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ORIGINAL ARTICLE

On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative

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Abstract In this manuscript, we investigate epidemic model of dengue fever disease under Caputo and Fabrizio fractional derivative abbreviated as (CFFD). The respective investigation is devoted to qualitative theory of existence of solution for the model under consideration by using fixed point theory. After the establishing the qualitative aspect, we apply Laplace transform coupled with Adomian decomposition method to develop an algorithm for semi analytical solution under CFFD. In same line, we also develop the semi analytical solution for the considered model under usual Caputo fractional derivative (CFD). By using Matlab, we present both type of solutions via graphs and hence give some comparative remarks about the nature of the solutions of both derivatives.

1. Introduction

Dengue fever is one of the most serious infectious disease caused by bacteria and viruses which have been affecting about 2.5 billion people around the globe, especially in the hot countries in present time [1]. Dengue fever is a strong epidemic disease in Southeast Asia [2]. This epidemic disease can explain climate change and there is need much more knowledge about the awareness of dengue fever. This dangerous disease is also a big problem of health in recent time in many countries of the world. So the creation of dengue fever model is needed. Mathematical models including Dengue fever have been analyzed and study in the last few decades increasingly (see [3–7]).

Due to mathematical models, we can be aware about the rate of change of the disease, how a disease can impact the susceptible, infected and recovered peoples. The area devoted to investigate biological model of infectious disease is warm area of research in recent time. Many studies about the mathematical models and applied problems are devoted to study stability theory, existence results and optimization of biological models, we refer few as ([8–16]). For instance authors [18] have presented numerical solution of Dengue fever disease by perturbation technique algorithm and compare the result with RK-4 for the system

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$$\begin{cases} \dot{x}(t) = \mu - [\mu + \alpha z(t)]x(t), \\ \dot{y}(t) = \alpha x(t)z(t) - \beta y(t), \\ \dot{z}(t) = \gamma y(t) - [\gamma y(t) + \delta]z(t), \end{cases}$$

where x is the human population that may call susceptible, y is the people which may be infected with Dengue virus and z is the people who get rid from Dengue virus in time t.

For last few decades the area of fractional calculus has got enormous attention from researchers. This is because fractional calculus has the ability to explain retention and heritable properties of various materials and process more accurately than integer order models. For more applications about fractional calculus, we provide ([19–22]). Therefore the mentioned area was investigated from different angles like qualitative theory, numerical analysis, etc, (see [23–26]).

Therefore, researchers extended the tools of classical calculus as used in ([27–34]) to FODEs. For instance, to handle nonlinear problems analytically, famous decomposition method was constructed by Adomian in 1980. After that the said method was increasingly applied as a strong tool to compute analytical or approximate solution to many problems of applied nature. In this regard, the mathematical models were greatly studied by using LADM, homotopy and variational techniques, (see [35–38]) and the references cited there. The mentioned methods were greatly used to handle linear and nonlinear FODEs [39–41]. Recently residual power series method, Fourier transform method, spectral methods and collocation method as well as some new type computational methods have been used for treating differential equations of fractional as well as classical order and their system, see for detail [42–46].

On the other hand, conventional fractional derivatives contain singular kernel which sometimes cause problem in explanation of some characteristics. To overcome this, Caputo and Fabrizio introduced a new definition of fractional integral and derivative which involve exponential kernel instead of singular, [47]. These operators were also given much attentions and have been proved to be better in adoption for mathematical models of many real world problems (see [48–50]).

Thanks to the aforementioned work, we take the given model to extend of (1) under CFFD as

$$\begin{cases} {}^{CF}_{0}\mathbf{D}^{r}_{t}\mathrm{x}(t) = \mu - [\mu + \alpha z(t)]x(t), \\ {}^{CF}_{0}\mathbf{D}^{r}_{t}\mathrm{y}(t) = \alpha x(t)z(t) - \beta y(t), \\ {}^{CF}_{0}\mathbf{D}^{r}_{t}\mathrm{z}(t) = \gamma y(t) - [\gamma y(t) + \delta]z(t) \end{cases} \tag{1}$$

with given initial conditions, $\mathrm{x}(0) = \mathrm{N}_1,\ \mathrm{y}(0) = \mathrm{N}_2,\ \mathrm{z}(0) = \mathrm{N}_3$, where $0 < r \leqslant 1$. Further the involve functions in the model obey $\mathrm{N}(t) = \mathrm{x}(t) + \mathrm{y}(t) + \mathrm{z}(t)$, where the total papulation is N. With the help of LADM, we handle the considered problem for semi analytical solution. The concerned techniques has very rarely studied regarding the aforementioned derivative of fractional order. Here we remark that a complete nomenclature is provided in Table 1 for the model (1).

In this work, we establish qualitative theory for the consider model because, by using fixed point theory it is ensured that wether the consider model has a solution or not. Further by fixed point results, existence of a physical phenomenon is ensured. Since each numerical or analytical technique has some merits and de-merits. For instance discretization of data is used in collocation methods which required extra memory and time consuming process. Also these methods are expensive. on other hand perturbation methods of homotopy involving axillary parameters which often controls the method and solution is dependent on that. Because for best approximate solution suitable value of the axillary parameters are required. Therefore, LADM is good among these method which needs no discretization of date neither required axillary parameter. Also it is efficient and produce the same solution as generated by homotopy analysis, homotopy perturbation and homotopy transform method [52].

Table 1 Description of the parameters involve in the consider Model (1)

Class/parameter	Description
x	Represents the susceptible people to catch infection
y	Represents the infected people with Dengue virus
z	Represents the recovered humans from Dengue virus
N	Represents the total papulation
β	It represents the infection rate
α	It represents the average number of bites per infected mosquito
μ	It represents death rate of the susceptible host
γ	It represents recover rate after infection
δ	It represents the number of deaths among the susceptible mosquito

2. Fundamental results

Here,we recall some definition about fractional calculus ([47,48,51]). We denote the exponential kernel as $\mathrm{K}(t,\rho) = \exp\left[-r\frac{t-\rho}{1-r}\right]$.

Definition 2.1. If $\mathrm{g} \in \mathrm{H}^1[0,T], T > 0, r \in (0,1)$, then the CFFD is defined as$\mathrm{U}^{234}{}_{10}$

$${}^{CF}_{0}\mathbf{D}^{r}_{t}[\mathrm{g}(t)] = \frac{\mathrm{M}(r)}{1-r}\int_0^t \mathrm{g}\prime(\rho)\mathrm{K}(t,\rho)d\rho,$$

$\mathrm{M}(r)$ is the normalization function with $\mathrm{M}(1) = \mathrm{M}(0) = 1$. If $\mathrm{g} \in \mathrm{H}^1[0,T]$, then the above derivative is recalled as

$${}^{CF}_{0}\mathbf{D}^{r}_{t}[\mathrm{g}(t)] = \frac{\mathrm{M}(r)}{1-r}\int_0^t (\mathrm{g}(t) - \mathrm{g}(\rho))\mathrm{K}(t,\rho)d\rho.$$

Definition 2.2. The fractional integral due to Caputo -Fabrizio $r \in (0,1)$ is given by

$${}^{CF}_{0}\mathbf{I}^{r}_{t}[\mathrm{g(t)}] = \frac{1-r}{\mathrm{M}(r)}\mathrm{g}(t) + \frac{r}{\mathrm{M}(r)}\int_0^t \mathrm{g}(\rho)d\rho,\ \ t \geqslant 0.$$

Taking the normalization function $\mathrm{M}(r) = 1$, we get the Laplace transform [49] as given bellow

Definition 2.3. We compute a general relation for Laplace transform of CFFD as

$$\mathscr{L}\left\{{}_0^{CF}\mathbf{D}_t^{r+M}[\mathrm{g}(t)]\right\} = \frac{1}{1-r}\mathscr{L}\left[\mathrm{g}^{(m+r)}(t)\right]\mathscr{L}\left[\exp\left(\frac{-rt}{1-r}\right)\right] = \frac{1}{s+r(1-s)}\left[s^{m+1}\mathscr{L}[\mathrm{g}(t)] + \sum_{k=1}^{m} s^{m-k}\mathrm{g}^{(k)}(0)\right].$$

If $m = 0, 1$, we have the following results respectively:

$$\mathscr{L}\left\{{}_0^{CF}\mathbf{D}_t^{r}[\mathrm{g}(t)]\right\} = \frac{s\mathscr{L}[\mathrm{g}(t)]}{s+r(1-s)}$$
$$\mathscr{L}\left\{{}_0^{CF}\mathbf{D}_t^{r+1}[\mathrm{g}(t)]\right\} = \frac{s\mathscr{L}[\mathrm{g}(t)]+s\mathrm{g}(0)-\mathrm{g}'(0)}{s+r(1-s)}.$$

Definition 2.4. The Laplace transform of CFD is provided by

$$\mathscr{L}\left\{{}_0^{C}\mathbf{D}_t^{r}[\mathrm{g}(t)]\right\} = s^r\mathscr{L}[\mathrm{g}(t)] - \sum_{k=0}^{p-1} s^{p-k-1}\mathrm{g}^{(k)}(0).$$

3. Qualitative analysis of the considered model

This portion deals with the qualitative analysis of model (1) using fixed point theorem due to Banach. To check, weather the problem arise after modeling a physical or biological phenomenon exist or not, this is guaranteed by existence theory. In this regard, fixed point theory is a powerful tool which provides information that weather the considered problem has a solution or not. Plenty of fixed point results for existence of solution have been constructed. One of the most wellknown result in fixed point theory is the Banach contraction theorem which provides information about existence and uniqueness of solution for the consider problem.

In this regard, we first define the following functions

$$\begin{aligned} \mathrm{h}_1(t,\mathrm{x},\mathrm{y},\mathrm{z}) &= \mu - [\mu + \alpha z(t)]x(t),\\ \mathrm{h}_2(t,\mathrm{x},\mathrm{y},\mathrm{z}) &= \alpha \mathrm{x}(t)\mathrm{z}(t) - \beta \mathrm{y}(t), \\ \mathrm{h}_3(t,\mathrm{x},\mathrm{y},\mathrm{z}) &= \gamma y(t) - [\gamma y(t) + \delta]z(t) \end{aligned} \tag{2}$$

$$\Delta_k = \sup_{C[d,b_k]} \|\mathrm{h}_1(t,\mathrm{x},\mathrm{y},\mathrm{z})\|, \text{ for } k = 1,2,3 \tag{3}$$

with

$$C[d,b_i] = [t-d,t+d]\times[u-c_k,u+c_k] = D\times D_k, \text{ for } k=1,2,3.$$

Now using Banach fixed point theorem by defining the norm on $C[d,d_k]$, for $k = 1,2,3$ as

$$\|\mathbf{W}\|_\infty = \sup_{t\in[t-d,t+b]} |\phi(t)|. \tag{4}$$

We define the Picard's operator as

$$\mathbf{A} : C(D,D_1,D_2,D_3) \to C(D,D_1,D_2,D_3). \tag{5}$$

In this regard, applying ${}^{CF}\mathbf{I}^r$ on both sides of all the equations of Model (1) inview of (2) respectively, we get

$$\begin{cases} \mathrm{x}(t) - \mathrm{x}(0) = {}_0^{CF}\mathbf{I}_t^r[\mathrm{h}_1(t,\mathrm{x},\mathrm{y},\mathrm{z})],\\ \mathrm{y}(t) - \mathrm{y}(0) = {}_0^{CF}\mathbf{I}_t^r[\mathrm{h}_2(t,\mathrm{x},\mathrm{y},\mathrm{z})],\\ \mathrm{z}(t) - \mathrm{z}(0) = {}_0^{CF}\mathbf{I}_t^r[\mathrm{h}_3(t,\mathrm{x},\mathrm{y},\mathrm{z})], \end{cases} \tag{6}$$

on evaluation of right hand sides and witting in simple form, we have

$$\mathbf{W}(t) = \mathbf{W}_0(t) + [\Theta(t,\mathbf{W}(t)) - \Theta_0(t)]\frac{(1-r)}{\mathrm{M}(r)} + \frac{r}{\mathrm{M}(r)}\int_0^t \Theta(\rho,\mathbf{W}(\rho))d\rho, \tag{7}$$

where

$$\mathbf{W}(t) = \begin{cases} \mathrm{x}(t)\\ \mathrm{y}(t)\\ \mathrm{z}(t) \end{cases}, \quad \mathbf{W}_0(t) = \begin{cases} \mathrm{x}(0)\\ \mathrm{y}(0)\\ \mathrm{z}(0) \end{cases},$$
$$\Theta(t,\mathbf{W}(t)) = \begin{cases} \mathrm{h}_1(t,\mathrm{x},\mathrm{y},\mathrm{z})\\ \mathrm{h}_2(t,\mathrm{x},\mathrm{y},\mathrm{z})\\ \mathrm{h}_3(t,\mathrm{x},\mathrm{y},\mathrm{z}). \end{cases} \tag{8}$$
$$\Theta_0(t) = \begin{cases} \mathrm{h}_1(0,\mathrm{x}(0),\mathrm{y}(0),\mathrm{z}(0))\\ \mathrm{h}_2(0,\mathrm{x}(0),\mathrm{y}(0),\mathrm{z}(0))\\ \mathrm{h}_3(0,\mathrm{x}(0),\mathrm{y}(0),\mathrm{z}(0)). \end{cases}$$

Due to (7) and (8), the operator in (5) is defined as

$$\mathbf{A}\mathbf{W}(t) = \mathbf{W}_0(t) + [\Theta(t,\mathbf{W}(t)) - \Theta_0(t)]\frac{(1-r)}{\mathrm{M}(r)} + \frac{r}{\mathrm{M}(r)} \times \int_0^t \Theta(\rho,\mathbf{W}(\rho))d\rho. \tag{9}$$

Assume that the considered problem obeys the following results:

$$\|\mathbf{W}\|_\infty \leqslant \max\{d_1,d_2,d_3\}. \tag{10}$$

Then, letting $\Delta = \max\{\Delta_i\}$ for $i = 1,2,3,$ and $t_0 = \max\{t\in D\}$, we have

$$\begin{aligned} \|\mathbf{A}\mathbf{W}(t) - \mathbf{W}_0(t)\| &= \left\|\Theta(t,\mathbf{W}(t))\tfrac{(1-r)}{\mathrm{M}(r)} + \tfrac{r}{\mathrm{M}(r)}\textstyle\int_0^t \Theta(\rho,\mathbf{W}(\rho))d\rho\right\|\\ &\leqslant \tfrac{(1-r)}{\mathrm{M}(r)}\|\Theta(t,\mathbf{W}(t))\| + \tfrac{r}{\mathrm{M}(r)}\textstyle\int_0^t \|\Theta(\rho,\mathbf{W}(\rho))\|d\rho\\ &\leqslant \tfrac{(1-r)}{\mathrm{M}(r)}\Delta + \tfrac{r}{\mathrm{M}(r)}\Delta t,\\ &\leqslant \Delta d \leqslant \max\{d_1,d_2,d_3\} = \bar{d}, \text{ where } d = \frac{1+rt_0}{\mathrm{M}(r)}, \end{aligned} \tag{11}$$

where, d satisfies the relation

$$d < \frac{\bar{d}}{\Delta}.$$

Furthermore to evaluate the equality given by

$$\|\mathbf{A}\mathbf{W}_1 - \mathbf{A}\mathbf{W}_2\|_\infty = \sup_{t\in D}|\mathbf{W}_1(t) - \mathbf{W}_2(t)| \tag{12}$$

we proceed as

$$\begin{aligned} \|\mathbf{A}\mathbf{W}_1 - \mathbf{A}\mathbf{W}_2\| &= \Big\|\tfrac{(1-r)}{\mathrm{M}(r)}(\Theta(\rho,\mathbf{W}_1(t)) - \Theta(\rho,\mathbf{W}_2(t)))\\ &\quad + \tfrac{r}{\mathrm{M}(r)}\textstyle\int_0^t(\Theta(\rho,\mathbf{W}_1(\rho)) - \Theta(\rho,\mathbf{W}_2(\rho)))d\rho\Big\|\\ &\leqslant \tfrac{(1-r)}{\mathrm{M}(r)}k\|\mathbf{W}_1(t) - \mathbf{W}_2(t)\| + \tfrac{rk}{\mathrm{M}(r)}\textstyle\int_0^t\|\mathbf{W}_1(t)\\ &\quad -\mathbf{W}_2(t)\|, \text{with } k<1\\ &\leqslant \left\{\tfrac{(1-r)}{\mathrm{M}(r)}k + \tfrac{rt_0}{\mathcal{M}(r)}k\right\}\|\mathbf{W}_1(t) - \mathbf{W}_2(t)\|\\ &\leqslant dk\|\mathbf{W}_1 - \mathbf{W}_2\|. \end{aligned} \tag{13}$$

For **A** to be a contraction we should have $kd < 1$. Thus the defined operator **A** is contraction. Therefore, the consider system (6) has unique solution. Here, we remark that we can show that the solutions are stable as proved in [12–14].

4. Solutions of model (1) under new derivative and convergence of LADM

In this section, we compute general series solution to the considered model under new fractional order derivatives. Keep in mind that onward, we use $\mathrm{M}(r) = 1$ in this paper. On Applying Laplace transform on both sides of Model (1), we get

$$\begin{cases} \mathscr{L}[x(t)] = x(0) + \frac{s+r(1-s)}{s}\mathscr{L}[\mu - \mu \mathrm{x}(t) - \alpha \mathrm{x}(t)\mathrm{z}(t)] \\ \mathscr{L}[y(t)] = y(0) + \frac{s+r(1-s)}{s}\mathscr{L}[\alpha \mathrm{x}(t)\mathrm{z}(t) - \beta \mathrm{y}(t)] \\ \mathscr{L}[z(t)] = z(0) + \frac{s+r(1-s)}{s}\mathscr{L}[\gamma \mathrm{y}(t) - \gamma \mathrm{z}(t)\mathrm{y}(t) - \delta \mathrm{z}(t)]. \end{cases} \tag{14}$$

Now assuming the solution in the series has the form

$$\mathrm{x}(t) = \sum_{p=0}^{\infty}\mathrm{x}_p(t),\ \ \mathrm{y}(t) = \sum_{p=0}^{\infty}\mathrm{y}_p(t),\ \ \mathrm{z}(t) = \sum_{p=0}^{\infty}\mathrm{z}_p(t). \tag{15}$$

Next, we decompose the nonlinear terms $\mathrm{x}(t)\mathrm{z}(t), \mathrm{y}(t)\mathrm{z}(t)$ in terms of Adomian polynomials as

$$\mathrm{x}(t)\mathrm{z}(t) = \sum_{p=0}^{\infty}\mathrm{P}_p(\mathrm{x},\mathrm{z}),\ \ \mathrm{y}(t)\mathrm{z}(t) = \sum_{p=0}^{\infty}\mathrm{Q}_p(\mathrm{x},\mathrm{z}), \tag{16}$$

where the Adomian polynomial $\mathrm{P}_p(\mathrm{x},\mathrm{z})$ can be defined as

$$\begin{aligned} \mathrm{P}_p(\mathrm{x},\mathrm{z}) &= \frac{1}{p!}\frac{d^p}{d\lambda^p}\left[\sum_{k=0}^{p}\lambda^k \mathrm{x}_k(t)\sum_{k=0}^{p}\lambda^k \mathrm{z}_k(t)\right]\Bigg|_{\lambda=0} \\ \mathrm{Q}_p(\mathrm{x},\mathrm{z}) &= \frac{1}{p!}\frac{d^p}{d\lambda^p}\left[\sum_{k=0}^{p}\lambda^k \mathrm{y}_k(t)\sum_{k=0}^{p}\lambda^k \mathrm{z}_k(t)\right]\Bigg|_{\lambda=0}. \end{aligned} \tag{17}$$

Hence, in view of (15) and (16), the system (14) becomes

$$\begin{cases} \mathscr{L}\left[\sum_{p=0}^{\infty}\mathrm{x}_p(t)\right] \\ = \mathrm{N}_1 + \frac{s+r(1-s)}{s}\mathscr{L}\left[\mu - \mu\sum_{p=0}^{\infty}\mathrm{x}_p(t) - \sum_{p=0}^{\infty}\mathrm{P}_p(\mathrm{x},\mathrm{z})\right] \\ \mathscr{L}\left[\sum_{p=0}^{\infty}\mathrm{y}_p(t)\right] \\ = \mathrm{N}_2 + \frac{s+r(1-s)}{s}\mathscr{L}\left[\alpha\sum_{p=0}^{\infty}\mathrm{P}_p(\mathrm{x},\mathrm{z}) - \beta\sum_{p=0}^{\infty}\mathrm{y}_p(t)\right] \\ \mathscr{L}\left[\sum_{p=0}^{\infty}\mathrm{z}_p(t)\right] \\ = \mathrm{N}_3 + \frac{s+r(1-s)}{s}\mathscr{L}\left[\gamma\sum_{p=0}^{\infty}\mathrm{y}_p(t) - \gamma\sum_{p=0}^{\infty}\mathrm{Q}_p(\mathrm{y},\mathrm{z}) - \delta\sum_{p=0}^{\infty}\mathrm{z}_p(t)\right]. \end{cases} \tag{18}$$

Now equating terms on both sides of (18), we have

$$\begin{cases} \mathscr{L}[\mathrm{x}_0(t)] = \mathrm{N}_1,\ \mathscr{L}[\mathrm{y}_0(t)] = \mathrm{N}_2,\ \mathscr{L}[\mathrm{z}_0(t)] = \mathrm{N}_3, \\ \mathscr{L}[\mathrm{x}_1(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\mu - \mu\mathrm{x}_0(t) - \mathrm{P}_0(\mathrm{x},\mathrm{z})], \\ \mathscr{L}[\mathrm{y}_1(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\alpha\mathrm{P}_0(\mathrm{x},\mathrm{z}) - \beta\mathrm{y}_0(t)], \\ \mathscr{L}[\mathrm{z}_1(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\gamma\mathrm{y}_0(t) - \gamma\mathrm{Q}_0(\mathrm{y},\mathrm{z}) - \delta\mathrm{z}_0(t)], \\ \mathscr{L}[\mathrm{x}_2(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\mu - \mu\mathrm{x}_1(t) - \mathrm{P}_1(\mathrm{x},\mathrm{z})], \\ \mathscr{L}[\mathrm{y}_2(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\alpha\mathrm{P}_1(\mathrm{x},\mathrm{z}) - \beta\mathrm{y}_1(t)], \\ \mathscr{L}[\mathrm{z}_2(t)] = \frac{s+r(1-s)}{s}\mathscr{L}[\gamma\mathrm{y}_1(t) - \gamma\mathrm{Q}_1(\mathrm{y},\mathrm{z}) - \delta\mathrm{z}_0(t)], \\ \vdots \\ \mathscr{L}\left[\mathrm{x}_{p+1}(t)\right] = \frac{s+r(1-s)}{s}\mathscr{L}\left[\mu - \mu\mathrm{x}_p(t) - \mathrm{P}_p(\mathrm{x},\mathrm{z})\right], \\ \mathscr{L}\left[\mathrm{y}_{p+1}(t)\right] = \frac{s+r(1-s)}{s}\mathscr{L}\left[\alpha\mathrm{P}_p(\mathrm{x},\mathrm{z}) - \beta\mathrm{y}_p(t)\right], \\ \mathscr{L}\left[\mathrm{z}_{p+1}(t)\right] = \frac{s+r(1-s)}{s}\mathscr{L}\left[\gamma\mathrm{y}_p(t) - \gamma\mathrm{Q}_p(\mathrm{y},\mathrm{z}) - \delta\mathrm{z}_p(t)\right],\ p \geqslant 0. \end{cases} \tag{19}$$

Upon computation of Laplace transform in (19), we have

$$\begin{cases} \mathrm{x}_0(t) = \mathrm{N}_1,\ \mathrm{y}_0(t) = \mathrm{N}_2,\ \mathrm{z}_0(t) = \mathrm{N}_1, \\ \mathrm{x}_1(t) = [\mu - \mu\mathrm{N}_1 - \mathrm{N}_1\mathrm{N}_2](1 + r(t-1)), \\ \mathrm{y}_1(t) = [\alpha\mathrm{N}_1\mathrm{N}_3 - \beta\mathrm{N}_2](1 + r(t-1)), \\ \mathrm{z}_1(t) = [\gamma\mathrm{N}_2 - \gamma\mathrm{N}_2\mathrm{N}_3 - \delta\mathrm{N}_3](1 + r(t-1)), \\ \mathrm{x}_2(t) = \mu[1 + r(t-1)] \\ -[(\mu + \mathrm{N}_2)(\mu - \mu\mathrm{N}_1 - \mathrm{N}_1\mathrm{N}_2) \\ +\mathrm{N}_1(\alpha\mathrm{N}_1\mathrm{N}_3 - \beta\mathrm{N}_3)](1 + r^2(t-1)\mathrm{y}_2(t) \\ = [\alpha\mathrm{N}_1(\gamma\mathrm{N}_2 - \gamma\mathrm{N}_2\mathrm{N}_3 - \delta\mathrm{N}_3) \\ -\beta^2(\mathrm{N}_1\mathrm{N}_3 - \mathrm{N}_2)](1 + r^2(t-1)), \mathrm{z}_2(t) = [\gamma(\alpha\mathrm{N}_1\mathrm{N}_3 - \beta\mathrm{N}_2) \\ -\gamma\mathrm{N}_2(\gamma\mathrm{N}_2 - \gamma\mathrm{N}_2\mathrm{N}_3 - \delta\mathrm{N}_3) - \gamma\mathrm{N}_3(\alpha\mathrm{N}_1\mathrm{N}_3 - \beta\mathrm{N}_2) \\ -\delta(\gamma\mathrm{N}_2 - \gamma\mathrm{N}_2\mathrm{N}_3 - \delta\mathrm{N}_3)](1 + r^2(t-1)), \end{cases} \tag{20}$$

and so on. On the same fashion the other terms can be computed. So, we get the series solution as

$$\begin{cases} \mathrm{x}(t) = \mathrm{x}_0(t) + \mathrm{x}_1(t) + \mathrm{x}_2(t) + \mathrm{x}_3(t) + \ldots, \\ \mathrm{y}(t) = \mathrm{y}_0(t) + \mathrm{y}_1(t) + \mathrm{y}_2(t) + \mathrm{y}_3(t) + \ldots, \\ \mathrm{z}(t) = \mathrm{z}_0 + \mathrm{z}_1(t) + \mathrm{z}_2(t) + \mathrm{z}_3(t) + \ldots. \end{cases} \tag{21}$$

Hence, in this way we can compute the series solution. To check the convergence of the series (20), we provide the following result.

Theorem 4.1. *Let* $\mathbf{Y} = C[d, d_k]$ *is the Banach spaces and* $\mathbf{A} : \mathbf{Y} \to \mathbf{Y}$ *be a contractive nonlinear operator such that for all* $\mathbf{W}, \overline{\mathbf{W}} \in \mathbf{Y}$, $\|\mathbf{A}(\mathbf{W}) - \mathbf{A}(\overline{\mathbf{W}})\|_\infty \leqslant \kappa\|\mathbf{W} - \overline{\mathbf{W}}\|_\infty, 0 < \kappa < 1$. *On the use of Banach contraction principle* **A** *has a unique fixed point* **W**, *such that* $\mathbf{AW} = \mathbf{W}$, *where* $\mathbf{W} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$. *The series given in (20) can be written by applying LADM as*

$$\mathbf{W}_p = \mathbf{AW}_{p-1}, \mathbf{W}_{p-1} = \sum_{j=0}^{p-1}\mathbf{W}_p, p = 1, 2, 3, \cdots.$$

Let $\mathbf{W}_0 = \mathbf{W}_0 \in B_\varepsilon(\mathbf{W})$, *where* $B_\varepsilon(\mathbf{W}) = \{\overline{\mathbf{W}} \in \mathbf{Y} : \|\overline{\mathbf{W}} - \mathbf{W}\|_{\mathbf{Y}} < \varepsilon\}$. *Then one has*

(i) $\mathbf{W}_p \in B_r(\mathbf{W})$;

(ii) $\lim_{p\to\infty} \mathbf{W}_p = \mathbf{W}$.

Proof. The proof can be similarly derived as in [17].

5. Numerical solution under CFFD and usual CFD

For the parameters involved in Model (1), we use the following numerical values from [18] to compute the series solution of the considered model in the frameworks of CFFD and conventional CFD. A detail nomenclature of the considered model (1) is given in Table 1.

If the total papulation $N = 5071126$, then

$$\begin{aligned} &N_1 = \frac{5070822}{5071126} = 0.9999400528,\\ &N_2 = \frac{304}{5071126} = 0.0000599472,\ N_3 = 0.1,\ \alpha = 0.0006,\\ &\beta = 0.333\gamma = 0.375,\ \delta = 0.02941,\ \mu = 0.0045. \end{aligned} \tag{22}$$

In view of these values, we have

$$\begin{cases} x_0(t) = 0.9999400528,\ y_0(t) = 0.0000599472,\ z_0(t) = 0.1,\\ x_1(t) = -0.185494(1 + r(t-1)),\\ y_1(t) = 0.00040033(1 + r(t-1)),\\ z_1(t) = -0.00292076(1 + r(t-1)),\\ x_2(t) = 0.0045(1 + r(t-1)) + 0.0223047(1 + r^2(t-1)),\\ y_2(t) = -0.0001426871(1 + r^2(t-1)),\\ z_2(t) = 0.000221076(1 + r^2(t-1)),\\ x_3(t) = 0.0045(1 + r(t-1)) - 0.000452025(1 + r^2(t-1))\\ -0.00262613\Big[(1-2r)(1 + r(t-1)) + 2r^2\left(\frac{t^2}{2} + r\left(\frac{t^3}{3!} - \frac{t^2}{2!}\right)\right)\\ +r(2-r)\left(t + r\left(\frac{t^2}{2} - t\right)\right)\Big],\\ y_3(t) = 0.00006\left(1 + r^2(t-1) + rt + r^3\left(\frac{t^2}{2!} - 2t + 1\right)\right.\\ +0.00000475576(1 + r^2(t-1))\\ -0.0000008084\Big[(1-2r)(1 + r(t-1)) + 2r^2\left(\frac{t^2}{2} + r\left(\frac{t^3}{3!} - \frac{t^2}{2!}\right)\right)\\ +r(2-r)\left(t + r\left(\frac{t^2}{2} - t\right)\right)\Big], z_3(t)\\ = -0.00005471\left(1 + r^2(t-1) + rt + r^3\left(\frac{t^2}{2!} - 2t + 1\right)\right.\\ +0.0000008769[(1-2r)(1 + r(t-1))\\ +2r^2\left(\frac{t^2}{2} + r\left(\frac{t^3}{3!} - \frac{t^2}{2!}\right)\right) + r(2-r)\left(t + r\left(\frac{t^2}{2} - t\right)\right)\Big]. \end{cases} \tag{23}$$

Analogously, we can compute the other terms of the series. On the other part, if we consider CFD, then the general series solution of the model (1) with given initial conditions given by

$$\begin{cases} x_0 = N_1,\ y_0 = N_2,\ z_0 = N_3\\ x_1 = \mu\frac{t^r}{\Gamma(r+1)} - \mu N_1(1+\alpha N_3)\frac{t^r}{\Gamma(r+1)} - \mu(\mu+\alpha N_3)\frac{t^{2r}}{\Gamma(2r+1)}\\ y_1 = (\alpha N_1 N_3 - \beta N_2)\frac{t^r}{\Gamma(r+1)} + \alpha\mu N_3\frac{t^{2r}}{\Gamma(2r+1)}\\ z_1 = (\gamma(1-N_3)N_2 - \delta N_3)\frac{t^r}{\Gamma(r+1)}\\ x_2 = [\mu^2 N_1(1+\alpha N_3) - \alpha N_1(\gamma N_2(1-N_3) - \delta N_3)]\frac{t^{2r}}{\Gamma(2r+1)}\\ \quad +\mu^2(\mu+\alpha N_3)\frac{t^{3r}}{\Gamma(3r+1)}\\ \quad -\alpha\mu(\gamma(1-N_3)N_2 - \delta N_3)\frac{t^{3r_1}}{\Gamma(3r+1)},\\ y_2 = \alpha(N_1\gamma(1-N_3)N_2 - \delta N_3)\frac{t^{2r}}{\Gamma(2r+1)} + \alpha\mu(\gamma(1-N_3) - \delta N_3)\frac{t^{3r+1}}{\Gamma(3r+2)}\\ \quad -\mu N_1 N_3(1+\alpha N_3)\frac{t^{2r}}{\Gamma(2r+1)} - N_3\mu(\mu+\alpha N_3)\frac{t^{3r}}{\Gamma(3r+1)}\\ \quad -(\beta\alpha N_1 N_3 - \beta N_2)\frac{t^{2r}}{\Gamma(2r+1)} - \beta\alpha\mu N_3\frac{t^{3r}}{\Gamma(3r+1)},\\ z_2 = \gamma(\alpha N_1 N_3 - \beta N_2)\frac{t^r}{\Gamma(r+1)} - \gamma N_2(\gamma(1-N_3)N_2 - \delta N_3)\frac{t^{2r}}{\Gamma(2r+1)}\\ \quad -\gamma(\alpha N_1 N_3 - \beta N_2)\frac{t^{2r}}{\Gamma(2r+1)} - \gamma\alpha\mu N_3\frac{t^{2r}}{\Gamma(2r+1)} \end{cases} \tag{24}$$

and so on. In this fashion, we can generate the series solutions of the consider model (1). After using the numerical values given in (22) in (24), we get few terms of the corresponding series solution as

$$\begin{cases} x(t) = 0.9999400528 - 2.21414\times 10^{-10}\frac{t^r}{\Gamma(r+1)}\\ +9.149431\times 10^{-7}\frac{t^{2r}}{\Gamma(2r+1)} + 9.63859\times 10^{-8}\frac{t^{3r}}{\Gamma(3r+1)},\\ y(t) = 5.9999\times 10^{-5} + 4.001598\times 10^{-5}\frac{t^r}{\Gamma(r+1)} - 4.49051\\ \times 10^{-4}\frac{t^{2r}}{\Gamma(2r+1)} - 1.2098295\times 10^{-6}\frac{t^{3r}}{\Gamma(3r+1)},\\ z(t) = 0.1 - 2.905755\times 10^{-3}\frac{t^r}{\Gamma(r+1)} + 7.68156\\ \times 10^{-3}\frac{t^{2r}}{\Gamma(2r+1)} - 2.2098295\times 10^{-3}\frac{t^{3r}}{\Gamma(3r+1)}. \end{cases} \tag{25}$$

6. Numerical interpretation and discussion

In the previous section, we have computed two types of series solution one for CFFD and the other for CFD. In both cases, we have obtained two different rational type expression given in (23) and (25) respectively by using the numerical values given in (22). In the current section, we provide the numerical plots of various compartments corresponding to different fractional order derivatives. In Figs. 1–3, we plotted the solutions up to four terms for different fractional order by using CFFD while in Figs. 4–6, we have plotted the resultant solutions up to three terms at various fractional order via CFD. Since fractional differential operators are global operators which produce greater degree of freedom as compared to classical derivative which is local in nature. From the figures, we deduce that CFFD gives more clear interpretation of dynamics instead of usual CFD. The global behavior of the fractional differential operator due to Caputo and Fabrizio is more clear than the usual CFD. In Figs. 1–3, we provide plots at different fractional order for (23) after four term solutions for the considered model.

From Figs. 1–3s, we see that when the Dengue virus starts spreading in a healthy community, then the number of susceptible human population is increasing as well as the infection also goes on increasing and if there is no cure then the recovered population is also decreasing. This increase or decease can be seen from Figs. 1–3 at different fractional order adopting

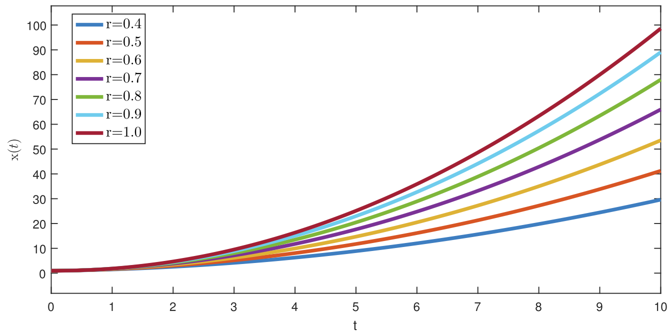

Fig. 1 Plot of transmission of Dengue virus in susceptible human population in Model (1) at different values of fractional order r using CFFD.

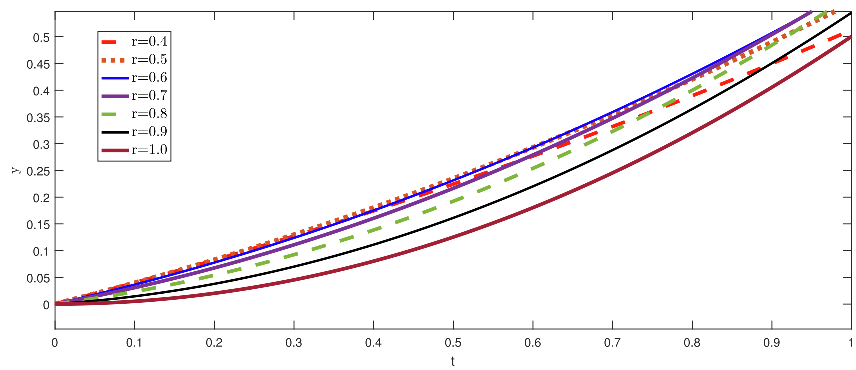

Fig. 2 Plot of transmission of Dengue virus of infected human population in proposed model (1) at different values of fractional order *r* using CFFD.

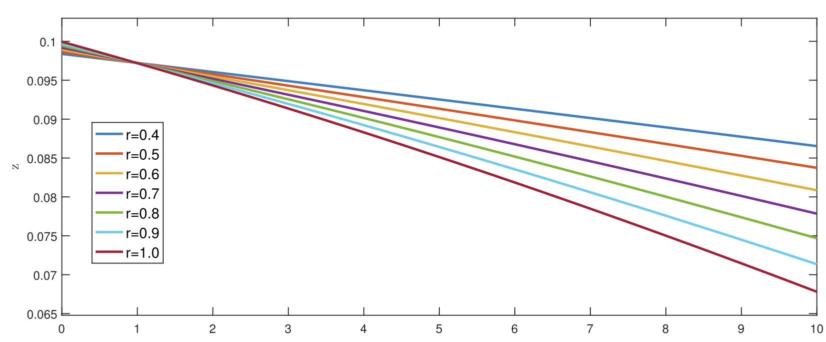

Fig. 3 Plot of transmission of Dengue fever in recovered human population in proposed model (1) at different values of fractional order *r* using CFFD.

CFFD. At lower order, the rate of increase and decrease of different compartment is faster, while as the order increases the process becomes slower and vice versa.

Infact numerical analysis plays a vital role in dealing real world problem. It is needed to solve engineering problems that lead to equations that cannot be solved analytically with simple formulas. Since biological models are nonlinear mostly and their exact solution is quiet difficult to find. Involving fractional derivative makes it more complicated. Therefore, by solving on some efficient method, we plot the solutions through graphs. From these graphical reforestation, we get information about how a disease transmit in a community which we assume susceptible about the infection, when virus attacks, the susceptible is going on decreasing as they are converting to infected papulation and hence decays of susceptible will results in growth of infected. If proper cure or vaccine is applied then some people will get back their health and hence their papulation is also increasing. Now fractional order derivatives provide all the possible ways of this increase decrease. Hence this geometry tell us maximum approaches at which transmission of a disease and its recovery take place.

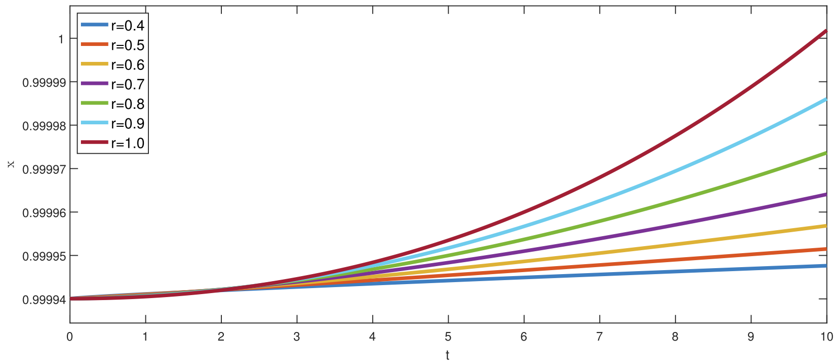

Fig. 4 Plot of transmission of Dengue virus in susceptible human population in Model (1) at different values of fractional order *r* using CFD.

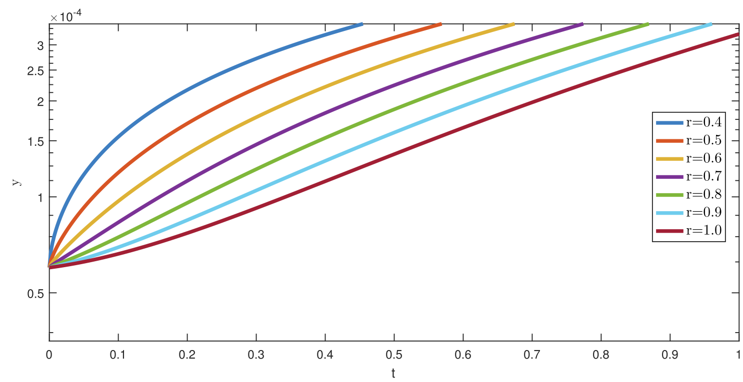

Fig. 5 Plot of transmission of Dengue virus of infected human population in proposed model (1) at different values of fractional order *r* using CFD.

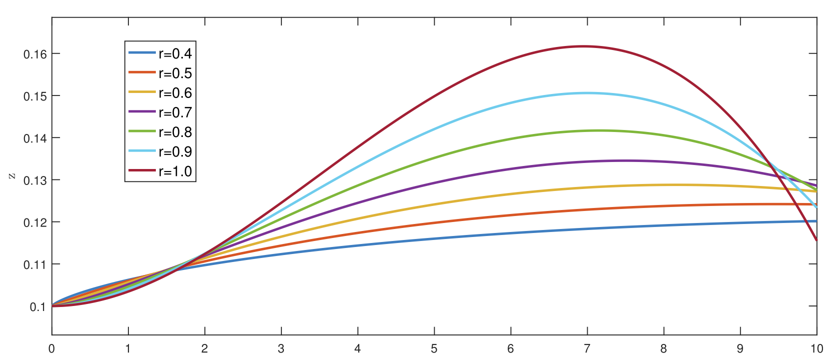

Fig. 6 Plot of transmission of Dengue fever in recovered human population in proposed model (1) at different values of fractional order *r* using CFD. Like from Figs. 1–3, in Figs. 4–6, the same phenomenon can be observed but the dynamics is slightly difficult to understand as compared by adopting usual CFD instead of CFFD. Thus we conclude that both produce same behavior but CFFD produce more clear physical interpretation as compared to usual CFD.

7. Conclusion

In this research work, we have developed qualitative and semi analytical analysis for fractional order model of Dengue fever disease by using CFFD. We have used fixed point approach and established some necessary conditions for the existence of solution to the consider model. Then we have established a general algorithm to develop series type solution for the considered model under CFFD. In this regard, we have applied famous Laplace transform together with Adomian decomposition method to obtain the required semi analytical solution for the fractional order model (1) under CFFD. Also, we have computed the series solution via using the same method under the usual Caputo derivative for the consider model. Both the series solutions have different in nature. The numerical plots indicates that CFFD is more better in using for mathematical modeling as compared to CFD. The aforementioned derivatives has the ability to produce excellent results as compared to usual Caputo derivative. Further, we have used LADM for nonlinear FODEs under CFFD which has been very rarely used for the said operator. For the numerical computations and implementation, we have used Matlab.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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