpha}\mathbf{T}_{0+}^{\alpha} x = x(t) + \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \text{ for } t \in (0,1], \tag{11}$$

where $c_k = \frac{x^{(k)}(0)}{k!}$ and n is the smallest integer greater than or equal to α ($n = [\alpha] + 1$).

Lemma 2.5. *Let $t_2 > t_1 \geq 0$ and $f : [t_1, t_2] \to \mathbb{R}$ be a function with the properties that*

(i) f is continuous on $[t_1, t_2]$

(ii) f is α-differentiable on (t_1, t_2) for some $\alpha \in (0,1)$. Then there exists $\tau \in (t_1, t_2)$ such that

$$\left(T_{0+}^{\alpha} f\right)(\tau) = \frac{f(t_2) - f(t_1)}{\frac{1}{\alpha}\left(t_2^{\alpha} - t_1^{\alpha}\right)}. \tag{12}$$

Lemma 2.6. *Let $\alpha \in (0,1]$ and f, g be α -differentiable at a point $t > 0$. Then*

(i) $\left(T_{0+}^{\alpha}\right)(r_1 f + r_2 g) = r_1\left(T_{0+}^{\alpha} f\right) + r_2\left(T_{0+}^{\alpha} g\right),\ r_1, r_2 \in \mathbb{R}$.

(ii) $\left(T_{0+}^{\alpha}\right)(r_1) = 0$ for all constant functions $f(t) = r_1$.

(iii) $\left(T_{0+}^{\alpha}\right)(fg) = g\left(T_{0+}^{\alpha} f\right) + f\left(T_{0+}^{\alpha} g\right)$.

It is well known that a powerful tool for proving existence results for nonlinear problems is the upper and lower solution method [5, 15]. Thus, we introduce the following definitions of a couple of the lower and upper solutions for fourth-order p-Laplacian conformable boundary value problem (6-7). Denote

$$E = \left\{x : x \in C^2([0,1]) \text{ and } \varphi_p\left(\mathbf{T}_{0+}^{\alpha} x\right) \in C^2([0,1])\right\}.$$

Definition 2.7. A function $\underline{x}$ is called a lower solution of the conformable boundary value problem (6-7), if $\underline{x}(t) \in E$ and $\underline{x}(t)$ satisfies

$$\begin{cases} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\underline{x}(t)\right)\right)(t) - f\left(t, \underline{x}(t), -\mathbf{T}_{0+}^{\alpha}\underline{x}(t)\right) \leq 0,\ t \in (0,1), \\ \underline{x}(0) \leq 0,\ \underline{x}(1) - b_1\underline{x}(\xi_1) \leq 0, \left(\mathbf{T}_{0+}^{\alpha}\right)\underline{x}(0) \geq 0,\ \left(\mathbf{T}_{0+}^{\alpha}\right)\underline{x}(1) - b_2\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{x}(\xi_2) \geq 0. \end{cases} \tag{13}$$

This condition is motivated by the fact that an lower solution for an equation of the type $\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{x}(t)\right)\right)(t) = f\left(t, \underline{x}(t), -\mathbf{T}_{0+}^{\alpha}\underline{x}(t)\right)$ can be obtained if one has an lower solution for another equation of the form $\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{x}(t)\right)\right)(t) = g\left(t, \underline{x}(t), -\mathbf{T}_{0+}^{\alpha}\underline{x}(t)\right)$ with $g\left(s, \underline{x}(s), -\mathbf{T}_{0+}^{\alpha}\underline{x}(s)\right) \leq f\left(s, \underline{x}(s), -\mathbf{T}_{0+}^{\alpha}\underline{x}(s)\right)$ for all s.

An upper solution $\overline{x}$ is defined by reversing inequalities in the previous definition.

The existence of a lower solution over the upper solution implies the existence of solutions lying between both functions.

Lemma 2.8 (Krasnosel'skii)**.** [14]*Let P be a positive cone in a real Banach space E. Let Ω_1, Ω_2 be bounded open balls of E centered at the origin with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to P$ is a completely continuous operator such that*

$$\|Ax\| \leq \|x\| \quad \forall x \in P \cap \partial\Omega_1 \quad \text{and} \quad \|Ax\| \geq \|x\| \quad \forall x \in P \cap \partial\Omega_2,$$

or

$$\|Ax\| \geq \|x\| \quad \text{for } P \cap \partial\Omega_1 \quad \text{and} \quad \|Ax\| \leq \|x\| \quad \text{for } P \cap \partial\Omega_2,$$

hold. Then A has a fixed point in $P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

For $x \in C[0,1]$, the corresponding norm is $\|x\|_0 = \max\{|x(t)| : t \in [0,1]\}$. And for $x \in C^2[0,1]$, the corresponding norm is

$$\|x\| = \max\left\{\|x\|_0, \left\|T_{0+}^2 x\right\|_0\right\}. \tag{14}$$

Definition 2.9. Let $\underline{x}(t), \overline{x}(t) \in E$, we say that $\underline{x}(t)$ is called a lower solution of operator A if $\underline{x}(t) \leq A\underline{x}(t)$,and $\overline{x}(t)$ is called an upper solution of operator A if $\overline{x}(t) \geq A\overline{x}(t)$.

Lemma 2.10. [32]*For any $t, \tau \in \mathbb{R}$ and $1 < p, q < \infty$, $1/p + 1/q = 1$, we have $\varphi_q = \varphi_p^{-1}$,*

(i) If $1 < q \leq 2$, then

$$|\varphi_q(t+\tau) - \varphi_q(\tau)| \leq 2^{2-q}|t|^{q-1}. \tag{15}$$

(ii) If $q > 2$, then

$$|\varphi_q(t+\tau) - \varphi_q(\tau)| \leq (q-1)(|t| + |\tau|)^{q-2}|t|. \tag{16}$$

Lemma 2.11. *If $a, b \geq 0$, $r > 0$ and λ, $\mu \in \mathbb{R}^+$ then*

$$(a+b)^r \leq \lambda^{r-1}a^r + \mu^{r-1}a^r, \ \lambda + \mu = 1. \tag{17}$$

Our assumptions on the nonlinearity f will be the following:

A: The function f fulfill a Lipschitz condition with respect to the second and third variables,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2|, \tag{18}$$

such that $L_1, L_2 > 0$, (f is locally Lipschitz in $(0,1) \times (0, +\infty) \times (-\infty, 0]$).

For convenience, we suppose that the following hypotheses are satisfied:

H: $f : ((0,1) \times (0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^+)$ is continuous. There exist constants λ_i, μ_i, $i = 1, 2$, $0 < \lambda_1 \leq \mu_1$, $0 \leq \lambda_2 \leq \mu_2 < p - 1, \lambda_1 + \lambda_2 > p - 1$, such that

$$\begin{array}{ll} \sigma^{\mu_1} f(t,x,y) \leq f(t,\sigma x,y) \leq \sigma^{\lambda_1} f(t,x,y) & \text{if} \quad 0 < \sigma \leq 1, \\ \sigma^{\mu_2} f(t,x,y) \leq f(t,x,\sigma y) \leq \sigma^{\lambda_2} f(t,x,y) & \text{if} \quad 0 < \sigma \leq 1. \end{array} \tag{19}$$

It can be easily seen that $f(t,x,y)$ is non-decreasing with respect to x, y, and (19) are equivalent to

$$\begin{array}{ll} \sigma^{\lambda_1} f(t,x,y) \leq f(t,\sigma x,y) \leq \sigma^{\mu_1} f(t,x,y) & \text{if} \quad \sigma \geq 1, \\ \sigma^{\lambda_2} f(t,x,y) \leq f(t,x,\sigma y) \leq \sigma^{\mu_2} f(t,x,y) & \text{if} \quad \sigma \geq 1. \end{array} \tag{20}$$

for $(t,x,y) \in (0,1) \times [0,+\infty) \times [0,+\infty)$. From (19-20), we have

$$f(t, x_1, y_1) \leq f(t, x_2, y_2) \ \text{ for } x_1 \leq x_2, \ t \in (0,1). \tag{21}$$

We say $x_1 \leq x_2$ if $x_1(t) \leq x_2(t)$ for $t \in [0,1]$.

Now we present the Green's function for boundary value problem of differential equation.

3. **Construction of Green's function of problem (6- 7).** In this section, we obtain Green's function corresponding to the differential equations (6) with $1 < \alpha, \beta \leq 2$ subject to four-point boundary conditions (7) and estimate bounds for Green's function that will be used to prove our main theorems.

To study the nonlinear problem (6-7), we first consider the associated linear problem and obtain its solution.

Lemma 3.1. *Suppose* $v(t) \geq 0, \alpha \in (1, 2]$. *The corresponding Green's function for the problem*

$$\begin{cases} \mathbf{T}^{\alpha}_{0+}x(t) + v(t) = 0, \\ x(0) = 0, \ x(1) = b_1 x(\xi_1), \end{cases} \quad t \in (0, 1), \tag{22}$$

is given by

$$G_\alpha(t, s) = \mathcal{G}_\alpha(t, s) + \frac{b_1 t}{1 - b_1 \xi_1} \mathcal{G}_\alpha(\xi_1, s), \tag{23}$$

where

$$\mathcal{G}_\alpha(t, s) = \begin{cases} (1-t)s^{\alpha-1} & \text{if} \quad 0 \leq s \leq t \leq 1, \\ t(1-s)s^{\alpha-2} & \text{if} \quad 0 \leq t \leq s \leq 1, \end{cases} \tag{24}$$

where we assume the parameters satisfy $0 \leq b_1 \leq 1$, $0 < \xi_1 < 1$.

Moreover, if v *is not identically* 0 *on* $(0, 1)$, *then* x *is concave, decreasing,* $x(t) > 0$ *and* $x(t) \geq \psi_1(t) \|x\|_0$ *for* $t \in [0, 1]$, *where* $\psi_1(t) = t\left(\frac{b_1(1-\xi_1)}{1-b_1\xi_1} t + 1\right)$.

Proof. We will show that

$$x(t) = \int_0^1 G_\alpha(t, s) v(s) \mathrm{d}s, \tag{25}$$

for G_α given by (23), is a unique solution to the linear boundary value problem (6-7).

By applying Lemma 2.4, we may reduce (6) to an equivalent integral equation

$$x(t) = -\mathbf{I}^\alpha v(t) + c_0 + c_1 t, \ c_0, c_1 \in \mathbb{R}. \tag{26}$$

From $x(0) = 0$ and (26), we have $c_0 = 0$. Consequently the general solution of (6) is

$$x(t) = -\mathbf{I}^\alpha_{0+} v(t) + c_1 t = -\int_0^t (t-s) s^{\alpha-2} v(s) \mathrm{d}s + c_1 t. \tag{27}$$

By (27), one has

$$x(1) = -\int_0^1 (1-s) s^{\alpha-2} v(s) \mathrm{d}s + c_1, \ x(\xi_1) = -\int_0^{\xi_1} (\xi_1 - s) s^{\alpha-2} v(s) \mathrm{d}s + c_1 \xi_1.$$

And from $x(1) = b_1 x(\xi_1)$, then we have

$$c_1 = \frac{1}{1 - b_1 \xi_1} \left[\int_0^1 (1-s) s^{\alpha-2} v(s) \mathrm{d}s - b_1 \int_0^{\xi_1} (\xi_1 - s) s^{\alpha-2} v(s) \mathrm{d}s \right].$$

So, the unique solution of problem (6-7) is

$$\begin{aligned}
x(t) &= -\int_0^t (t-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} + \frac{t}{1-b_1\xi_1}\Big[\int_0^1 (1-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} \\
&- b_1 \int_0^{\xi_1} (\xi_1 - s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}}\Big] \\
&= -\int_0^t (t-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} + \frac{1}{1-b_1\xi_1}\int_0^1 t(1-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} \\
&- b_1 \int_0^{\xi_1} (\xi_1 - s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} \\
&= -\int_0^t (t-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} \\
&+ \frac{1}{1-b_1\xi_1}\left(\int_0^t t(1-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}} + \int_t^1 t(1-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}}\right) \\
&+ \frac{b_1 t}{1-b_1\xi_1}\left[\int_0^{\xi_1} (\xi_1(1-s) - (\xi_1 - s))\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}}\right. \\
&+ \left.\int_{\xi_1}^1 \xi_1 (1-s)\, v(s) \frac{\mathrm{d}s}{s^{2-\alpha}}\right] \\
&= \int_0^t \frac{(1-t)\, v(s)\, \mathrm{d}s}{s^{1-\alpha}} + \int_t^1 \frac{t(1-s)\, v(s)\, \mathrm{d}s}{s^{2-\alpha}} \\
&+ \frac{b_1 t}{1-b_1\xi_1}\left[\int_0^{\xi_1} \frac{(1-\xi_1)\, v(s)\, \mathrm{d}s}{s^{1-\alpha}} + \int_{\xi_1}^1 \frac{\xi_1(1-s)\, v(s)\, \mathrm{d}s}{s^{2-\alpha}}\right] \\
&= \int_0^1 \left[\mathcal{G}_\alpha(t,s) + \frac{b_1 t}{1-b_1\xi_1}\mathcal{G}_\alpha(\xi_1, s)\right] v(s)\, \mathrm{d}s \\
&= \int_0^1 G_\alpha(t,s) v(s)\, \mathrm{d}s,
\end{aligned}$$

where $G_\alpha(t,s)$ is defined in (23).

For $t \in [0,1]$ we have $\mathcal{G}_\alpha(t,s) \geq 0$ and $G_\alpha(t,s) \geq 0$ for $s \in (0,1)$, hence, when v is not identically 0 on $[0,1]$, it follows that $x(t) > 0$.

Furthermore,

$$G_\alpha(t,s) = \begin{cases} bt(1-s)\, s^{\alpha-2} & \text{if } \ 0 \leq t \leq s \leq 1, \\ (ct+1)\, s^{\alpha-1} & \text{if } \ 0 \leq s \leq t \leq 1, \end{cases} \tag{28}$$

where

$$a = \frac{b_1}{1-b_1\xi_1},\ b = 1 + a\xi_1,\ c = a(1-\xi_1) - 1. \tag{29}$$

When $0 \leq s \leq t \leq 1$, we have

$$G_\alpha(t,s) = (ct+1)\, s^{\alpha-1} \geq (ct+1)(cs+1)\, s^{\alpha-1} = (ct+1)\, G_\alpha(s), \tag{30}$$

where $0 < (ct+1) < 1$. Now, when $0 \leq t \leq s \leq 1$ then

$$G_\alpha(t,s) = bt(1-s)\, s^{\alpha-2} \geq bts(1-s)\, s^{\alpha-2} = tG_\alpha(s),\ s \in (0,1). \tag{31}$$

From (30) and (31), we get

$$G_\alpha(t,s) \geq \psi_1(t)\, G_\alpha(s), \text{with } \psi_1(t) = t(ct+1), \tag{32}$$

and

$$x(t)=\int_0^1 G_\alpha(t,s)v(s)\mathrm{d}s\geq\psi_1(t)\int_0^1 G_\alpha(s)\,v(s)\mathrm{d}s\geq\psi_1(t)\,\|x\|_0\,,\tag{33}$$

where

$$\|x\|_0=\max|x(t):t\in[0,1]|\leq\int_0^1 G_\alpha(s)\,v(s)\mathrm{d}s,\tag{34}$$

with $G_\alpha(.)\equiv G_\alpha(.,.)$ as in (23. □

Lemma 3.2. *Suppose $u(t)\geq 0,\ 1<\alpha,\beta\leq 2, 0\leq b_1,b_2\leq 1$ and $0<\xi_1,\xi_2<1$. Then the unique solution of the following conformable boundary value problem with the p-Laplacian operator*

$$\begin{cases}\mathbf{T}^{\beta}_{0+}(\varphi_p\left(\mathbf{T}^{\alpha}_{0+}x\right)(t))+u(t)=0, \\ \mathbf{T}^{\alpha}_{0+}x\,(0)=0,\mathbf{T}^{\alpha}_{0+}x\,(1)=b_2\mathbf{T}^{\alpha}_{0+}x\,(\xi_2)\end{cases}\quad t\in(0,1)\,,\tag{35}$$

is given by

$$\varphi_p\left(\mathbf{T}^{\alpha}_{0+}x\right)(t)=\int_0^1 G_\beta\,(t,s)\,u\,(s)\,\mathrm{d}s,$$

where

$$G_\beta(t,s)=\mathcal{G}_\beta(t,s)+\frac{b_0t}{1-b_0\xi_2}\mathcal{G}_\beta(\xi_2,s)\tag{36}$$

and $b_0=b_2^{p-1}$ and $\mathcal{G}_\beta(t,s)$ is defined in (24) with α replaced by β.

Moreover, if u is not identically 0 on $(0,1)$, then x is concave, decreasing, $x(t)>0$ and $x(t)\geq\psi_2(t)\,\|x\|_0$ for $t\in[0,1]$, where $\psi_2(t)=t\left(\frac{b_0(1-\xi_2)}{1-b_0\xi_2}t+1\right)$.

Proof. By a similar argument in the proof of Lemma 3.2, we can get Lemma 3.1. □

Lemma 3.3. *Suppose $x(t)\geq 0,\ 1<\alpha,\beta\leq 2, 0\leq b_1,b_2\leq 1$ and $0<\xi_1,\xi_2<1$. Then the unique solution of the following conformable boundary value problem with the p-Laplacian operator (6-7) given by*

$$x(t)=\int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau,x\,(\tau)\,,-\mathbf{T}^{\alpha}_{0+}x\,(\tau))\mathrm{d}\tau\right)\mathrm{d}s,\tag{37}$$

where $G_\alpha(t,s)$ and $G_\beta(t,s)$ defined by (23) and (36) respectively.

Remark 2. From the expression of (25) and (37), we can see that if all the conditions in Lemmas 3.1, 3.2 and 3.3 are satisfied, the solution is a $C^2\,[0,1]$ solution of the boundary value problem (6-7). Furthermore, if we denote $-\varphi_p\left(\mathbf{T}^{\alpha}_{0+}x\right)(t)=u\,(t)$, there holds $u\geq\psi_1(t)\,\|u\|_0$ for $t\in[0,1]$, where $\psi_1(t)$ is defined in Lemma 3.1.

The properties of $\mathcal{G}_\alpha$ and G_α are collected in the following lemma.

Lemma 3.4. *Let $1<\alpha\leq 2, 0\leq b_1\leq 1$ and $0<\xi_1<1$. Then*
(i) *Let $\mathcal{G}_\alpha$ be as in (24). Then $\mathcal{G}_\alpha(0,s)=0=\mathcal{G}_\alpha(1,s)$ for $s\in[0,1]$.*

$$\mathcal{G}_\alpha(t,s)\leq\mathcal{G}_\alpha\,(s)\ \text{ or }\ \mathcal{G}_\alpha\,(t)\ \text{ for all } t,s\in[0,1]\tag{38}$$

and if $\delta\in\left(0,\frac{1}{2}\right)$ then

$$\min\left\{\mathcal{G}_\alpha(t,s):\delta\leq t\leq 1-\delta\right\}\geq\mathcal{G}_\alpha\,(1-\delta)\,\mathcal{G}_\alpha\,(s)\,,\tag{39}$$

with $\mathcal{G}_\alpha(.)\equiv\mathcal{G}_\alpha(.,.)$ is defined in (24).

(ii) *Function G_α defined by (23) is continuous on $[0,1]\times[0,1]$ satisfying*

(a) *For all $t,s\in[0,1]$, $G_\alpha(t,s)\geq 0$ and*

$$\phi_1(s)\,t\leq G_\alpha(t,s)\leq\phi_2(s)\,t, \tag{40}$$

where

$$\phi_1(s)=\frac{b_1\mathcal{G}_\alpha(\xi_1,s)}{1-b_1\xi_1}\quad\text{and}\quad\phi_2(s)=s^{\alpha-2}+\phi_1(s). \tag{41}$$

(b) *For all $t,s\in[0,1]$,*

$$G_\alpha(t)\,G_\alpha(s)\leq G_\alpha(t,s)\leq G_\alpha(s)\ \ \text{or}\ \ G_\alpha(t) \tag{42}$$

and

$$G_\alpha(t,s)\leq\left(1+\frac{b_1t}{1-b_1\xi_1}\right)\ \mathcal{G}_\alpha(s)\leq\left(\frac{1-b_1(1-\xi_1)}{1-b_1\xi_1}\right)\mathcal{G}_\alpha(s). \tag{43}$$

(c) *If $\delta\in\left(0,\frac{1}{2}\right)$ then*

$$\min\{G_\alpha(t,s):\delta\leq t\leq 1-\delta\}\geq G_\alpha(1-\delta)\,G_\alpha(s). \tag{44}$$

Proof. (**i**) From the expression of $\mathcal{G}_\alpha(t,s)$, it is clear that $\mathcal{G}_\alpha(t,s)>0$ for $t,s\in(0,1)$, with

$$\mathcal{G}_\alpha(t,s)=\begin{cases}\mathcal{G}^1_\alpha(t,s) & \text{if}\quad 0\leq s\leq t\leq 1,\\ \mathcal{G}^2_\alpha(t,s) & \text{if}\quad 0\leq t\leq s\leq 1.\end{cases} \tag{45}$$

Next, for given $s\in(0,1)$ we consider the partial derivative of $\mathcal{G}_\alpha(t,s)$ with respect to t,

$$\partial_t\mathcal{G}_\alpha(t,s)=\begin{cases}-s^{\alpha-1} & \text{if}\quad 0\leq s\leq t\leq 1,\\ (1-s)s^{\alpha-2} & \text{if}\quad 0\leq t\leq s\leq 1.\end{cases} \tag{46}$$

This shows that $\mathcal{G}_\alpha(t,s)$ is decreasing with respect to t for $s\leq t$, and increasing for $t\leq s$. So,

$$\mathcal{G}_\alpha(t,s)\leq\mathcal{G}_\alpha(s)\ \text{ or }\mathcal{G}_\alpha(t)\ \text{ for all }t,s\in(0,1)$$

and if $\delta\in\left(0,\frac{1}{2}\right)$ then it is easily to see that

$$\begin{aligned}&\min\{\mathcal{G}_\alpha(t,s):\delta\leq t\leq 1-\delta\}\\ &=\begin{cases}\mathcal{G}^1_\alpha(1-\delta,s), & \text{if}\quad s\in[0,\delta],\\ \min\left\{\mathcal{G}^1_\alpha(1-\delta,s),\mathcal{G}^2_\alpha(\delta,s)\right\}, & \text{if}\quad s\in[\delta,1-\delta],\\ \mathcal{G}^2_\alpha(\delta,s), & \text{if}\quad s\in[1-\delta,1],\end{cases}\end{aligned} \tag{47}$$

or

$$\min_{t\in[\delta,1-\delta]}\left\{\mathcal{G}_\alpha(t,s):\delta\in\left(0,\tfrac{1}{2}\right)\right\}=\begin{cases}\mathcal{G}^1_\alpha(1-\delta,s), & \text{if}\quad s\in[0,\theta],\\ \mathcal{G}^2_\alpha(\delta,s), & \text{if}\quad s\in[\theta,1],\end{cases} \tag{48}$$

with $\theta\in[\delta,1-\delta]$ is a solution of the equation $\mathcal{G}_\alpha(1-\delta,\theta)=0$.

Consequently,

$$\min\{\mathcal{G}_\alpha(t,s):\delta\leq t\leq 1-\delta\}\geq\mathcal{G}_\alpha(1-\delta)\,\mathcal{G}_\alpha(s).$$

(**ii**) It is easy to verify properties (**ii-a**).

(**ii-b$_1$**) If $\mathcal{G}_\alpha(t,s)=t(1-s)s^{\alpha-2}$ for all $t\in[0,s]$, then from (28) and (29), we have

$$G^1_\alpha(t,s)=bt\,(1-s)\,s^{\alpha-2},\ \ s\in(0,1). \tag{49}$$

Continuity of G^1_α clearly follows from the definition of G^1_α. We start by differentiation $G^1_\alpha(t,s)$ with respect to $s\in[t,1]$ for every fixed $t\in(0,1)$, we can get

$$\partial_sG^1_\alpha(t,s)=(1+a\xi_1)\,ts^{\alpha-3}\left((1-\alpha)\,s+(\alpha-2)\right).$$

This together with the fact that $G^1_\alpha(t,1)=0$ imply that $G^1_\alpha(t,s)<0$.

By fixing an arbitrary $s\in(0,1)$. Differentiating $G^1_\alpha(t,s)$ with respect to t, we get

$$\partial_t G^1_\alpha(t,s)=(1+a\xi_1)\,ts^{\alpha-3}\left((1-\alpha)\,s+(\alpha-2)\right)>0,\ G^1_\alpha(0,s)=0.$$

Hence, $G^1_\alpha(t,s)$ has a maximum at point $t=s$. Observe that

$$G^1_\alpha(t,s)\le G^1_\alpha(t,t) \text{ or } G^1_\alpha(t,s)\le G^1_\alpha(s,s) \tag{50}$$

and

$$G^1_\alpha(t,s)=bt\,(1-s)\,s^{\alpha-2}\ge bts\,(1-s)\,s^{\alpha-2}=tG(s,s),\ \forall t\le s,\ s\in[0,1]\,. \tag{51}$$

To prove **(ii-c$_2$)** if $\mathcal{G}_\alpha(t,s)=(1-t)s^{\alpha-1}$ for all $s\in[0,t]$, then we have

$$G^2_\alpha(t,s)=(ct+1)\,s^{\alpha-1},\ 0<(ct+1)<1,\ s\le t. \tag{52}$$

In an entirely similar manner to **(ii-c$_1$)**, we get

$$\partial_s G^2_\alpha(t,s)=(\alpha-1)\,(ct+1)\,s^{\alpha-2}>0,\ G^2_\alpha(t,0)=0.$$

The other cases can be dealt similarly. Now, from

$$\partial_t G^2_\alpha(t,s)=(\alpha-1)\,(ct+1)\,s^{\alpha-2}<0,\ G^2_\alpha(1,s)>0$$

we deduce that

$$G^2_\alpha(t,s)\le G^2_\alpha(t,t) \text{ or } G^2_\alpha(t,s)\le G^2_\alpha(s,s) \tag{53}$$

and observe that

$$G^2_\alpha(t,s)=(ct+1)\,s^{\alpha-1}\ge(ct+1)\,(cs+1)\,s^{\alpha-1}=(ct+1)\,G_\alpha(s)\,, \tag{54}$$

where $0<(ct+1)<1$. So, from **(ii-b$_1$)** and **(ii-b$_2$)**, we conclude that

$$\psi_1(t)\,G_\alpha(s)\le G_\alpha(t,s)\le G_\alpha(s)\ \text{ where } \psi_1(t)=t\,(ct+1)\,\forall t,s\in[0,1]\,. \tag{55}$$

To prove **(c),** if $\delta\in\left(0,\frac{1}{2}\right)$ then from (**i**), it is easily to see that

$$\min_{t\in[\delta,1-\delta]}\left\{G_\alpha(t,s):\delta\in\left(0,\tfrac{1}{2}\right)\right\}=\begin{cases} G^1_\alpha(1-\delta,s), & \text{if}\quad s\in[0,\theta]\,,\\ G^2_\alpha(\delta,s), & \text{if}\quad s\in[\theta,1]\,,\end{cases}$$

where $\theta\in[\delta,1-\delta]$. Consequently,

$$\min\left\{G_\alpha(t,s):\delta\le t\le 1-\delta \text{ when } \delta\in\left(0,\tfrac{1}{2}\right)\right\}\ge G_\alpha(1-\delta)\,G_\alpha(s)\,. \tag{56}$$

□

Lemma 3.5. *Suppose that (H) holds. Let $x(t)$ be a $C^2([0,1])$ positive solution of (6-7). Then there are constants a_1 and a_2, $0<a_1<1<a_2$ such that*

$$a_1G_\alpha(t)\le x(t)\le a_2G_\alpha(t)\quad \textit{or}\quad a_1t\le x(t)\le a_2t\quad \textit{for}\quad t\in[0,1]\,. \tag{57}$$

Proof. Assume that $x(t)$ is a $C^2([0,1])$ positive solution of (6-7) By Lemma 3.1, $x(t)$ given by

$$x(t)=\int_0^1 G_\alpha(t,s)\left(-\mathbf{T}^\alpha_{0+}x(s)\right)\mathrm{d}s, \tag{58}$$

or

$$x(t)=\int_0^1 \mathcal{G}_\alpha(t,s)\left(-\mathbf{T}^\alpha_{0+}x(s)\right)\mathrm{d}s+\frac{b_1t}{1-b_1\xi_1}\int_0^1 \mathcal{G}_\alpha(\xi_1,s)\left(-\mathbf{T}^\alpha_{0+}x(s)\right)\mathrm{d}s. \tag{59}$$

From (59) and (7), we have

$$x(0) = \int_0^1 \mathcal{G}_\alpha(0,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s = 0, \tag{60}$$

$$\begin{aligned} x(1) &= \int_0^1 \mathcal{G}_\alpha(1,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s \\ &+ \frac{b_1}{1-b_1\xi_1}\int_0^1 \mathcal{G}_\alpha(\xi_1,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s = b_1 x(\xi_1). \end{aligned} \tag{61}$$

Thus, it follows from (60) and (61) that

$$\frac{b_1}{1-b_1\xi_1}\int_0^1 \mathcal{G}_\alpha(\xi_1,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s = b_1 x(\xi_1),\ \mathcal{G}_\alpha(0,s) = \mathcal{G}_\alpha(1,s). \tag{62}$$

Noticing $\left(\mathbf{T}_{0+}^\alpha x(s)\right) \le 0,\ t\in[0,1]$ and (62), we have

$$x(t) = \int_0^1 \mathcal{G}_\alpha(t,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s + t b_1 x(\xi_1) \ge t b_1 x(\xi_1).$$

On the other hand, from (42) and (58), we have

$$x(t) = \int_0^1 G_\alpha(t,s)\left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s \le G_\alpha(t)\int_0^1 \left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s.$$

Now we choose

$$a_1 < \min\{1, b_1 x(\xi_1)\} \text{ and } a_2 > \max\left\{1, \int_0^1 \left(-\mathbf{T}_{0+}^\alpha x(s)\right) \mathrm{d}s \ge 1\right\}. \tag{63}$$

From Lemmas 3.1 and 3.2, it is easy to obtain the following lemma. □

Lemma 3.6. *Suppose that $x\in C^2[0,1]$ is a function with $\mathbf{T}_{0+}^\alpha x(t)\ge 0,\ 1<\alpha\le 2, x(0)\ge 0,\ \mathbf{T}_{0+}^\alpha x(0)\ge 0,\ x(1)\le b_1 x(\xi_1)$ and $\mathbf{T}_{0+}^\alpha x(1)\le b_2\mathbf{T}_{0+}^\alpha x(\xi_2)$. Then $x(t)\ge 0$ and $\mathbf{T}_{0+}^\alpha x(t)\le 0$ for any $t\in[0,1]$.*

4. **A necessary and sufficient condition for the existence of $C^2[0,1]$ positive solution for (6-7).** In this section, by using the upper and lower solutions technique, Arzela-Ascoli theorem and Krasnosel'skii fixed-point theorem, we establish the existence of positive solution to conformable boundary value problem (6-7).

Theorem 4.1. *Suppose that (H) holds, $f\left(s, G_\alpha(s), s^{2-\alpha}\right)$ does not vanish identically on $(0,1)$. Then a necessary and sufficient condition for problem (6-7) to have $C^2[0,1]$ positive solutions is that the following integral condition holds*

$$0 < \int_0^1 \mathcal{G}_\beta(s) f\left(s, G_\alpha(s), s^{2-\alpha}\right) \mathrm{d}s < +\infty, \tag{64}$$

where $G_\alpha(.)\equiv G_\alpha(.,.)$ and $\mathcal{G}_\beta(.)\equiv\mathcal{G}_\beta(.,.)$ defined by (23) and (24) respectively.

Proof. The proof is divided into two parts, necessity and sufficiency.

□

Necessary. First we prove $\int_0^1 \mathcal{G}_\beta(s) f\left(s, G_\alpha(s), s^{2-\alpha}\right) ds < \infty$. Assume that x is a $C^2[0,1]$ positive solution of (6-7). By Lemma 3.5, there exist constants a_1 and a_2, $0<a_1<1<a_2$ such that (57) holds. Choose $\sigma>0$ such that

$$0<\sigma\le 1,\quad M = \sup\nolimits_{s\in[0,1]}\left|T_{0+}^2 x(s)\right| \quad \text{and} \quad \sigma M \le 1. \tag{65}$$

Then, from (H) and (65) , we have

$$\begin{aligned} f\left(s, x(s), -\mathbf{T}_{0+}^{\alpha} x(s)\right) &\geq f\left(s, a_1 G_\alpha(s), -s^{2-\alpha} T_{0+}^{2} x(s)\right) \\ &= f\left(s, a_1 G_\alpha(s), -\sigma^{-1} \sigma s^{2-\alpha} T_{0+}^{2} x(s)\right) \\ &\geq a_1^{\mu_1} \sigma^{-\lambda_2} \left(-\sigma T_{0+}^{2} x(s)\right)^{\mu_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right) \\ &\geq a_1^{\mu_1} \sigma^{-\lambda_2} (\sigma M)^{\mu_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right), \end{aligned}$$

which implies for $s \in (0, 1)$

$$f\left(s, x(s), -\mathbf{T}_{0+}^{\alpha} x(s)\right) \geq \omega_1 f\left(s, G_\alpha(s), s^{2-\alpha}\right), \tag{66}$$

where $\omega_1 = a_1^{\mu_1} \sigma^{\mu_2 - \lambda_2} M^{\mu_2} > 0$. We also have

$$\begin{aligned} f\left(s, x(s), -\mathbf{T}_{0+}^{\alpha} x(s)\right) &\leq f\left(s, a_2 G_\alpha(s), -s^{2-\alpha} T_{0+}^{2} x(s)\right) \\ &\leq a_2^{\mu_1} f\left(s, G_\alpha(s), -\sigma^{-1} \sigma s^{2-\alpha} T_{0+}^{2} x(s)\right) \\ &\leq a_2^{\mu_1} \sigma^{\lambda_2 - \mu_2} \left(-T_{0+}^{2} x(s)\right)^{\lambda_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right) \\ &\leq a_2^{\mu_1} \sigma^{\lambda_2 - \mu_2} (M)^{\lambda_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right), \end{aligned}$$

which implies for $s \in (0, 1)$

$$f\left(s, x(s), -\mathbf{T}_{0+}^{\alpha} x(s)\right) \leq \omega_2 f\left(s, G_\alpha(s), s^{2-\alpha}\right), \tag{67}$$

where $\omega_2 = a_2^{\mu_1} \sigma^{\lambda_2 - \mu_2} M^{\lambda_2} > 0$. According to (6), we have

$$\omega_1 f\left(s, G_\alpha(s), s^{2-\alpha}\right) \leq f\left(s, x(s), -\mathbf{T}_{0+}^{\alpha} x(s)\right) = \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x\right)\right)(s). \tag{68}$$

By applying Lemmas 2.4 and 2.6, we have

$$\begin{aligned} \omega_1 \mathbf{I}_{0+}^{\beta}\left(f\left(s, G_\alpha(s), s^{2-\alpha}\right)\right)(t) &\leq \left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x\right)\right)(t) + t \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(s)\right)\right)(t)\Big|_{t=0^+} \\ &+ \left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x\right)\right)(t)\Big|_{t=0^+}. \end{aligned}$$

Moreover

$$\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(t)\right) = -\int_0^1 G_\beta(t, \tau) f(\tau, x(\tau), -\mathbf{T}_{0+}^{\alpha} x(\tau)) \mathrm{d}\tau \text{ for } t \in (0, 1), \tag{69}$$

which implies that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(t)\right)\right) &= -\int_0^t \frac{\partial}{\partial t} G_\beta(t, \tau) f(\tau, x, -\mathbf{T}_{0+}^{\alpha} x) \mathrm{d}\tau \\ &- \int_t^1 \frac{\partial}{\partial t} G_\beta(t, \tau) f(\tau, x, -\mathbf{T}_{0+}^{\alpha} x) \mathrm{d}\tau. \end{aligned} \tag{70}$$

By (6), we have

$$\partial_t G_\beta(t, s) \leq 1 + \frac{b_0}{1 - b_0 \xi_2},$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi\left({}_p\left(\mathbf{T}_{0+}^{\alpha} x(t)\right)\right) \leq \left(\tfrac{1 + b_0(1 - \xi_2)}{1 - b_0 \xi_2}\right) \int_0^1 \mathcal{G}_\beta(\tau) f(\tau, x(\tau), -\mathbf{T}_{0+}^{\alpha} x(\tau)) \mathrm{d}\tau.$$

Thus, from (6) and (69), we have $\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(s)\right) < 0$ and $\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x\right)\right)(s) \geq 0$ for $t \in (0, 1)$, combining this with $\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(s)\right)(t) \in C^1[0, 1]$, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(s)\right)\right)(t)\Big|_{t=0^+} < 0 \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} x(s)\right)\right)(t)\Big|_{t=1^-} > 0,$$

with

$$\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)\right)(t)\Big|_{t=0^+}=0\Longleftrightarrow \mathbf{T}_{0+}^{\alpha}x\left(0\right)=0,\varphi_p^{-1}\left(0\right)=0.$$

We deduce that

$$\begin{aligned}
\mathbf{I}_{0+}^{\beta}\left(f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\right)(t) &= \int_0^t (t-s)\,s^{\beta-2}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\mathrm{d}s && (71)\\
&\le \frac{1}{\omega_1}\left[\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)(t)\right. && (72)\\
&+ \left. t\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\left(s\right)\right)\right)(t)\Big|_{t=0^+}\right].
\end{aligned}$$

Letting $t\longrightarrow 1$ in (71) we have

$$\begin{aligned}
\int_0^1 (1-s)\,s^{\beta-2}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\mathrm{d}s &\le \frac{1}{\omega_1}\left[\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)(t)\Big|_{t=1^-}\right.\\
&+ \left. t\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\left(s\right)\right)\right)(t)\Big|_{t=0^+}\right]\\
&< \infty.
\end{aligned}$$

Second, we prove that $\int_0^1(1-s)\,s^{\beta-2}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)ds>0$.
The function $f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\neq 0$ for all $s\in(0,1)$ yield

$$\int_0^1 \mathcal{G}_{\beta}\left(s\right)f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\mathrm{d}s>0.$$

Therefore, we immediately get (64).

Sufficiency. Suppose that (64) holds, we will divide our proof into tow steps.

Step 1: Auxiliary problem of (6-7)

$\forall x\left(t\right)\in C^2\left[0,1\right]\cap C^4\left[0,1\right]$ we define an auxiliary function

$$F\left(x\right)(t)\equiv F\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)\right)$$

$$=\begin{cases} f\left(t,\underline{x}\left(t\right),-\mathbf{T}_{0+}^{\alpha}\underline{x}\left(t\right)\right) & \text{if} \quad x\left(t\right)<\underline{x}\left(t\right),\\ f\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)\right) & \text{if} \quad x\left(t\right)\in\left[\underline{x}\left(t\right),\overline{x}\left(t\right)\right],\\ f\left(t,\overline{x}\left(t\right),-\mathbf{T}_{0+}^{\alpha}\overline{x}\left(t\right)\right) & \text{if} \quad x\left(t\right)>\overline{x}\left(t\right).\end{cases} \tag{73}$$

The function $F\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)\right)$ is called a modification of $f\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)\right)$ associated with the coupled of lower and upper solutions $\underline{x}\left(t\right)$ and $\bar{x}\left(t\right)$. By the hypotheses (H) we have $F:\longrightarrow[0,+\infty)$ is continuous.(i.e., $F:(0,1)\times(0,+\infty)\times(-\infty,0)\longrightarrow\mathbb{R}^+$ is continuous). Consider the auxiliary problem of (6-7)

$$\begin{cases} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)\right)(t)=F\left(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\right),\ t\in(0,1),\\ x\left(0\right)=0,\ x\left(1\right)=b_1x\left(\xi_1\right),\mathbf{T}_{0+}^{\alpha}x\left(0\right)=0,\ \mathbf{T}_{0+}^{\alpha}x\left(1\right)=b_2\mathbf{T}_{0+}^{\alpha}x\left(\xi_2\right).\end{cases} \tag{74}$$

For convenience, we define linear operators as follows [23],

$$A_2x\left(t\right)=\int_0^1 G_{\beta}\left(t,s\right)x\left(s\right)\mathrm{d}s \text{ and } A_1x(t)=\int_0^1 G_{\alpha}(t,s)x(s)\mathrm{d}s. \tag{75}$$

Obviously, by the proof of Lemma 3.3, the problem (74) is equivalent to the integral equation

$$x\left(t\right)=\left(A_1\varphi_q\left(A_2F\right)\right)x\left(t\right),\ t\in(0,1). \tag{76}$$

By the definition (73) of F, we can get that $A_1\varphi_q(A_2f)$ and F is bounded. By the continuity of $G_\alpha(t,s)$, we can show that $A_1\varphi_q(A_2)$ is a compact operator. So, $(A_1\varphi_q(A_2F))$ is a relatively compact set. So $A_1\varphi_q(A_2)$ is a compact operator. Moreover, x is a solution of (74) if and only if $(A_1\varphi_q(A_2F))\,x = x$. Using the Schauder's fixed point theorem, we assert that $A_1\varphi_q(A_2F)$ has at least one fixed point $x \in C^2[0,1]$, by $x(t) = (A_1\varphi_q(A_2F))\,x(t)$, we can get $x \in C^4[0,1]$.

1.1: Consider the problem

$$\begin{cases} \mathbf{T}^{\beta}_{0+}\left(\varphi_p\left(\mathbf{T}^{\alpha}_{0+}y\right)\right)(t) = f(t, G_\alpha(t), t^{2-\alpha}),\ t \in [0,1], \\ y(0) = 0,\ y(1) = b_1 y(\xi_1), \mathbf{T}^{\alpha}_{0+}y(0) = 0,\ \mathbf{T}^{\alpha}_{0+}y(1) = b_2\mathbf{T}^{\alpha}_{0+}y(\xi_2). \end{cases} \tag{77}$$

Let

$$y(t) = \int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right)\mathrm{d}s,\ t \in [0,1].$$

From the Lemma 3.5, (64) implies that

$$\exists 0 < a_3 < 1 < a_4 : 0 < y(t) < \infty \text{ and } a_3 G_\alpha(t) \le y(t) \le a_4 G_\alpha(t). \tag{78}$$

We will prove that the functions

$$\underline{x}(t) = k_1 y(t),\ \bar{x}(t) = k_2 y(t),\ t \in [0,1], \tag{79}$$

are lower and upper solutions of (6-7), respectively, here

$$k_1 \le \min\left\{1, \frac{1}{a_3}, \frac{1}{a_4}, \left(a_3^{\mu_1}\sigma^{\mu_2-\lambda_2}M^{\mu_2}\right)^{\frac{1}{1-\mu_1+\mu_2}}\right\}, \tag{80}$$

$$k_2 \ge \max\left\{1, \frac{1}{a_3}, \frac{1}{a_4}, \left(a_4^{\mu_2}\sigma^{\lambda_2-\mu_2}M^{\lambda_2}\right)^{\frac{1}{1-\lambda_1+\lambda_2}}\right\}. \tag{81}$$

This, by virtue of the assumption of the Lemma 3.4, (78) and (79), shows that

$$k_1 a_3 G_\alpha(t) \le \underline{x}(t) \le k_1 a_4 G_\alpha(t) \tag{82}$$

and

$$k_1 a_3 \le \frac{\underline{x}(t)}{G_\alpha(t)} \le k_1 a_4 \le 1,\ \frac{1}{k_2 a_4} \le \frac{G_\alpha(t)}{\bar{x}(t)} \le \frac{1}{k_2 a_3}.$$

By Lemma 2.6, shows that

$$-k_1 a_3 \mathbf{T}^{\alpha}_{0+}G_\alpha(t) \le -\ \mathbf{T}^{\alpha}_{0+}\underline{x}(t) \le -k_1 a_4 \mathbf{T}^{\alpha}_{0+}G_\alpha(t) \tag{83}$$

and

$$-k_2 a_4 \mathbf{T}^{\alpha}_{0+}G_\alpha(t) \le -\ \mathbf{T}^{\alpha}_{0+}\bar{x}(t) \le -k_2 a_3 \mathbf{T}^{\alpha}_{0+}G_\alpha(t). \tag{84}$$

Choose $0 < k_1 < 1$ small enough, and from (65), (82), (83) and (H) yield that

$$\begin{aligned} f\left(t, \underline{x}(t), -\mathbf{T}^{\alpha}_{0+}\underline{x}(t)\right) &= f(t, \left(\frac{\underline{x}(t)}{G_\alpha(t)}\right)G_\alpha(t), -\underline{x}''(t)\,t^{2-\alpha}) \\ &\ge f(t, k_1 a_3 G_\alpha(t), -\sigma^{-1}\sigma k_1 y t^{2-\alpha}) \\ &\ge k_1^{\mu_1+\mu_2}a_3^{\mu_1}\sigma^{\mu_2-\lambda_2}M^{\mu_2}f(t, G_\alpha(t), t^{2-\alpha}) \\ &\ge k_1 f(t, G_\alpha(t), t^{2-\alpha}),\ t \in (0,1). \end{aligned} \tag{85}$$

Similarly, choose $k_2 > 1$ large enough, we have

$$\begin{aligned} f\left(t, \bar{x}(t), -\mathbf{T}_{0+}^{\alpha}\bar{x}(t)\right) &\leq f(t, \frac{\bar{x}(t)}{G_\alpha(t)} G_\alpha(t), -\bar{x}''(t) t^{2-\alpha}) \\ &\leq f(t, k_1 a_4 G_\alpha(t), -\sigma^{-1}\sigma k_1 y t^{2-\alpha}) \\ &\leq k_1^{\lambda_1+\lambda_2} a_4^{\mu_2} \sigma^{\lambda_2-\mu_2} M^{\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\leq k_2 f(t, G_\alpha(t), t^{2-\alpha}),\ t \in (0,1). \end{aligned} \tag{86}$$

Consequently, by Lemma 2.6, for $t \in (0,1)$

$$\begin{aligned} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\underline{x}\right)\right)(t) &= k_1 \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}y\right)\right)(t) = k_1 f(t, G_\alpha(t), t^{2-\alpha}) \\ &\leq f\left(t, \underline{x}, -\mathbf{T}_{0+}^{\alpha}\underline{x}\right) \end{aligned} \tag{87}$$

and

$$\begin{aligned} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\bar{x}\right)\right)(t) &= k_2 \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}y\right)\right)(t) = k_2 f(t, G_\alpha(t), t^{2-\alpha}) \\ &\geq f\left(t, \bar{x}, -\mathbf{T}_{0+}^{\alpha}\bar{x}\right). \end{aligned} \tag{88}$$

From (87) and (88), we obtain that for such choice of k_1 and k_2, $\underline{x}(t)$ and $\bar{x}(t)$ are, respectively, lower and upper solutions of (6-7) satisfying $0 < \underline{x}(t) \leq \bar{x}(t)$ for $t \in (0,1)$.

1.2 Let X be the Banach space $C^2[0,1]$ and the cone P in X be

$$P = \left\{ \begin{array}{l} x : x \in X, \varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right) \in X, -\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right) \text{ is concave on } t \in (0,1), \\ x(t) \geq 0, -\mathbf{T}_{0+}^{\alpha}x(t) \geq 0 \text{ for } t \in [0,1],\ x \text{ satisfies } (7) \end{array} \right\}$$

If $x \in P$, then, it follows from Lemmas 3.1, 3.3 and (H) that

$$\begin{aligned} x(t) &= \int_0^1 G_\alpha(t,s)\left(-\mathbf{T}_{0+}^{\alpha}x(s)\right) \mathrm{d}s \\ &\leq \int_0^1 G_\alpha(s)\left(s^{2-\alpha}\|x''\|_0\right) \mathrm{d}s \leq \|x''\|_0 \int_0^1 G_\alpha(s)\left(s^{2-\alpha}\right) \mathrm{d}s. \end{aligned} \tag{89}$$

Thus, it is clear that

$$\|x\| = \|x''\|_0 \ \ \forall x \in P. \tag{90}$$

From Lemma 3.3 we have

$$\begin{aligned} x(t) &= \int_0^1 G_\alpha(t,s)\left(-\mathbf{T}_{0+}^{\alpha}x(s)\right) \mathrm{d}s \\ &\leq \int_0^1 G_\alpha(t)\left(s^{2-\alpha}\|x''\|_0\right) \mathrm{d}s \leq \frac{1}{3-\alpha} G_\alpha(t)\|x''\|_0 \ \text{ for } t \in [0,1]. \end{aligned} \tag{91}$$

Moreover, Remark 2 implies that

$$-\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)(t) \geq \psi(t) t^{2-\alpha} \varphi_p\left(\|x''\|_0\right) \ \text{ for } t \in [0,1]. \tag{92}$$

From (64), there exists an interval $[\delta, 1-\delta] \subset (0,1)$ such that

$$0 < \int_\delta^{1-\delta} \mathcal{G}_\beta(s) f\left(s, G_\alpha(s), s^{2-\alpha}\right) \mathrm{d}s < +\infty, \text{ where } \delta \in \left(0, \tfrac{1}{2}\right). \tag{93}$$

Note that from (23) and (24) we have

$$G_\alpha(t,s) \geq \mathcal{G}_\alpha(t,s) \geq \delta \text{ for } t, s \in [\delta, 1-\delta]. \tag{94}$$

Noting the continuity of G_β and F, we can choose $[\delta_1,\delta_2]\subset(0,1)$ such that

$$\int_0^1 G_\beta(s,\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\ \ \mathrm{d}\tau \geq \int_{\delta_1}^{\delta_2} G_\beta(s,\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau>0.$$

Then, from (92) and (94), we obtain

$$x(t)=\int_0^1 G_\alpha\left(t,s\right)\left(-\mathbf{T}_{0+}^{\alpha}x\left(s\right)\right)\mathrm{d}s\geq\delta\left\|x''\right\|_0\int_\delta^{1-\delta}\varphi_q\left(s^{2-\alpha}\right)\mathrm{d}s\geq m_1\left\|x''\right\|_0, \tag{95}$$

for $t\in[\delta,1-\delta]$, where

$$m_1=\delta\int_\delta^{1-\delta}\varphi_q\left(s^{2-\alpha}\right)\mathrm{d}s\in(0,1). \tag{96}$$

For any fixed $x\in P$, choose a positive number $\theta=\frac{1}{\|x\|+1}<1$ Then, (94) yields

$$\frac{\theta x\left(t\right)}{G_\alpha\left(t\right)}\leq\theta\left\|x''\right\|_0=\theta\left\|x\right\|\leq1 \text{ for } t\in(0,1). \tag{97}$$

Thus, (65) and (H), imply

$$\begin{aligned} f(t,x\left(t\right),-\mathbf{T}_{0+}^{\alpha}x\left(t\right)) &= f(t,\frac{\theta x\left(t\right)}{\theta G_\alpha\left(t\right)}G_\alpha\left(t\right),\theta^{-1}\theta\left(-x''\left(t\right)\right)t^{2-\alpha}) \\ &\leq \theta^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}f(t,G_\alpha\left(t\right),t^{2-\alpha}) \text{ for } t\in(0,1). \end{aligned}$$

Therefore, from (42), we have

$$\begin{aligned} &\int_0^1 G_\beta(t,s)f(s,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}s\leq \\ &\theta^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}\left(\tfrac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)\int_0^1\mathcal{G}_\beta(s)f(s,G_\alpha\left(s\right),s^{2-\alpha})<\infty. \end{aligned} \tag{98}$$

Now we prove that problem (74) has a positive solution $x^*\in X$ with $0<\underline{x}\left(t\right)\leq x^*\leq\bar{x}\left(t\right)$. We consider the operator $A:X\longrightarrow X$ defined as follows

$$\left(Ax\right)\left(t\right)=\int_0^1 G_\alpha\left(t,s\right)\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,x\left(\tau\right),-\mathbf{T}_{0+}^{\alpha}x\left(\tau\right))\mathrm{d}\tau\right)\mathrm{d}s. \tag{99}$$

It is well known that a fixed point of the operator A is a solution of the problem (74). The following fixed point result of cone compression type due to Krasnosel'skii is fundamental for the solvability of problem (74).

From (99), Lemmas 3.1, 3.1 and 3.3 it is easy to see that $x\in P$ is a $C^2[0,1]$ nonnegative solution of the problem (74) if and only if x is a fixed point of A. Moreover

$$-\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\left(t\right)=\varphi_q\left(\int_0^1 G_\beta(t,\tau)F(\tau,x\left(\tau\right),-\mathbf{T}_{0+}^{\alpha}x\left(\tau\right))\mathrm{d}\tau\right) \text{ for } t\in[0,1]. \tag{100}$$

In the following, we divide the proof of the existence of fixed point of $A:X\longrightarrow X$ into three steps.

$(\mathbf{S_1})$ The operator $A:P\longrightarrow P$ is completely continuous.

$(\mathbf{S_{11}})$ $A:P\longrightarrow P$

If $x\in P$, it is clear that $Ax\in X$, for $t\in[0,1]$,

$$\begin{aligned} &\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\right)\right)\left(t\right)\geq0, \text{for } t\in(0,1) \text{ and } \left(\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\right)\left(t\right)\leq0, \\ &\left(Ax\right)\left(t\right)\geq0, \left(Ax\right)\left(0\right)=0,\ \left(Ax\right)\left(1\right)=b_1\left(Ax\right)\left(\xi_1\right), \\ &\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\left(0\right)=0,\ \mathbf{T}_{0+}^{\alpha}\left(Ax\right)\left(1\right)=b_2\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\left(\xi_2\right). \end{aligned} \tag{101}$$

($\mathbf{S_{12}}$) A is pre-compact in P.

Let Ω be a bounded set on x. Then there is $\rho > 0$ such that $\|x\| \leq \rho$ for all $x \in \Omega$. We show that $(A\Omega)$ is a pre-compact set in P. Denote $\theta_\rho = \frac{1}{1+\rho}$. For all $x \in \Omega$

$$\begin{aligned}
|(Ax)(t)| &= \left|\int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^\alpha x)\mathrm{d}\tau\right)\mathrm{d}s\right| \qquad (102)\\
&\leq \left|\int_0^1 G_\alpha(s)\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,x(\tau),\left(-\mathbf{T}_{0+}^\alpha x\right)(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right|\\
&\leq \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\left(\varphi_q\left(\int_0^1 G_\beta(\tau)f(\tau,\bar{x}(\tau),\left(-\mathbf{T}_{0+}^\alpha \bar{x}\right)(\tau))\mathrm{d}\tau\right)\right)\\
&\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)^{\frac{1}{p-1}}\\
&\times \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\varphi_q\left(\int_0^1 G_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\leq \mathcal{L}_\rho\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right) < \infty,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_\rho &= \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\tfrac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}}\\
&\times \varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right) \qquad (103)
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathbf{T}_{0+}^\alpha(Ax)(t)\right| &= \left|\varphi_q\left(\int_0^1 G_\beta(t,\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^\alpha x(\tau))\mathrm{d}\tau\right)\right| \qquad (104)\\
&\leq \varphi_q\left(\int_0^1 G_\beta(\tau)f(\tau,\bar{x}(\tau),\left(-\mathbf{T}_{0+}^\alpha \bar{x}\right)(\tau))\mathrm{d}\tau\right)\\
&\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)^{\frac{1}{p-1}}\varphi_q\left(\int_0^1 G_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\leq \mathcal{L}_\rho < \infty.
\end{aligned}$$

Then (Ax) is uniformly bounded in X.

For all $x \in \Omega,\ t_1,t_2 \in [0,1]$,

$$\begin{aligned}
|(Ax)(t_2)-(Ax)(t_1)| &= \int_0^1 |G_\alpha(t_2,s)-G_\alpha(t_1,s)|\,\varphi_q\\
&\times \left(\int_0^1 G_\beta(s,\tau)F(\tau,x,-\mathbf{T}_{0+}^\alpha x)\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq \int_0^1 |G_\alpha(t_2,s)-G_\alpha(t_1,s)|\,\varphi_q \qquad (105)\\
&\times \left(\int_0^1 G_\beta(s,\tau)f(\tau,\bar{x},\left(-\mathbf{T}_{0+}^\alpha \bar{x}\right)\mathrm{d}\tau\right)\mathrm{d}s\\
&\leq \mathcal{L}_\rho\left(\int_0^1 |G_\alpha(t_2,s)-G_\alpha(t_1,s)|\,\mathrm{d}s\right)\\
&\leq \mathcal{L}_\rho\,|t_2-t_1| < \infty.
\end{aligned}$$

For arbitrary $\epsilon > 0$. Let $\delta_1 = \frac{\epsilon}{\mathcal{L}_\rho}$, then

$$|(Ax)(t_2) - (Ax)(t_1)| < \epsilon \quad \text{if} \quad |t_2 - t_1| < \delta_1. \tag{106}$$

At the same time, from

$$\begin{aligned} \left|\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(t)\right| &= \left|\int_0^1 G_\beta(s,\tau)F(\tau, x(\tau), -\mathbf{T}^\alpha_{0+}x)\mathrm{d}\tau\right| \\ &\leq \left|\int_0^1 G_\beta(s,\tau)f\left(\tau, \bar{x}(\tau), \left(-\mathbf{T}^\alpha_{0+}\bar{x}\right)(\tau)\right)\mathrm{d}\tau\right| \\ &\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)\left(\int_0^1 G_\beta(\tau)f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right) \\ &< \infty, \end{aligned} \tag{107}$$

it follows that for $\epsilon > 0$, there is $\delta_2 > 0$ such that $\forall x \in \Omega, t \in [0,1]$

$$\begin{aligned} \int_{1-\delta_2}^1 G_\beta(s,\tau)F(\tau, x, -\mathbf{T}^\alpha_{0+}x)\mathrm{d}\tau &< \int_0^1 G_\beta(s,\tau)f\left(\tau, \bar{x}(\tau), \left(-\mathbf{T}^\alpha_{0+}\bar{x}\right)(\tau)\right)\mathrm{d}\tau \\ &< \frac{\epsilon}{3}. \end{aligned} \tag{108}$$

On the other hand, (64) imply that for $x \in \Omega, t \in [0, 1-\delta_2]$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(t) &= \int_0^t \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}\tau^{2-\beta}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}\mathbf{T}^\beta_{0+}\left(\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}F(\tau, (Ax)(\tau), -\mathbf{T}^\alpha_{0+}(Ax))\mathrm{d}\tau \\ &\leq \int_0^t \tau^{\beta-2}f(\tau, (A\bar{x})(\tau), -\mathbf{T}^\alpha_{0+}(A\bar{x}))\mathrm{d}\tau \\ &\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)\left(\int_0^{1-\delta_2}\tau^{\beta-2}f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right) \\ &= m_2. \end{aligned} \tag{109}$$

where k_2 is defined in (81).

Let $\delta = \min\left\{\delta_1, \delta_2, \frac{\epsilon}{3m_2}\right\}$. Then for $x \in \Omega, t_1, t_2 \in [0,1], 0 \leq t_2 - t_1 \leq \delta$.

Moreover, by mean value theorem, Lemma 2.5 ensures that, for any t_2, t_1 in $[0,1]$ with $t_1 < t_2$, there exists a point t in (t_1, t_2) such that

$$\begin{aligned} &\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(t_2) - \varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)(t_1) \\ &= \frac{1}{\beta-1}\left(t_2^{\beta-1} - t_1^{\beta-1}\right)\left[T^{\beta-1}_{0+}\varphi_p\left(\mathbf{T}^\alpha_{0+}(Ax)\right)\right](t). \end{aligned}$$

This, together with (109), yields

$$\begin{aligned}
\mathfrak{W} &= \left|\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t_2)-\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t_1)\right| \\
&= \left|\frac{1}{\beta-1}\left(t_2^{\beta-1}-t_1^{\beta-1}\right)\left[T_{0+}^{\beta-1}\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)\right](t)\right| \\
&= \left|\frac{1}{\beta-1}\left(t_2^{\beta-1}-t_1^{\beta-1}\right)\left[t^{\beta-2}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t)\right]\right| \\
&\leq \left|\frac{1}{\beta-1}\left(t_2^{\beta-1}-t_1^{\beta-1}\right)\left[t^{\beta-2}\int_0^t\tau^{\beta-2}f(\tau,(A\bar{x})(\tau),-\mathbf{T}_{0+}^{\alpha}(A\bar{x})(\tau))\mathrm{d}\tau\right]\right| \\
&\leq M_1k_2\left|t_2^{\beta-1}-t_1^{\beta-1}\right|\leq\frac{\epsilon}{3}<\epsilon,\ t_1\in[0,1-\delta_2].
\end{aligned}$$

In the same way, we can show that

$$\begin{aligned}
\mathfrak{W} &= \left|\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t_2)-\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t_1)\right| \qquad (110)\\
&\leq \int_0^1\left|G_\beta(t_2,\tau)-G_\beta(t_1,\tau)\right|F(\tau,x(\tau),-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau \\
&\leq \left(\int_0^{1-\delta_2}+\int_{1-\delta_2}^1\right)\left|G_\beta(t_2,\tau)-G_\beta(t_1,\tau)\right|F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau \\
&\leq \int_0^{1-\delta_2}|t_2-t_1|F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau+\int_{1-\delta_2}^1|G_\beta(t_2,\tau)|F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau \\
&\quad+\int_{1-\delta_2}^1|G_\beta(t_1,\tau)|F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau \\
&\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\varepsilon,\ t_1\in[1-\delta_2,1].
\end{aligned}$$

Then $\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t)$ is equi-continuous on $[0,1]$. Since φ is uniformly continuous on arbitrary closed interval of $\mathbb{R}$. $\left(\mathbf{T}_{0+}^{\alpha}(Ax)\right)(t)$ is also equi-continuous on $[0,1]$.

($\mathbf{S_{13}}$) A is continuous in P: Let $x_n\longrightarrow x$ as $n\longrightarrow\infty$ in P. Then there exists $\bar{\rho}>0$ such that $\sup_{n\in\mathbb{N}}\|x_n\|\leq\bar{\rho}$. Then by (73), there holds

$$(Ax_n)(t)=\int_0^1G_\alpha(t,s)\varphi_q\left(\int_0^1G_\beta(s,\tau)F(\tau,x_n(\tau),-\mathbf{T}_{0+}^{\alpha}x_n)\mathrm{d}\tau\right)\mathrm{d}s \qquad (111)$$

$$\begin{aligned}
&\leq \int_0^1G_\alpha(s)\varphi_q\left(\int_0^1G_\beta(\tau)F(\tau,x_n(\tau),-\mathbf{T}_{0+}^{\alpha}x_n)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \int_0^1G_\alpha(s)\varphi_q\left(\int_0^1G_\beta(\tau)f(\tau,\bar{x}_n(\tau),-\mathbf{T}_{0+}^{\alpha}\bar{x}_n(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \left(\theta_{\bar{\rho}}^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\int_0^1G_\alpha(s)\,\mathrm{d}s\right) \\
&\times\left(\int_0^{1-\delta_2}G_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right) \\
&\leq \mathcal{L}_{\bar{\rho}}\left(\int_0^1G_\alpha(s)\,\mathrm{d}s\right)<\infty,
\end{aligned}$$

where $\mathcal{L}_{\bar{\rho}}$ is given by (103) with ρ replaced by $\bar{\rho}$. By Lebesgue's dominated convergence theorem,

$$(Ax_n)(t) = \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau, x_n(\tau), -\mathbf{T}_{0+}^\alpha x_n)\mathrm{d}\tau\right)\mathrm{d}s \tag{112}$$

$$\begin{aligned} &\longrightarrow \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau, x(\tau), -\mathbf{T}_{0+}^\alpha x)\mathrm{d}\tau\right) \\ &= (Ax)(t) \text{ as } n \longrightarrow \infty, t \in [0,1]. \end{aligned}$$

which means $(Ax_n) \longrightarrow (Ax)$ as $n \longrightarrow \infty$.

Similarly, $\mathbf{T}_{0+}^\alpha(Ax_n) \longrightarrow \mathbf{T}_{0+}^\alpha(Ax)$ as $n \longrightarrow \infty$. Therefore, by Arzelá Ascoli Theorem, $A : P \longrightarrow P$ is completely continuous.

(S₂) If $\Omega_1 = \{x \in x : \|x\| < r_1\}$, then

$$\|Ax\| \le \|x\| \text{ for } x \in P \cap \partial\Omega_1, \tag{113}$$

where $r_1 > 0$ satisfies

$$r_1 < \min\left\{1,\ \tilde{r} \times \left(k_2^{\frac{1}{p-1}}\left(\tfrac{1}{3-\alpha}\right)^{\frac{\lambda_1}{p-1}}\right)^{\frac{p-1}{p-(\lambda_1+\lambda_2+1)}}\right\}, \tag{114}$$

where

$$\begin{aligned} \tilde{r} &= \left(\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\left(\tfrac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}}\right. \\ &\times \left.\varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right)\right)^{\frac{p-1}{p-(\lambda_1+\lambda_2+1)}}. \end{aligned}$$

If $x \in P \cap \partial\Omega_1$, then from (H), (74), (80) and (114), we have

$$\begin{aligned} F(t, x(t), -\mathbf{T}_{0+}^\alpha x(t)) &\le F(t, \bar{x}(t), t^{2-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{x}(t)) \\ &= k_2 f(t, y(t), t^{2-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)) \\ &\le k_2 f(t, \frac{1}{3-\alpha}G_\alpha(t)\|y''\|_0, t^{2-\alpha}\|y''\|_0) \\ &\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1}(\|y''\|_0)^{\lambda_1+\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1}\|y\|^{\lambda_1+\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}),\ \text{for } t \in (0,1), \end{aligned} \tag{115}$$

where k_2 is defined in (81). From Lemmas 3.4,3.5, (114) and (115), we obtain

$$\begin{aligned}
(Ax)(t) &= \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\mathrm{d}s \qquad (116)\\
&\le \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\\
&\le M_b\times\left(\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\\
&\times\ \varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\le r_1=\|x\|
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathbf{T}_{0+}^{\alpha}(Ax)(t)\right| &= \left|\varphi_q\left(\int_0^1 G_\beta(t,\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\right|\\
&\le \varphi_q\left(\int_0^1 G_\beta(t,\tau)F(\tau,x,-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\\
&\le M_b\times\left(\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2}k_2\right)^{\frac{1}{p-1}}\\
&\times\ \varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\le r_1=\|x\|,\ \text{where } M_b=\left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}},
\end{aligned}$$

for $t\in[0,1]$ and thus $.(Ax)$ satisfies (113).

(S$_3$) If $\Omega_2=\{x\in x:\|x\|<r_2\}$, then

$$\|Ax\|\ge\|x\| \text{ for } x\in P\cap\partial\Omega_2, \qquad (117)$$

where $r_2>0$ satisfies

$$r_2>\max\left\{1,\frac{1}{\varepsilon m_1},\frac{1}{d},\frac{1}{(\delta^{2\alpha})^{\frac{1}{p-1}}}\right\}, \qquad (118)$$

m_1 is as defined in (96), $\epsilon\in(0,1)$ is a fixed number small enough such that

$$\delta\left(c\left(1-\delta\right)+1\right)\le G_\alpha\left(t\right)\le\left(1-\delta\right)\left(c\delta+1\right),c<0$$

and

$$\psi_1(t)=t\,(ct+1)\ge\epsilon G_\alpha(t) \text{ for } t\in[\delta,1-\delta]. \qquad (119)$$

Setting

$$d=d_m\times\left(\int_\delta^{1-\delta}\mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha}\mathrm{d}\tau\right)^{\frac{1}{\lambda_1+\lambda_2+p-1}}. \qquad (120)$$

$$d_m=k_1^{\frac{1}{p-1}}(1-2\delta)^{p-1}\left(\delta^{2\alpha}\right)^{\frac{1}{\lambda_1+\lambda_2+p-1}}(\epsilon m_1)^{\lambda_1}\left(\delta\left(c\left(1-\delta\right)+1\right)\right)^{\frac{\lambda_2}{p-1}}$$

Now if $x \in P \cap \partial\Omega_2$,, from (90), 92) and (118), we have

$$\begin{aligned} -\left(\mathbf{T}_{0+}^{\alpha} x\right)(t) &\geq (\psi_1(t))^{\frac{1}{p-1}} \|x''\|_0 \\ &\geq (\delta(c(1-\delta)+1))^{\frac{1}{p-1}} \|x''\|_0 \\ &\geq (\delta(c(1-\delta)+1))^{\frac{1}{p-1}} \|x\| > 1 \text{ for } t \in [\delta, 1-\delta]. \end{aligned} \tag{121}$$

Similarly, from (95) and (118), we have

$$\epsilon x(t) \geq \epsilon m_1 \|x''\|_0 = \epsilon m_1 \|x\| > 1 \text{ for } t \in [\delta, 1-\delta]. \tag{122}$$

Moreover, for $t \in [\delta, 1-\delta]$ by Lemma 3.1 and (119),

$$x(t) \geq \psi(t) \|x\|_0 \geq \epsilon G_\alpha(t) \|x\|_0. \tag{123}$$

Hence, for $t \in [\delta, 1-\delta]$, from (H), (121), (122) and (123) imply

$$\begin{aligned} F(t,x,-\mathbf{T}_{0+}^{\alpha}x) &\geq f(t,\underline{x},-\mathbf{T}_{0+}^{\alpha}\underline{x}) = f(t,k_1 y,-k_1\mathbf{T}_{0+}^{\alpha}y) \\ &\geq k_1 f(t,y(t),-t^{2-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)) \\ &\geq k_1 f(t,\epsilon G_\alpha(t)\|y\|_0,(\delta(c(1-\delta)+1))^{\frac{1}{p-1}} t^{2-\alpha}\|y\|) \\ &\geq k_1(\epsilon\|y\|_0)^{\lambda_1}\left((\delta(c(1-\delta)+1))^{\frac{1}{p-1}}\|y\|\right)^{\lambda_2} f(t,G_\alpha(t),t^{2-\alpha}) \\ &\geq k_1(\epsilon m_1)^{\lambda_1}(\delta(c(1-\delta)+1))^{\frac{\lambda_2}{p-1}}\|y\|^{\lambda_1+\lambda_2} f(t,G_\alpha(t),t^{2-\alpha}) \\ &= k_1(\epsilon m_1)^{\lambda_1}(\delta(c(1-\delta)+1))^{\frac{\lambda_2}{p-1}} r_2^{\lambda_1+\lambda_2} f(t,G_\alpha(t),t^{2-\alpha}), \end{aligned} \tag{124}$$

where $m_1 \in [0,1]$ and k_1 is defined in (80).

Since $G_\alpha(t,s) \geq \mathcal{G}_\alpha(t,s) \geq \delta\delta^{\alpha-1} = \delta^\alpha \geq \delta^\alpha G_\alpha(s)$ for $t,s \in [\delta,1-\delta]$, it then follows from (99), (118) and (124), that

$$\begin{aligned} (Ax)(t) &= \int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \delta^\alpha\int_\delta^{1-\delta}\varphi_q\left(\int_\delta^{1-\delta} G_\beta(s,\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \delta^\alpha(\delta^\alpha)^{\frac{1}{p-1}}\left(\int_\delta^{1-\delta}\mathrm{d}s\right) \\ &\times \left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau)f(\tau,\underline{x}(\tau),-\mathbf{T}_{0+}^{\alpha}\underline{x}(\tau))\mathrm{d}\tau\right)\right) \\ &\geq M_m\times(1-2\delta)\,r_2^{\frac{\lambda_1+\lambda_2}{p-1}}\varphi_q\left(\int_\delta^{1-\delta}\mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha}\mathrm{d}\tau\right) \\ &\geq r_2 = \|x\|, \end{aligned} \tag{125}$$

where $M_m = k_1\delta^\alpha(\delta^\alpha)^{\frac{1}{p-1}}(\epsilon m_1)^{\frac{\lambda_1}{p-1}}(\delta(c(1-\delta)+1))^{\frac{\lambda_2}{(p-1)^2}}$. Hence,

$$\|Ax\|_0 \geq \|x\|. \tag{126}$$

From (100), (118), (120) and (124), we have

$$\begin{aligned}\mathbf{T}_{0+}^{\alpha}\left(Ax\right)(t) &= \varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right) \\ &\geq (\delta^\alpha)^{\frac{1}{p-1}}\left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau)F(\tau,x(\tau),-\mathbf{T}_{0+}^{\alpha}x)\mathrm{d}\tau\right)\right) \\ &\geq (\delta^\alpha)^{\frac{1}{p-1}}\left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau)f(\tau,\underline{x}(\tau),-\mathbf{T}_{0+}^{\alpha}\underline{x}(\tau))\mathrm{d}\tau\right)\right) \\ &\geq M_m\times r_2^{\frac{\lambda_1+\lambda_2}{p-1}}\varphi_q\left(\int_\delta^{1-\delta}\mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha}\mathrm{d}\tau\right)) \\ &\geq r_2=\|x\|.\end{aligned} \tag{127}$$

Hence,

$$\left\|\mathbf{T}_{0+}^{\alpha}\left(Ax\right)\right\|\geq\|x\|. \tag{128}$$

In view of (126) and (128), we see that (117) holds.

Therefore, by steps one to three, and Lemma 4.1, we see that A has at least one fixed point $x\in P\cap\bar{\Omega}_2\backslash\Omega_1$. It can be verified that for $x\in P$, there holds $x(t)\geq G_\alpha(t)\|x\|_0$. Thus, $x(t)$ is a solution of problem (74).

Step 2 **:** Finally, we will prove that the conformable boundary value problem (6-7) has at least one positive solution. Suppose that $x^*(t)$ is a solution of (6-7), we only need to prove that $\underline{x}(t)\leq x^*(t)\leq\overline{x}(t)$, $t\in[0,1]$. The method is similar for the two inequalities. We only prove $x^*(t)\leq\overline{x}(t)$ for $t\in[0,1]$.

In fact, since x^* is fixed point of A and (101), we get

$$\begin{aligned} x^*(0)=0\ x^*(1)=b_1x^*(\xi_1)\ \ \mathbf{T}_{0+}^{\alpha}x^*(0)=0\ \mathbf{T}_{0+}^{\alpha}x^*(1)=b_2\mathbf{T}_{0+}^{\alpha}x^*(\xi_2), \\ \overline{x}(0)=0\ \overline{x}(1)=b_1\overline{x}(\xi_1)\ \ \mathbf{T}_{0+}^{\alpha}\overline{x}(0)=0\ \mathbf{T}_{0+}^{\alpha}\overline{x}(1)=b_2\mathbf{T}_{0+}^{\alpha}\overline{x}(\xi_2).\end{aligned} \tag{129}$$

Otherwise, suppose by contradiction that $x^*(t)>\overline{x}(t)$. According to the definition of F, one verifies that for $t\in(0,1)$

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x^*\right)\right)(t)=F\left(t,x^*(t),-\mathbf{T}_{0+}^{\alpha}x^*(t)\right)=f\left(t,\overline{x}(t),-\mathbf{T}_{0+}^{\alpha}\bar{x}(t)\right). \tag{130}$$

On the other hand, since $\overline{x}$ is an upper solution to (6), we obviously have

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{x}\right)\right)(t)\geq f\left(t,\overline{x}(t),-\mathbf{T}_{0+}^{\alpha}\bar{x}(t)\right),\ t\in(0,1). \tag{131}$$

Setting

$$z(t)=\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{x}(t)\right)-\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x^*(t)\right),\ t\in(0,1). \tag{132}$$

From (130) and (131), we can get

$$\begin{aligned}\mathbf{T}_{0+}^{\beta}z(t) &= \mathbf{T}_{0+}^{\beta}\left(\phi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{x}\right)\right)(t)-\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x^*\right)\right)(t) \\ &\geq f\left(t,\overline{x}(t),-\mathbf{T}_{0+}^{\alpha}x(t)\right)-f\left(t,\overline{x}(t),-\mathbf{T}_{0+}^{\alpha}x(t)\right)=0,\ t\in(0,1),\end{aligned}$$

with

$$z(0)=0\quad\text{and}\quad z(1)=\varphi_p(b_2)z(\zeta_1).$$

Thus, by Lemma 3.6, we have $z(t)\leq 0$, $t\in[0,1]$, which implies that

$$\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{x}\right)(t)\leq\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x^*\right)(t),t\in[0,1].$$

Since φ_p is monotone increasing, we obtain

$$\mathbf{T}_{0+}^{\alpha}\overline{x}(t)\leq\mathbf{T}_{0+}^{\alpha}x^*(t)\Longrightarrow\mathbf{T}_{0+}^{\alpha}\left(\overline{x}(t)-x^*(t)\right)\leq 0,\ t\in[0,1].$$

Combining Lemma 3.6 and (129), we have $\overline{x}(t) \geq x^*(t)$ This contradiction proves the validity of $\overline{x}(t) < x^*(t)\,,\ t \in [0,1]\,.$

Similarly, suppose by contradiction that $\underline{x}(t) > x^*(t)$. By the same way, we also have $x^*(t) \geq \underline{x}(t)$ on $t \in [0,1]$ so,

$$\underline{x}(t) \leq x^*(t) \leq \overline{x}(t)\,,\ t \in [0,1]\,, \tag{133}$$

that is, $x^*(t)$ is a positive solution of the conformable boundary value problem (6-7). Furthermore, $\underline{x}(t)\,,\ \overline{x}(t) \in P$ implies that there exist two positive constants $0 < a_1 < 1 < a_2$ such that

$$0 < a_1 G_\alpha(t) \leq \underline{x}(t) < x(t) < \overline{x}(t) \leq a_2 G_\alpha(t)\,,\ t \in [0,1]\,.$$

Thus, we have finished the proof of Theorem 4.1. From the Theorem 4.1, we can easily derive the following corollary.

Corollary 1. *Suppose that condition (H) are satisfied, then the conformable boundary value problem (6-7) with the Lidstone boundary conditions*

$$x(0) = 0,\ x(1) = 0, \mathbf{T}_{0+}^{\alpha} x(0) = 0,\ \mathbf{T}_{0+}^{\alpha} x(1) = 0,$$

has at least one positive solution x.

we present the following theorem without proof because the proof are similar to Theorem 4.1.

Theorem 4.2. *If $f\left(t, x, \mathbf{T}_{0+}^{\alpha} x(t)\right) \in C([0,1] \times (0,+\infty) \times (-\infty, 0), [0,+\infty))$ is creasing in x and $f\left(t, G_\alpha(t), t^{2-\alpha}\right) \neq 0$ for any $G_\alpha(t) > 0$, then the boundary value problem (6-7) has at least one positive solution x, and there exist two positive constants $0 < a_1 < 1 < a_2$ such that $a_1 G_\alpha(t) < x(t) < a_2 G_\alpha(t)$.*

4.1. **Example.** We conclude this section with an example as an application of our discussion. Typical functions that satisfy the above sub-linear hypothesis are those taking the form

$$f(t,x,y) = \sum_{l}^{n} \sum_{k}^{m} Q_{l,k}(t)\, x^{\mu_k} y^{\mu_l},$$

here $Q_{l,k}(t) \in C(0,1)\,,\ Q_{l,k}(t) > 0$ on $(0,1)\,,\ \mu_k \in \mathbb{R},\ \mu_l < 1.$

To obtain the approximate solutions of (6-7), the assumption of $f(t,x,y) \leq N_0(t)N_1(x)N_2(y)$ is usually needed. Now, we give one example to illustrate the above results. Consider the following p-Laplacian conformable boundary value problem (6-7) where

$$f(t,x,y) = t^{\mu_0}(1-t)^{\mu_1} x^{\mu_2} y^{\mu_3},\ 0 < t < 1, \tag{134}$$

where $p > 1$, $\mu_0, \mu_1 \in \mathbb{R}$ and $\mu_2, \mu_3 > 0$, $\mu_2 + \mu_3 > p - 1$.

Clearly, f is nonincreasing relative to x. This shows that (H) holds. Theorem 4.1 implies that the boundary value problem (134) has at least one positive solution.

Conclusion. A necessary and sufficient condition for problem (134) to have at least one $C^2[0,1]$ positive solution is

$$(\beta - 1) + \mu_0 + (2-\alpha)\mu_3 + (\alpha - 1)\mu_2 > -1 \text{ and } \ \mu_1 > -2,$$

or

$$(\beta - 1) + \mu_0 + (2-\alpha)\mu_3 + (\alpha - 1)\mu_2 > -1 \text{ and } \mu_1 + \mu_2 > -2.$$

5. **Dependence of solution on the parameters.** For f Lipschitz in the second and third variables, the solution's dependence on the order of the differential operator, the boundary values, and the nonlinear term f are also discussed.

In the following, suppose that (A) holds and for any $x \in X$ and $t \in (0,1)$, we let

$$(fx)(t) := f\left(t, x(t), -\mathbf{T}_{0+}^{\alpha} x(t)\right), (fx_\epsilon)(t) := f\left(t, x_\epsilon(t), -\mathbf{T}_{0+}^{\alpha} x_\epsilon(t)\right). \tag{135}$$

5.1. **The dependence on parameters of the left-hand side of (6).** We show that the solutions of two equations with neighbouring orders will (under suitable conditions on their right hand sides f) lie close to one another.

Theorem 5.1. *Suppose that the conditions of Theorem 4.1 hold. Let $x(t)$, $x_\epsilon(t)$ be the solutions, respectively, of the problems (6-7) and*

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(T^{\alpha-\epsilon}x\right)\right)(t) = f\left(t, x(t), -\mathbf{T}_{0+}^{\alpha} x(t)\right),\ t \in (0,1), \epsilon > 0, \tag{136}$$

with the boundary conditions (7), where $1 < \alpha - \epsilon < \alpha \leq 2$. Then, $\|x - x_\epsilon\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof. By the above theorems, we can obtain the following results. Let

$$x_\epsilon(t) = \int_0^1 G_{\alpha\epsilon}(t,s)\varphi_q\left(\int_0^1 G_\beta(\tau,s)(fx_\epsilon)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{137}$$

be the solution of (6-7), where

$$G_{\alpha\epsilon}(t,s) = k_{\alpha-\epsilon}(t,s) + \frac{b_1 t}{1 - b_1\xi_1} k_{\alpha-\epsilon}(\xi_1, s). \tag{138}$$

On one hand, from (37) and (137) yields

$$\begin{aligned}
|x(t) - x_\epsilon(t)| &= \left|\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx)\,\mathrm{d}\tau\right)\mathrm{d}s\right. \\
&- \left.\int_0^1 G_{\alpha\epsilon(t,s)}\varphi_q\left(\int_0^1 G_{\beta(t,s)}(fx_\epsilon)\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\
&= \left|\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right. \\
&- \int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx_\epsilon)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&+ \int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx_\epsilon)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&- \left.\int_0^1 G_{\alpha\epsilon(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx_\epsilon)\,\mathrm{d}\tau\right)\mathrm{d}s\right|.
\end{aligned} \tag{139}$$

Thus,

$$\begin{aligned}
|x(t) - x_\epsilon(t)| &\leq \int_0^1 G_{\alpha(t,s)}\left|\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx)\,\mathrm{d}\tau\right)\right. \\
&- \left.\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx_\epsilon)\,\mathrm{d}\tau\right)\right|\mathrm{d}s \\
&+ \int_0^1 \left|G_{\alpha(t,s)} - G_{\alpha\epsilon(t,s)}\right|\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}(fx_\epsilon)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned} \tag{140}$$

On the other hand, in a similar manner, we can get

$$\begin{aligned}\left|\mathbf{T}_{0+}^{\alpha}x\left(t\right)-\mathbf{T}_{0+}^{\alpha}x_{\epsilon}\left(t\right)\right| &= \left|t^{1-\alpha}\left(\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right.\right.\\ &- \left.\left.\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right)\mathrm{d}s\right|\\ &= \left|t^{1-\alpha}\left(\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right.\right.\\ &- \left.\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right)\mathrm{d}s\\ &+ t^{1-\alpha}\left(\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right.\\ &- \left.\left.\int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha\epsilon(t,s)}\varphi_{q}\left(\int_{0}^{1}G_{\beta(\tau,s)}\left(fx_{\epsilon}\right)\mathrm{d}\tau\right)\right)\mathrm{d}s\right|.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\mathbf{T}_{0+}^{\alpha}x\left(t\right)-\mathbf{T}_{0+}^{\alpha}x_{\epsilon}\left(t\right)\right| \\ \leq\ & \int_{0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\left|\varphi_{q}\left(\int_{0}^{1}G_{\beta(t,s)}\left(fx\right)\left(t\right)\mathrm{d}\tau\right)\right.\\ -\ & \left.\varphi_{q}\left(\int_{0}^{1}G_{\beta(t,s)}\left(fx_{\epsilon}\right)\left(t\right)\right)\mathrm{d}\tau\right|\mathrm{d}s\\ +\ & \int_{0}^{1}\left|\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}-\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha\epsilon(t,s)}\right|\varphi_{q}\left(\int_{0}^{1}G_{\beta(t,s)}\left(fx_{\epsilon}\right)\left(t\right)\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned} \tag{141}$$

Moreover, from (140), (141), we have

$$\begin{aligned}\left\|x-x_{\epsilon}\right\| &\leq \varphi_{q}\left(\int_{0}^{1}G_{\beta(t,s)}\mathrm{d}s\right)\\ &\times \left[\int_{0}^{1}\left|G_{\alpha(t,s)}+\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\right|\left|\varphi_{q}\left(\left(fx\right)\left(t\right)\right)-\varphi_{q}\left(\left(fx_{\epsilon}\right)\left(t\right)\right)\mathrm{d}s\right|\right.\\ &+ \left.\int_{0}^{1}\left|\left(G_{\alpha(t,s)}-G_{\alpha\epsilon(t,s)}\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}-\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha\epsilon(t,s)}\right)\right|\varphi_{q}\left(fx_{\epsilon}\right)\mathrm{d}s.\right]\end{aligned} \tag{142}$$

From (36), we have

$$\begin{aligned}\int_{0}^{1}\mathcal{G}_{\beta}(t,s)\mathrm{d}s &\leq \int_{0}^{t}\left(1-t\right)s^{\beta-1}\mathrm{d}s+\int_{t}^{1}t\left(1-s\right)s^{\beta-2}\mathrm{d}s\\ &\leq \tfrac{1-t}{\beta}+\tfrac{t}{\beta(\beta-1)}\leq\tfrac{1}{\beta-1}.\end{aligned}$$

Using an analogous argument it holds that

$$\begin{gathered}\int_{0}^{1}\left|G_{\alpha(t,s)}\right|\mathrm{d}s\leq\tfrac{1}{\alpha-1}\left(1+\tfrac{b_{1}}{1-b_{1}\xi_{1}}\right)\ \text{and}\\ \int_{0}^{1}\left|\frac{\mathrm{d}}{\mathrm{d}t}G_{\alpha(t,s)}\right|\mathrm{d}s\leq\frac{1}{\alpha-1}\left(1+\tfrac{b_{1}}{1-b_{1}\xi_{1}}\right).\end{gathered} \tag{143}$$

Similarly, it holds that

$$\int_0^1 \left|G_{\alpha(t,s)} - G_{\alpha\epsilon}(t,s)\right| \mathrm{d}s \leq \tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)$$

and

$$\int_0^1 G_{\beta(t,s)}\mathrm{d}s \leq \tfrac{1}{\beta-1}\left[1+\tfrac{b_0}{1-b_0\xi_2}\right] \text{ and } \int_0^1 G_{\beta(\xi_2,s)}\mathrm{d}s \leq \tfrac{1}{\beta-1}\left[1+\tfrac{b_0\xi_2}{1-b_0\xi_2}\right]. \quad (144)$$

i) In case $1 < q \leq 2$ we apply (15). From (140),we have

$$\begin{aligned} &\left|\varphi_q\left(\left(fx\right)(t)\right) - \varphi_q\left(\left(fx_\epsilon\right)(t)\right)\right| \\ &\qquad \leq 2^{2-q}\left|\left(fx\right)(t) - \left(fx_\epsilon\right)(t)\right|^{q-1} \\ &\qquad \leq 2^{2-q}L^{q-1}\left\|x-x_\epsilon\right\|^{q-1}, \end{aligned} \quad (145)$$

where

$$\left|\left(fx\right)(t) - \left(fx_\epsilon\right)(t)\right| \leq L\left\|x-x_\epsilon\right\|.$$

From (140), we get

$$\begin{aligned} \left\|x-x_\epsilon\right\| \leq\ & \tfrac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\left\|x-x_\epsilon\right\|^{q-1} \quad (146) \\ & +2^{2-q}\left(\|f\|\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\left\|x-x_\epsilon\right\| \leq \quad (147) \\ &\frac{2^{2-q}\left(\frac{|||f|||}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\left(1+\frac{b_1}{1-b_1\xi_1}\right)}{\left[1-\frac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\frac{b_1}{1-b_1\xi_1}\right)\left(\frac{1}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\right]}\frac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}, \end{aligned}$$

where

$$|||f||| = \sup_{0<\epsilon<\alpha-1}\left\{\max\left|f\left(t,x_\epsilon,-\mathbf{T}^\alpha_{0+}x_\epsilon\right)\right| : t\in(0,1)\right\}$$

and

$$0 < \left[1-\tfrac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\right] \leq 1.$$

ii) In case $q > 2$, we apply (16) and the inequality (17) to obtain

$$\left|\varphi_q\left(\left(fx\right)(t)\right) - \varphi_q\left(\left(fx_\epsilon\right)(t)\right)\right|$$

$$\begin{aligned} \leq\ & (q-1)\left[\left|(fx)-(fx_\epsilon)\right| + \left|(fx_\epsilon)\right|^{q-2}\left(\left|(fx)-(fx_\epsilon)\right|\right)\right] \\ \leq\ & (q-1)\left[\left|(fx)(t)-(fx_\epsilon)(t)\right| + \left|(fx_\epsilon)(t)\right|^{q-2}\left(\left|(fx)(t)-(fx_\epsilon)(t)\right|\right)\right] \\ \leq\ & (q-1)\left[L\left\|x-x_\epsilon\right\| + \|f\|\right]^{q-2}\left(L\left\|x-x_\epsilon\right\|\right) \\ \leq\ & (q-1)\left(\lambda^{3-q}L^{q-2}\left\|x-x_\epsilon\right\|^{q-2} + \mu^{3-q}\|f\|^{q-2}\right)\left(L\left\|x-x_\epsilon\right\|\right). \end{aligned}$$

Hence,

$$\begin{aligned} \left|\varphi_q\left(\left(fx\right)(t)\right) - \varphi_q\left(\left(fx_\epsilon\right)(t)\right)\right| \leq\ & (q-1)\left(\lambda^{3-q}L^{q-2} + \mu^{3-q}\|f\|^{q-2}\right) \\ \times\ & \left(L\left\|x-x_\epsilon\right\|\right). \end{aligned}$$

Thus, by (140), we have

$$\begin{aligned}\|x-x_\epsilon\| &\le C_1\left(\lambda^{3-q}L^{q-2}+\mu^{3-q}\|f\|^{q-2}\right) \\ &\quad +\tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\|f\|\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1},\end{aligned} \tag{148}$$

where

$$C_1=\tfrac{(q-1)}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\frac{1}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-2}.$$

Thus, in accordance with (147) and (148) we obtain $\|x-x_\epsilon\|=O(\epsilon)$. □

Theorem 5.2. *Suppose that the conditions of Theorem 4.1 hold. Let $x(t),x_\epsilon(t)$ be the solutions, respectively, of the problems (6-7) and*

$$T_t^{\beta-\epsilon}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)\right)(t)=f\left(t,x(t),-\mathbf{T}_{0+}^{\alpha}x(t)\right),\ t\in(0,1),\ \epsilon>0, \tag{149}$$

with the boundary conditions (7), where $1<\beta-\epsilon<\beta<2$. Then,

$$\|x-x_\epsilon\|=O(\epsilon).$$

Proof. Let

$$x_\epsilon(t)=\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta\epsilon(t,s)}(fx_\epsilon)(t)\right)\mathrm{d}s, \tag{150}$$

be the solution of (6-7), where

$$G_{\beta\epsilon}(t,s)=k_{\beta-\epsilon}(t,s)+\frac{b_0t}{1-b_0\xi_2}k_{\beta-\epsilon}(\xi_2,s),\ b_0=b_2^{p-1}. \tag{151}$$

Then (150) and (151) yields.
Observing that

$$\int_0^1 G_{\beta(t,s)}\mathrm{d}s\le\tfrac{1}{\beta-1}\left[1+\tfrac{b_0}{1-b_0\xi_2}\right]\ \text{and}\ \int_0^1 G_{\beta(\xi_2,s)}\mathrm{d}s\le\tfrac{1}{\beta-1}\left[1+\tfrac{b_0\xi_2}{1-b_0\xi_2}\right]. \tag{152}$$

Similarly of Theorem 5.1, we also have $\|x-x_\epsilon\|=O(\epsilon)$. □

5.2. **The dependence on parameters of initial conditions.** Let us introduce small changes in the initial conditions of (6-7) and consider (6) with boundary conditions

$$\begin{aligned} x(0)=0,\ x(1)=b_1x(\xi_1), \\ T_{0+}^{\alpha-\epsilon}x(0)=0,\ T_{0+}^{\alpha-\epsilon}x(1)=b_2T_{0+}^{\alpha-\epsilon}x(\xi_2),\ 1<\alpha-\epsilon<\alpha.\end{aligned} \tag{153}$$

Theorem 5.3. *Assume the conditions of Theorem 4.1 hold. Let $x(t),x_\epsilon(t)$ be respective solutions, of the problems (6-7) and the boundary conditions (6-153). Then $\|x-x_\epsilon\|=O(\epsilon)$.*

Proof. Let

$$x_\epsilon(t)=\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta\epsilon}(\tau,s)(fx_\epsilon)(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \tag{154}$$

solutions of the problem (6-153), where

$$G_{\beta\epsilon}(t,s)=\mathcal{G}_\beta(t,s)+\tfrac{b_0t}{1-b_0\xi_2}\mathcal{G}_\beta(\xi_2,s),b_0=\varphi_p\left(b_2\xi_2^\epsilon\right). \tag{155}$$

It is easy to see that $\|x-x_\epsilon\|=O(\epsilon)$. □

5.3. **The dependence on parameters of the right-hand side of (6).**

Theorem 5.4. *Suppose that the conditions of Theorem 4.1 hold.*
Let $x(t), x_\epsilon(t)$ be the solutions, respectively, of the problems (6-7) and

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}x\right)\right)(t) = f\left(t, x(t), -\mathbf{T}_{0+}^{\alpha}x(t)\right) + \epsilon,\ t \in (0,1), \tag{156}$$

with boundary conditions (7), where $1 < \alpha \leq 2$. Then $\|x - x_\epsilon\| = O(\epsilon)$.

Proof. In accordance with Lemma 3.1, we have

$$x_\epsilon(t) = \int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(\tau,s)}\left[(fx_\epsilon)(\tau) + \epsilon\right]\mathrm{d}\tau\right)\mathrm{d}s. \tag{157}$$

i) In case $1 < q \leq 2$ we apply (15). From (140), we have

$$\begin{aligned}
&\left|\varphi_q\left((fx)(t)\right) - \varphi_q\left((fx_\epsilon)(t) + \epsilon\right)\right| \\
&\qquad \leq 2^{2-q}\left(\left|(fx)(t) - (fx_\epsilon)(t)\right| + \epsilon\right)^{q-1} \\
&\qquad \leq 2^{2-q}\left(\lambda^{2-q}L^{q-1}\|x - x_\epsilon\|^{q-1} + \mu^{2-q}\epsilon^{q-1}\right).
\end{aligned} \tag{158}$$

Therefore, by taking

$$\begin{aligned}
R_{01} &= (q-1)\left[(\lambda_1\mu_1)^{q-3} + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fx_\epsilon)(t)\right|\right], \\
R_{02} &= (q-1)\left[(\lambda_1\mu_1)^{q-3} + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fx_\epsilon)(t)\right|\right],
\end{aligned}$$

we get

$$\left|\varphi_q\left((fx)(t)\right) - \varphi_q\left((fx_\epsilon)(t) + \epsilon\right)\right| \leq R_{01}\|x - x_\epsilon\| + R_{02}.$$

ii) When $q > 2$:

$$\begin{aligned}
&\left|\varphi_q\left((fx)(t)\right) - \varphi_q\left((fx_\epsilon)(t) + \epsilon\right)\right| \\
&\leq (q-1)\left(\left|(fx)(t) - (fx_\epsilon)(t) - \epsilon\right| + \left|(fx_\epsilon)(t) + \epsilon\right|\right)^{q-2} \\
&\times\left(\left|(fx) - (fx_\epsilon) - \epsilon\right|\right) \\
&\leq (q-1)\left(\lambda^{3-q}\left|(fx) - (fx_\epsilon) - \epsilon\right|^{q-2} + \mu^{3-q}\left|(fx_\epsilon) + \epsilon\right|^{q-2}\right) \\
&\times\left(\left|(fx) - (fx_\epsilon) - \epsilon\right|\right) \\
&\leq (q-1)\left(\lambda^{3-q}\left|(fx) - (fx_\epsilon) - \epsilon\right|^{q-2} + \mu^{3-q}\left|(fx_\epsilon) + \epsilon\right|^{q-2}\right) \\
&\times\left(\left|(fx) - (fx_\epsilon) - \epsilon\right|\right) \\
&\leq R_{11}\|x - x_\epsilon\| + R_{12}.
\end{aligned} \tag{159}$$

Therefore, by taking

$$\begin{aligned}
R_{11} &= (q-1)\left[(\lambda_1\mu_1)^{q-3}(1+\epsilon) + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fx_\epsilon)\right|\right], \\
R_{12} &= (q-1)\left[(\lambda_1\mu_2)^{q-3}\epsilon^{q-1} + \lambda_2^{q-3}\epsilon^2 + \lambda_2^{q-3}\left|(fx_\epsilon)(t)\right|\epsilon\right],
\end{aligned}$$

we get

$$\left|\varphi_q\left((fx)(t)\right) - \varphi_q\left((fx_\epsilon)(t) + \epsilon\right)\right| \leq R_{11}\|x - x_\epsilon\| + R_{12}.$$

It is easy to see that $\|x - x_\epsilon\| = O(\epsilon)$. $\square$

6. **Conclusion.** Differential equations with the p-Laplacian operator arise in modeling different physical and natural phenomena, which can be encountered in, for example, non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow laws and system of Monge–Kantorovich partial differential equations. Thus, the consideration of p-Laplacian equation in frame of derivatives becomes compulsory. As a result of this interest, several results have been revealed and different versions of p-Laplacian equation have been under study.

In this article, we have studied a class of fourth point singular boundary value problem of p-Laplacian operator in the setting of a local fraction derivative, namely a newly defined conformable derivative. By using the upper and lower solutions method and fixed point theorems on cones, necessary and sufficient conditions for the existence of positive solutions were obtained.

We present an example to demonstrate the consistency to the theoretical findings. We have also investigated the continuous dependence of solutions all on its right side function, initial value condition. Using these results, the properties of the solution process can be discussed through numerical simulation. We hope to consider this problem in a future work.

We claim that the results of this paper is new and generalize some earlier results. For example, by taking $p = \alpha = \beta = 2$ and $b_2 = 0$, in the results of this paper which can be considered a special case of a simple Jerk Chaotic circuit equation see [20].

At the end of this article, we also remark that the extension of the previous results to the nonlinearities depending on the time delayed differential system or impulsive differential equation taking into account that sometimes the corresponding research when the fractional derivative with non-singular kernels are considered is interesting, in the future work, we will focus our concentration on the Caputo-Fabrizio derivative, Atangana-Baleanu, fractional derivatives of a function with respect to another function and try to mix idea of this work with q-fractional derivatives. Also, the reader can find some new methods for approximate solutions of fractional integro-differential equations involving the Caputo-Fabrizio derivative or extended fractional Caputo-Fabrizio derivative. The approximation solutions are interesting and need more concentration.

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