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Research Article

Nonunique Coincidence Point Results via Admissible Mappings

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This paper is aimed at presenting some coincidence point results using admissible mapping in the framework of the partial b-metric spaces. Observed results of the article cover a number of existing works on the topic of "investigation of nonunique fixed points." We express an example to indicate the validity of the observed outcomes.

1. Introduction and Preliminaries

In 1974, Ćirić [1] published the first paper on nonunique fixed point theory. Despite Banach's theorem, Ćirić [1] focused only on the existence of a fixed point, but not the uniqueness. The motivation of Ćirić [1] was inspired by Banach's motivation. As it is known, Banach's fixed point theorem is abstracted from Picard's paper, in which Picard [2] analyzed both the existence and uniqueness of the solution of the certain differential equation (see [3–5]). On the other hand, not all differential or integral equations have a unique solution. In the differential/integral equations, nonunique solutions are also crucial, for example, periodic solutions. Consequently, Ćirić [1] investigated the corresponding fixed point theorems that would be a tool in finding periodic solutions of the differential/integral equations. In the last five decades, a number of nonunique fixed point results have been reported in two ways: either proposing a new contraction type or changing the structure. The first example for the changing the contraction inequality, in the standard set-up, was given by Achari [6] in 1976 and Pachpatte [7] in 1973. Fifteen years later, Ćirić and Jotić [8] proposed a new type of contraction inequalities in the context of complete metric space. This trend was followed by the attractive results [9–13]. On the other hand side, in [14–17], the authors observed several characterizations of the unique fixed point results in the setting of complete b-metric spaces. Indeed, among the several extensions of metric structure, the true extension is the b-metric space. For this reason, observed nonunique fixed theorems in the context of b-metric space is very interesting and important, see also [18–20]. In addition, in [21–23], the characterization of fixed point theorems in partial metric spaces is crucial due to the potential application in the domain theory of computer science. Regarding the applied mathematics, nonunique fixed point results in cone metric spaces have taken attention [24].

In this paper, we consider a nonunique fixed point theorem in the context of the very general frame, partial b-metric spaces. An illustrative example is a set-up to indicate the validity of the main theorem.

Let M be a nonempty set, a real number $s \geq 1$, and $\mathbb{N} = \{1, 2, 3, \cdots\}$.In this case, the triplet (M, p_b, s) forms a partial b *-metric space*, on short p_b-ms.Undoubtedly, b-metric spaces (and ordinary metric spaces) are closely related to partial b -metric spaces. Definitely, a b-metric space ($s \geq 1$) is a partial b-metric space with zero self-distance and a partial metric space is a partial b-metric space with $s = 1$. Moreover, a partial b-metric can define a b-metric space. Indeed, for example, let p_b be a partial b-metric on M. Then, the functions b'_p, b_p, $b_{p,m} : M \longrightarrow M$, where

$$b_p(u,v) = \begin{cases} p_b(u,v), & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases} \tag{1}$$

$$b'_p(u,y) = 2p_b(u,y) - p_b(u,u) - p_b(y,y), \tag{2}$$

$$b_{p,m}(u,y) = p_b(u,y) - \min\{p_b(u,u), p_b(y,y)\} \tag{3}$$

are b-metrics on M.

Definition 1. A function $p_b : M \times M \longrightarrow [0,\infty)$ is a partial b-metric on M if for all $u, y, w \in M$, it satisfies the following conditions:

$(p_b)_1 u = y \Longleftrightarrow p_b(u,u) = p_b(u,y) = p_b(y,y)$
$(p_b)_2 p_b(u,u) \leq p_b b(u,y)$
$(p_b)_3 p_b(u,y) \leq p(y,u)$
$(p_b)_4 p_b(u,y) \leq s[p_b(u,w) + p_b(w,u)] - p_b(w,w)$

Example 1. (see [25]). Let p_b be a partial metric on the set M. Then, the functions $p_b : M \times M \longrightarrow [0,\infty)$ are given for all $u, y \in M$ by

(1) $p_b(u,y) = p(u,y) + b(u,y)$ is a partial b-metric on M (where b is a b-metric $(s > 1)$ on M)

(2) $p_b(u,y) = [p(u,y)]^r$ for $r \geq 1$, define a partial b-metrics on M with coefficient $s = 2^{r-1}$

Remark 2. From $(pb)_1$ and $(pb)_2$, it follows that if $u, y \in M$ are such that $p_b(u,y) = 0$, then $u = y$.

Definition 3. (see [26, 27]). Let $\{u_n\}$ be a sequence on the p_b-$ms(M, \mathrm{p}_b, s \geq 1)$

(1) $\{u_n\}$ is p_b-convergent to $u \in M$ if $\lim_{n\to\infty} p_b(u, u_n) = p_b(u,u)$

(2) $\{u_n\}$ is p_b-Cauchy if $\lim_{n,q\to\infty} p_b(\mathrm{u_n}, \mathrm{u_q})$ exists and is finite

(3) $\{u_n\}$ is 0-p_b-Cauchy if $\lim_{n,q\to\infty} p_b(u_n, u_q) = 0$

(4) $(M, p_b, s \geq 1)$ is p_b-complete if every p_b-Cauchy sequence in M is p_b-convergent

$$\lim_{n,q\to\infty} p_b(u_n, u_q) = \lim_{n\to\infty} p_b(u_n, u) = p_b(u,u) \tag{4}$$

(5) $(M, p_b, s \geq 1)$ is 0-p_b-complete if every 0-p_b-Cauchy sequence we can find $u \in M$ such that

$$\lim_{n,q\to\infty} p_b(u_n, u_q) = \lim_{n\to\infty} p_b(u_n, u) = p_b(u,u) = 0 \tag{5}$$

Moreover, in [26], the following interesting results were proved.

Lemma 4. *(see [26]). Every p_b-complete p_b-ms $(M, p_b, s \geq 1)$ is 0 - p_b -complete.*

Lemma 5. *(see [26]). The p_b-ms $(M, p_b, s \geq 1)$ is 0 - p_b -complete if and only if the b -metric space $(M, b_p, s \geq 1)$ is complete, where the b -metric b_p was defined in (3).*

They also showed that the converse affirmation does not hold.

Let R, S to self-mappings on the set M. We say that

(i) S commutes with R on M if $RSu = SRu$ for all $u \in M$

(ii) a point $z \in M$ is a point of coincidence of R and S if we can find $u^* \in M$ such that $z = Ru^* = Su^*$

(iii) a point $u^* \in M$ is a common fixed point of R and S if $Ru^* = u^* = Su^*$

We will use the following notations:

$$C_c(R,S)_M = \{u \in M \mid Ru = Su\} M^* = M \setminus C_c(R,S)_M. \tag{6}$$

In [28], the notion of R-β-admissible mapping was introduced as follows:

(i) Let the function $\beta : M \times M \longrightarrow [0,\infty)$ and $R, S : M \longrightarrow M$. The mapping S is said to be R-β-admissible if

$$\beta(Ru, Ry) \geq 1 \text{ implies } \beta(Su, Sy) \geq 1, \tag{7}$$

for all $u, y \in M$.

In case that $R = I_M$, the mapping S is said to be β -admissible.

Let $(M, p_b, s \geq 1)$ be a p_b-ms and $\beta ; M \times M \longrightarrow [0,+\infty)$. The space M is β-regular if for every sequence $\{z_n\}$ in M such that $z_n \longrightarrow z$ and $\beta(z_n, z_{n+1}) \geq 1$, there exists a subsequence $\{z_{n_l}\}$ of $\{z_n\}$ such that

$$\beta(z_{n_l}, z_*) \geq 1, \tag{8}$$

for all $l \in \mathbb{N}$.

Lemma 6. *Let $R, S : M \longrightarrow M$ such that S is a R-β-admissible. If there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$, then*

$$\beta(Ru_n, Ru_{n+1}) \geq 1, \tag{9}$$

where the sequence $\{u_n\}$ in M is defined by $Su_n = Ru_{n+1}$, for each $n \in \mathbb{N} \cup \{0\}$.

Proof. By the assumption $\beta(Ru_0, Su_0) \geq 1$, since the mapping S is R-β-admissible, we get

$$\beta(Ru_0, Ru_1) = \beta(Ru_0, Su_0) \geq 1 \text{ implies } \beta(Ru_1, Ru_2) = \beta(Su_0, Su_1) \geq 1, \tag{10}$$

and by induction, it follows that

$$\beta(Ru_n, Ru_{n+1}) \geq 1, \tag{11}$$

for $n \in \mathbb{N} \cup \{0\}$.

2. Main Results

Following the idea in [29], we state the following results useful in the sequel.

Lemma 7. *Let $(M, p_b, s \geq 1)$ be a p_b-ms. If $\{u_n\}$ is a sequence in M such that there exists $\{z_n\}$ in M, satisfying the inequality*

$$p_b(u_n, u_{n+1}) \leq c p_b(u_{n-1}, u_n), \tag{12}$$

for any $n \in \mathbb{N}$, then the sequence is $\{u_n\}$ and is 0-p_b-Cauchy.

Proof. First of all, by (12), we get

$$p_b(u_n, u_{n+1}) \leq c^n p_b(u_0, u_1), \tag{13}$$

for all $n \in \mathbb{N}$. On the other hand, by using $(pb)_4$, we can derive that

$$\begin{aligned}
p_b\left(u_n, u_{n+q}\right) &\leq s(p_b\left(u_n, u_{n+1}\right) \\
&\quad + p_b\left(u_{n+1}, u_{n+q}\right)) - p_b\left(u_{n+1}, u_{n+1}\right) \\
&\leq s p_b\left(u_n, u_{n+1}\right) \\
&\quad + s^2\left(p_b\left(u_{n+1}, u_{n+2}\right) + p_b\left(u_{n+2}, u_{n+2}, u_{n+q}\right)\right. \\
&\quad -- p_b\left(u_{n+1}, u_{n+1}\right) - p_b\left(u_{n+2}, u_{n+2}\right) \cdots \\
&\leq s p_b\left(u_n, u_{n+1}\right) + s^2 p_b\left(u_{n+1}, u_{n+2}\right) + \cdots \\
&\quad + s^q p_b\left(u_{n+q-1}, u_{n+q}\right) -- \sum_{l=1}^{q-1} p_b\left(u_{n+l}, u_{n+l}\right) \\
&\leq s^q [p_b\left(u_n, u_{n+1}\right) + p_b\left(u_{n+1}, u_{n+2}\right) + \cdots \\
&\quad + p_b\left(u_{n+q-1}, u_{n+q}\right)] -- \sum_{l=1}^{q-1} p_b\left(u_{n+l}, u_{n+l}\right).
\end{aligned} \tag{14}$$

(1) If $c \in [0, 1/s)$, by (13) and (14), we get

$$\begin{aligned}
p_b\left(u_n, u_{n+q}\right) &\leq \sum_{l=0}^{q-1} s^{l+1} c^{n+l} p_b\left(u_0, u_1\right) - \sum_{l=1}^{q-1} p_b\left(u_{n+l}, u_{n+l}\right) \\
&\leq s c^n \sum_{l=0}^{q-1} (sc)^l p_b\left(u_0, u_1\right) \\
&= s c^n \frac{1 - (sc)^q}{1 - sc} \longrightarrow 0 \text{ as } n, q \longrightarrow \infty.
\end{aligned} \tag{15}$$

Therefore, $\{u_n\}$ is a 0-p_b-Cauchy sequence.

(2) If $c \in [1/s, 1)$, thus $c^n \longrightarrow 0$ (as $n \longrightarrow \infty$). Moreover, there exits $l \in \mathbb{N}$ such that $c^l < 1/s$. This means $l > -\log s/\log c$. Again, by (13) together with (14), we have

$$\begin{aligned}
p_b\left(u_{nl}, u_{(n+1)l}\right) &\leq s^l \left[p_b(unl, u_{nl+1}) + \cdots + p_b\left(u_{nl+l-1}, u_{(n+1)l}\right)\right] \\
&\quad -- \sum_{j=1}^{l-1} p_b\left(u_{nl+j}, u_{nl+j}\right) \\
&\leq s^l \sum_{j=0}^{l-1} c^{nl+j} p_b\left(u_0, u_1\right) - \sum_{j=1}^{l-1} p_b\left(u_{nl+j}, u_{nl+j}\right) \\
&\leq s^l c^{nl} \sum_{j=0}^{l-1} p_b\left(u_0, u_1\right) \\
&\leq c^{nl} \frac{s^l \cdot p_b\left(u_0, u_1\right)}{1 - c} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{16}$$

Thereby, letting $\lambda = c^l < 1/s$ by Case (i), we get that the sequence $\{u_{nl}\}$ is 0-p_b-Cauchy sequence, which means that

$$\lim_{n,q \to \infty} p_b\left(u_{nl}, u_{ql}\right) = 0. \tag{17}$$

On the other hand,

$$\begin{aligned}
p_b\left(u_{l[n/l]}, u_n\right) &\leq s\left(p_b\left(u_{l[n/l]}, u_{l[n/l]+1}\right)\right. \\
&\quad \left. + p_b\left(u_{l[n/l]+1}, u_n\right)\right) - p_b\left(u_{l[n/l]+1}, u_{l[n/l]+1}\right) \\
&\leq s^l \left[p_b\left(u_{l[n/l]}, u_{l[n/l]+1}\right) + \cdots + p_b\left(u_{n-1}, u_n\right)\right] \\
&\quad -- \left(p_b\left(u_{l[n/l]+1}, u_{l[n/l]+1}\right) + \cdots + p_b\left(u_{n-1}, u_{n-1}\right)\right),
\end{aligned} \tag{18}$$

and using (13), we have

$$\begin{aligned}
p_b\left(u_{l[n/l]}, u_n\right) &\leq s^l \left[c^{l[n/l]} + \cdots + c^{n-1}\right] p_b\left(u_0, u_1\right) \\
&\leq s^l c^{l[n/l]} \frac{p_b\left(u_0, u_1\right)}{1 - c} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{19}$$

Finally, combining relations (19) and (17) and keeping in mind $(pb)_4$, we have

$$\begin{aligned}
p_b\left(u_n, u_q\right) &\leq s\left[p_b\left(u_n, u_{l[n/l]}\right)\right. \\
&\quad \left. + p_b\left(u_{l[n/l]}, u_q\right)\right] - p_b\left(u_{l[n/l]}, u_{l[n/l]}\right) \\
&\leq s p_b\left(u_n, u_{l[n/l]}\right) \\
&\quad + s^2 p_b\left(u_{l[n/l]}, u_{l[q/l]}\right) + s^2 p_b\left(u_{l[q/l]}, u_q\right) \\
&\quad -- p_b\left(u_{l[n/l]}, u_{l[n/l]}\right) - p_b\left(u_{l[q/l]}, u_{l[q/l]}\right) \\
&\leq s p_b\left(u_n, u_{l[n/l]}\right) \\
&\quad + s^2 p_b\left(u_{l[n/l]}, u_{l[q/l]}\right) + s^2 p_b\left(u_{l[q/l]}, u_q\right) \longrightarrow 0 \text{ as } n, q \longrightarrow \infty.
\end{aligned} \tag{20}$$

Thereupon, the sequence $\{u_n\}$ is 0-p_b-Cauchy.

Theorem 8. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned}&\beta(Ru, Ry) \min \{p_b(Su, Sy), p_b(Sy, Ry)\}\\&\quad -- \min \left\{b_p(Su, Ry), b_p(Sy, Ru)\right\}\\&\quad \leq \kappa \max \{p_b(Ru, Ry), p_b(Su, Ru)\},\end{aligned} \tag{21}$$

for all $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R, S)_M$. Suppose also that

(a) *$S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms*

(b) *S is R-β-admissible, and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$*

(c) *M is β-regular*

Then, the mappings S and R have a point of coincidence.

Proof. Let u_0 be an arbitrary point in M, such that $\beta(Ru_0, S u_0) \geq 1$. Thus, since $S(M) \subset R(M)$, there exists $u_1 \in M$ such that $Su_0 = Ru_1$. Thereupon, $Su_1 \in S(M) \subset R(M)$ and we can find $u_2 \in M$ such that $Su_1 = Ru_2$. In this way, we can build a sequence $\{u_n\} \subseteq M$ as follows:

$$\text{having defined } u_n \in M, \text{ we let } u_{n+1} \in M \text{ such that } Su_n = Ru_{n+1}, \tag{22}$$

for all $n \in \mathbb{N} \cup \{0\}$. Letting $u = u_n$ and $y = u_{n+1}$ in (ref1T1) and taking into account Lemma 6, we have

$$\begin{aligned}&\min \{p_b(Su_n, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1})\}\\&\quad -- \min \left\{b_{p,m}(Su_n, Ru_{n+1}), b_{p,m}(Su_{n+1}, Ru_n)\right\}\\&\leq \beta(Ru_n, Ru_{n+1}) \min \{p_b(Su_n, Su_{n+1}), p_b(Su_{n+1}, Ru_{n+1})\}\\&\quad -- \min \left\{b_p(Su_{n+1}, Ru_n)\right\}\\&\leq \kappa \max \{p_b(Ru_n, Ru_{n+1}), p_b(Su_n, Ru_n)\}.\end{aligned} \tag{23}$$

Keeping in mind (22), we get

$$\begin{aligned}&\min \{p_b(Ru_{n+1}, Ru_{n+2}), p_b(Ru_{n+1}, Ru_n), p_b(Ru_{n+2}, Ru_{n+1})\}\\&\quad -- \min \left\{b_p(Ru_{n+1}, Ru_{n+1}), b_p(Ru_{n+2}, Ru_n)\right\}\\&\leq \kappa \max \{p_b(Ru_n, Ru_{n+1}), p_b(Ru_{n+1}, R(u_n))\}\\&= \kappa p_b(Ru_n, Ru_{n+1}),\end{aligned} \tag{24}$$

which is equivalent with

$$\begin{aligned}&\min \{p_b(Ru_{n+1}, Ru_{n+2}), p_b(Ru_{n+1}, Ru_n)\}\\&\quad -- \min \left\{b_p(Ru_{n+1}, Ru_{n+1}), b_p(Ru_{n+2}, Ru_n)\right\}\\&\leq \kappa p_b(Ru_n, Ru_{n+1}).\end{aligned} \tag{25}$$

Therefore, we get

$$p_b(Ru_{n+1}, Ru_{n+2}) \leq \kappa p_b(Ru_n, Ru_{n+1}), \tag{26}$$

for any $n \in \mathbb{N} \cup \{0\}$.Let now $\{z_n\}$ be a sequence in M, with $z_n = Ru_{n+1} = Su_n$, $n \in \mathbb{N} \cup \{0\}$. First of all, we mention that $z_n \neq z_{n+1}$ for every $n \in \mathbb{N}$. Indeed, if we suppose that there exists $m_0 \in \mathbb{N} \cup \{0\}$ such that $z_{m_0} = z_{m_0+1}$, thus by (22), we have

$$Ru_{m_0+1} = Su_{m_0} = z_{m_0} = z_{m_0+1} = Su_{m_0+1}, \tag{27}$$

so that z_{m_0+1} is a point of coincidence. Thus, $z_n \neq z_{n+1}$ for every $\mathbb{N} \cup \{0\}$ and (28) can be rewritten as

$$p_b(z_n, z_{n+1}) \leq \kappa p_b(z_{n-1}, z_n). \tag{28}$$

Therefore, according to Lemma 7, the sequence $\{z_n\}$ is 0 -p_b-Cauchy. Since the space is 0-p_b-complete, it follows that there is $z \in M$ such that

$$\lim_{n,q\to\infty} p_b\left(z_n, z_q\right) = \lim_{n\to\infty} p_b(z_n, z) = \mathrm{p}_b(z, z) = 0. \tag{29}$$

But, on the other hand, since $z_n = Ru_{n+1}$ and the space $(R(M), p_b, s)$ is 0-p_b-complete, we can find $u_* \in M$, with $z = Ru_*$. Thus,

$$\lim_{n\to\infty} p_b(Su_n, Ru_*) = \lim_{n\to\infty} p_b(Ru_n, Ru_*) = p_b(Ru_*, Ru_*) = 0. \tag{30}$$

Supposing that $Ru_* \neq Su_*$ for $u = u_{n_l}$ and $y = u_*$ and taking into account the β-regularity of the space M, we have

$$\begin{aligned}&\min \left\{p_b\left(Su_{n_l}, Su_*\right), p_b(Su_*, Ru_*)\right\}\\&\quad - \min \left\{b_p\left(Su_{n_l}, Ru_*\right), b_p\left(Su_*, Ru_{n_l}\right)\right\}\\&\leq \leq \beta\left(z_{n_l}, z\right) \min \left\{p_b\left(Su_{n_l}, Su_*\right), p_b(Su_*, Ru_*)\right\}\\&\quad - \min \left\{b_p\left(Su_{n_l}, Ru_*\right), b_p\left(Su_*, Ru_{n_l}\right)\right\}\\&= \beta\left(Ru_{n_l}, Ru_*\right) \min \left\{p_b\left(Su_{n_l}, Su_*\right), p_b\left(Su_*, Ru_*\right)\right\}\\&\quad - \min \left\{b_p\left(Su_{n_l}, Ru_*\right), b_p\left(Su_*, Ru_{n_l}\right)\right\}\\&\leq \kappa \max \left\{p_b\left(Ru_{n_l}, Ru_*\right), p_b\left(Su_{n_l}, Ru_{n_l}\right)\right\}.\end{aligned} \tag{31}$$

If $\min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} = p_b(Su_*, Ru_*)$, the above inequality becomes

$$\begin{aligned}&p_b(Su_*, Ru_*) - \min \left\{p_b\left(Su_{n_l}, Su_*\right), p_b\left(Su_*, Ru_*\right)\right\}\\&\quad - \min \left\{b_p\left(Su_{n_l}, Ru_*\right), b_p\left(Su_*, Ru_{n_l}\right)\right\}\\&\leq \kappa \max \left\{p_b\left(Ru_{n_l}, Ru_*\right), p_b\left(Su_{n_l}, Ru_{n_l}\right)\right\}.\end{aligned} \tag{32}$$

Letting $l \longrightarrow \infty$ and taking into account (28) and (30), we get

$$p_b(Su_*, Ru_*) = 0, \tag{33}$$

and by $(pb)_1$, $(pb)_1$, we have $Su_* = Ru_*$.If $\min \{p_b(Su_{n_l}, Su_*), p_b(Su_*, Ru_*)\} = p_b(Su_{n_l}, Su_*)$, we find that $\lim_{l\to\infty} p_b(Su_{n_l}, Su_*) = 0$. On the other hand, by $(pb)_4$,

$$p_b(Su_*, Ru_*) \le s\left[p_b\left(Su_*, Su_{n_l}\right) + p_b\left(S\left(u_{n_l}, Ru_*\right)\right] - p_b\left(S\left(u_{n_l}, Su_{n_l}\right),\right.\right. \tag{34}$$

and then, $p_b(Su_*, Ru_*) = 0$, as $l \longrightarrow \infty$. This proves that $z = Su_* = Ru_*$, that is, z is a point of coincidence for S and R.

Example 2. Let $M = [0,\infty)$ and $p_b : M \times M \longrightarrow [0,\infty)$ be a partial b-metric, where $p_b(u, y) = (\max \{u, y\})^2$. Let the mappings $S, R : M \longrightarrow M$,

$$\begin{gathered} Su = \begin{cases} \dfrac{u+1}{2}, & \text{if } u \in [0,1], \\ 3, & \text{if } u > 1, \end{cases} \\ Ru = \begin{cases} \dfrac{u+2}{4}, & \text{if } u \in [0,1], \\ \dfrac{u+5}{10}, & \text{if } u > 1, \end{cases} \end{gathered} \tag{35}$$

and the function $\beta : M \times M \longrightarrow [0,\infty)$,

$$\beta(x, v) = \begin{cases} 2, & \text{for } x = v = \dfrac{1}{2}, \\ 3, & \text{for } x = v = 3, \\ 1, & \text{for } x, v \ge 4, \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Obviously, since $x = Ru \ge 4$ for $u \ge 35$ we have

(i) For $u, y \ge 35$

$$\begin{gathered} \beta(Ru, Ry) = 1 \Longrightarrow \beta(Su, Sv) = \beta(3, 3) = 3 > 1, \\ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \beta(R(0), R(0)) = 2 \Longrightarrow \beta(S(0), S(0)) = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2, \\ \beta(3, 3) = \beta(R(25), R(25)) = 3 \Longrightarrow \beta(S(25), S(25)) = \beta(3, 3) = 3. \end{gathered} \tag{37}$$

Moreover,

$$\begin{aligned} &\beta(Ru, Ry) \min \{p_b(Su, Sy), p_b b(Sy, Ry)\} \\ &\quad -- \min \left\{b_p(Su, Ry), b_p(Sy, Ru)\right\} \\ &\quad \le \min \{p_b(3, 3), p_b(3, Ry)\} \\ &\quad = 9, \le \kappa \cdot 16 \le \kappa \cdot \max \{p_b(Ru, Ry), p_b(Su, Ru)\}, \end{aligned} \tag{38}$$

for any $9/16 < \kappa < 1$.

(ii) All other cases are uninteresting due to the way the function β was defined

Consequently, by Theorem 8, the mappings S, R have points of coincidence. These are $1/2 = S(0) = R(0)$, respectively, $3 = S(25) = R(25)$.

Corollary 9. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\min \{p_b(Su, Sy), p_b(Sy, Ry)\} \\ &\quad -- \min \left\{b_p(Su, Ry), b_p(Sy, Ru)\right\} \\ &\quad \le \kappa \max \{p_b(Ru, Rv), p_b(Su, Ru)\}, \end{aligned} \tag{39}$$

for every $u, y \in M$, such that $u \ne y$ when $u, y \in C_c(R, S)_M$. If $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms, then the mappings S and R have a point of coincidence.

Proof. It is enough to choose $\beta(u, y) = 1$ in Theorem 8.

Theorem 10. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\beta(u, y) \min \{p_b(Su, Sy), p_b(Sy, y)\} \\ &\quad -- \min \left\{b_p(Su, y), b_p(Sy, u)\right\} \\ &\quad \le \kappa \max \{p_b(u, y), p_b(Su, u)\}, \end{aligned} \tag{40}$$

for every $u, y \in M$, such that $u \ne y$. Suppose also that

(a) S is β-admissible, and there exists $u_0 \in M$ such that $\beta(u_0, Su_0) \ge 1$

(b) M is β-regular

Then, the mapping S has a fixed point.

Proof. Put $R = I_M$ in Theorem 8.

Corollary 11. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and a mapping $S : M \longrightarrow M$. Suppose that there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned} &\min \{p_b(Su, Sy), p_b(Sy, y)\} \\ &\quad -- \min \left\{b_p(Su, y), b_p(Sy, u)\right\} \\ &\quad \le \kappa \max \{p_b(u, y), p_b(Su, u)\}, \end{aligned} \tag{41}$$

for every $u, y \in M$,$u \ne y$. Then, the mapping S has a fixed point.

Proof. It is enough to choose $\beta(u, y) = 1$ in Theorem 10.

Theorem 12. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \longrightarrow M$. Suppose that there exist $\kappa \in (0, 1)$*

and $a > 0$ such that

$$\beta(Ru, Ry)M^1_S(u, y) - a \cdot N^1_{S,R}(u, y) \le \kappa p_b(Su, Ru)p_b(Sy, Ry), \tag{42}$$

where

$$M^1_{S,R}(u, y) = \min\left\{[p_b(Su, Sy)]^2, [p_b(Sy, Ry)]^2\right\},$$
$$N^1_{S,R}(u, y) = \min\left\{b_p(Su, Ry)b_p(Sy, Ru), p_b(Su, Ry)p_b(Su, Sy), p_b(Sy, Ru)p_b(Ru, Ry)\right\}, \tag{43}$$

for every $u, y \in M$, such that $u \ne y$ when $u, y \in C_c(R, S)_M$. Suppose also that:

(a) $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms

(b) S is R-β-admissible, and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \ge 1$

(c) M is β-regular

Then, the mappings S and R have a point of coincidence.

Proof. Starting with a point $u_0 \in M$ such that $\beta(Ru_0, Su_0) \ge 1$, we build the sequences $\{u_n\}$, $\{z_n\}$ as in Theorem 8,

$$z_n = Ru_{n+1} = Su_n, \text{ for all } n \in \mathbb{N}. \tag{44}$$

Using the same arguments, we can assume that $z_n \ne z_{n+1}$, also, for all $n \in \mathbb{N}$. Thus, for $u = u_n$, $y = u_{n+1}$,

$$\begin{aligned} M^1_{S,R}(u_n, u_{n+1}) &= \min\left\{[p_b(Su_n, Su_{n+1})]^2, [p_b(Su_{n+1}, Ru_{n+1})]^2\right\} \\ &= \min\left\{p_b(z_n, z_{n+1})]^2, [p_b(z_{n+1}, z_n)]^2\right\} \\ &= [p_b(z_{n+1}, z_n)]^2, \end{aligned} \tag{45}$$

$$\begin{aligned} N^1_{S,R}(u_n, u_{n+1}) &= \min\left\{\begin{array}{c} b_p(Su_n, Ru_{n+1})b_p(Su_{n+1}, Ru_n), p_b(Su_n, Ru_{n+1})p_b(Su_n, Su_{n+1}), \\ p_b(Su_{n+1}, Ru_n)p_b(Ru_n, Ru_{n+1}) \end{array}\right\} \\ &= \min\left\{\begin{array}{c} b_p(z_n, z_n)b_p(z_{n+1}, z_{n-1}), p_b(z_n, z_{n+1})p_b(z_n, z_{n+1}), \\ p_b(z_{n+1}, u_{n-1})p_b(u_{n-1}, z_n) \end{array}\right\} = 0, \end{aligned} \tag{46}$$

and taking into account Lemma 6, (42) becomes

$$\begin{aligned} M^1_{S,R}(u_n, u_{n+1}) &\le \beta(Ru_n, Ru_{n+1})M^1_{S,R}(u_n, u_{n+1}) - a \cdot N^1_{S,R}(u_n, u_{n+1}) \\ &\le \kappa p_b(Su_n, Ru_n) \cdot p_b(Su_{n+1}, Ru_{n+1}). \end{aligned} \tag{47}$$

Taking into account (46), the above inequality turns into

$$[p_b(z_n, z_{n+1})]^2 \le \kappa p_b(z_n, z_{n-1})p_b(z_{n+1}, z_n), \tag{48}$$

or equivalent (since $z_n \ne z_{n+1}$)

$$p_b(z_n, z_{n+1}) \le \kappa p_b(z_n, z_{n-1}). \tag{49}$$

Accordingly, from Lemma 7, it follows that the sequence $\{z_n\}$ is 0-p_b-Cauchy and due to the completeness of the space, there exists $z \in M$ such that $\lim n \longrightarrow \infty p_b(z_n), z) = p_b(z, z) = 0$. Following the corresponding lines in Theorem 8, we can find $u_* \in M$ such that $Ru_* = z$. Supposing that $Ru_* \ne Su^*$ for $u = u_{n_l}$ and $y = u_*$ and taking into account the assumption (c),

$$\begin{aligned} M^1_{S,R}(u_{n_l}, u_*) &\le \beta(Ru_{n_l}, Ru_*)M^1_{S,R}(u_{n_l}, u_*) - a \cdot N^1_{S,R}(u_{n_l}, u_*) \\ &\le \kappa p_b(Su_{n_l}, Ru_{n_l}) \cdot p_b(Su_*, Ru_*), \end{aligned} \tag{50}$$

where

$$\begin{aligned} M^1_{S,R}(u_{n_l}, u_*) &= \min\left\{[p_b(Su_{n_l}, Su_*)]^2, [p_b(Su_*, Ru_*)]^2\right\}, \\ N^1_{S,R}(u_{n_l}, u_*) &= \min\left\{b_p(Su_{n_l}, Ru_*)b_p(Su_*, Ru_{n_l}), p_b(Su_{n_l}, Ru_*)p_b(Su_*, Su_{n_l}), p_b(Su_*, Ru_{n_l})p_b(Ru_*, Ru_{n_l})\right\}. \end{aligned} \tag{51}$$

Since $\lim_{l\longrightarrow\infty} N^1_{S,R}(u_{n_l}, u_*) = 0$ and $\lim_{l\longrightarrow\infty} = p_b(Su_{n_l}, Ru_{n_l}) \cdot p_b(Su_*, Ru_*) = 0$ (by) letting $l \longrightarrow \infty$ in (50), we have

$$\text{either } [p_b(Su_*, Ru_*)]^2 = 0 \text{ or} \lim_{l\longrightarrow\infty} \left[p_b\left(Su_{n_l}, Su_*\right)\right]^2 = 0. \quad (52)$$

(1) If $[p_b(Su_*, Ru_*)]^2 = 0$, it follows that $Su_* = Ru_*$.

(2) If $\lim_{l\longrightarrow\infty}[p_b(Su_{n_l}, Su_*)]^2 = 0$, by $(pb)_4$

$$\begin{aligned} p_b(Ru_*, Su_*) &\le s\left[p_b\left(Ru_*, Su_{n_l}\right) + p_b\left(Su_{n_l}, Su_*\right)\right] - p_b\left(Su_{n_l}, Su_{n_l}\right) \\ &\le s\left[p_b\left(Ru_*, Su_{n_l}\right) + p_b\left(Su_{n_l}, Su_*\right)\right] \longrightarrow 0 \text{ as } l \longrightarrow \infty, \end{aligned} \quad (53)$$

so $p_b(Ru_*, Su_*) = 0$.

Thereupon, $Ru_* = Su_* = z$ and z is a point of coincidence of R and S.

Example 3. Let $M = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and the partial b-metric $p_b : M \times M \longrightarrow [0,+\infty)$ defined as follows (Table 1).

Let the function $\beta : M \times M \longrightarrow [0,+\infty)$, with

$$\beta(u, y) = \begin{cases} 1, & \text{for } (u, y) \in \{(\alpha_5, \alpha_3), (\alpha_3, \alpha_2)\}, \\ 2, & \text{for } (u, y) = (\alpha_2, \alpha_2), \\ 0, & \text{otherwise}, \end{cases} \quad (54)$$

and two mappings $S, R : M \longrightarrow M$ (Table 2).

First of all, we remark that

$$\begin{aligned} \beta(\alpha_5, \alpha_3) &= \beta(R\alpha_2, R\alpha_5) = 1 \Longrightarrow \beta(S\alpha_2, S\alpha_5) = \beta(\alpha_3, \alpha_2) = 1, \\ \beta(\alpha_3, \alpha_2) &= \beta(R\alpha_5, R\alpha_4) = 1 \Longrightarrow \beta(S\alpha_5, S\alpha_4) = \beta(\alpha_2, \alpha_2) = 2, \\ \beta(\alpha_2, \alpha_2) &= \beta(R\alpha_4, R\alpha_4) = 2 \Longrightarrow \beta(S\alpha_4, S\alpha_4) = \beta(\alpha_2, \alpha_2) = 2, \end{aligned} \quad (55)$$

which shows as that (b) holds. Also, it is easy to see that (a) and (c) are satisfied, so it remains to be verified (42). We distinguish two cases as follows:

(1) $(u, y) = (\alpha_2, \alpha_5)$

$$\begin{aligned} M^1_{S,R}(\alpha_2, \alpha_5) &= \min\left\{[p_b(S\alpha_2, S\alpha_5)]^2, [p_b(S\alpha_5, R\alpha_5)]^2\right\} = \min\left\{[p_b(\alpha_3, \alpha_2)]^2, [p_b b(\alpha_2, \alpha_3)]^2\right\} = 9, \\ N^1_{S,R}(\alpha_2, \alpha_5) &= \min\left\{b_p(S\alpha_2, R\alpha_5) b_p(S\alpha_5, R\alpha_2), \cdots\right\} = \min\left\{b_p(\alpha_3, \alpha_3) b_p(\alpha_2, \alpha_5), \cdots\right\} = 0, \\ p_b(S\alpha_2, R\alpha_2) p_b(S\alpha_5, R\alpha_5) &= p_b(\alpha_3, \alpha_5) p_b(\alpha_2, \alpha_3) = 22 \cdot 3 = 66. \end{aligned} \quad (56)$$

(2) $(u, y) = (\alpha_5, \alpha_4)$

$$\begin{aligned} M^1_{S,R}(\alpha_5, \alpha_4) &= \min\left\{[p_b(S\alpha_5, S\alpha_4)]^2, [p_b(S\alpha_4, R\alpha_4)]^2\right\} = \min\left\{[p_b(\alpha_2, \alpha_2)]^2, [p_b(\alpha_2, \alpha_2)]^2\right\} = 1, \\ N^1_{S,R}(\alpha_5, \alpha_4) &= \min\left\{b_p(S\alpha_5, R\alpha_4) b_p(S\alpha_4, R\alpha_5), \cdots\right\} = \min\left\{b_p(\alpha_2, \alpha_2) b_p(\alpha_2, \alpha_3), \cdots\right\} = 0, \\ p_b(S\alpha_5, R\alpha_5) p_b(S\alpha_4, R\alpha_4) &= p_b(\alpha_2, \alpha_3) p_b(\alpha_2, \alpha_2) = 3 \cdot 1 = 3. \end{aligned} \quad (57)$$

So, for any $\kappa \in (0, 1)$, the inequality (42) holds. Therefore, the mappings S, R have a point of coincidence, which is $z = \alpha_2$.

Corollary 13. *Let $(M, p_b, s \ge 1)$ be a complete p_b-ms and two mappings $R, S : M \to M$. Suppose that there exist $\kappa \in (0, 1)$ and $a > 0$ such that*

$$M^1_{S,R}(u, y) - a \cdot N^1_{S,R}(u, y) \le \kappa p_b(Su, Ru) p_b(Sy, Ry), \quad (58)$$

where

$$\begin{aligned} M^1_{S,R}(u, y) &= \min\left\{[p_b(Su, Sy)]^2, [p_b(Sy, Ry)]^{22}\right\}, \\ N^1_{S,R}(u, y) &= \min\left\{b_p(Su, Ry) b_p(Sy, Ru), p_b(Su, Ry) p_b(Su, Sy), p_b(Sy, Ru) p_b(Ru, Ry)\right\}, \end{aligned} \quad (59)$$

Table 1

$p_b(u,y)$	α_1	α_2	α_3	α_4	α_5
α_1	0	2	6	30	42
α_2	2	1	3	21	31
α_3	6	3	2	14	22
α_4	30	21	14	5	7
α_5	42	31	22	7	6

Table 2

	α_1	α_2	α_3	α_4	α_5
S	α_5	α_3	α_2	α_2	α_2
R	α_1	α_5	α_1	α_2	α_3

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R,S)_M$. Then, the mappings S and R have a point of coincidence providing that $S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms.

Proof. Put $\beta(u,y)=1$ in Theorem 12.

Theorem 14. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms a mapping $S: M \longrightarrow M$. Suppose that there exists $\kappa \in (0,1)$ and $a>0$ such that*

$$\beta(u,y) M_S^1(u,y) - a \cdot N_S^1(u,y) \leq \kappa p_b(Su,u) p_b(Sy,y), \quad (60)$$

where

$$\begin{aligned} M_S^1(u,y) &= \min\left\{[p_b(Su,Sy)]^2, [p_b(Sy,y)]^2\right\}, \\ N_S^1(u,y) &= \min\left\{b_p(Su,y) b_p(Sy,u), p_b(Su,y) p_b(Su,Sy), p_b(Sy,u) p_b(u,y)\right\}, \end{aligned} \quad (61)$$

for every $u, y \in M, u \neq y$. Suppose also that

(a) *S is β-admissible, and there exists $u_0 \in M$ such that $\beta(u_0, Su_0) \geq 1$*

(b) *M is β-regular*

Then, the mapping S possesses a fixed point.

Proof. Choose $R = I_M$ in Theorem 12.

Corollary 15. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms a mapping $S: M \longrightarrow M$. Suppose that there exists $\kappa \in (0,1)$ and $a>0$ such that*

$$M_S^1(u,y) - a \cdot N_S^1(u,y) \leq \kappa p_b(Su,u) p_b(Sy,y), \quad (62)$$

where

$$\begin{aligned} M_S^1(u,y) &= \min\left\{[p_b(Su,Sy)]^2, [p_b(Sy,y)]^2\right\}, \\ N_S^1(u,y) &= \min\left\{b_p(Su,y) b_p(Sy,u), p_b(Su,y) p_b(Su,Sy), p_b(Sy,u) p_b(u,y)\right\}, \end{aligned} \quad (63)$$

for every $u, y \in M$, $u \neq y$. Then, the mapping S possesses a fixed point.

Proof. Put $\beta(u,y)=1$ in Theorem 14.

Theorem 16. *Let $(M, p_b, s \geq 1)$ be a complete p_b-ms and two mappings $R, S: M \longrightarrow M$. Suppose that there exist $\kappa \in (0,1)$ and $a>0$ such that*

$$\beta(Ru, Ry) M_{S,R}^2(u,y) \leq \kappa \cdot N_{S,R}^2(u,y), \quad (64)$$

where

$$\begin{aligned} M_{S,R}^2(u,y) &= p_b(Su,Sy) p_b(Sy,Ry) - a \cdot \min\left\{b_p(Su,Ry), b_p(Sy,Ru)\right\}, \\ N_{S,R}^2(u,y) &= p_b(Ru,Ry) \cdot \max\left\{p_b(Su,Ru), p_b(Sy,Ry), \frac{p_b(Su,Ry)+p_b(Sy,Ru)}{2s}\right\}, \end{aligned} \quad (65)$$

for every $u, y \in M$, such that $u \neq y$ when $u, y \in C_c(R,S)_M$. Suppose also that

(a) *$S(M) \subset R(M)$ and $(R(M), p_b, s)$ is a 0-p_b-complete p_b-ms*

(b) *S is R-β-admissible and there exists $u_0 \in M$ such that $\beta(Ru_0, Su_0) \geq 1$*

(c) *M is β-regular*

Then, the mappings S and R have a point of coincidence.

Proof. We will only sketch the proof, because, basically, we use the same technique that was used in the above theorems. Indeed, for $u = u_n$, $y = u_{n+1}$, where the sequences $\{z_n\}$, $\{u_n\}$ are defined in Theorem 8, we have

$$\begin{aligned} M_{S,R}^2(u_n, u_{n+1}) &= p_b(Su_n, Su_{n+1}) p_b(Su_{n+1}, Ru_{n+1}) \\ &\quad --a \cdot \min\left\{b_p(Su_n, Ru_{n+1}), b_p(Su_{n+1}, Ru_n)\right\} \\ &= p_b(z_n, z_{n+1}) p_b(z_{n+1}, z_n) \\ &\quad - a \cdot \min\left\{b_p(z_n, z_n), b_p(z_{n+1}, z_{n-1})\right\} \\ &= [p_b(z_n, z_{n+1})]^2, \end{aligned}$$

$$\begin{aligned}N^2_{S,R}(u_n,u_{n+1}) &= p_b(Ru_n,Ru_{n+1})\cdot\max\left\{\begin{array}{c}p_b(Su_n,Ru_n),p_b(Su_{n+1},Ru_{n+1}),\\ \dfrac{p_b(Su_n,Ru_{n+1})+p_b(Su_{n+1},Ru_n)}{2s}\end{array}\right\}\\
&= \max\left\{\begin{array}{c}p_b(z_{n-1},z_n),p_b(z_n,z_{n+1}),\\ \dfrac{p_b(z_n,z_n)+p_b(z_{n+1},z_{n-1})}{2s}\end{array}\right\}\\
&\le p_b(z_{n-1},z_n)\cdot \mathrm{m}p_b(z_{n-1},z_n)\cdot\max\\
&\quad\cdot\left\{\begin{array}{c}p_b(z_{n-1},z_n),p_b(z_n,z_{n+1}),\\ \dfrac{p_b(z_n,z_n)+s[p_b(z_{n+1},z_n)+p_b(z_n,z_{n+1})]-p_b(z_n,z_n)}{2s}\end{array}\right\}\\
&= p_b(z_{n-1},z_n)\cdot\max\\
&\quad\cdot\left\{p_b(z_{n-1},z_n),p_b(z_n,z_{n+1}),\frac{p_b(z_{n+1},z_n)+p_b(z_n,z_{n+1})}{2}\right\}\\
&= p_b(z_{n-1},z_n)\cdot\max\{p_b(z_{n-1},z_n),p_b(z_n,z_{n+1})\}.\end{aligned}\tag{66}$$

Thus, the inequality (64) becomes

$$[p_b(z_n,z_{n+1})]^2\le\kappa p_b(z_{n-1},z_n)\cdot\max\{p_b(z_{n-1},z_n),p_b(z_n,z_{n+1})\}.\tag{67}$$

Since for the case $\max\{p_b(z_{n-1},z_n),p_b(z_n,z_{n+1})\}=p_b(z_n,z_{n+1})$ we get $[p_b(z_n,z_{n+1})]^2\le\kappa p_b(z_{n-1},z_n)\cdot p_b(z_n,z_{n+1})$, or $p_b(z_n,z_{n+1})\le\kappa p_b(z_{n-1},z_n)<p_b(z_{n-1},z_n)$, which is a contradiction, we conclude that $\max\{p_b(z_{n-1},z_n),p_b(z_n,z_{n+1})\}=p_b(z_{n-1},z_n)$ and then (67) becomes

$$p_b(z_n,z_{n+1})\le\kappa p_b(z_{n-1},z_n),\tag{68}$$

for any $n\in\mathbb{N}$. Therefore, by Lemma L2A and using similar arguments as in Theorems 8 and 12, there exists $u_*\in M$ such that

$$\lim_{n\to\infty}p_b(Su_n,Ru_*)=\lim_{n\to\infty}p_b(Ru_n,Ru_*)=p_b(Ru_*,Ru_*)=0.\tag{69}$$

Finally, we claim that $Su_*=Ru_*$. From the assumptions (c), there exists a subsequences $\{u_{n_l}\}$ of $\{u_n\}$ such that $\beta(u_{n_l},u_*)\ge 1$. Thus, replacing u by u_{n_l} and y by u_*, we get (as $l\longrightarrow\infty$)

$$\begin{aligned}\lim_{n\to\infty}M^2_{S,R}(u_{n_l},u_*) &= \lim_{n\to\infty}\left[p_b(Su_{n_l},Su_*)p_b(Su_*,Ru_*)\right.\\
&\quad\left.--a\cdot\min\left\{b_p(Su_{n_l},Ru_*),b_p(Su_*,Ru_{n_l})\right\}\right]\\
&= p_b(Su_*,Ru_*)\cdot\lim_{n\to\infty}\left[p_b(Su_{n_l},Su_*)\right),\\
\lim_{n\to\infty}N^2_{S,R}(u_{n_l},u_*) &= \lim_{n\to\infty}p_b(Ru_{n_l},Ru_*)\\
&\quad\cdot\max\left\{\begin{array}{c}p_b(Su_{n_l},Ru_{n_l}),p_b(Su_*,Ru_*),\\ \dfrac{p_b(Su_{n_l},Ru_*)+p_b(Su_*,Ru_{n_l})}{2s}\end{array}\right\}=0.\end{aligned}\tag{70}$$

Consequently, (64) becomes $p_b(Su_*,Ru_*)\cdot\lim_{n\to\infty}[p_b(Su_{n_l},Su_*)=0$ and the rest is just a verbatim repetition of the lines in the previous proofs.

Corollary 17. *Let $(M,p_b,s\ge 1)$ be a complete p_b-ms and two mappings $R,S:M\longrightarrow M$. Suppose that there exist $\kappa\in(0,1)$ and $a>0$ such that*

$$M^2_{S,R}(u,y)\le\kappa\cdot N^2_{S,R}(u,y),\tag{71}$$

where

$$\begin{aligned}M^2_{S,R}(u,y)&=p_b(Su,Sy)p_b(Sy,Ry)-a\cdot\min\left\{b_p(Su,Ry),b_p(Sy,Ru)\right\},\\
N^2_{S,R}(u,y)&=p_b(Ru,Ry)\cdot\max\left\{p_b(Su,Ru),p_b(Sy,Ry),\frac{p_b(Su,Ry)+p_b(Sy,Ru)}{2s}\right\},\end{aligned}\tag{72}$$

for every $u,y\in M$, such that $u\ne y$ when $u,y\in C_c(R,S)_M$. If $S(M)\subset R(M)$ and $(R(M),p_b,s)$ is a 0-p_b-complete p_b-ms, then, the mappings S and R have a point of coincidence.

Proof. Let $\beta(u,y)=1$ in Theorem 16.

Theorem 18. *Let $(M,p_b,s\ge 1)$ be a complete p_b-ms and a mapping $S:M\longrightarrow M$. Suppose that there exist $\kappa\in(0,1)$ and $a>0$ such that*

$$\beta(u,y)M^2_S(u,y)\le\kappa\cdot N^2_S(u,y),\tag{73}$$

where

$$\begin{aligned}M^2_S(u,y)&=p_b(Su,Sy)p_b(Sy,y)-a\cdot\min\left\{b_p(Su,y),b_p(Sy,u)\right\},\\
N^2_S(u,y)&=p_b(u,y)\cdot\max\left\{p_b(Su,u),p_b(Sy,y),\frac{p_b(Su,y)+p_b(Sy,u)}{2s}\right\},\end{aligned}\tag{74}$$

for every $u,y\in M$. Suppose also that

(i) S is β-admissible, and there exists $u_0\in M$ such that $\beta(Ru_0,Su_0)\ge 1$

(ii) M is β-regular

Then, the mapping S admits a fixed point.

Proof. Choose $R=I_M$.

Corollary 19. *Let $(M,p_b,s\ge 1)$ be a complete p_b-ms and two mappings $R,S:M\longrightarrow M$. Suppose that there exist $\kappa\in(0,1)$ and $a>0$ such that*

$$M^2_S(u,y)\le\kappa\cdot N^2_S(u,y),\tag{75}$$

where

$$M_S^2(u,y)M_S^2(u,y) = p_b(Su,Sy)p_b(Sy,y) - a \cdot \min\left\{b_p(Su,y), b_p(Sy,u)\right\},$$
$$N_S^2(u,y) = p_b(u,y) \cdot \max\left\{p_b(Su,u), p_b(Sy,y), \frac{p_b(Su,y)+p_b(Sy,u)}{2s}\right\}, \tag{76}$$

for every $u, y \in M$. Then, the mapping S has a fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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