Nonlinear discrete fractional sum inequalities related to the theory of discrete fractional calculus with applications

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Abstract

By means of ς fractional sum operator, certain discrete fractional nonlinear inequalities are replicated in this text. Considering the methodology of discrete fractional calculus, we establish estimations of Gronwall type inequalities for unknown functions. These inequalities are of a new form comparative with the current writing discoveries up until this point and can be viewed as a supportive strategy to assess the solutions of discrete partial differential equations numerically. We show a couple of employments of the compensated inequalities to reflect the benefits of our work. The main outcomes might be demonstrated by the use of the examination procedure and the approach of the mean value hypothesis.

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1 Introduction

Fractional calculus consisting of a derivative and an integral component of noninteger order is a natural increase in the regular integer order calculus. With different analysts and experts devoting themselves to this area, fractional analytic is apparently widespread considering its intriguing applications concerning various fields of science, for instance, viscoelasticity, dispersion, nervous system science, control hypothesis, and statistics [1–9].

The justification for this paper is to implement discrete fractional sum equations in terms of creating a method for interpreting such equations and to derive the related Gronwall form of inequality. Particularly Gronwall's inequality is pointed out as one of the central inequalities in the premise of differential form equations. Starting now and into the foreseeable future, various speculations and growth of these inequalities ended up being a bit of the composition. In 1969, Sugiyama [10] claimed to have turned up and created the discrete Gronwall inequality. In the associated structure he carried out the most precise and complete discrete module of Gronwall inequality as follows.

Theorem 1.1 *Let $h(t_1)$ and $r(t_1)$ be real-valued functions defined for $t_1 \in \mathbb{N}_0$, and suppose that $r(t_1) > 0$ for every $t_1 \in \mathbb{N}_0$. If*

$$h(t_1) \leq \eta_0 + \sum_{t_2=0}^{t_1-1} r(t_2)h(t_2), \quad t_1 \in \mathbb{N}_0,$$

where η_0 is a nonnegative constant, then

$$h(t_1) \leq \eta_0 \prod_{t_2=0}^{t_1-1} \big[1 + r(t_2)\big].$$

For difference and integral equations, Theorem 1.1 is as yet used to possess the integration of the discrete factor models.

Despite the existence of a systematic mathematical theory of continuous fractional calculus, the potential progress of the discrete fractional calculus (DFC) has been insufficient until very recently. The discrete counterpart of the hypothesis in the presence of a fractional sum of order $\varsigma > 0$ was described by Miller and Ross [11] who discussed solutions to linear difference equation and checked some basic features of this operator. Additionally, Atici and Eloe [12] actualized a discrete Laplace transform strategy for a sequence of fractional difference equations. The triggers of the initial value in the discrete fractional calculus were established by Atici and Eloe [13]. Atici and Eloe [14] explored the layout of a discrete fractional calculus with the nabla operator. They generated exponential laws and the item rule for the forward fractional calculus. Atici and Sengul [15] set up the law of Leibniz and summation by parts equation in a discrete fractional principle. Bastos and Torres [16] created a more wide-ranging, discrete fractional operator, which has been calculated by delta and nabla fractional sums. Holm [17] introduced fractional sums and difference operators and extended this concept to resolve the issue of fractional initial value. Anastassiou [18] defined the privilege of the discrete nabla fractional Taylor equation. The science that came about because of this depiction was charming to a few perusers, and now it is a subject of extreme examination in various ways: existence and accuracy of discrete fractional equations, modeling of tumor growth [19], stability of tumor-based solutions to the order of Legendre's derivative ς [20], Euler–Lagrange equation, and optimal status for calculus of variations problems [21]. The impression of discrete fractional calculus is being presented just more as of late, generally attributable to the blast of work in the analysis of fractional differential (see, for example, the books [22, 23]). Usually, within a particular fractional system, there are several derivative analogues, so experts choose those that are generally appropriate in a specific sense (see [24–28]).

Finite difference inequalities showing unique limits of unknown functions provide a thoroughly valuable and important method for improving the perception of finite differential equations. During the recent years, guided and motivated by their characterizations in different parts of difference equations, several of these inequalities have been linked up [29–34]. Hence, difference equations arise as logical constructs that describe such real-life situations, e.g., queueing problems, electrical networks, financial dimensions, etc., and this defense is sufficient to seek such a framework. There are many representations for these sorts of inequalities at the point where one wants to evaluate several properties of a differential equation. Basically reliant on the capacity of the above investigation, in this

material, we can search for the discrete fractional nonlinear inequalities related to ς fractional sum operator that has been built up to explain fractional inequalities and integrate some proven literature trials.

To represent the theoretical aspects, it was seen that the inequalities delivered may be used to analyze various classes of discrete fractional differential equations. So as to investigate the uniqueness and boundedness of the usage of fractional sum difference equations, two theorems are secured throughout this manuscript.

Definitive parts of the record are situated as such. In Sect. 2, we portray significant real factors and fundamental hypotheses which can be key setups for our main impacts. Section 3 is devoted to abstract discussions of nonlinear discrete fractional inequalities with a few conducting remarks. The last bit is considered in fulfilling the theoretical examination necessities.

2 Material history

Throughout this endeavor, without lack of broad declaration, let $C^{\alpha}(L, K)$ be the class of functions of continuously differentiable with all α times from a set L into a set K, P be a constant, $\mathbb{N}_{t_1} = \{t_1, t_1 + 1, t_1 + 2, \ldots\}$, $M_{t_1} = [t_1, P] \cap \mathbb{N}_{t_1}$, where $P, t_1 \in \mathbb{N}_{t_1}$, $\sum_{t_2=c}^{\vartheta} r(t_2) = 0$, $\mathbb{R}_+ = [0, \infty)$ and difference operator of q be assigned as $\Delta q(\vartheta) = q(\vartheta) - q(\vartheta - 1)$, $\vartheta \in \mathbb{N}_{t_1}$.

A portion of the basic necessities and theorems in the assessment of discrete fractional are accounted for as follows.

Definition 2.1 ([15]) Let $\varsigma > 0$, l be any real number, and $\sigma(t_1) = t_1 + 1$, then ςth fractional sum of r is defined for $t_2 = l \pmod 1$ by

$$\Delta_l^{-\varsigma} r(t_1) = \frac{1}{\Gamma(\varsigma)} \sum_{t_2=l}^{t_1-\varsigma} \big(t_1 - \sigma(t_2)\big)^{\varsigma-1} r(t_2)$$

so that $t_1^{\varsigma} = \frac{\Gamma(t_1+1)}{\Gamma(t_1+1-\varsigma)}$, $\Delta_l^{-\varsigma} r$ is defined for $t_2 = l + \varsigma \pmod 1$, and $\Delta_l^{-\varsigma} : \mathbb{N}_l \to \mathbb{N}_{l+\varsigma}$.

Definition 2.2 ([15]) Let $\delta > 0$ and $\gamma - 1 < \delta < \gamma$. Then the δth fractional difference of r is characterized as

$$\Delta^{\delta} r(t_1) = \Delta^{\gamma-\varsigma} r(t_1) = \Delta^{\gamma}\big(\Delta^{-\varsigma} r(t_1)\big),$$

where γ is a positive integer and $-\varsigma = \delta - \gamma$.

Theorem 2.3 ([13]) *If a real-valued function r is prescribed on $\mathbb{N}_l$, such that $\delta, \varsigma > 0$, then*

$$\Delta^{-\varsigma}\big(\Delta^{-\delta} r(t_1)\big) = \Delta^{-(\varsigma+\delta)} r(t_1) = \Delta^{-\delta}\big(\Delta^{-\varsigma} r(t_1)\big).$$

Theorem 2.4 ([13]) *Let $\varsigma > 0$ and r be a function which is real valued on $\mathbb{N}_l$, then*

$$\Delta^{-\varsigma} \Delta r(t_1) = \Delta \Delta^{-\varsigma} r(t_1) = \frac{(t_1 - l)^{\varsigma-1}}{\Gamma(\varsigma)} r(l).$$

On the discrete fractional principle, the reader can turn their attention to further important properties to [13, 15].

In this article, based on the Riemann–Liouville definition of fractional difference pioneered by Miller and Ross [11] and generated by Atici and Eloe [13], we explore certain new nonlinear discrete fractional sum inequalities which lead to the generalizations of Gronwall–Bellman types.

3 Outcome assertion

Now we are going to round off the simple tests.

Theorem 3.1 *Suppose that $h \in \mathbb{N}_{\varsigma-1} \to \mathbb{R}_+, W : \mathbb{N}_\varsigma \to \mathbb{R}_+$ are functions, $0 < \varsigma \le 1, j > g > 0$ are constants, and η is a positive nondecreasing function defined on $\mathbb{N}_{\varsigma-1}$. If*

$$h^j(t_1) \le \eta^j(t_1) + \Delta_0^{-\varsigma}\big[W(t_1)h^g(t_1 + \varsigma - 1)\big], \quad t_1 \in M_{\varsigma-1}, \tag{1}$$

is satisfied, then

$$h(t_1) \le \eta(t_1)\left[1 + \frac{j-g}{j}\sum_{t_2=\varsigma}^{t_1} w(t_2 - \varsigma)\eta^{g-j}(t_2)\right]^{\frac{1}{j-g}}, \quad t_1 \in M_{\varsigma-1}, \tag{2}$$

where

$$W(t_2, t_1) = \frac{1}{\Gamma(\varsigma)}(t_1 - t_2 - 1)^{\varsigma-1}W(t_2). \tag{3}$$

Proof Since $\eta(t_1)$ is a positive nondecreasing function, from (1) we have

$$\frac{h^j(t_1)}{\eta^j(t_1)} \le 1 + \Delta_0^{-\varsigma}\left[W(t_1)\frac{h^g(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}\right]. \tag{4}$$

Defining

$$y(t_1) = 1 + \Delta_0^{-\varsigma}\left[W(t_1)\frac{h^g(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}\right], \tag{5}$$

from (4) and (5), we attain

$$\frac{h^j(t_1)}{\eta^j(t_1)} \le y(t_1) \quad \Rightarrow \quad h(t_1) \le \eta(t_1)y^{\frac{1}{j}}(t_1), \quad t_1 \in M_{\varsigma-1}. \tag{6}$$

Inequality (5) with Definition 2.1 imply that

$$y(t_1) = 1 + \frac{1}{\Gamma(\varsigma)}\sum_{t_2=0}^{t_1-\varsigma}(t_1 - t_2 - 1)^{\varsigma-1}W(t_2)\frac{h^g(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}$$

$$= 1 + \sum_{t_2=0}^{t_1-\varsigma} W(t_2, t_1)\eta^{g-\hat{j}}(t_2 + \varsigma - 1)\left[\frac{h(t_2 + \varsigma - 1)}{\eta(t_2 + \varsigma - 1)}\right]^g, \tag{7}$$

where $W(t_2, t_1)$ is given as in (3). Now $W(t_2, t_1), t_1^\varsigma$ by their definition and $W(t_2, t_1))$ is decreasing in t_1 for each $t_2 \in \mathbb{N}_0$. Using straightforward computation, for $t_1 \in M_\varsigma$ and (6),

we get

$$y(t_1) - y(t_1 - 1) = W(t_1 - \varsigma, t_1)\eta^{g-j}(t_1 - 1)\left[\frac{h(t_2 - 1)}{\eta(t_2 - 1)}\right]^j$$

$$+ \sum_{t_2=0}^{t_1-\varsigma-1}\left[W(t_2, t_1) - W(t_2, t_1 - 1)\right]\eta^{g-j}(t_2 + \varsigma - 1)\left[\frac{h(t_2 + \varsigma - 1)}{\eta(t_2 + \varsigma - 1)}\right]^g$$

$$\leq W(t_1 - \varsigma, t_1)\eta^{g-j}(t_1 - 1)\left[\frac{h(t_2 - 1)}{\eta(t_2 - 1)}\right]^g$$

$$\leq w(t_1 - \varsigma)\eta^{g-j}(t_1 - 1)y^{\frac{\hat{g}}{j}}(t_1 - 1)$$

leads to

$$\frac{y(t_1) - y(t_1 - 1)}{y^{\frac{g}{j}}(t_1 - 1)} \leq w(t_1 - \varsigma)\eta^{g-j}(t_1 - 1), \quad t_1 \in M_\varsigma. \tag{8}$$

By the mean value theorem, it can be seen that

$$\Delta\left(\frac{j}{j-g}y^{\frac{g-j}{g}}(t_1 - 1)\right) = \frac{j}{j-g}y^{\frac{j-g}{j}}(t_1) - \frac{j}{j-g}y^{\frac{j-g}{j}}(t_1 - 1)$$

$$= \rho^{\frac{-g}{j}}\Delta y(t_1 - 1) = \frac{\Delta y(t_1 - 1)}{\rho^{\frac{g}{j}}}$$

$$\leq \frac{\Delta y(t_1 - 1)}{y^{\frac{g}{j}}(t_1 - 1)}; \quad \rho \in \left[y(t_1 - 1), y(t_1)\right]. \tag{9}$$

In view of (8) and (9), we conclude

$$\Delta\left(\frac{j}{j-g}y^{\frac{j-g}{j}}(t_1 - 1)\right) \leq w(t_1 - \varsigma)\eta^{g-j}(t_1 - 1). \tag{10}$$

Summing (10) from ς to $t_1 - 1$ and $y(\varsigma - 1) = 1$, we obtain

$$\sum_{t_2=\varsigma}^{t_1-1}\Delta\left(\frac{j}{j-g}y^{\frac{j-g}{j}}(t_1 - 1)\right) \leq \sum_{t_2=\varsigma}^{t_1-1}w(t_2 - \varsigma)\eta^{g-j}(t_2 - 1),$$

that is,

$$y^{\frac{j-g}{j}}(t_1 - 1) \leq 1 + \frac{j-g}{j}\sum_{t_2=\varsigma}^{t_1-1}w(t_2 - \varsigma)\eta^{g-j}(t_2 - 1)$$

or

$$y^{\frac{1}{j}}(t_1 - 1) \leq \left[1 + \frac{j-g}{j}\sum_{t_2=\varsigma}^{t_1-1}w(t_2 - \varsigma)\eta^{g-j}(t_2 - 1)\right]^{\frac{1}{j-g}}, \quad t_1 \in M_\varsigma,$$

so that

$$y^{\frac{1}{j}}(t_1) \leq \left[1 + \frac{j-g}{j} \sum_{t_2=\varsigma}^{t_1} w(t_2 - \varsigma)\eta^{g-j}(t_2)\right]^{\frac{1}{j-g}}, \quad t_1 \in M_{\varsigma-1}, \tag{11}$$

the conclusion of (2) can be obtained from (6) and (11). $\qquad\square$

Remark 3.2 By inserting $\varsigma = 1$, $\Gamma(1) = 1$ (the property of gamma function), $\eta(t_1) = \eta_0$, and $j = g = 1$ in (1), then Theorem 3.1 shifts to Theorem 1.1 [10].

Theorem 3.3 *If* $T : \mathbb{N}_\varsigma \to \mathbb{R}_+$, $j \neq 1$, $j > 1$ *is a constant and the inequality*

$$h^j(t_1) \leq \eta^j(t_1) + \Delta_0^{-\varsigma}\left[W(t_1)h(t_1 + \varsigma - 1)\right]$$
$$\times \left[h(t_1 + \varsigma - 1) + \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma}(t_1 - t_2 - 1)^{\varsigma-1}T(t_2)h(t_2 + \varsigma - 1)\right],$$
$$t_1 \in M_{\varsigma-1}, \tag{12}$$

is satisfied under the same suppositions of h, W, ς, η, $W(t_2, t_1)$ *of Theorem* 3.1, *then*

$$h(t_1) \leq \eta(t_1)\left[1 + \frac{j-1}{j} \sum_{t_2=\varsigma}^{t_1} w(t_2 - \varsigma)\eta(t_2) \prod_{p=\varsigma}^{t_2}\left[1 + \frac{1}{j}w(p - \varsigma)\eta^{2-j}(p) + t(p - \varsigma)\right]\right]^{\frac{1}{j-1}},$$
$$t_1 \in M_{\varsigma-1}, \tag{13}$$

such that

$$T(t_2, t_1) = \frac{1}{\Gamma(\varsigma)}(t_1 - t_2 - 1)^{\varsigma-1}T(t_2). \tag{14}$$

Proof Obviously, by the positive and nondecreasing nature of $\eta(t_1)$, inequality (12) takes the form

$$\frac{h^j(t_1)}{\eta^j(t_1)} \leq 1 + \Delta_0^{-\varsigma}\left[W(t_1)\frac{h(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}\right]$$
$$\times \left[h(t_2 + \varsigma - 1) + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma}(t_2 - p - 1)^{\varsigma-1}T(p)h(p + \varsigma - 1)\right], \tag{15}$$

denoting

$$y_1(t_1) = 1 + \Delta_0^{-\varsigma}\left[W(t_1)\frac{h(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}\right]$$
$$\times \left[h(t_2 + \varsigma - 1) + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma}(t_2 - p - 1)^{\varsigma-1}T(p)h(p + \varsigma - 1)\right], \tag{16}$$

(15), (16) produce

$$\frac{h^j(t_1)}{\eta^j(t_1)} \leq y_1(t_1) \quad \Rightarrow \quad h(t_1) \leq \eta(t_1)y_1^{\frac{1}{j}}(t_1). \tag{17}$$

Utilizing Definition 2.1 to (16), we deduce

$$y_1(t_1) = 1 + \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} W(t_2) \frac{h(t_2 + \varsigma - 1)}{\eta^j(t_2 + \varsigma - 1)}$$

$$\times \left[h(t_2 + \varsigma - 1) + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma} (t_2 - p - 1)^{\varsigma-1} T(p) h(p + \varsigma - 1) \right], \quad t_1 \in M_{\varsigma-1},$$

the last inequality with (17) turns out to be

$$y_1(t_1) \leq 1 + \sum_{t_2=0}^{t_1-\varsigma} W(t_2, t_1) \eta^{2-j}(t_2 + \varsigma - 1) y_1^{\frac{1}{j}}(t_2 + \varsigma - 1)$$

$$\times \left[y_1^{\frac{1}{j}}(t_2 + \varsigma - 1) + \sum_{p=0}^{t_2-\varsigma} T(p, t_2) y_1^{\frac{1}{j}}(p + \varsigma - 1) \right]$$

$$\leq R(t_1), \tag{18}$$

where $T(t_2, t_1)$ is given as in (14) and

$$R(t_1) = 1 + \sum_{t_2=0}^{t_1-\varsigma} W(t_2, t_1) \eta^{2-j}(t_2 + \varsigma - 1) y_1^{\frac{1}{j}}(t_2 + \varsigma - 1)$$

$$\times \left[y_1^{\frac{1}{j}}(t_2 + \varsigma - 1) + \sum_{p=0}^{t_2-\varsigma} T(p, t_2) y_1^{\frac{1}{j}}(p + \varsigma - 1) \right].$$

As $R(t_1) \geq 0$ is nondecreasing and with the support of straightforward computation for $t_1 \in M_\varsigma$, the decreasing nature of $W(t_2, t_1), T(t_2, t_1)$ for $t_2 \in \mathbb{N}_0$, the definition of $W(t_2, t_1)$, $T(t_2, t_1), t_1^\varsigma$, and (18), we get

$$R(t_1) - R(t_1 - 1)$$

$$= W(t_1 - \varsigma, t_1) \eta^{2-j}(t_1 - 1) y_1^{\frac{1}{j}}(t_1 - 1) \left[y_1^{\frac{1}{j}}(t_1 - 1) + \sum_{t_2=0}^{t_1-\varsigma} T(t_2, t_1) y_1^{\frac{1}{j}}(t_2 + \varsigma - 1) \right],$$

$$\leq w(t_1 - \varsigma) \eta^{2-j}(t_1 - 1) R^{\frac{1}{j}}(t_1 - 1) \left[R^{\frac{1}{j}}(t_1 - 1) + \sum_{t_2=0}^{t_1-\varsigma} T(t_2, t_1) R^{\frac{1}{j}}(t_2 + \varsigma - 1) \right].$$

On the other hand, by the mean value theorem, we attain

$$\frac{j}{j-1} \left[R^{1-\frac{1}{j}}(t_1) - R^{1-\frac{1}{j}}(t_1 - 1) \right] = \frac{R^{1-\frac{1}{j}}(t_1) - R^{1-\frac{1}{j}}(t_1 - 1)}{[\varrho(t_1)]^{\frac{1}{j}}}$$

for some $\varrho(t_1) \in [R(t_1 - 1), R(t_1)]$. Furthermore

$$R^{1-\frac{1}{j}}(t_1) - R^{1-\frac{1}{j}}(t_1 - 1)$$

$$\leq \frac{j-1}{j} \left[\frac{R(t_1 - 1), R(t_1)}{R^{\frac{1}{j}}(t_1 - 1)} \right]$$

$$\leq \frac{j-1}{j} w(t_1 - \varsigma)\eta^{2-j}(t_1 - 1)\left[R^{\frac{1}{j}}(t_1 - 1) + \sum_{t_2=0}^{t_1-\varsigma} T(t_2, t_1)R^{\frac{1}{j}}(t_2 + \varsigma - 1)\right], \quad t_1 \in M_\varsigma,$$

the above inequality, by summing from ς to $t_1 - 1$ and with $R(\varsigma - 1) = 1$, implies

$$R^{\frac{j-1}{j}}(t_1 - 1)$$

$$\leq 1 + \frac{j-1}{j} \sum_{t_2=\varsigma}^{t_1-1} w(t_2 - \varsigma)\eta^{2-j}(t_2 - 1)\left[R^{\frac{1}{j}}(t_2 - 1) + \sum_{p=0}^{t_2-\varsigma} T(p, t_2)R^{\frac{1}{j}}(p + \varsigma - 1)\right]. \quad (19)$$

Consider

$$B(t_1) = 1 + \frac{j-1}{j} \sum_{t_2=\varsigma}^{t_1-1} w(t_2 - \varsigma)\eta^{2-j}(t_2 - 1)\left[R^{\frac{1}{j}}(t_2 - 1) + \sum_{p=0}^{t_2-\varsigma} T(p, t_2)R^{\frac{1}{j}}(p + \varsigma - 1)\right], \quad (20)$$

from (19) and (20), one has

$$R^{\frac{j-1}{j}}(t_1 - 1) \leq B(t_1). \tag{21}$$

We proceed from (19) and (20) to

$$B(t_1) - B(t_1 - 1)$$

$$= \frac{j-1}{j} w(t_1 - \varsigma - 1)\eta^{2-j}(t_1 - 2)\left[R^{\frac{1}{j}}(t_1 - 2) + \sum_{t_2=0}^{t_1-\varsigma-1} T(t_2, t_1 - 1)R^{\frac{1}{j}}(t_2 + \varsigma - 1)\right]$$

$$\leq \frac{j-1}{j} w(t_1 - \varsigma - 1)\eta^{2-j}(t_1 - 2)\left[B^{\frac{1}{j-1}}(t_1 - 1) + \sum_{t_2=0}^{t_1-\varsigma-1} T(t_2, t_1 - 1)B^{\frac{1}{j-1}}(t_2 + \varsigma - 1)\right]$$

$$\leq \frac{j-1}{j} w(t_1 - \varsigma - 1)\eta^{2-j}(t_1 - 2)S(t_1), \tag{22}$$

where

$$S(t_1) = B^{\frac{1}{j-1}}(t_1 - 1) + \sum_{t_2=0}^{t_1-\varsigma-1} T(t_2, t_1 - 1)B^{\frac{1}{j-1}}(t_2 + \varsigma - 1) \tag{23}$$

and

$$B^{\frac{1}{j-1}}(t_1 - 1) \leq S(t_1). \tag{24}$$

Equation (23) with inequality (24) becomes

$$S(t_1) - S(t_1 - 1) = B^{\frac{1}{j-1}}(t_1 - 1) - B^{\frac{1}{j-1}}(t_1 - 2) + t(t_1 - \varsigma - 2)B^{\frac{1}{j-1}}(t_1 - 2),$$

$$\leq B^{\frac{1}{j-1}}(t_1 - 1) - B^{\frac{1}{j-1}}(t_1 - 2) + t(t_1 - \varsigma - 2)S(t_1 - 1). \tag{25}$$

A similar analysis of the mean value theorem as before yields

$$\left(\frac{j-1}{j-2}\right)\left[B^{1-\frac{1}{j-1}}(t_1 - 1) - B^{1-\frac{1}{j-1}}(t_1 - 2)\right] = \frac{B(t_1 - 1) - B(t_1 - 2)}{[\varrho(t_1 - 1)]^{\frac{1}{j-1}}}$$

gives

$$B^{1-\frac{1}{j-1}}(t_1-1) - B^{1-\frac{1}{j-1}}(t_1-2) \le \frac{j-2}{j-1}\left[\frac{B(t_1-1)-B(t_1-2)}{\varrho^{\frac{1}{j-1}}(t_1-1)}\right]$$

$$\le \frac{1}{j}w(t_1-\varsigma-2)\eta^{2-j}(t_1-3)S(t_1-1). \tag{26}$$

Substituting (26) in (25), we get

$$S(t_1) - S(t_1-1) \le \frac{1}{j}w(t_1-\varsigma-2)\eta^{2-j}(t_1-3)S(t_1-1) + t(t_1-\varsigma-2)S(t_1-1),$$

which with $S(\varsigma-1) = 1$ offers the estimation

$$S(t_1-1) \le \prod_{t_2=\varsigma}^{t_1-2}\left[1 + \frac{1}{j}w(t_2-\varsigma)\eta^{2-j}(t_2-1)S(t_1-1) + t(t_2-\varsigma)\right], \quad t_1 \in M_\varsigma.$$

Also,

$$S(t_1) \le \prod_{t_2=\varsigma}^{t_1-1}\left[1 + \frac{1}{j}w(t_2-\varsigma)\eta^{2-j}(t_2)S(t_1-1) + t(t_2-\varsigma)\right].$$

From the last inequality and (22), we observe that

$$B(t_1) - B(t_1-1) = \frac{j-1}{j}w(t_1-\varsigma-1)\eta^{2-j}(t_1-2)$$

$$\times \prod_{t_2=\varsigma}^{t_1-1}\left[1 + \frac{1}{j}w(t_2-\varsigma)\eta^{2-j}(t_2)S(t_1-1) + t(t_2-\varsigma)\right]. \tag{27}$$

Summing (27) from ς to t_1-1 and utilizing $B(\varsigma-1) = 1$, we acquire

$$B(t_1-1) \le B(\varsigma-1) + \frac{j-1}{j}\sum_{t_2=\varsigma}^{t_1-1}w(t_2-\varsigma)\eta^{2-j}(t_2-1)$$

$$\times \prod_{p=\varsigma}^{t_2-1}\left[1 + \frac{1}{j}w(p-\varsigma)\eta^{2-j}(p) + t(p-\varsigma)\right], \quad t_1 \in M_\varsigma,$$

or

$$B(t_1) \le 1 + \frac{j-1}{j}\sum_{t_2=\varsigma}^{t_1}w(t_2-\varsigma)\eta^{2-j}(t_2)\prod_{p=\varsigma}^{t_2}\left[1 + \frac{1}{j}w(p-\varsigma)\eta^{2-j}(p) + t(p-\varsigma)\right],$$

$$t_1 \in M_{\varsigma-1}. \tag{28}$$

The acquired bound in (13) can be carried out by substituting (28) in (21), (18), and (17) with $t_1 \in M_{\varsigma-1}$ simultaneously. □

Remark 3.4 If $j = 1$, $\varsigma = 1$, $\Gamma(1) = 1$, $T(t_2) = k(s,\sigma)$ and $h^2(t_1) = u(n)$, $k(n,s)$, $\Delta_1 k(n,s)$, $0 < s < n < 1$, $n,s \in \mathbb{N}_0$ in (12), then Theorem 3.3 can be modified into [35] Theorem 2.3(c2).

Corollary 3.5 *Suppose that h, ς, W, T, η, $W(t_2, t_1)$, $T(t_2, t_1)$ of Theorem 3.3 and $\hat{j}$, g of Theorem 3.1 with the inequality*

$$h^j(t_1) \leq \eta(t_1) + \Delta_0^{-\varsigma}\big[W(t_1)h^g(t_1 + \varsigma - 1)\big]$$

$$+ \frac{1}{\Gamma(\varsigma)}\sum_{t_2=0}^{t_1-\varsigma}(t_1 - t_2 - 1)^{\varsigma-1}T(t_2)h^j(t_2 + \varsigma - 1), \quad t_1 \in M_{\varsigma-1},$$

are fulfilled. Then

$$h(t_1) \leq \eta^{\frac{1}{j}}(t_1)\prod_{t_2=\varsigma}^{t_1}\big[1 + t(t_2 - \varsigma)\big]^{\frac{1}{j}}$$

$$\times \left[1 + \frac{\hat{g} - \hat{j}}{\hat{j}}\sum_{t_2=\varsigma}^{t_1}w(t_2 - \varsigma)\eta^{\frac{g-j}{j}}(t_2)\prod_{p=\varsigma}^{t_2}\big[1 + t(p - \varsigma)^{\frac{g-j}{j}}\big]\right]^{\frac{1}{j-g}}, \quad t_1 \in M_{\varsigma-1}.$$

Remark 3.6 Corollary 3.5 alters into [36], Lemma 2.5 (β_1) by letting $j = 1$, $\varsigma = 1$, $\Gamma(1) = 1$, $T(t_2) = 0$, and $w(t_1) = b(s)$.

4 Boundedness and uniqueness

The boundedness and uniqueness of the discrete fractional inequalities can be evaluated by a relevant practice of Theorem 3.3 in this segment. Consider the IVP of fractional difference equation of the form

$$\begin{cases} \Delta^{\varsigma}h^j(t_1) = K(t_1, h(t_1 + \varsigma - 1), \Delta^{-\varsigma}[V(t_1, h(t_1 + \varsigma - 1))]), & t_1 \in M_{\varsigma-1}, \\ \Delta^{\varsigma-1}h(t_1)|_{t_1=0} = h_0, \end{cases} \tag{29}$$

where $K, V : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ are functions, h_0 is a constant, and t_1, ς, j, h are mentioned as in Theorem 3.3.

The ensuing theorem can illustrate the boundedness on the solutions of (29).

Theorem 4.1 *Suppose that*

$$\big|K(t_1, h, v)\big| \leq W(t_1)|h|\big[|h| + |v|\big], \tag{30}$$

$$\big|V(t_1, h)\big| \leq T(t_1)|h|. \tag{31}$$

If $h(t_1)$ is a solution of (29), then

$$\big|h(t_1)\big| \leq \eta(t_1)\left[1 + \frac{j-1}{j}\sum_{t_2=\varsigma}^{t_1}w(t_2 - \varsigma)\eta(t_2)\prod_{p=\varsigma}^{t_2}\left[1 + \frac{1}{j}w(p - \varsigma)\eta^{2-j}(p) + t(p - \varsigma)\right]\right]^{\frac{1}{j-1}},$$

$$t_1 \in M_{\varsigma-1}. \tag{32}$$

Proof Equation (29) is transformed into

$$h^j(t_1) \leq \frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)}h_0$$

$$+ \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} \hat{K}\big(t_2, h(t_2 + \varsigma - 1), \Delta^\varsigma\big[V\big(t_2, h(t_2 + \varsigma - 1)\big)\big]\big).$$

Apparently, equation (29) with Definition 2.1 and the combination of (30) and (31) approaches to

$$\big|h^j(t_1)\big| \leq \frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)} |h_0| + \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} W(t_2) \big|h(t_2 + \varsigma - 1)\big|$$

$$\times \left[\big|h(t_2 + \varsigma - 1)\big| + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma} (t_2 - p - 1)^{\varsigma-1} T(p) \big|h(p + \varsigma - 1)\big| \right],$$

$$\leq \big|\eta^j(t_1)\big| + \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} W(t_2) \big|h(t_2 + \varsigma - 1)\big|$$

$$\times \left[\big|h(t_2 + \varsigma - 1)\big| + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma} (t_2 - p - 1)^{\varsigma-1} T(p) \big|h(p + \varsigma - 1)\big| \right],$$

where $\frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)} |h_0| \leq |\eta^j(t_1)|$. The remaining calculations can be done through the assumption of correct composition of Theorem 3.3 to get the required inequality (32). □

The uniqueness of solutions of (29) can be identified by the following theorem.

Theorem 4.2 *Let*

$$\big|K(t_1, h_1, v_1) - K(t_1, h_2, v_2)\big| \leq W(t_1)|h_1 - h_2|\big[|h_1 - h_2| + |v_1 - v_2|\big], \tag{33}$$

$$\big|V(t_1, h_1) - V(t_1, h_2)\big| \leq T(t_1)|h_1 - h_2|. \tag{34}$$

Then (29) *has at most one solution.*

Proof IVP (29) with solutions $h_1(t_1)$ and $h_2(t_1)$ is restated as follows:

$$h_1^j(t_1) - h_2^j(t_1)$$

$$= \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} K\big(t_2, h_1(t_2 + \varsigma - 1), \Delta^\varsigma\big[V\big(t_2, h_1(t_2 + \varsigma - 1)\big)\big]\big)$$

$$- \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} K\big(t_2, h_2(t_2 + \varsigma - 1), \Delta^\varsigma\big[V\big(t_2, h_2(t_2 + \varsigma - 1)\big)\big]\big).$$

The last equation with speculations (33), (34) provides

$$\big|h_1^j(t_1) - h_2^j(t_1)\big| \leq \frac{1}{\Gamma(\varsigma)} \sum_{t_2=0}^{t_1-\varsigma} (t_1 - t_2 - 1)^{\varsigma-1} W(t_2) \big|h_1(t_2 + \varsigma - 1) - h_2(t_2 + \varsigma - 1)\big|$$

$$\times \left[\big|h_1(t_2 + \varsigma - 1) - h_2(t_2 + \varsigma - 1)\big| + \frac{1}{\Gamma(\varsigma)} \sum_{p=0}^{t_2-\varsigma} (t_2 - p - 1)^{\varsigma-1} T(p) \right.$$

$$\times \left| h_1(p + \varsigma - 1) - h_2(p + \varsigma - 1) \right| \Bigg].$$

The prior inequality by making a few modifications in the technique for Theorem 3.3 to $|h_1^j(t_1) - h_2^j(t_1)|$ induces

$$\left| h_1^j(t_1) - h_2^j(t_1) \right| \le 0.$$

Subsequently $h_1(t_1) = h_2(t_1)$, and one positive solution of fractional difference equation (29) exists. □

5 Concluding remarks

Fixated on the guideline of discrete fractional analytic and with the advantage of fractional sum inequalities, we suggested new varieties of discrete Gronwall fractional inequalities in this paper. Such inequalities can be seen not exclusively to remember explicit estimations for solutions of fractional difference equations in discrete type yet additionally in the investigation to the uniqueness and continuous dependency on the initial value for the solutions in the analysis.

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