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Nonlinear self-adjointness, conserved vectors, and traveling wave structures for the kinetics of phase separation dependent on ternary alloys in iron (Fe-Cr-Y (Y = Mo, Cu))

Muhammad Bilal Riaz [a], Dumitru Baleanu [b,c,d,*], Adil Jhangeer [e], Naseem Abbas [f]

[a] Department of Mathematics, University of Management and Technology, Pakistan
[b] Department of Mathematics, Cankaya University, 06530 Ankara, Turkey
[c] Institute of Space Sciences, Magurele, 077125 Bucharest, Romania
[d] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
[e] Department of Mathematics, Namal Institute, Talagang Road, Mianwali 42250, Pakistan
[f] Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan

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ABSTRACT

The present exploration is concerned with fundamental elements corresponding to the phase decomposition in (Fe-Cr-Mo) and (Fe-Cr-Cu) ternary composites. For the ternary composites of iron, we examine the dynamical behavior of the phase separation. The dynamic of this separation is depicted by a model known as the Cahn-Hilliard equation. The nonlinear self-adjointness for the model under consideration is taken into account. The conserved quantities are calculated with the help of the direct method. For each symmetry generator, we have reduced the considered equation into non-linear ordinary differential equations (ODEs). Also, we have computed the optimal system of the equation under study to find the similarity reduction. Also, the traveling wave structures of the Cahn-Hilliard equation are obtained with the modified simple equation (MSE) technique. Moreover, solitary wave structures is exhibited graphically in the form of 3D, 2D and contour plots.

Introduction

Many physical phenomena in nature can be modeled [1] or best stated by the differential equations. The solutions of non-linear differential equations impact great value in many fields of engineering and science [2]. The exploration of the traveling wave structures has of great value role in physics, mathematics, and other non-linear physical phenomena. It is not an easy job to find out the solutions to non-linear problems. In literature, there exist many techniques for the evaluation of the correct results for such a class of problems, few of them are given such as, the improved F-expansion method [5], inverse scattering technique [4], the tanh function technique [3], (G'/G)-expansion technique [7] and the Jacobi elliptic function expansion technique [6].

From few past years, the physicists and mathematicians have found the potential of the nonlinear wave equations for speaking to the nonlinear phenomena in a different zone of science, for example, ecology, geography, human science, zoology, designing, medication, applied mathematics, applied physics, medication, and software engineering [8–12].

The numerical and analytical simulations of Cahn-Hilliard equation are discussed with two well-known techniques Modified auxiliary equation technique and cubic B-spline method [13]. In [18], authors have studied Fourier spectral approximation for the Cahn-Hilliard model in a 2D case. P.O. Mchedlov-Petrosyan [20], found the exact solutions of the convective viscous Cahn-Hilliard equation. A. Scheel, Spinodal [21] discussed the decay and coarsening fronts in the Cahn-Hilliard equation. The Instability of traveling waves of the considered model is mentioned in literature [22]. In present work, we inquire the nonlinear self-adjointness, conservation laws, and traveling wave structures of the model under consideration. The considered model has the formula of the type [14–23]:

$$\left.\begin{aligned}(U_{Cr})_\tau &= M_{Cr}\big(V_{U_{Cr}} - K_{Cr}(U_{Cr})_{YY} - L_{CrY}(U_Y)_{\xi\xi}\big)_{\xi\xi},\\ (U_Y)_\tau &= M_Y\big(V_{U_\xi} - L_Y(U_{Cr})_{\xi\xi} - K_\xi(U_Y)_{\xi\xi}\big)_{\xi\xi},\end{aligned}\right\} \quad (1)$$

* Corresponding author.
E-mail addresses: bilalsehole@gmail.com (M.B. Riaz), dumitru.baleanu@gmail.com (D. Baleanu), adil.jhangeer@namal.edu.pk (A. Jhangeer), naseemabbas@math.qau.edu.pk (N. Abbas).

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where $U_Y(\xi,\tau), U_{Cr}(\xi,\tau)$ are attentiveness arenas of Y and Cr features, respectively [42,43]. System (1) gives

$$\left.\begin{aligned}(U_{Cr})_\tau = \; & M_{Cr}\Big[V_{U_{Cr}^2}(U_{Cr})_{\xi\xi} + V_{U_{Cr}U_\xi}(U_Y)_{\xi\xi} + 2V_{U_{Cr}^2U_\xi}(U_{Cr})_\xi(U_Y)_\xi + V_{U_{Cr}^3}((U_{Cr})_\xi)^2 \\ & + V_{U_{Cr}U_\xi^2}((U_Y)_\xi)^2 - K_{Cr}(U_{Cr})_{\xi\xi\xi} - L_{CrY}(U_Y)_{\xi\xi\xi\xi}\Big], \\ (U_Y)_\tau = \; & M_Y\Big[V_{U_{Cr}^2}(U_{Cr})_{\xi\xi} + V_{U_Y^2}(U_Y)_{\xi\xi} + 2V_{U_{Cr}^2U_Y^2}(U_{Cr})_\xi(U_Y)_\xi + V_{U_{Cr}^2U_\xi}((U_{Cr})_\xi)^2 \\ & + V_{U_Y^3}((U_Y)_\xi)^2 - L_{YCr}(U_{Cr})_{\xi\xi\xi\xi} - K_Y(U_Y)_{\xi\xi\xi\xi}\Big].\end{aligned}\right\} \quad (2)$$

The systemic solution model allows indigenous free energy to be written in the following formula

$$\begin{aligned}V = \; & V^\star(1-U_{Cr}-U_Y) + V^{\star\star}U_{Cr} + V^{\star\star\star}U_Y + \Theta_{FeY}U_Y(1-U_{Cr}-U_Y) \\ & + \Theta_{FeCr}U_{Cr}(1-U_{Cr}-U_Y) + \Theta_{CrY}U_{Cr}U_Y + RT[(1-U_{Cr}-U_Y) \\ & \ln(1-U_{Cr}-U_Y) + U_{Cr}\ln(U_{Cr}) + U_Y\ln(U_Y)],\end{aligned} \quad (3)$$

where $V^\star, V^{\star\star}, V^{\star\star\star}$ indicate the energies of pure Fe, pure Cr, and pure Y components, while the parameters of interaction are $\Theta_{FeX}, \Theta_{FeCr}, \Theta_{CrY}$. The singular nature of the logarithmic term around the values of -1 and 1 prevents the solution from reaching these singular values, and this subtle fact indicates that the proposed algorithm has a unique solution with preserved positivity for the logarithmic arguments. [24,25] In addition, R and T represent the constant and absolute temperature of the gas, respectively. Eq. (3) gives

$$\left.\begin{aligned} & V_{U_{Cr}^2} + 2\Theta_{FeCr} - RT\left(\frac{1}{U_{Cr}} + \frac{1}{1-U_{Cr}-U_Y}\right) = 0, \\ & V_{U_Y^2} + 2\Theta_{FeY} - RT\left(\frac{1}{U_Y} + \frac{1}{1-U_{Cr}-U_Y}\right) = 0, \\ & V_{U_{Cr}U_Y} - \Theta_{CrY} + \Theta_{FeCr} + \Theta_{FeY} - \frac{RT}{1-U_{Cr}-U_Y} = 0.\end{aligned}\right\} \quad (4)$$

The application of the Cahn-Hilliard equations for binary alloys of (Fe-Cr) and (Fe-Y) can find mobility and gradient energy. These equations are linearized as

$$(U_i)_\tau + D_i(U_i)_{\xi\xi} + M_iK_i(U_i)_{\xi\xi\xi\xi} = 0, \quad (5)$$

where $U_i = \begin{cases} U_{Cr} \\ U_Y, \end{cases} D_i = M_i(V)_{U_i^2}$ and M_i shows the uphill diffusion. Also, we can mention the Cahn-Hilliard equation as

$$\begin{cases} \Psi_\tau = \nabla.M(\Psi)\nabla[f(\Psi) - \epsilon^2\Delta\Psi], & (\xi,\tau)\in\Theta\times R^+ \\ n.\nabla\Psi = n.M(\Psi)\nabla[f(\Psi) - \epsilon^2\Delta\Psi], & (\xi,\tau)\in\partial\Theta\times R^+. \end{cases} \quad (6)$$

In the view of the above system, we can mark the Convective-Diffusive Cahn-Hilliard Equation

$$\Psi_\tau = \nabla.[M(\Psi)\nabla(f(\Psi))_\Psi - K\nabla^2\Psi], \quad (7)$$

here $\Psi(\xi,\tau)$ is the concentration, $M(\Psi)$ shows the mobility and $f(\Psi)$ indicates the double-well potential (polynomial approximation). Moreover, above equation gives

$$\Psi_\tau + D^4\Psi = D^2A(\Psi) + lD(\Psi), \quad l > 0 \quad (8)$$

here $A(\Psi(\xi,\tau))$ is an intrinsic chemical potential with a particular value as $A(\Psi(\xi,\tau)) = \Psi^3(\xi,\tau) - \Psi(\xi,\tau)$ and $\Psi(\xi,\tau)$ shows the concentration of one of two phases in a system and $(lD\Psi(\xi,\tau))$ is the phase transition affected by the steady fluid flow [40,41].

The pattern of the present article is arranged in this order: Section (), is the introduction of the model. Section (), describes the nonlinear self-adjointness of the considered model, conserved vectors by the direct method, optimal system, and similarity reduction of the convective-diffusive Cahn-Hilliard model. In Section (), we use the MSE method to find the traveling wave structures of the model under study and present their graphical interpretation of obtained solutions. In Section (), we write results and discussion for obtained solutions. Section () is left for the conclusion.

Nonlinear self-adjointness and conserved vectors

A general review

Suppose the general form of the m-th order PDE of the form

$$G = G(\xi, \Psi, \Psi_1, \Psi_2, \ldots, \Psi_m), \quad (9)$$

where dependent variable $\Psi = \Psi(\xi)$ and independent variable $\xi = \xi(\xi_1, \xi_2, \ldots, \xi_n)$. Also Ψ_1 and Ψ_m show the first and m-th order derivative of Ψ respectively. Now the formal Lagrangian $\mathscr{L} = vG$ is assumed such that for Eq. (9) adjoint equation has the following form

$$G^\star \equiv \frac{\delta}{\delta\Psi}(vG), \quad (10)$$

where

$$\frac{\delta}{\delta\Psi} = \frac{\partial}{\partial\Psi} + \sum_{i=1}^{\infty}(-1)^s D_{i^1}\ldots D_{i^s}\frac{\partial}{\partial\Psi_{i^1\ldots i^s}}, \quad (11)$$

is termed as the Euler-Lagrange operator with D_i as total derivative operators and given by

$$D_i = \frac{\partial}{\partial\xi_i} + \Psi_i\frac{\partial}{\partial\Psi} + \Psi_{ij}\frac{\partial}{\partial\Psi_j} + \ldots. \quad (12)$$

Definition 1. Eq. (8) is termed as strictly self-adjoint when the equation attained from its adjoint equation by putting $v = \Psi$ is same as to Eq. (9), if

$$G^\star|_{v=\Psi} = \mu(\xi, \Psi, \ldots)G, \quad (13)$$

for some $\mu \in \mathscr{D}$

Definition 2. Eq. (8) is known as quasi-self-adjoint when the equation attained from its adjoint equation after putting $v = \Phi(\Psi) \neq 0$ is same as that of Eq. (9), such that

$$G^\star|_{v=\Phi(\Psi)} = \mu(\xi, \Psi, \ldots)G, \quad (14)$$

where $\mu \in \mathscr{D}$.

Definition 3. Eq. (8) is called weak self-adjoint if the equation attained from its adjoint equation after putting $v = \Phi(\xi, \Psi) \neq 0$ is same as Eq. (9) for a particular function Φ such that $\Phi_\Psi \neq 0$ and $\Phi_{\xi^i} \neq 0$ for some ξ^i, such that

$$G^\star|_{v=\Phi(\xi,\Psi)} = \mu(\xi, \Psi, \ldots)G, \tag{15}$$

where $\mu \in \mathscr{D}$.

Definition 4. Eq. (8) is called nonlinearly self-adjoint when the equation attained from its adjoint equation by the substitution $v = \Phi(\xi, \Psi)$, with a particular function such that $\Phi(\xi, \Psi) \neq 0$, Eq. (9) fulfill the condition,

$$G^\star|_{v=\Phi(\xi,\Psi)} = \mu(\xi, \Psi, \ldots)G, \tag{16}$$

where $\mu \in \mathscr{D}$. It is good to mention that Ibragimov [26–28] gave the concept of above three Defs. (1), (2), (4) and Gandarias [31] introduced the idea of Def. (3).

Theorem 1. Suppose Lie point, Lie-Backlund or nonlocal symmetry of Eq. (9) of the form

$$Z = \phi^i \frac{\partial}{\partial \xi^i} + \eta \frac{\partial}{\partial \Psi}, \tag{17}$$

with a formal Lagrangian $\mathscr{L}$. Then the conserved vectors for systems (8) and (10) can be written as

$$\begin{aligned} C^{\xi^i} = {} & \phi^i \mathscr{L} + W\left[\frac{\partial \mathscr{L}}{\partial \Psi_i} - D_j\left(\frac{\partial \mathscr{L}}{\partial \Psi_{ij}}\right) + D_j D_k\left(\frac{\partial \mathscr{L}}{\partial \Psi_{ijk}}\right)\right] + D_j(W)\left[\frac{\partial \mathscr{L}}{\partial \Psi_{ij}} - D_k\left(\frac{\partial \mathscr{L}}{\partial \Psi_{ijk}}\right)\right] \\ & + D_j D_k(W)\frac{\partial \mathscr{L}}{\partial \Psi_{ijk}}, \end{aligned} \tag{18}$$

here W is termed as the Lie characteristic function and can be acquired from

$$W = \eta - \phi^i \Psi_i, \tag{19}$$

while $D_i(C^{\xi^i}) = 0$.

$$G^\star = \begin{aligned} & -6v\Psi\Psi_{\xi\xi} - 6v\Psi_\xi^2 - v_\tau + lv_\xi + 12v_\xi\Psi\Psi_\xi + 12v\Psi_\xi^2 + 12v\Psi\Psi_{\xi\xi} + v_{\xi\xi} \\ & -3v_{\xi\xi} - 6v_\xi\Psi\Psi_\xi - 6v\Psi_\xi^2 - 6v_\xi\Psi\Psi_\xi - 6v_\xi\Psi\Psi_\xi - 6v\Psi\Psi_{\xi\xi}. \end{aligned} \tag{22}$$

Now, by applying Definitions 1 –4 and after performing some computations, we put the theorem as follows:

Theorem 2. *Eq. (8) is not strictly self-adjoint, quasi-self-adjoint or weak self-adjoint. Nevertheless, Eq. (8) is nonlinearly self-adjoint for $v = \Phi$, while $\Phi(\tau, \xi)$ is the solution of the following equation:*

$$-\Phi_\tau + l\Phi_\xi - 2\Phi_{\xi\xi} = 0. \tag{23}$$

Conserved vectors and connected Lie symmetries

In the present subsection, we calculate the conserved quantities with the help of symmetries for an arbitrary value of the function $A(\Psi)$. The symmetries of Eq. (8) are [32–35]

$$Z_1 = \frac{\partial}{\partial \tau}, \qquad Z_2 = \frac{\partial}{\partial \xi}. \tag{24}$$

If we take $A(\Psi) = \Psi^3 - \Psi$, then there are infinite many conservation laws presented below:

(I) The conservation laws for Z_1 are

$$\begin{aligned} C^\tau &= \mathscr{L} - \Phi\Psi_\tau, \\ C^\xi &= \Psi_\tau\left[\Phi(l + 12\Psi\Psi_\xi) + \Phi_\xi - 3\Psi^2\Phi_\xi - 6\Psi\Psi_\xi\Phi + \Phi_{\xi\xi\xi}\right] \\ &\quad + \Psi_{\xi\tau}\left[\Phi - 3\Psi^2\Phi - \Phi_{\xi\xi\xi}\right] + \Psi_{\xi\xi\tau}\Phi_{\xi\xi\xi}, \end{aligned}$$

where Φ satisfies Eq. (23).

(II) The conserved quantities corresponding to symmetry Z_2 are

$$\begin{aligned} C^\tau &= -\Phi\Psi_\xi, \\ C^\xi &= \mathscr{L} - \Psi_\xi\left[-\Phi(l + 12\Psi\Psi_\xi) - \Phi_\xi + 3\Psi^2\Phi_\xi + 6\Psi\Psi_\xi\Phi - \Phi_{\xi\xi\xi}\right] - \Psi_{\xi\xi}\left[3\Psi^2\Phi - \Phi + \Phi_{\xi\xi\xi}\right] \\ &\quad -\Psi_{\xi\xi\xi}\Phi_{\xi\xi\xi}, \end{aligned}$$

where Φ satisfies Eq. (23).

Nonlinear self-adjointness classification

In current subsection, we will describe the classification of nonlinear self-adjointness [30] of Eq. (8). Now, suppose the formal Lagrangian $\mathscr{L}$ with value of $A(\Psi) = \Psi^3(\xi, \tau) - \Psi(\xi, \tau)$ of the form:

$$\mathscr{L} = v\left[\Psi_\tau + \Psi_{\xi\xi} + \Psi_{\xi\xi\xi\xi} - l\Psi_\xi - 3\Psi^2\Psi_{\xi\xi} - 6\Psi\Psi_\xi^2\right]. \tag{20}$$

Eq. (10) yields:

$$G^\star \equiv \frac{\delta}{\delta \Psi}\left[v\left(\Psi_\tau + \Psi_{\xi\xi} + \Psi_{\xi\xi\xi\xi} - l\Psi_\xi - 3\Psi^2\Psi_{\xi\xi} - 6\Psi\Psi_\xi^2\right)\right], \tag{21}$$

which gives as follows

Conserved vectors by multiplier technique

Anco and Bluman [29] built up a precise technique for developing non-trivial conserved vectors. For this, there is a need of searching the multipliers Λ of definite order for a discussed differential equation which is additionally used to find their related fluxes utilizing suitable strategies. For determining multiplier equations $\Lambda = \Lambda(\tau, \xi, \Psi)$, we apply the Euler-Lagrange operator as

$$\frac{\partial}{\partial \Psi}\left[\Lambda(\Psi_\tau + D^4\Psi - D^2A(\Psi) - lD(\Psi))\right] = 0. \tag{25}$$

After, solving Eq. (25) we get a system of determining equations and has cases given below:

$$(\mathbf{i})\Lambda = \frac{l\tau + \xi}{l}, \qquad (\mathbf{ii})\Lambda = 1.$$

Now, corresponding to the multiplier Λ^1 we have the following conservation laws:

$$C^{\tau} = \frac{1}{l}(l\tau + \xi)\Psi,$$

$$C^{\xi} = -\frac{1}{l}\left[3l\tau\Psi^2\Psi_{\xi} + l^2\tau\Psi + 3\Psi^2\xi\Psi_{\xi} - l\tau\Psi_{\xi} - l\tau\Psi_{\xi\xi\xi} + l\Psi_{\xi} - \Psi^3 - \xi\Psi_{\xi} - \xi\Psi_{\xi\xi\xi} + \Psi + \Psi_{\xi\xi}\right],$$

while multiplier Λ^2 yields the following conserved quantity

$$C^{\tau} = \Psi,$$
$$C^{\xi} = \Psi_{\xi} + \Psi_{\xi\xi\xi} - 3\Psi^2\Psi_{\xi} - l\Psi,$$

for the choice of the arbitrary function $A(\Psi) = \Psi^3 - \Psi$.

Optimal system

Since, Lie algebra $L = \{\mathbf{Z}_1, \mathbf{Z}_2\}$ satisfies the followings:

$$[\mathbf{Z}_1, \mathbf{Z}_2] = 0, \tag{26}$$

where [,] is termed as Lie bracket and defined as follows:

$$[\mathbf{Z}_i, \mathbf{Z}_j] = \mathbf{Z}_i(\mathbf{Z}_j) - \mathbf{Z}_j(\mathbf{Z}_i), \tag{27}$$

where $\mathbf{Z}_i$ and $\mathbf{Z}_j$ are the symmetry generators.

For $\mathbf{Z}_1$ and $\mathbf{Z}_2$ which satisfy (26), the optimal system [32] of one-dimension is given by

$$\langle \mathbf{Z}_1 \rangle, \langle \mathbf{Z}_2 \rangle, \langle \mathbf{Z}_1 + c\mathbf{Z}_2 \rangle$$

Reduction of equation

In this subsection, the similarity reductions are done with each element of optimal system for Eq. (8).

(1) For $\pounds_1 = \langle \mathbf{Z}_1 \rangle$. In this case, we can get

$$\varrho = \xi, \qquad \Psi = g(\varrho), \tag{28}$$

here g satisfies ODE of the following form

$$g'''' = (g^3 - g)'' + lg'. \tag{29}$$

Now, integrating two times Eq. (29), and ignoring constants of integration

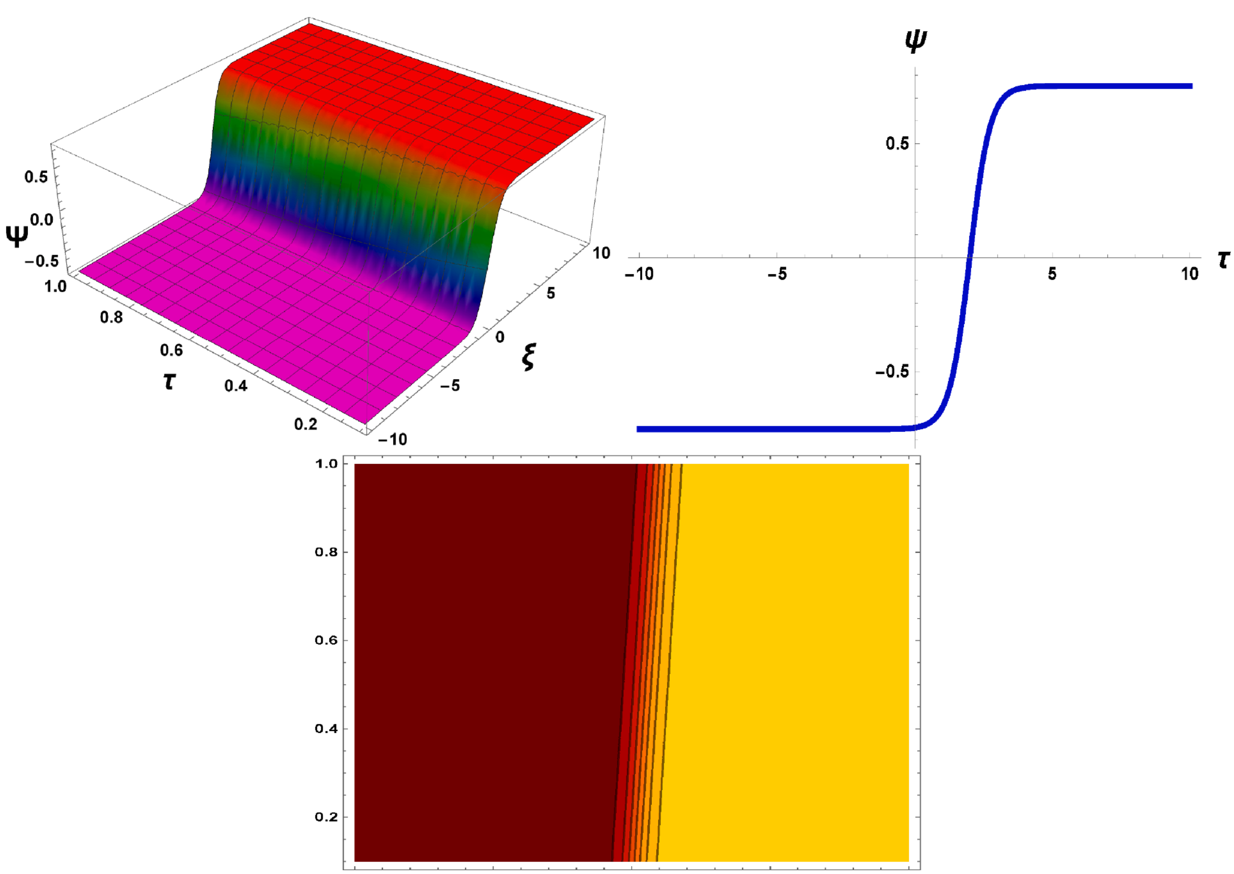

Fig. 1. (a) Graphical representation of Eq. (56) with suitable values of the parameters.

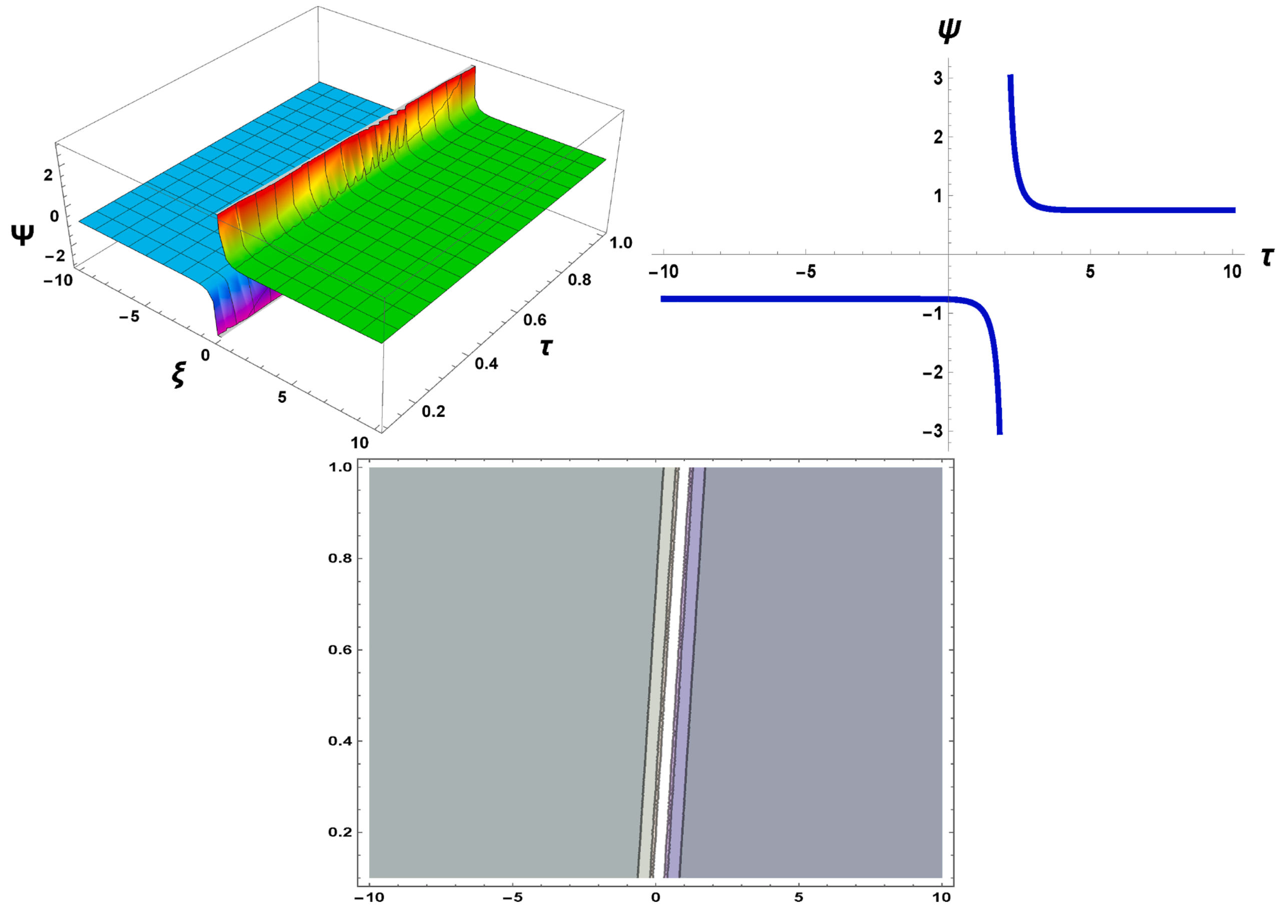

Fig. 2. (a) Solution Eq. (57) for $c = 1, k = 2$ and $l = 1.3$ (b) 2D graph with $\xi = 2$ (c) Contour plot.

$$-\frac{1}{2}(l)g^2 + g'' - g^3 + g = 0. \tag{30}$$

(2) For $\pounds_2 = \langle \mathbf{Z}_2 \rangle$. In this case, we get

$$\varrho = \tau, \qquad \Psi = g(\varrho), \tag{31}$$

which gives a constant solution.

(3) For $\pounds_3 = \langle \mathbf{Z}_1 + c\mathbf{Z}_2 \rangle$. For this case, we can easily obtain

$$\varrho = \xi - c\tau, \Psi = g(\varrho), \tag{32}$$

here c is the wave velocity. We have

$$-cg' + g'''' = (g^3 - g)'' + lg'. \tag{33}$$

Integrating Eq. (33) twice and ignoring constants of integration

$$-\frac{1}{2}(c + l)g^2 + g'' - g^3 + g = 0. \tag{34}$$

Traveling wave structures

Traveling wave structures of Eq. (34), we used the modified simple equation (MSE) method.

Description of MSE Method

Assume a non-linear equation of the type

$$\mathfrak{F}(\Psi, \Psi_\tau, \Psi_\xi, \Psi_{\xi\xi}, \Psi_{\tau\tau}, \ldots) = 0, \ \xi \in \mathbb{R} \ \& \tau \geqslant 0, \tag{35}$$

where $\mathfrak{F}$ is function of $\Psi(\xi, \tau)$ with its partial derivatives. The important steps of this method [36–38] are summarized below:

Step:1: The traveling wave transformation [39],

$$\Psi(\xi, \tau) = \Psi(\varrho), \ \varrho = k(\xi \pm c\tau) \tag{36}$$

where c and k are the speed of traveling wave and wave number respectively.

The above Eq. (36) transforms Eq. (35) into the following ODE:

$$\Re(\Psi, \Psi', \Psi'', \ldots) = 0, \tag{37}$$

where $\Re$ is a function of $\Psi(\varrho)$ with its derivatives. Also, for simplicity $\Psi' = \frac{d\Psi}{d\varrho}$.

Step 2: Suppose the solution of Eq. (35) can be presented as:

$$\Psi(\varrho) = B_0 + \sum_{i=1}^{M} B_i \left(\frac{\phi(\varrho)'}{\phi(\varrho)} \right)^i, \tag{38}$$

where M is a positive integer, $B_M \neq 0$, and $B_i (i = 1, 2, 3, \ldots, M)$ are arbitrary constants to be found later, and $\phi(\varrho)$ is a function to be found after this, such that $\phi'(\varrho) \neq 0$.

Step 3: The value of M in Eq. (38) can be found by balancing the highest order derivatives and the non-linear terms occurring in Eq. (35) or Eq. (37).

Step 4: Incorporate Eq. (38) into (37), and find all the derivatives Ψ', Ψ'', ... of the function $\Psi(\varrho)$ and then account the function $\phi(\varrho)$. After putting this, a polynomial of $\phi^{-j}, (j = 0, 1, 2, \ldots)$ with the derivatives of $\phi(\varrho)$ will be obtained. By comparing the coefficients of ϕ^{-j} to zero, where $j = 0, 1, 2, \ldots$. Then as a result, we have a system of algebraic and ODEs. By solving the algebraic equations, we get the values of B'_Ms, and value of $\phi(\varrho)$ can be obtained by solving the ODEs. Hence, the complete solution of Eq. (35) is obtained by putting the values of B_M and $\phi(\varrho)$ into

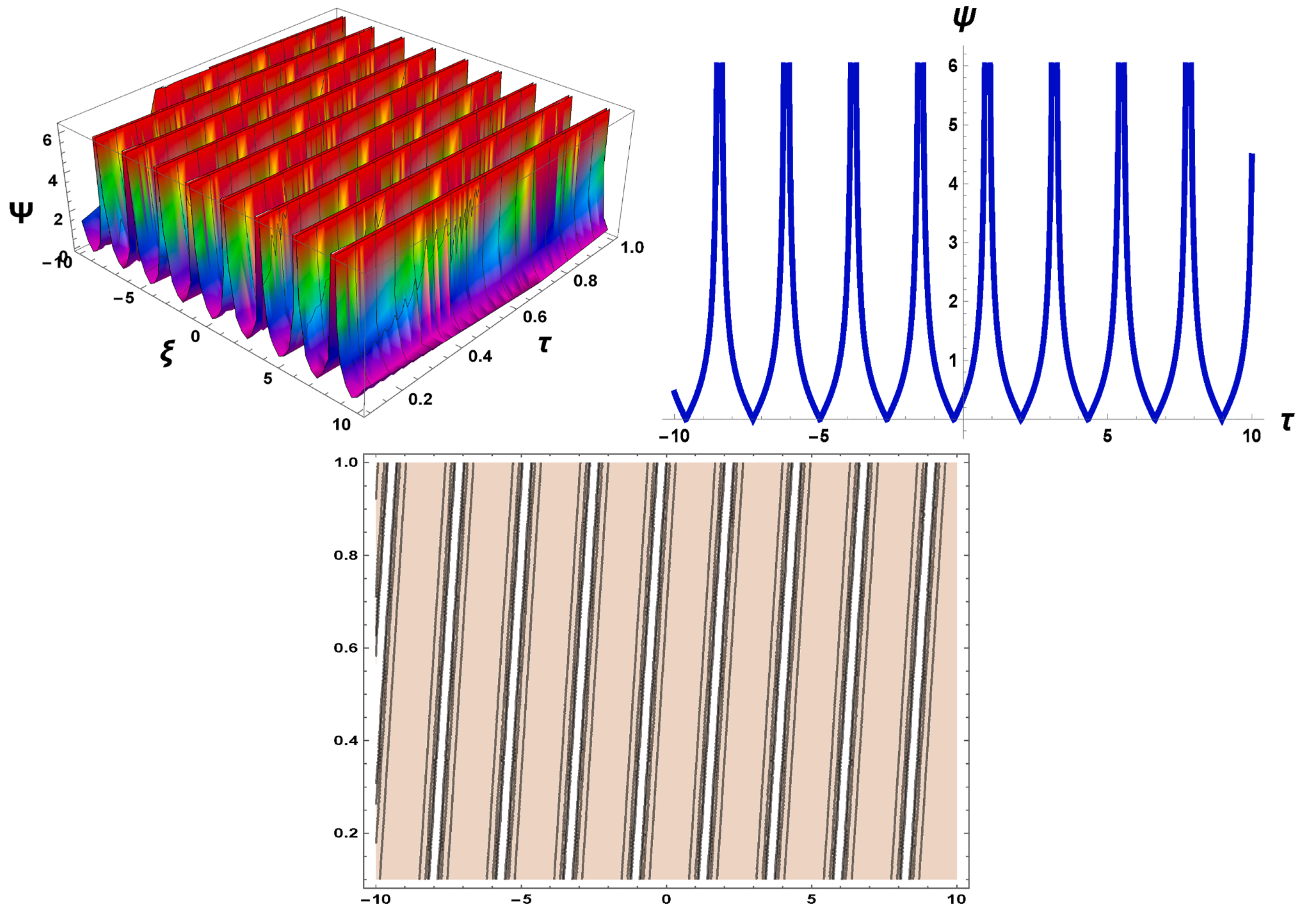

Fig. 3. (a) Solitary wave solution of Cahn-Hilliard equation obtained from Eq. (58) for $c = 1, k = 2$ and $l = 1.3$ (b) 2D graph with $\xi = 2$ (c) Contour plot.

Eq. (38).

Applications

The traveling wave transformation,

$$\Psi = g(\varrho), \qquad \varrho = k(\xi - c\tau) \tag{39}$$

converts Eq. (8) into this ODE

$$-ckg' + k^4 g'''' = k^2(g^3 - g)'' + lkg'. \tag{40}$$

Two times integrating Eq. (40) and ignoring constants of integration, we have

$$-\frac{k}{2}(c+l)g^2 + k^4 g'' + k^2(g - g^3) = 0. \tag{41}$$

Now, by balancing the terms g'' and g^3, gives $M+2 = 3M$ which yields $M = 1$. Hence, the solution Eq. (38) takes the form,

$$g(\varrho) = B_o + B_1\left(\frac{\phi'}{\phi}\right), \tag{42}$$

where B_0 and B_1 are constants and $B_1 \neq 0$, and $\phi(\varrho)$ is a function to be found later. It is not difficult to calculate that,

$$g' = B_1\left[\frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2\right], \tag{43}$$

$$g'' = B_1\frac{\phi'''}{\phi} - 3B_1\frac{\phi''\phi'}{\phi^2} + 2B_1\left(\frac{\phi'}{\phi}\right)^3. \tag{44}$$

Incorporating the values of g'' and g into Eq. (41) and comparing the coefficients of $\phi^0, \phi^{-1}, \phi^{-2}, \phi^{-3}$ to zero, gives:

$$k^2(B_0 - B_0^3) - \frac{k}{2}(c+l)B_0^2 = 0, \tag{45}$$

$$k^4 B_1\phi''' + (k^2B_1 - 3k^2B_0^2B_1 - kcB_0B_1 - klB_0B_1)\phi' = 0, \tag{46}$$

$$-3k^4B_1\phi'\phi'' - \left(3k^2B_0B_1^2 + \frac{kc}{2}B_1^2 + \frac{kl}{2}B_1^2\right)\phi'^2 = 0, \tag{47}$$

$$(2k^4B_1 - k^2B_1^3)\phi'^3 = 0. \tag{48}$$

By solving Eqs. (45) and (48), we get:

$$B_0 = 0, \pm\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k}, B_1 = \pm\sqrt{2}k, \ B_1 \neq 0. \tag{49}$$

Case-I: When $B_0 = 0$, putting this value into Eqs. (46) and (47) gives an inappropriate solution. Therefore, this case is overruled.

Case-II: When $B_0 = \pm\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k}$, the Eqs. (46) and (47) give,

$$\frac{\phi'''}{\phi''} + \alpha = 0, \tag{50}$$

where $\alpha = \frac{3\left(k^2 - 3k^2B_0^2 - k(c+l)B_0\right)}{\left(3k^2B_0 + \frac{k}{2}(c+l)\right)B_1}$.

Integrating, Eq. (50) with respect to ϱ, gives

$$\phi'' = b_1\exp(-\alpha\varrho). \tag{51}$$

Eqs. (51) and (47), give

$$\phi' = -b_1 \frac{3k^4}{\left(3k^2B_0 + \frac{k}{2}(c+l)\right)B_1} \exp(-\alpha\varrho). \tag{52}$$

Therefore, by integrating, we get:

$$\phi = b_2 + b_1 \frac{3k^4}{3(k^2 - 3k^2B_0^2 - k(c+l)B_0)} \exp(-\alpha\varrho), \tag{53}$$

where b_1 and b_2 are arbitrary constants. Putting the values of ϕ and ϕ' into Eq. (42) gives the solution,

$$g(\varrho) = B_0 + B_1\beta \frac{-b_1 3k^4 \exp(-\alpha\varrho)}{3(k^2 - 3k^2B_0^2 - k(c+l)B_0)b_2 + 3k^4 b_1 \exp(-\alpha\varrho)}, \tag{54}$$

where $\beta = \frac{3(k^2 - 3k^2B_0^2 - k(c+l)B_0)}{\left(3k^2B_0 + \frac{k}{2}(c+l)\right)B_1}$. Substituting the values of B_0, B_1 and α and simplifying, we get

$$\begin{aligned}\Psi(\xi,\tau) &= \pm\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k} \\ &\pm\sqrt{2}k\beta' \frac{3k^4 b_1\{cosh(\alpha'(\xi-c\tau)) + sinh(\alpha'(\xi-c\tau))\}}{(\omega b_2 + 3k^4 b_1)cosh(\alpha'(\xi-c\tau)) + (\omega b_2 - 3k^4 b_1)sinh(\alpha'(\xi-c\tau))},\end{aligned} \tag{55}$$

where $\alpha' = \beta' = -\frac{3}{4}\frac{\sqrt{2}(5c\sqrt{c^2+2cl+16k^2+l^2}+5l\sqrt{c^2+2cl+16k^2+l^2}+5c^2+10cl+16k^2+5l^2)}{k^2(5c+5l+3\sqrt{c^2+2cl+16k^2+l^2})}$

and

$$\omega = -\frac{15}{8}(c^2+l^2) - \frac{15}{4}cl - 6k^2 - \frac{15}{8}(c+l)\sqrt{c^2+2cl+16k^2+l^2}$$

Putting $b_1 = \frac{\omega}{3k^4}b_2$ and $b_1 = -\frac{\omega}{3k^4}b_2$ into Eq. (55). For $c < 0$, we have the following solitary wave solutions in respective order.

$$\Psi_{1,2}(\xi,\tau) = \pm\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k} tanh((\alpha'(\xi-c\tau))), \tag{56}$$

$$\Psi_{3,4}(\xi,\tau) = \pm\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k} coth((\alpha'(\xi-c\tau))). \tag{57}$$

Now, by using the hyperbolic functions identities, Eqs. (56) and (57) give the following periodic traveling wave structures for $c > 0$

$$\Psi_{5,6}(\xi,\tau) = \pm i\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k} tan((\alpha'(\xi-c\tau))), \tag{58}$$

$$\Psi_{7,8}(\xi,\tau) = \pm i\frac{1}{4}\frac{-c-l+\sqrt{c^2+2cl+16k^2+l^2}}{k} cot((\alpha'(\xi-c\tau))). \tag{59}$$

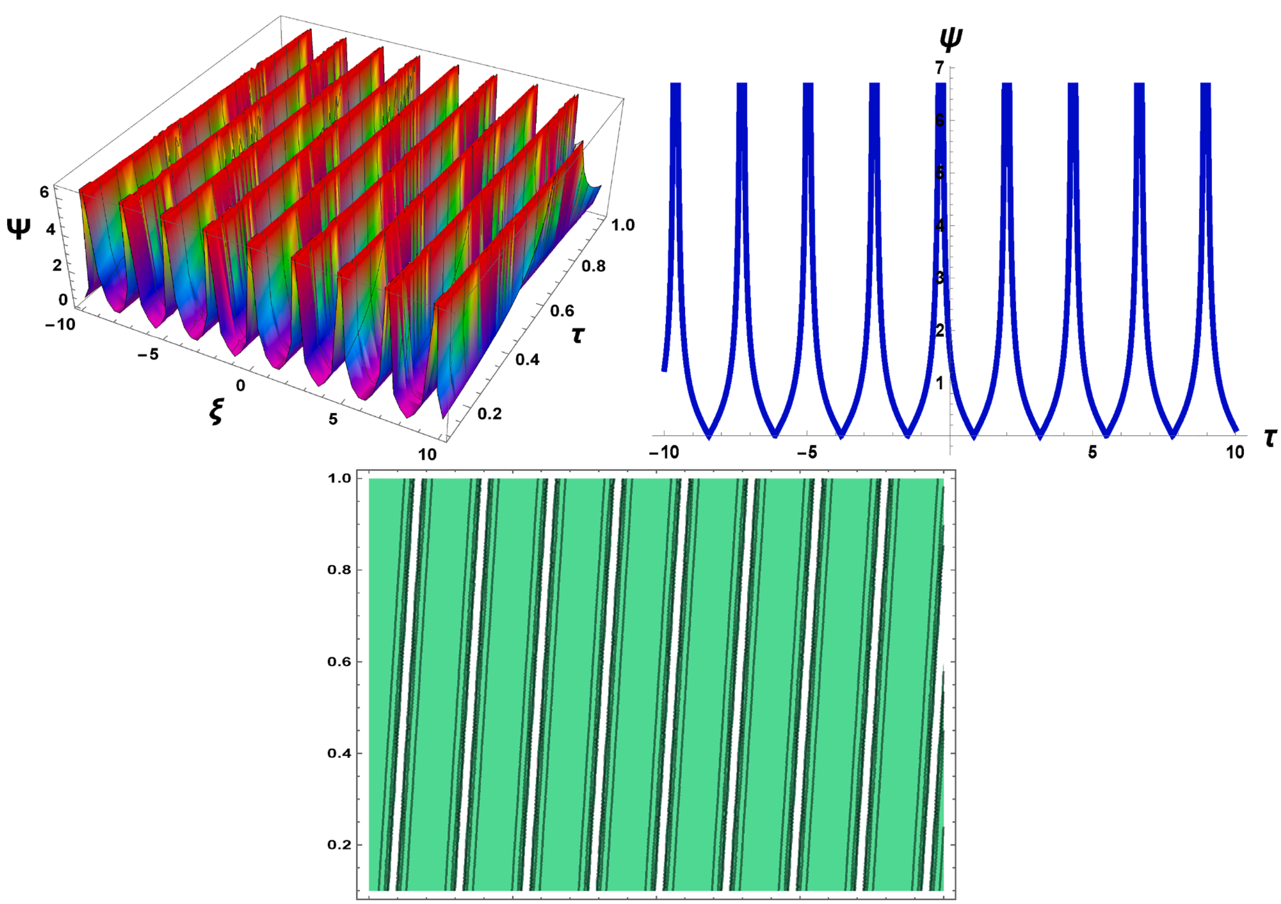

Fig. 4. (a) Solitary wave solution Eq. (58) for $c = 1, k = 2$ and $l = 1.3$ (b) 2D graph with $\xi = 2$ (c) Contour plot.

The solutions $\Psi(\xi,\tau)$ acquired in Eqs. (56) and (57) are presented in the Figs. 1–4.

Results and discussion

Fig. (1): In this figure, we discuss the 3D, 2D and contour plots for the $\Psi_{1,2}$ with constant parameter as (a): 3D graph for $c = 1, k = 2$ and $l = 1.3$ (b): 2D with $\xi = 2$ (c): contour plot with taking τ and ξ as variables. We have found the solitary wave solution which is of kink type solution.

Fig. (2): In this figure working on the same lines as in Fig. (1), we also see the physical appearance of the solution by different types of the graphs and found that the solitary wave solution is of the singular soliton type.

Fig. (3): In Fig. (3), we report the various type of the graphs by taking the same values of parameters as Fig. (1). This representation of the obtained solution shows the periodic wave solution.

Fig. (4): In this graph, we draw the obtained solution by fixing the values of parameters as in Fig. (1). This figure indicates the solitary wave of periodic wave type.

Conclusion

In the current study, the Lie analysis technique and multiplier method is utilized for the formation of some exact solution and conserved vectors. The optimal system of Lie algebras has been calculated and applied to find the conserved quantities of the equation under study. The symmetry reductions for each element of the optimal system is presented. Also, the exact traveling wave structures of the Cahn-Hilliard equation are easily obtained with the implementation of the MSE method. To understand the physical interpretation solitary wave structures are exhibited graphically. We claim that the MSE method is a simple, concise, and efficient method to find traveling wave structures of nonlinear equations as compared to other existing methods. It is simple in the sense that it does not require software like Maple or Mathematica for symbolic computations as compared to other existing methods i-e the Exp-function technique and the tanh-function etc. does. The main drawback of this used method is that it cannot solve the nonlinear equations with a balanced number $M\geqslant2$. To search out the reason for its failure is the future direction of study.

CRediT authorship contribution statement

Muhammad Bilal Riaz: Conceptualization, Investigation, Methodology. **Dumitru Baleanu:** Data curation. **Adil Jhangeer:** Software, Visualization, Supervision, Formal analysis. **Naseem Abbas:** Validation, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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