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On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators

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Abstract We investigate the existence of solutions for a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions. In this way, we use a generalization of the hybrid Dhage's fixed point result for sum of three fractional operators. Finally, we give an example to illustrate our main result.

1. Introduction and preliminaries

As you know, most of the advanced technologies in various sciences are somehow related to computer systems and applied softwares. This allows computer programmers to try to use more sophisticated algorithms to design such softwares, so that not to encounter many errors when programs are running. In order to achieve high accuracy and low error in the implementation of all software applications, it is necessary to write computer instructions and algorithms based on exact arithmetic calculations and real mathematical modelings of economic, physical and medical processes and phenomena.

For this purpose, we need strong mathematical tools in the first step. In other words, if one can make exact patterns for natural phenomena and processes by using new mathematical formulas and operators, then more flexible algorithms can be written in the software programming based on such relations and formulas. This results in accurate computer calculations with the least error in the shortest time. In this way, many researchers from different scientific fields are currently studying various types of advanced mathematical modelings using fractional operators and fractional differential equations with more general boundary value conditions. Indeed, they try to model the processes such that it covers many applied cases and in this situation, mathematicians would like to solve a wide range of these boundary value problems with advanced and complicate boundary conditions. They are studying advanced fractional modelings and its related existence results and qualitative behaviors of solutions for distinct fractional problems (see for example, [1,3–8,11,12,15–24,29,32,33,38,41,43,45]).Even some researchers have been published applied

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papers in different fields in which the main results are proved using the numerical and analytical techniques, simultaneously (see for example, [28,30,35,36]). In recent decades, a new class of mathematical modelings based on fractional hybrid differential equations with hybrid or non-hybrid boundary value conditions have achieved a great deal of interest and attention of many researchers (see for example, [9,13,14,31,39,42]).

Dhage and Lakshmikantham [27] started working on hybrid equations. They introduced a new category of nonlinear differential equation called ordinary hybrid differential equation and studied the existence of extremal solutions for this boundary value problem by establishing some fundamental differential inequalities [27]. Zhao et al. [46] provided an extension for the Dhage's work to fractional order and considered a boundary value problem of fractional hybrid differential equations. Ahmad et al. [2] studied the existence of solutions for the hybrid nonlocal boundary value inclusion problem

$$\begin{cases} {}^c\mathcal{D}_0^{\alpha}\left(\dfrac{k(t)-\sum_{i=1}^{m}\mathcal{I}_0^{\beta_i}h_i(t,k(t))}{g(t,k(t))}\right) \in \mathcal{G}(t,k(t)) = 0, \quad (t \in J=[0,1]) \\ k(0) = \mu(x), \qquad k(1) = A \in \mathbb{R}, \end{cases}$$

where ${}^c\mathcal{D}_0^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (1,2]$ and $\mathcal{I}_0^{\phi}$is the Riemann–Liouville fractional integral of order $\phi > 0$ with $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$. In 2018,Ullah et al. derived an existence result for the fractional hybrid boundary value problem

$$\begin{cases} \mathcal{D}_0^{\alpha}\left(\frac{k(t)-f(t,k(t))}{h(t,k(t))}\right) = g(t,k(t)), \quad (t\in[0,1]) \\ \left(\frac{k(t)-f(t,k(t))}{h(t,k(t))}\right)|_{t=0} = 0, \qquad \left(\frac{k(t)-f(t,k(t))}{h(t,k(t))}\right)|_{t=1} = 0, \end{cases}$$

where $h \in C([0,1]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$,$f$ and g are continuous real-valued functions on $[0,1]\times\mathbb{R}$ and $\mathcal{D}_0^{\alpha}$ denotes the fractional Riemann–Liouville derivative of order $\alpha\in(0,1]$ [44]. In 2019, Derbazi et al. proved the existence and uniqueness of solutions for the fractional hybrid boundary value problem

$${}^c\mathcal{D}_0^{\alpha}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right) = \Theta(t,k(t)), \quad (t\in[0,T])$$

with the fractional hybrid boundary value conditions

$$\begin{cases} a_1\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=0} + b_1\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=T} = \lambda_1, \\ a_2\,{}^c\mathcal{D}_0^{\beta}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=\eta} + b_2\,{}^c\mathcal{D}_0^{\beta}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=T} = \lambda_2, \end{cases}$$

where $\alpha\in(1,2], \beta\in(0,1], \eta\in(0,T)$ and a_1, a_2, b_1, b_2 and λ_1, λ_2 are real constants. Moreover, two fractional derivatives appeared in the above problem are of Caputo type [25]. Recently, Samei et al. discussed the existence of solutions for the fractional hybrid Caputo-Hadamard differential inclusion

$$\begin{cases} {}^{CH}\mathcal{D}_{1^+}^{\alpha}\left(\frac{k(t)-f\left(t,k(t),\mathcal{I}^{\beta_1}h_1(t,k(t)),\ldots,\mathcal{I}^{\beta_n}h_n(t,k(t))\right)}{g(t,k(t),\mathcal{I}^{\gamma_1}k(t),\ldots,\mathcal{I}^{\gamma_m}k(t))}\right) \in K(t,k(t)), \quad (t\in J=[1,e]) \\ k(1)=\mu(t), \qquad k(e)=\eta(t), \end{cases}$$

where $\alpha\in(1,2], n,m\in\mathbb{N}, \beta_i>0$ for $i=1,2,\ldots,n, \gamma_i>0$ for $i=1,2,\ldots,m$, every three functions $g: J\times\mathbb{R}^{m+1}\to\mathbb{R}\setminus\{0\}$, $f: J\times\mathbb{R}^{n+1}\to\mathbb{R}$ and $h_i: J\times\mathbb{R}\to\mathbb{R}$ are continuous. Also, $\mu,\eta\in C(J,\mathbb{R})$ and $K: J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a set-valued map with certain conditions. The operators ${}^{CH}\mathcal{D}_{1^+}^{(\cdot)}$ and $\mathcal{I}^{(\cdot)}$ denote the fractional Caputo-Hadamard derivative and the fractional Hadamard integral of order $(\cdot)$, respectively. [39].

More recently, Baleanu et al. [10] formulated a hybrid model of thermostat insulated at $t=0$ with the controller at $t=1$ as

$${}^c\mathcal{D}_0^{p}\left(\frac{z(t)}{h(t,z(t))}\right) + \Phi(t,z(t)) = 0 (t\in[0,1], p\in(1,2])$$

with the fractional hybrid boundary value problems

$$\begin{cases} \mathcal{D}\left(\frac{z(t)}{h(t,z(t))}\right)|_{t=0} = 0, \\ \lambda\,{}^c\mathcal{D}_0^{p-1}\left(\frac{z(t)}{h(t,z(t))}\right)|_{t=1} + \left(\frac{z(t)}{h(t,z(t))}\right)|_{t=\eta} = 0, \end{cases}$$

where $\eta\in[0,1], \lambda>0$ is a parameter, $p-1\in(0,1]$, $\mathcal{D}=\frac{d}{dt}, {}^c\mathcal{D}_0^{\gamma}$is the Caputo derivative of fractional order $\gamma\in\{p,p-1\}$, the function $\Phi:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and the map h belongs to $C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$. According to this fractional model, the thermostat adds or discharges heat depending on the temperature detected by the sensor at $t=\eta$. This applied model of thermostat can be the basic idea for the next researches in the mechanical engineering.

By mixing idea of the above works, we derived an existence result for the fractional hybrid integro-differential equation

$$\begin{aligned} &{}^c\mathcal{D}_0^{\omega}\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right) \\ &= \Upsilon(t,k(t)), \quad (t\in[0,1]) \end{aligned} \tag{1.1}$$

with mixed integral hybrid boundary value conditions

$$\begin{cases} \lambda_1\int_0^1 {}^c\mathcal{D}_0^{\beta_1}\left(\frac{k(s)-\varphi\left(s,k(s),\mathcal{I}_0^{\gamma_1}k(s),\mathcal{I}_0^{\gamma_2}k(s),\ldots,\mathcal{I}_0^{\gamma_n}k(s)\right)}{\phi\left(s,k(s),\mathcal{I}_0^{\mu_1}k(s),\mathcal{I}_0^{\mu_2}k(s),\ldots,\mathcal{I}_0^{\mu_m}k(s)\right)}\right)ds \\ \quad +\lambda_2\,{}^c\mathcal{D}_0^{\alpha_1}\left(\frac{k(t)-\varphi\left(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t)\right)}{\phi\left(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t)\right)}\right)|_{t=1} \\ \quad +\lambda_3\left(\frac{k(t)-\varphi\left(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t)\right)}{\phi\left(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t)\right)}\right)|_{t=0} = 0, \\ \lambda_4\int_0^1 {}^c\mathcal{D}_0^{\beta_2}\left(\frac{k(s)-\varphi\left(s,k(s),\mathcal{I}_0^{\gamma_1}k(s),\mathcal{I}_0^{\gamma_2}k(s),\ldots,\mathcal{I}_0^{\gamma_n}k(s)\right)}{\phi\left(s,k(s),\mathcal{I}_0^{\mu_1}k(s),\mathcal{I}_0^{\mu_2}k(s),\ldots,\mathcal{I}_0^{\mu_m}k(s)\right)}\right)ds \\ \quad +\lambda_5\,{}^c\mathcal{D}_0^{\alpha_2}\left(\frac{k(t)-\varphi\left(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t)\right)}{\phi\left(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t)\right)}\right)|_{t=1} \\ \quad +\lambda_6\left(\frac{k(t)-\varphi\left(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t)\right)}{\phi\left(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t)\right)}\right)|_{t=0} = 0, \end{cases} \tag{1.2}$$

where $\omega\in(1,2],\beta_1,\beta_2\in(0,1],\alpha_1,\alpha_2\in(0,1],\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\in\mathbb{R}^+$. Also, ${}^c\mathcal{D}_0^{v}$ denotes the fractional Caputo derivative of order $v\in\{\omega,\beta_1,\beta_2,\alpha_1,\alpha_2\}$,$\mathcal{I}_0^{\eta}$ denotes the fractional Riemann–Liouville integral of order $\eta\in\{\gamma_i,\mu_i\}$ with $\gamma_1,\ldots,\gamma_n>0$ and $\mu_1,\ldots,\mu_m>0$ and the maps $\varphi:[0,1]\times\mathbb{R}^{n+1}\to\mathbb{R},\phi:[0,1]\times\mathbb{R}^{m+1}\to\mathbb{R}\setminus\{0\}$ and $\Upsilon:[0,1]\times\mathbb{R}\to\mathbb{R}$are continuous.In this model, the boundary value conditions show the linear combinations of values of the unknown function and its derivative and integral at two end of the interval [0,1]. If one of the coefficients λ_i's be zero, then the mentioned fractional hybrid boundary value problem (1.1) and (1.2) reduces to a simple hybrid problem.

It is notable that the fractional hybrid integro-differential equation presented in this paper is the novel in the sense that the boundary value conditions are as mixed Caputo integro-derivative hybrid conditions and also all dependent functions φ and ϕ are considered in the form of multi-term. This complicate modeling of the hybrid type is general and leads to cover many fractional dynamical systems as special cases. For instance, it is clear that if we put $\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))=0$ and $\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))=1$, then the fractional hybrid boundary value problem (1.1) and (1.2) reduces to the fractional differential equation ${}^c\mathcal{D}_0^{\omega}k(t)=\Upsilon(t,k(t))$with fractional integral boundary value conditions

$$\lambda_1\int_0^1 {}^c\mathcal{D}_0^{\beta_1}k(s)\,\mathrm{d}s+\lambda_2\,{}^c\mathcal{D}_0^{\alpha_1}k(1)+\lambda_3k(0)=0,$$
$$\lambda_4\int_0^1 {}^c\mathcal{D}_0^{\beta_2}k(s)\,\mathrm{d}s+\lambda_5\,{}^c\mathcal{D}_0^{\alpha_2}k(1)+\lambda_6k(0)=0.$$

Hence, this is an abstract and unique idea and is the original subject which can be useful in the future. In this way, we use a generalization of the hybrid Dhage fixed point theorem for sum of three fractional operators in our existence theorem. First of all, we recall some notions and results on the fractional calculus which are needed in the sequel.

Let $\omega>0$. The fractional Riemann–Liouville integral of a function $k:[a,b]\to\mathbb{R}$ is defined by $\mathcal{I}_0^{\omega}k(t)=\int_0^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}k(s)\mathrm{d}s$ provided that the right-hand side integral exists [37,40]. Now, let $n-1<\omega<n$ and $n=[\omega]+1$. The fractional Caputo derivative of a function $k\in C^{(n)}([a,b],\mathbb{R})$ is defined by ${}^c\mathcal{D}_0^{\omega}k(t)=\int_0^t\frac{(t-s)^{n-\omega-1}}{\Gamma(n-\omega)}k^{(n)}(s)\mathrm{d}s$ provided that the right-hand side integral exists [37,40]. It has been proved that the general solution for the homogeneous fractional differential equation ${}^c\mathcal{D}_0^{\omega}k(t)=0$ is in the form $k(t)=m_0^*+m_1^*t+m_2^*t^2+\ldots+m_{n-1}^*t^{n-1}$ and we have

$$\mathcal{I}_0^{\omega}\left({}^c\mathcal{D}_0^{\omega}k(t)\right)=k(t)+\sum_{j=0}^{n-1}m_j^*t^j$$
$$=k(t)+m_0^*+m_1^*t+m_2^*t^2+\ldots+m_{n-1}^*t^{n-1},$$

where $m_0^*,\ldots,m_{n-1}^*$ are some real constants and $n=[\omega]+1$ [34]. We need next result.

Theorem 1.1 [26]. *Let $\mathcal{X}$ be a Banach algebra and E be a closed convex bounded nonempty subset of $\mathcal{X}$. Suppose that three operators $\Phi_1,\Phi_2:\mathcal{X}\to\mathcal{X}$ and $\Phi_3:E\to\mathcal{X}$ satisfy the following conditions:*

(i) *Φ_1 and Φ_2 are Lipschitzian with Lipschitz constants l_1^* and l_2^*, respectively,*
(ii) *Φ_3 is compact and continuous,*
(iii) *$l_1^*\hat{\Delta}+l_2^*<1$, where $\hat{\Delta}=\|\Phi_3(E)\|_{\mathcal{X}}=\sup\{\|\Phi_3k\|_{\mathcal{X}}:k\in E\}$.*

Then either (a) the operator equation $(\Phi_1k)(\Phi_3k)+(\Phi_2k)=k$ has a solution in E or (b) for each number $r>0$, there is an element $v^*\in\mathcal{X}$ with $\|v^*\|_{\mathcal{X}}=r$such that we have $\alpha_0(\Phi_1v^*)(\Phi_3v^*)+\alpha_0(\Phi_2v^*)=v^*$ for some constant $\alpha_0\in(0,1)$.

The paper is organized as follows. In Section 2, we first present the structure of the solution function for the hybrid fractional integro-differential Eq. (1.1) and (1.2) and then prove our main existence results using the generalized Dhage's fixed point theorem. In Section 3, we provide an illustrative example about the corresponding existence results.

2. Main results

In this moment, we are ready to state the main existence result. Consider $\mathcal{X}=\{k(t):k(t)\in C_{\mathbb{R}}([0,1])\}$ with the supremum norm $\|k\|_{\mathcal{X}}=\sup_{t\in[0,1]}|tk(t)|t$and the multiplication action on $\mathcal{X}$ by $(k\cdot k')(t)=k(t)k'(t)$ for all $k,k'\in\mathcal{X}$. Then an ordered triple $(\mathcal{X},\|\cdot\|_{\mathcal{X}},\cdot)$ is a Banach algebra.

Lemma 2.1. *Let $z\in\mathcal{X}$. Then a function k_0 is a solution for the hybrid fractional integro-differential equation*

$${}^c\mathcal{D}_0^{\omega}\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right)=z(t)\tag{2.1}$$

with mixed integral hybrid boundary value conditions

$$\begin{cases}\lambda_1\int_0^1{}^c\mathcal{D}_0^{\beta_1}\left(\frac{k(s)-\varphi(s,k(s),\mathcal{I}_0^{\gamma_1}k(s),\mathcal{I}_0^{\gamma_2}k(s),\ldots,\mathcal{I}_0^{\gamma_n}k(s))}{\phi(s,k(s),\mathcal{I}_0^{\mu_1}k(s),\mathcal{I}_0^{\mu_2}k(s),\ldots,\mathcal{I}_0^{\mu_m}k(s))}\right)\mathrm{d}s\\ \quad+\lambda_2\,{}^c\mathcal{D}_0^{\alpha_1}\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right)\Big|_{t=1}\\ \quad+\lambda_3\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right)\Big|_{t=0}=0,\\ \lambda_4\int_0^1{}^c\mathcal{D}_0^{\beta_2}\left(\frac{k(s)-\varphi(s,k(s),\mathcal{I}_0^{\gamma_1}k(s),\mathcal{I}_0^{\gamma_2}k(s),\ldots,\mathcal{I}_0^{\gamma_n}k(s))}{\phi(s,k(s),\mathcal{I}_0^{\mu_1}k(s),\mathcal{I}_0^{\mu_2}k(s),\ldots,\mathcal{I}_0^{\mu_m}k(s))}\right)\mathrm{d}s\\ \quad+\lambda_5\,{}^c\mathcal{D}_0^{\alpha_2}\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right)\Big|_{t=1}\\ \quad+\lambda_6\left(\frac{k(t)-\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))}{\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))}\right)\Big|_{t=0}=0,\end{cases}\tag{2.2}$$

if and only if k_0 is a solution for the fractional integral equation

$$\begin{aligned}k(t)&=\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))\\&\quad+\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))\\&\quad\times\left[\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}z(s)\,\mathrm{d}s+\frac{\lambda_2\Delta_1\Delta_3-\lambda_1\lambda_3\lambda_6\Delta_2\Delta_5t}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\alpha_1)}\right.\\&\quad\times\int_0^1(1-s)^{\omega-\alpha_1-1}z(s)\,\mathrm{d}s+\frac{\lambda_5[\lambda_3\Delta_2\Delta_5t-\Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\alpha_2)}\\&\quad\times\int_0^1(1-s)^{\omega-\alpha_2-1}z(s)\,\mathrm{d}s+\frac{\lambda_1[\Delta_1\Delta_3-\lambda_3\lambda_6\Delta_2\Delta_5t]}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\beta_1)}\\&\quad\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta_1-1}z(\tau)\,\mathrm{d}\tau\mathrm{d}s+\frac{\lambda_4[\lambda_3\Delta_2\Delta_5t-\Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\beta_2)}\\&\quad\left.\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta_2-1}z(\tau)\,\mathrm{d}\tau\mathrm{d}s\right],\end{aligned}\tag{2.3}$$

where

$$\Delta_1=[\lambda_2\lambda_6\Gamma(2-\alpha_2)-\lambda_3\lambda_5\Gamma(2-\alpha_1)](3-\beta_1)(3-\beta_2)$$
$$+[\lambda_1\lambda_6(3-\beta_2)-\lambda_3\lambda_4(3-\beta_1)]\Gamma(2-\alpha_1)\Gamma(2-\alpha_2),$$

$$\Delta_2=(3-\beta_1)(3-\beta_2)\Gamma(2-\alpha_1)\Gamma(2-\alpha_2),$$

$$\Delta_3=\lambda_6\Delta_2(\lambda_1\Gamma(2-\alpha_1)+\lambda_2(3-\beta_1))-\Delta_1(3-\beta_1)\Gamma(2-\alpha_1),$$

$$\Delta_4=\Delta_2[\lambda_1\Gamma(2-\alpha_1)+\lambda_2(3-\beta_1)],$$

$$\Delta_5=\Delta_1(3-\beta_1)\Gamma(2-\alpha_1).\tag{2.4}$$

Proof. First, assume that k_0 is a solution for differential Eq. (2.1). Then, there exist constants $m_0^*, m_1^* \in \mathbb{R}$ such that

$$\frac{k_0(t) - \varphi(t, k_0(t), \mathcal{I}_0^{\gamma_1} k_0(t), \mathcal{I}_0^{\gamma_2} k_0(t), \ldots, \mathcal{I}_0^{\gamma_n} k_0(t))}{\phi(t, k_0(t), \mathcal{I}_0^{\mu_1} k_0(t), \mathcal{I}_0^{\mu_2} k_0(t), \ldots, \mathcal{I}_0^{\mu_m} k_0(t))} = \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} z(s)\,ds + m_0^* + m_1^* t$$

and so

$$\begin{aligned} k_0(t) &= \varphi(t, k_0(t), \mathcal{I}_0^{\gamma_1} k_0(t), \mathcal{I}_0^{\gamma_2} k_0(t), \ldots, \mathcal{I}_0^{\gamma_n} k_0(t)) \\ &\quad + \phi(t, k_0(t), \mathcal{I}_0^{\mu_1} k_0(t), \mathcal{I}_0^{\mu_2} k_0(t), \ldots, \mathcal{I}_0^{\mu_m} k_0(t)) \\ &\quad \times \left[\frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} z(s)\,ds + m_0^* + m_1^* t\right]. \end{aligned} \tag{2.5}$$

For $i = 1, 2$, we get

$${}^c\mathcal{D}_0^{\alpha_i}\left(\frac{k_0(t) - \varphi(t, k_0(t), \mathcal{I}_0^{\gamma_1} k_0(t), \ldots, \mathcal{I}_0^{\gamma_n} k_0(t))}{\phi(t, k_0(t), \mathcal{I}_0^{\mu_1} k_0(t), \ldots, \mathcal{I}_0^{\mu_m} k_0(t))}\right) = \int_0^t \frac{(t-s)^{\omega-\alpha_i-1}}{\Gamma(\omega-\alpha_i)} z(s)\,ds + m_1^* \frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)},$$

$${}^c\mathcal{D}_0^{\beta_i}\left(\frac{k_0(t) - \varphi(t, k_0(t), \mathcal{I}_0^{\gamma_1} k_0(t), \ldots, \mathcal{I}_0^{\gamma_n} k_0(t))}{\phi(t, k_0(t), \mathcal{I}_0^{\mu_1} k_0(t), \ldots, \mathcal{I}_0^{\mu_m} k_0(t))}\right) = \int_0^t \frac{(t-s)^{\omega-\beta_i-1}}{\Gamma(\omega-\beta_i)} z(s)\,ds + m_1^* \frac{t^{1-\beta_i}}{\Gamma(2-\beta_i)}$$

and

$$\int_0^1 {}^c\mathcal{D}_0^{\beta_i}\left(\frac{k_0(s) - \varphi(s, k_0(s), \mathcal{I}_0^{\gamma_1} k_0(s), \ldots, \mathcal{I}_0^{\gamma_n} k_0(s))}{\phi(s, k_0(s), \mathcal{I}_0^{\mu_1} k_0(s), \ldots, \mathcal{I}_0^{\mu_m} k_0(s))}\right) ds = \int_0^1 \int_0^s \frac{(s-\tau)^{\omega-\beta_i-1}}{\Gamma(\omega-\beta_i)} z(\tau)\,d\tau ds + m_1^* \frac{1}{(3-\beta_i)}.$$

By using the hybrid boundary value conditions, we obtain

$$\begin{aligned} m_0^* &= \frac{\lambda_2\Delta_3}{\lambda_3\Delta_5\Gamma(\omega-\alpha_1)} \int_0^1 (1-s)^{\omega-\alpha_1-1} z(s)\,ds - \frac{\lambda_5\Delta_4}{\Delta_5\Gamma(\omega-\alpha_2)} \\ &\quad \times \int_0^1 (1-s)^{\omega-\alpha_2-1} z(s)\,ds + \frac{\lambda_1\Delta_3}{\lambda_3\Delta_5\Gamma(\omega-\beta_1)} \int_0^1 \\ &\quad \times \int_0^s (s-\tau)^{\omega-\beta_1-1} z(\tau)\,d\tau ds - \frac{\lambda_4\Delta_4}{\Delta_5\Gamma(\omega-\beta_2)} \int_0^1 \\ &\quad \times \int_0^s (s-\tau)^{\omega-\beta_2-1} z(\tau)\,d\tau ds \end{aligned}$$

and

$$\begin{aligned} m_1^* &= -\frac{\lambda_2\lambda_6\Delta_2}{\Delta_1\Gamma(\omega-\alpha_1)} \int_0^1 (1-s)^{\omega-\alpha_1-1} z(s)\,ds + \frac{\lambda_3\lambda_5\Delta_2}{\Delta_1\Gamma(\omega-\alpha_2)} \\ &\quad \times \int_0^1 (1-s)^{\omega-\alpha_2-1} z(s)\,ds - \frac{\lambda_1\lambda_6\Delta_2}{\Delta_1\Gamma(\omega-\beta_1)} \int_0^1 \\ &\quad \times \int_0^s (s-\tau)^{\omega-\beta_1-1} z(\tau)\,d\tau ds + \frac{\lambda_3\lambda_4\Delta_2}{\Delta_1\Gamma(\omega-\beta_2)} \int_0^1 \\ &\quad \times \int_0^s (s-\tau)^{\omega-\beta_2-1} z(\tau)\,d\tau ds. \end{aligned}$$

By substituting the values m_0^* and m_1^* in (2.5), we get

$$\begin{aligned} k_0(t) &= \varphi(t, k_0(t), \mathcal{I}_0^{\gamma_1} k_0(t), \mathcal{I}_0^{\gamma_2} k_0(t), \ldots, \mathcal{I}_0^{\gamma_n} k_0(t)) \\ &\quad + \phi(t, k_0(t), \mathcal{I}_0^{\mu_1} k_0(t), \mathcal{I}_0^{\mu_2} k_0(t), \ldots, \mathcal{I}_0^{\mu_m} k_0(t)) \\ &\quad \times \left[\frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} z(s)\,ds + \frac{\lambda_2\Delta_1\Delta_3 - \lambda_1\lambda_3\lambda_6\Delta_2\Delta_5 t}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\alpha_1)} \right. \\ &\quad \times \int_0^1 (1-s)^{\omega-\alpha_1-1} z(s)\,ds + \frac{\lambda_5[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\alpha_2)} \\ &\quad \times \int_0^1 (1-s)^{\omega-\alpha_2-1} z(s)\,ds + \frac{\lambda_1[\Delta_1\Delta_3 - \lambda_3\lambda_6\Delta_2\Delta_5 t]}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\beta_1)} \\ &\quad \times \int_0^1 \int_0^s (s-\tau)^{\omega-\beta_1-1} z(\tau)\,d\tau ds + \frac{\lambda_4[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\beta_2)} \\ &\quad \left. \times \int_0^1 \int_0^s (s-\tau)^{\omega-\beta_2-1} z(\tau)\,d\tau ds \right]. \end{aligned}$$

This follows that the function k_0 is a solution for the fractional integral Eq. (2.3). Conversely, one can easily check that k_0 is a solution for the problem (2.1) and (2.2) whenever k_0 is a solution function for the fractional integral Eq. (2.3). □

Now, we provide our main result about the existence of solutions of the problem (1.1) and (1.2).

Theorem 2.2. *Suppose that functions* $\phi : [0,1] \times \mathcal{X}^{m+1} \to \mathcal{X} \setminus \{0\}$ *and* $\varphi : [0,1] \times \mathcal{X}^{n+1} \to \mathcal{X}$ *and* $\Upsilon : [0,1] \times \mathcal{X} \to \mathcal{X}$ *are continuous. Assume that*

($\mathcal{C}1$) *there exists a bounded mapping* $\theta : [0,1] \to \mathbb{R}^+$ *such that for all* $k_i, k_i' \in \mathcal{X}$,

$$|\phi(t, k_1(t), k_2(t), \ldots, k_{m+1}(t)) - \phi(t, k_1'(t), k_2'(t), \ldots, k_{m+1}'(t))| \leqslant \theta(t) \sum_{i=1}^{m+1} |k_i(t) - k_i'(t)|,$$

($\mathcal{C}2$) *there exists a bounded mapping* $\sigma : [0,1] \to \mathbb{R}^+$ *such that for all* $k_i, k_i' \in \mathcal{X}$,

$$|\varphi(t, k_1(t), k_2(t), \ldots, k_{n+1}(t)) - \varphi(t, k_1'(t), k_2'(t), \ldots, k_{n+1}'(t))| \leqslant \sigma(t) \sum_{i=1}^{n+1} |k_i(t) - k_i'(t)|,$$

($\mathcal{C}3$) *there exist a continuous function* $\psi : [0,1] \to \mathbb{R}^+$ *and a continuous non-decreasing map* $\xi : [0,\infty) \to (0,\infty)$ *such that* $|\Upsilon(t,k)| \leqslant \psi(t)\xi(\|k\|_{\mathcal{X}})$ *for all* $t \in [0,1]$ *and* $k \in \mathcal{X}$,

($\mathcal{C}4$) *there exists a real number* $\rho > 0$ *such that*

$$\rho > \frac{\phi^* \widetilde{M} \psi^* \xi(\|k\|_{\mathcal{X}}) + \varphi^*}{1 - \theta^*\left[1 + \frac{1}{\Gamma(\mu_1+1)} + \ldots + \frac{1}{\Gamma(\mu_m+1)}\right] \psi^* \xi(\|k\|_{\mathcal{X}}) \widetilde{M} - \sigma^*\left[1 + \frac{1}{\Gamma(\gamma_1+1)} + \ldots + \frac{1}{\Gamma(\gamma_n+1)}\right]}, \tag{2.6}$$

where $\phi^* = \sup_{t\in[0,1]} |\phi(t, \overbrace{0, 0, \ldots, 0}^{m+1\,times})|$, $\varphi^* = \sup_{t\in[0,1]} |\varphi(t, \overbrace{0, 0, \ldots, 0}^{n+1\,times})|$, $\psi^* = \sup_{t\in[0,1]} |\psi(t)|$, $\theta^* = \sup_{t\in[0,1]} |\theta(t)|$, $\sigma^* = \sup_{t\in[0,1]} |\sigma(t)|$ *and*

$$\widetilde{M} = \frac{1}{\Gamma(\omega+1)} + \frac{\lambda_2|\Delta_1||\Delta_3| + \lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1+1)} + \frac{\lambda_5[\lambda_3|\Delta_2||\Delta_5| + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2+1)} + \frac{\lambda_1[|\Delta_1||\Delta_3| + \lambda_3\lambda_6|\Delta_2||\Delta_5|]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1+2)} + \frac{\lambda_4[\lambda_3|\Delta_2||\Delta_5| + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2+2)}. \quad (2.7)$$

Then the hybrid problem (1.1) and (1.2) has at least one solution whenever

$$\theta^*\left[1 + \frac{1}{\Gamma(\mu_1+1)} + \ldots + \frac{1}{\Gamma(\mu_m+1)}\right]\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M} + \sigma^*\left[1 + \frac{1}{\Gamma(\gamma_1+1)} + \ldots + \frac{1}{\Gamma(\gamma_n+1)}\right] < 1.$$

Proof. For every positive number $\rho \in \mathbb{R}$, consider the ball $\overline{\mathcal{V}_\rho(0)} := \{k(t) \in \mathcal{X} : \|k\|_{\mathcal{X}} \leqslant \rho\}$ in the Banach algebra $\mathcal{X}$, where ρsatisfies (2.6). It is clear that the ball $\overline{\mathcal{V}_\rho(0)}$is a closed convex bounded subset of the Banach algebra $\mathcal{X}$. By using Lemma 2.1, we define three fractional operators $\Phi_1, \Phi_2 : \mathcal{X} \to \mathcal{X}$and $\Phi_3 : \overline{\mathcal{V}_\rho(0)} \to \mathcal{X}$ by

$$(\Phi_1 k)(t) = \phi(t, k(t), \mathcal{I}_0^{\mu_1}k(t), \mathcal{I}_0^{\mu_2}k(t), \ldots, \mathcal{I}_0^{\mu_m}k(t)),$$
$$(\Phi_2 k)(t) = \varphi(t, k(t), \mathcal{I}_0^{\gamma_1}k(t), \mathcal{I}_0^{\gamma_2}k(t), \ldots, \mathcal{I}_0^{\gamma_n}k(t))$$

and

$$(\Phi_3 k)(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\Upsilon(s,k(s))\,\mathrm{d}s + \frac{\lambda_2\Delta_1\Delta_3 - \lambda_1\lambda_3\lambda_6\Delta_2\Delta_5 t}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\alpha_1)}\int_0^1 (1-s)^{\omega-\alpha_1-1}\Upsilon(s,k(s))\,\mathrm{d}s + \frac{\lambda_5[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\alpha_2)}\int_0^1 (1-s)^{\omega-\alpha_2-1}\Upsilon(s,k(s))\,\mathrm{d}s + \frac{\lambda_1[\Delta_1\Delta_3 - \lambda_3\lambda_6\Delta_2\Delta_5 t]}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\beta_1)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_1-1}\Upsilon(\tau,k(\tau))\,\mathrm{d}\tau\mathrm{d}s + \frac{\lambda_4[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\beta_2)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_2-1}\Upsilon(\tau,k(\tau))\,\mathrm{d}\tau\mathrm{d}s,$$

for all $t \in [0,1]$. It is clear that a function $k_0 \in \mathcal{X}$ is a solution for the hybrid problem (1.1) and (1.2) whenever it satisfies the operator equation $(\Phi_1 k_0)(\Phi_3 k_0) + (\Phi_2 k_0) = k_0$. We will show that the operators Φ_1, Φ_2 and Φ_3 satisfy all conditions of Theorem 1.1 and so by considering assumptions of Theorem 1.1, we will deduce that such a solution function exists. First, we show that the operator Φ_1 is Lipschitzian. Let $k_1, k_2 \in \mathcal{X}$. The assumption $(\mathcal{C}1)$ implies that

$$|(\Phi_1 k_1)(t) - (\Phi_1 k_2)(t)| = |\phi(t, k_1(t), \mathcal{I}_0^{\mu_1}k_1(t), \ldots, \mathcal{I}_0^{\mu_m}k_1(t)) - \phi(t, k_2(t), \mathcal{I}_0^{\mu_1}k_2(t), \ldots, \mathcal{I}_0^{\mu_m}k_2(t))| \leqslant \theta(t)\left[1 + \frac{t^{\mu_1}}{\Gamma(\mu_1+1)} + \frac{t^{\mu_2}}{\Gamma(\mu_2+1)} + \ldots + \frac{t^{\mu_m}}{\Gamma(\mu_m+1)}\right]|k_1(t) - k_2(t)|$$

for all $t \in [0,1]$. Hence, we get

$$\|\Phi_1 k_1 - \Phi_1 k_2\|_{\mathcal{X}} \leqslant \theta^*\left[1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_m+1)}\right]\|k_1 - k_2\|_{\mathcal{X}}$$

for all $k_1, k_2 \in \mathcal{X}$. This shows that the operator Φ_1 is Lipschitzian with a Lipschitz constant $\theta^*\left[1 + \frac{1}{\Gamma(\mu_1+1)} + \frac{1}{\Gamma(\mu_2+1)} + \ldots + \frac{1}{\Gamma(\mu_m+1)}\right] > 0$. Similarly, by using the assumption $(\mathcal{C}2)$, one can prove that the operator Φ_2 is Lipschitzian on $\mathcal{X}$. We have

$$|(\Phi_2 k_1)(t) - (\Phi_2 k_2)(t)| = |\varphi(t, k_1(t), \mathcal{I}_0^{\gamma_1}k_1(t), \ldots, \mathcal{I}_0^{\gamma_n}k_1(t)) - \varphi(t, k_2(t), \mathcal{I}_0^{\gamma_1}k_2(t), \ldots, \mathcal{I}_0^{\gamma_n}k_2(t))| \leqslant \sigma(t)\left[1 + \frac{t^{\gamma_1}}{\Gamma(\gamma_1+1)} + \frac{t^{\gamma_2}}{\Gamma(\gamma_2+1)} + \ldots + \frac{t^{\gamma_n}}{\Gamma(\gamma_n+1)}\right]|k_1(t) - k_2(t)|$$

Hence, we get

$$\|\Phi_2 k_1 - \Phi_2 k_2\|_{\mathcal{X}} \leqslant \sigma^*\left[1 + \frac{1}{\Gamma(\gamma_1+1)} + \frac{1}{\Gamma(\gamma_2+1)} + \ldots + \frac{1}{\Gamma(\gamma_n+1)}\right]\|k_1 - k_2\|_{\mathcal{X}}$$

for all $k_1, k_2 \in \mathcal{X}$. This shows that the operator Φ_2 is Lipschitzian with a Lipschitz constant $\sigma^*\left[1 + \frac{1}{\Gamma(\gamma_1+1)} + \frac{1}{\Gamma(\gamma_2+1)} + \ldots + \frac{1}{\Gamma(\gamma_n+1)}\right] > 0$. Now, we establish the complete continuity of the operator Φ_3 on the closed ball $\overline{\mathcal{V}_\rho(0)}$. We first prove that the operator Φ_3 is continuous on $\overline{\mathcal{V}_\rho(0)}$. For this aim, let $\{k_n\}$be a convergent sequence in $\overline{\mathcal{V}_\rho(0)}$ so that $k_n \to k$, where $k \in \overline{\mathcal{V}_\rho(0)}$ is an arbitrary element. Since the function $\Upsilon$is continuous on $[0,1] \times \mathcal{X}$, we conclude that $\lim_{n\to\infty}\Upsilon(t, k_n(t)) = \Upsilon(t, k(t))$. By using the Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty}(\Phi_3 k_n)(t) = &\ \tfrac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\lim_{n\to\infty}\Upsilon(s,k_n(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_2\Delta_1\Delta_3 - \lambda_1\lambda_3\lambda_6\Delta_2\Delta_5 t}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\alpha_1)}\int_0^1 (1-s)^{\omega-\alpha_1-1}\lim_{n\to\infty}\Upsilon(s,k_n(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_5[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\alpha_2)}\int_0^1 (1-s)^{\omega-\alpha_2-1}\lim_{n\to\infty}\Upsilon(s,k_n(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_1[\Delta_1\Delta_3 - \lambda_3\lambda_6\Delta_2\Delta_5 t]}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\beta_1)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_1-1}\lim_{n\to\infty}\Upsilon(\tau,k_n(\tau))\,\mathrm{d}\tau\mathrm{d}s \\
&+ \tfrac{\lambda_4[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\beta_2)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_2-1}\lim_{n\to\infty}\Upsilon(\tau,k_n(\tau))\,\mathrm{d}\tau\mathrm{d}s \\
= &\ \tfrac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\Upsilon(s,k(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_2\Delta_1\Delta_3 - \lambda_1\lambda_3\lambda_6\Delta_2\Delta_5 t}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\alpha_1)}\int_0^1 (1-s)^{\omega-\alpha_1-1}\Upsilon(s,k(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_5[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\alpha_2)}\int_0^1 (1-s)^{\omega-\alpha_2-1}\Upsilon(s,k(s))\,\mathrm{d}s \\
&+ \tfrac{\lambda_1[\Delta_1\Delta_3 - \lambda_3\lambda_6\Delta_2\Delta_5 t]}{\lambda_3\Delta_1\Delta_5\Gamma(\omega-\beta_1)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_1-1}\Upsilon(\tau,k(\tau))\,\mathrm{d}\tau\mathrm{d}s \\
&+ \tfrac{\lambda_4[\lambda_3\Delta_2\Delta_5 t - \Delta_1\Delta_4]}{\Delta_1\Delta_5\Gamma(\omega-\beta_2)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_2-1}\Upsilon(\tau,k(\tau))\,\mathrm{d}\tau\mathrm{d}s \\
= &\ (\Phi_3 k)(t)
\end{aligned}$$

for all $t \in [0,1]$. Hence, we see that $\Phi_3 k_n \to \Phi_3 k$ as $n \to \infty$ and so Φ_3 is continuous on $\overline{\mathcal{V}_\rho(0)}$. Here, we prove that the operator Φ_3 is uniformly bounded on $\overline{\mathcal{V}_\rho(0)}$. Let $k \in \overline{\mathcal{V}_\rho(0)}$. By using the assumption $(\mathcal{C}3)$, we have

$$\begin{aligned}
|(\Phi_3 k)(t)| = &\ \tfrac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}|\Upsilon(s,k(s))|\,\mathrm{d}s \\
&+ \tfrac{\lambda_2|\Delta_1||\Delta_3| + \lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5| t}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1)}\int_0^1 (1-s)^{\omega-\alpha_1-1}|\Upsilon(s,k(s))|\,\mathrm{d}s \\
&+ \tfrac{\lambda_5[\lambda_3|\Delta_2||\Delta_5| t + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2)}\int_0^1 (1-s)^{\omega-\alpha_2-1}|\Upsilon(s,k(s))|\,\mathrm{d}s \\
&+ \tfrac{\lambda_1[|\Delta_1||\Delta_3| + \lambda_3\lambda_6|\Delta_2||\Delta_5| t]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_1-1}|\Upsilon(\tau,k(\tau))|\,\mathrm{d}\tau\mathrm{d}s \\
&+ \tfrac{\lambda_4[\lambda_3|\Delta_2||\Delta_5| t + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2)}\int_0^1\int_0^s (s-\tau)^{\omega-\beta_2-1}|\Upsilon(\tau,k(\tau))|\,\mathrm{d}\tau\mathrm{d}s \\
\leqslant &\ \tfrac{t^\omega}{\Gamma(\omega+1)}\psi(t)\xi(\|k\|_{\mathcal{X}}) + \tfrac{\lambda_2|\Delta_1||\Delta_3| + \lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5| t}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1+1)}\psi(t)\xi(\|k\|_{\mathcal{X}}) \\
&+ \tfrac{\lambda_5[\lambda_3|\Delta_2||\Delta_5| t + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2+1)}\psi(t)\xi(\|k\|_{\mathcal{X}}) \\
&+ \tfrac{\lambda_1[|\Delta_1||\Delta_3| + \lambda_3\lambda_6|\Delta_2||\Delta_5| t]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1+2)}\psi(t)\xi(\|k\|_{\mathcal{X}}) \\
&+ \tfrac{\lambda_4[\lambda_3|\Delta_2||\Delta_5| t + |\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2+2)}\psi(t)\xi(\|k\|_{\mathcal{X}})
\end{aligned}$$

for all $t\in[0,1]$. Hence $\|\Phi_3 k\|_{\mathcal{X}}\leqslant\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M}$, where $\widetilde{M}$is given in (2.7). This shows that $\Phi_3\left(\overline{\mathcal{V}_\rho(0)}\right)$ is an uniformly bounded subset of $\mathcal{X}$. In the following, we show that the operator Φ_3 is equi-continuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $k\in\overline{\mathcal{V}_\rho(0)}$. Then, we obtain

$$|(\Phi_3k)(t_2)-(\Phi_3k)(t_1)|\leqslant\frac{1}{\Gamma(\omega)}\int_0^{t_1}\left[(t_2-s)^{\omega-1}-(t_1-s)^{\omega-1}\right]$$
$$|\Upsilon(s,k(s))|\mathrm{d}s+\frac{1}{\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-s)^{\omega-1}|\Upsilon(s,k(s))|\mathrm{d}s$$
$$+\frac{\lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|(t_2-t_1)}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1)}\int_0^1(1-s)^{\omega-\alpha_1-1}|\Upsilon(s,k(s))|\mathrm{d}s$$
$$+\frac{\lambda_5\lambda_3|\Delta_2||\Delta_5|(t_2-t_1)}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2)}\int_0^1(1-s)^{\omega-\alpha_2-1}|\Upsilon(s,k(s))|\mathrm{d}s$$
$$+\frac{\lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|(t_2-t_1)}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1)}\int_0^1\int_0^s(s-\tau)^{\omega-\beta_1-1}|\Upsilon(\tau,k(\tau))|\mathrm{d}\tau\mathrm{d}s$$
$$+\frac{\lambda_4\lambda_3|\Delta_2||\Delta_5|(t_2-t_1)}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2)}\int_0^1\int_0^s(s-\tau)^{\omega-\beta_2-1}|\Upsilon(\tau,k(\tau))|\mathrm{d}\tau\mathrm{d}s.$$

It is seen that the right-hand side of the inequality converges to zero independently of $k\in\overline{\mathcal{V}_\rho(0)}$ as $t_1\to t_2$. Thus, $|(\Phi_3k)(t_2)-(\Phi_3k)(t_1)|\to0$ as $t_1\to t_2$ and so the operator Φ_3 is equi-continuous. Now, the Arzela-Ascoli theorem implies that the operator Φ_3 is completely continuous on $\overline{\mathcal{V}_\rho(0)}$. On the other hand, by using the assumption ($\mathcal{C}3$), we have

$$\hat\Delta=\|\Phi_3\left(\overline{\mathcal{V}_\rho(0)}\right)\|_{\mathcal{X}}=\sup_{t\in[0,1]}\left\{|(\Phi_3k)(t)|:k\in\overline{\mathcal{V}_\rho(0)}\right\}\leqslant\psi^*\xi(\|k\|_{\mathcal{X}})$$
$$\times\left[\frac{1}{\Gamma(\omega+1)}+\frac{\lambda_2|\Delta_1||\Delta_3|+\lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1+1)}+\frac{\lambda_5[\lambda_3|\Delta_2||\Delta_5|+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2+1)}\right.$$
$$\left.+\frac{\lambda_1[|\Delta_1||\Delta_3|+\lambda_3\lambda_6|\Delta_2||\Delta_5|]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1+2)}+\frac{\lambda_4[\lambda_3|\Delta_2||\Delta_5|+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2+2)}\right]=\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M}.$$

Since $\hat\Delta\leqslant\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M}$, thus we get

$$\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\hat\Delta$$
$$+\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]$$
$$\leqslant\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M}$$
$$+\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]<1.$$

Now, put $l_1^*=\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]$ and $l_2^*=\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]$. Then, $l_1^*\hat\Delta+l_2^*<1$. Hence, all assumptions of Theorem 1.1 are satisfied. Thus, one of the conditions (a) or (b) in Theorem 1.1 holds. We first investigate the condition (b). To do this, let $\alpha_0\in(0,1)$. Assume that there exists a function $k\in\mathcal{X}$ with $\|k\|_{\mathcal{X}}=\rho$ such that the operator equation $k=\alpha_0(\Phi_1k)(\Phi_3k)+\alpha_0(\Phi_2k)$ holds. Then, one can write

$$|k(t)|\leqslant\alpha_0|(\Phi_1k)(t)||(\Phi_3k)(t)|+\alpha_0|(\Phi_2k)(t)|$$
$$=\alpha_0|\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,\mathcal{I}_0^{\mu_m}k(t))|$$
$$\times\left(\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}|\Upsilon(s,k(s))|\mathrm{d}s+\frac{\lambda_2|\Delta_1||\Delta_3|+\lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|t}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1)}\right.$$
$$\times\int_0^1(1-s)^{\omega-\alpha_1-1}|\Upsilon(s,k(s))|\mathrm{d}s+\frac{\lambda_5[\lambda_3|\Delta_2||\Delta_5|t+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2)}$$
$$\times\int_0^1(1-s)^{\omega-\alpha_2-1}|\Upsilon(s,k(s))|\mathrm{d}s+\frac{\lambda_1[|\Delta_1||\Delta_3|+\lambda_3\lambda_6|\Delta_2||\Delta_5|t]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1)}$$
$$\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta_1-1}|\Upsilon(\tau,k(\tau))|\mathrm{d}\tau\mathrm{d}s+\frac{\lambda_4[\lambda_3|\Delta_2||\Delta_5|t+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2)}$$
$$\left.\times\int_0^1\int_0^s(s-\tau)^{\omega-\beta_2-1}|\Upsilon(\tau,k(\tau))|\mathrm{d}\tau\mathrm{d}s\right)+\alpha_0|\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),$$
$$\times\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))|\leqslant(|\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\ldots,$$
$$\times\mathcal{I}_0^{\mu_m}k(t))-\phi(t,0,0,\ldots,0)|+|\phi(t,0,0,\ldots,0)|)$$
$$\times\left(\frac{t^\omega}{\Gamma(\omega+1)}\psi(t)\xi(\|k\|_{\mathcal{X}})+\frac{\lambda_2|\Delta_1||\Delta_3|+\lambda_1\lambda_3\lambda_6|\Delta_2||\Delta_5|t}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_1+1)}\right.$$
$$\psi(t)\xi(\|k\|_{\mathcal{X}})+\frac{\lambda_5[\lambda_3|\Delta_2||\Delta_5|t+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\alpha_2+1)}\psi(t)\xi(\|k\|_{\mathcal{X}})$$
$$+\frac{\lambda_1[|\Delta_1||\Delta_3|+\lambda_3\lambda_6|\Delta_2||\Delta_5|t]}{\lambda_3|\Delta_1||\Delta_5|\Gamma(\omega-\beta_1+2)}\psi(t)\xi(\|k\|_{\mathcal{X}})$$
$$\left.+\frac{\lambda_4[\lambda_3|\Delta_2||\Delta_5|t+|\Delta_1||\Delta_4|]}{|\Delta_1||\Delta_5|\Gamma(\omega-\beta_2+2)}\psi(t)\xi(\|k\|_{\mathcal{X}})\right)$$
$$+(|\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t),\ldots,\mathcal{I}_0^{\gamma_n}k(t))$$
$$-\varphi(t,0,0,\ldots,0)|+|\varphi(t,0,0,\ldots,0)|)$$
$$\leqslant\left(\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\frac{1}{\Gamma(\mu_2+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\|k\|_{\mathcal{X}}+\phi^*\right)$$
$$\widetilde{M}\psi^*\xi(\|k\|_{\mathcal{X}})+\left(\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\frac{1}{\Gamma(\gamma_2+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]\right.$$
$$\left.\|k\|_{\mathcal{X}}+\varphi^*\right).$$

This yields

$$\rho=\|k\|_{\mathcal{X}}\leqslant\left(\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\frac{1}{\Gamma(\mu_2+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\rho+\phi^*\right)$$
$$\widetilde{M}\psi^*\xi(\rho)+\left(\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\frac{1}{\Gamma(\gamma_2+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]\rho+\varphi^*\right)$$

and so

$$\rho\leqslant\frac{\phi^*\widetilde{M}\psi^*\xi(\rho)+\varphi^*}{1-\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\psi^*\xi(\rho)\widetilde{M}-\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]}$$

which contradicts to the inequality (2.6). Hence, the condition (b) in Theorem 1.1 is impossible. Thus, the condition (a) in Theorem 1.1 holds and so the operator equation $(\Phi_1k)(\Phi_3k)+(\Phi_2k)=k$has a solution. This means that the hybrid problem (1.1) and (1.2) has a solution. □

3. Example

In this section, we provide an example to illustrate our main result.

Example 3.1. Consider the fractional hybrid integro-differential equation

$$ {}^{c}\mathcal{D}_0^{1.51}\left(\frac{k(t)-\frac{2}{7621(1+t)}\left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009}{\frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right)+0.012}\right) = (0.07+t)^2\cos(k(t)) \tag{3.1} $$

with mixed integral hybrid boundary value conditions

$$ \begin{cases} 0.6\int_0^1 {}^{c}\mathcal{D}_0^{0.38}\left(\frac{k(s)-\frac{2}{7621(1+s)}\left(k(s)+\arctan\left(\mathcal{I}_0^{0.91}k(s)\right)+\sin\left(\mathcal{I}_0^{0.23}k(s)\right)\right)+0.009}{\frac{s}{651}\left(k(s)+\frac{|\mathcal{I}_0^{0.37}k(s)|}{1+|\mathcal{I}_0^{0.37}k(s)|}+\sin\left(\mathcal{I}_0^{0.49}k(s)\right)+\mathcal{I}_0^{0.61}k(s)\right)+0.012}\right)\mathrm{d}s \\ \quad +0.71\,{}^{c}\mathcal{D}_0^{0.45}\left(\frac{k(t)-\frac{2}{7621(1+t)}\left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009}{\frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right)+0.012}\right)\Big|_{t=1} \\ \quad +0.63\left(\frac{k(t)-\frac{2}{7621(1+t)}\left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009}{\frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right)+0.012}\right)\Big|_{t=0}=0, \\ 0.17\int_0^1 {}^{c}\mathcal{D}_0^{0.39}\left(\frac{k(s)-\frac{2}{7621(1+s)}\left(k(s)+\arctan\left(\mathcal{I}_0^{0.91}k(s)\right)+\sin\left(\mathcal{I}_0^{0.23}k(s)\right)\right)+0.009}{\frac{s}{651}\left(k(s)+\frac{|\mathcal{I}_0^{0.37}k(s)|}{1+|\mathcal{I}_0^{0.37}k(s)|}+\sin\left(\mathcal{I}_0^{0.49}k(s)\right)+\mathcal{I}_0^{0.61}k(s)\right)+0.012}\right)\mathrm{d}s \\ \quad +0.46\,{}^{c}\mathcal{D}_0^{0.22}\left(\frac{k(t)-\frac{2}{7621(1+t)}\left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009}{\frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right)+0.012}\right)\Big|_{t=1} \\ \quad +0.89\left(\frac{k(t)-\frac{2}{7621(1+t)}\left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009}{\frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right)+0.012}\right)\Big|_{t=0}=0, \end{cases} \tag{3.2} $$

where $t\in[0,1], \omega=1.51, \alpha_1=0.45, \alpha_2=0.22, \beta_1=0.38, \beta_2=0.39, \lambda_1=0.6, \lambda_2=0.71, \lambda_3=0.63, \lambda_4=0.17, \lambda_5=0.46$ and $\lambda_6=0.89$. Also, for $n=2$ and $m=3$, let $\gamma_1=0.91, \gamma_2=0.23, \mu_1=0.37, \mu_2=0.49$ and $\mu_3=0.61$. Consider the continuous function $\Upsilon:[0,1]\times\mathbb{R}\to\mathbb{R}$ defined by $\Upsilon(t,k(t))=(0.07+t)^2\cos(k(t))$. Now, put $\psi(t)=(0.07+t)^2$and $\xi(\|k\|)=1$. Thus, $\psi^*=\sup_{t\in[0,1]}\psi(t)\simeq 1.1449$. Consider two continuous maps $\varphi:[0,1]\times\mathbb{R}^{2+1}\to\mathbb{R}$ and $\phi:[0,1]\times\mathbb{R}^{3+1}\to\mathbb{R}\setminus\{0\}$ defined by

$$ \varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t)) = \frac{2}{7621(1+t)} \times \left(k(t)+\arctan\left(\mathcal{I}_0^{0.91}k(t)\right)+\sin\left(\mathcal{I}_0^{0.23}k(t)\right)\right)+0.009, $$

$$ \phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\mathcal{I}_0^{\mu_3}k(t)) = \frac{t}{651}\left(k(t)+\frac{|\mathcal{I}_0^{0.37}k(t)|}{1+|\mathcal{I}_0^{0.37}k(t)|}+\sin\left(\mathcal{I}_0^{0.49}k(t)\right)+\mathcal{I}_0^{0.61}k(t)\right) + 0.012. $$

Note that, $\varphi^*=\sup_{t\in[0,1]}|\varphi(t,0,0,0)|=0.009$ and $\phi^*=\sup_{t\in[0,1]}|\phi(t,0,0,0,0)|=0.012$. Also, the function $\varphi$is Lipschitzian, so for each $k,k'\in\mathbb{R}$, we have

$$ |\varphi(t,k(t),\mathcal{I}_0^{\gamma_1}k(t),\mathcal{I}_0^{\gamma_2}k(t))-\varphi(t,k'(t),\mathcal{I}_0^{\gamma_1}k'(t),\mathcal{I}_0^{\gamma_2}k'(t))| \leqslant \sigma(t)\left[1+\frac{t^{\gamma_1}}{\Gamma(\gamma_1+1)}+\frac{t^{\gamma_2}}{\Gamma(\gamma_2+1)}\right]|k(t)-k'(t)| = \frac{2}{7621(1+t)}\left[1+\frac{t^{0.91}}{\Gamma(1.91)}+\frac{t^{0.23}}{\Gamma(1.23)}\right]|k(t)-k'(t)| $$

such that $\sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\frac{1}{\Gamma(\gamma_2+1)}\right]=\frac{2}{7621}\left[1+\frac{1}{\Gamma(1.91)}+\frac{1}{\Gamma(1.23)}\right]\simeq 0.082248$. On the other hand, the function $\phi$is Lipschitzian, so for each $k,k'\in\mathbb{R}$, we have

$$ |\phi(t,k(t),\mathcal{I}_0^{\mu_1}k(t),\mathcal{I}_0^{\mu_2}k(t),\mathcal{I}_0^{\mu_3}k(t))-\phi(t,k'(t), \times\mathcal{I}_0^{\mu_1}k'(t),\mathcal{I}_0^{\mu_2}k'(t),\mathcal{I}_0^{\mu_3}k'(t))| \leqslant \theta(t) \times\left[1+\frac{t^{\mu_1}}{\Gamma(\mu_1+1)}+\frac{t^{\mu_2}}{\Gamma(\mu_2+1)}+\frac{t^{\mu_3}}{\Gamma(\mu_3+1)}\right]|k(t)-k'(t)| = \frac{t}{651}\left[1+\frac{t^{0.37}}{\Gamma(1.37)}+\frac{t^{0.49}}{\Gamma(1.49)}+\frac{t^{0.61}}{\Gamma(1.61)}\right]|k(t)-k'(t)| $$

such that $\theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\frac{1}{\Gamma(\mu_2+1)}+\frac{1}{\Gamma(\mu_3+1)}\right]=\frac{1}{651}\left[1+\frac{1}{\Gamma(1.37)}+\frac{1}{\Gamma(1.49)}+\frac{1}{\Gamma(1.61)}\right]\simeq 0.006687$. Finally, we obtain $\Delta_1\simeq 3.157109, \Delta_2\simeq 5.6292, \Delta_3\simeq 4.66862, \Delta_4\simeq 13.47337, \Delta_5\simeq 7.35182$ and $\widetilde{M}\simeq 5.48514$. Choose $\rho>0.096326$. Note that,

$$ \theta^*\left[1+\frac{1}{\Gamma(\mu_1+1)}+\ldots+\frac{1}{\Gamma(\mu_m+1)}\right]\psi^*\xi(\|k\|_{\mathcal{X}})\widetilde{M}+, \sigma^*\left[1+\frac{1}{\Gamma(\gamma_1+1)}+\ldots+\frac{1}{\Gamma(\gamma_n+1)}\right]\simeq 0.124241<1 $$

Now by using Theorem 2.2, it is deduced that the fractional hybrid integro-differential problem (3.1) and (3.2) has a solution.

4. Conclusion

It is known that most naturalphenomena is modeled by different types of fractional differential equations. This diversity in investigating complicate fractional differential equations increases our ability for exact modelings of different phenomena. This is useful in making modern softwares which help us to allow for more cost-free testing and less material consumption. In this work, we investigate a fractional hybrid integro-differential equation with mixed integral hybrid boundary value conditions. In this way, we use a generalization of the Dhage fixed point result for three operators. We provided an example to illustrate our main result.

Declarations

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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