

RESEARCH

Open Access



On Caputo–Hadamard type coupled systems of nonconvex fractional differential inclusions

Samih Belmor¹, Fahd Jarad^{2,3*}  and Thabet Abdeljawad^{4,5*}

*Correspondence:

fahd@cankaya.edu.tr;

tabdeljawad@psu.edu.sa

²Department of Mathematics,
Çankaya University, 06790
Etimesgut, Ankara, Turkey

⁴Department of Mathematics and
General Sciences, Prince Sultan
University, P. O. Box 66833, Riyadh
11586, Saudi Arabia

Full list of author information is
available at the end of the article

Abstract

This research article is mainly concerned with the existence of solutions for a coupled Caputo–Hadamard of nonconvex fractional differential inclusions equipped with boundary conditions. We derive our main result by applying Mizoguchi–Takahashi’s fixed point theorem with the help of \mathcal{P} -function characterizations.

Keywords: Hadamard fractional integral; Hadamard–Caputo fractional derivative; \mathcal{MT} -function; \mathcal{P} -function; Mizoguchi–Takahashi’s condition

1 Introduction

In the previous two decades, fractional calculus has earned sizeable importance owing to diverse applications in scientific and engineering problems. Fractional-order boundary value problems, in particular, have become a rapidly growing area due to features of fractional derivatives which make the systems of fractional-order practical and realistic than the corresponding classical systems. For some current work, we suggest [1–10]. There are numerous definitions of fractional differentiation operators in the literature, the most common is the classical Riemann–Liouville type fractional derivative after which a beneficial alternative has been introduced to cope with disadvantages caused by the Riemann–Liouville expression, the so-called Caputo derivative. Fractional derivatives within the frame of Hadamard type differ from the Riemann–Liouville type and the Caputo type due to the appearance of a logarithmic function in the definition of the Hadamard derivatives. One can find manifold monographs and articles devoted exclusively to the theory of fractional derivatives, not merely on mathematical subjects but also physics, applied sciences, engineering, etc.; see [11–14].

This article involves the so-called Caputo–Hadamard fractional derivatives which modifies the Hadamard derivative into a more beneficial type using Caputo approach [15, 16]

Differential inclusions are found to be of great advantage as the field of study for these inclusions covers theoretical treatment, inequalities, and applications in a variety of disciplines in physical and industrial sciences. Examples cover optimal control systems [17], isothermal dynamics with stochastic velocities [18], control problems [19] and sweeping processes [20]. The study of fractional-order differential inclusions was first launched by Sayed and Ibrahim [21]. Since then the literature on fractional-order differential inclusions

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

has found various qualitative results. We refer the reader to Ref. [22–25] for the recent advancement on the topic.

The study of coupled systems of fractional-order differential equations has also received great attention as such systems emerge in a diversity of problems of biological phenomena and environmental issues. For details and examples, the reader is referred to [26–28] and the references mentioned therein.

Recently, a class of coupled fractional-order differential inclusion was discussed in [29], of the form

$$\begin{aligned} {}^c\mathcal{D}^\gamma w(y) &\in W(y, w(y), z(y)), \quad y \in J = [0, T], \\ {}^c\mathcal{D}^\zeta z(y) &\in Z(y, w(y), z(y)) \quad y \in J = [0, T], \end{aligned} \tag{1.1}$$

subject to the coupled boundary condition

$$\begin{aligned} w(0) &= \nu_1 z(T), & w'(0) &= \nu_2 z'(T), \\ z(1) &= \mu_1 w(T), & z'(0) &= \mu_2 w'(T), \end{aligned} \tag{1.2}$$

where ${}^c\mathcal{D}^z$ is the Caputo–Liouville fractional derivative of order $1 < z \leq 2$, $z \in (\gamma, \zeta)$, W and Z are given multivalued maps. The authors investigated the existence criteria for solutions by applying standard fixed-point theorems for multivalued maps.

Motivated by the above and inspired by the work in [26–28], in this paper, we study the following coupled fractional differential inclusions:

$$\begin{aligned} {}^{Hc}\mathcal{D}^\gamma w(y) &\in W(y, w(y), z(y)), \quad y \in J = [1, T], \\ {}^{Hc}\mathcal{D}^\zeta z(y) &\in Z(y, w(y), z(y)), \quad y \in J = [1, T], \end{aligned} \tag{1.3}$$

with uncoupled boundary conditions of the form

$$\begin{aligned} w(1) &= 0, & \delta w(T) &= \delta w(1) = 0, \\ z(1) &= 0, & \delta z(T) &= \delta z(1) = 0, \end{aligned} \tag{1.4}$$

where ${}^{Hc}\mathcal{D}^r$ is the Caputo–Hadamard fractional derivative of order $1 < r \leq 2$, $r \in (\gamma, \zeta)$, $\delta = y \frac{d}{dy}$, and $W, Z : [1, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(\mathbb{R})$ are multivalued maps, $\mathcal{L}(\mathbb{R})$ is the family of all nonempty subsets.

The objective of the present paper is to establish new existence criteria of solutions for the problem (1.3)–(1.4) by applying Mizoguchi–Takahashi’s fixed point theorem for multivalued maps. To the best of our knowledge, the application of fixed-point theorem due to Mizoguchi and Takahashi to the framework of the current problem is new and has not been investigated elsewhere.

The article is designed as follows. Some introductory materials that we need in the sequel are presented in the next section. The main results are derived in Sect. 3. An example is provided to illustrate the theory in Sect. 4.

2 Axillary results

Let J be a finite interval on \mathbb{R} . We denote by $\Sigma = C([1, T], \mathbb{R})$ the set of continuous functions on $[1, T]$ supplied with the norm $\|w\| = \max_{\theta \in [1, T]} |w(\theta)|$. The product set

$(\Sigma \times \Sigma, \|(w, z)\|)$ is a Banach space endowed with the norm

$$\|(w, z)\| = \|w\| + \|z\|.$$

We define $AC_\delta^n([1, T], \mathbb{R})$ as

$$AC_\delta^n([1, T], \mathbb{R}) = \left\{ w : [1, T] \rightarrow \mathbb{R} \text{ and } \delta^{n-1}w(y) \in AC([1, T], \mathbb{R}), \delta = y \frac{d}{dy} \right\},$$

where $AC([1, T], \mathbb{R})$ is the set of absolute continuous functions from J into \mathbb{R} . $L^1([1, T], \mathbb{R})$ is the set of those Lebesgue measurable functions $w : [1, T] \rightarrow \mathbb{R}$ with the norm

$$\|w\|_1 = \int_1^T |w(y)| dy.$$

Now we recall some essential outlines on multivalued maps [30]. For a normed space $(\Sigma, \|\cdot\|)$, let

$$\begin{aligned} \mathcal{CL}(\Sigma) &= \{Q \in \mathcal{L}(\Sigma) : Q \text{ is closed}\}, \\ \mathcal{K}(\Sigma) &= \{Q \in \mathcal{L}(\Sigma) : Q \text{ is compact}\}, \\ \mathcal{CB}(\Sigma) &= \{Q \in \mathcal{L}(\Sigma) : Q \text{ is closed and bounded}\}. \end{aligned}$$

A multivalued operator $\mathfrak{G} \rightarrow \mathcal{CL}(\mathbb{R})$ is said to be measurable if for every $\zeta \in \mathbb{R}$, the function

$$y \rightarrow \inf\{|\zeta - z|, z \in \mathfrak{G}(y)\},$$

is measurable.

Next, we shall recall some known results concerning fractional operators.

Definition 2.1 ([31]) The fractional-order integral operator of Hadamard type of a function $f \in L^1([1, T], \mathbb{R})$ is given as

$${}^H I^r f(y) = \frac{1}{\Gamma(r)} \int_1^y \left(\log \frac{y}{\theta}\right)^{r-1} f(\theta) \frac{d\theta}{\theta}, \tag{2.1}$$

provided the integral exists.

Definition 2.2 ([15]) For a given function $f \in AC_\delta^n([1, T], \mathbb{R})$, the Caputo–Hadamard fractional derivative of order $r > 0$ is defined as follows:

$$({}^{Hc} \mathcal{D}^r f)(y) = \frac{1}{\Gamma(n-r)} \int_1^y \left(\log \frac{y}{\theta}\right)^{n-r-1} \delta^n f(\theta) \frac{d\theta}{\theta}, \tag{2.2}$$

where $n = [r] + 1$, $[r]$ is the integer part of r and $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(w) = \int_0^\infty y^{w-1} e^{-y} dy$.

If $r = n \in \mathbb{N}$ we have

$${}^{Hc} \mathcal{D}^r f(y) = (\delta^n f)(y).$$

Lemma 2.3 ([15]) *For a given function $f \in AC_\delta^n([1, T], \mathbb{R})$ or $f \in C_\delta^n([1, T], \mathbb{R})$, and $r \in \mathbb{C}$ we have*

$${}^H I^{rHc} \mathcal{D}^r f(y) = f(y) - \sum_{j=0}^{n-1} \frac{(\delta^j f)(1)}{j!} (\log(y))^j, \tag{2.3}$$

particularly, for $0 < r < 1$, we obtain

$${}^H I^{rHc} \mathcal{D}^r f(y) = f(y) - f(1).$$

The following lemma is useful in the forthcoming analysis related to the problem (1.3)–(1.4).

Lemma 2.4 *Let $\rho_1, \rho_2 : [1, T] \rightarrow \mathbb{R}$ be continuous functions, and $r \in]1, 2]$, $r \in (\gamma, \zeta)$. Then the fractional problem*

$${}^{Hc} \mathcal{D}^\gamma w(y) = \rho_1(y), \quad 1 \leq y \leq T, \tag{2.4}$$

$${}^{Hc} \mathcal{D}^\zeta z(y) = \rho_2(y), \quad 1 \leq y \leq T,$$

$$w(1) = 0, \quad \delta w(T) = \delta w(1) = 0, \tag{2.5}$$

$$z(1) = 0, \quad \delta z(T) = \delta z(1) = 0,$$

is equivalent to the system of integral equations

$$w(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \rho_1(\theta) \frac{d\theta}{\theta}, \tag{2.6}$$

$$z(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \rho_2(\theta) \frac{d\theta}{\theta}. \tag{2.7}$$

Proof Performing the Hadamard operator of order γ on the first equation in (2.4) and using Lemma 2.3, we get

$$w(t) = c_1 + c_2 \log(y) + {}^H I^\gamma \rho_1(y), \tag{2.8}$$

where c_1, c_2 are arbitrary constants. Taking the δ -derivative in (2.8) we get

$$(\delta w)(y) = c_2 + {}^H I^{\gamma-1} \rho_1(y). \tag{2.9}$$

Using the boundary conditions $(\delta w)(T) = (\delta w)(1) = 0$ in (2.9), we get $c_2 = 0$, then using the condition $w(1) = 0$ to (2.8), gives us $c_1 = 0$, therefor we get the solution described in (2.6). In the same manner we solve the second equation of (2.4) for z we get (2.7). This ends the proof. \square

3 Existence results

Let (Σ, ρ) be a metric space and $H_\rho(\cdot, \cdot)$ denote the Hausdorff metric on $\mathcal{CB}(\Sigma)$ defined as

$$H(Q, D) := \max \left\{ \sup_{q \in Q} \rho(q, D), \sup_{d \in D} \rho(d, Q) \right\},$$

where $\rho(Q, d) = \inf_{q \in Q} \rho(q, d)$ and $\rho(q, D) = \inf_{d \in D} \rho(q, d)$. Then $(\mathcal{CB}(\Sigma), H_\rho)$ is a metric space [32]. For each $(w, z) \in (\Sigma \times \Sigma)$, define the sets of selections of W, Z by

$$S_{W,(w,z)} = \{ \sigma \in L^1([1, T], \mathbb{R}), \sigma(y) \in W(y, w(y), z(y)), \text{ a.e. } y \in [1, T] \}$$

and

$$S_{Z,(w,z)} = \{ \vartheta \in L^1([1, T], \mathbb{R}), \vartheta(y) \in G(y, w(y), z(y)), \text{ a.e. } y \in [1, T] \}.$$

Definition 3.1 ([33]) We call a function $\phi : \mathbb{R}_0^+ \rightarrow [0, \frac{1}{2})$ a \mathcal{P} -function if it satisfies the conditions

$$\limsup_{\theta \rightarrow y^+} \phi(\theta) < \frac{1}{2}, \quad \text{for every } y \in \mathbb{R}_0^+. \tag{3.1}$$

Definition 3.2 ([34, 35]) We call a function $\alpha : \mathbb{R}_0^+ \rightarrow [0, 1)$ an \mathcal{MT} -function or (\mathcal{R} -function) if it fulfills the Mizoguchi–Takahashi’s condition i.e.

$$\limsup_{\theta \rightarrow y^+} \alpha(\theta) < 1, \quad \text{for every } y \in \mathbb{R}_0^+.$$

Remark 3.3

- $\phi : \mathbb{R}_0^+ \rightarrow [0, \frac{1}{2})$ is a \mathcal{P} -function if and only if
 - for any nonincreasing sequence $(z_n)_{n \geq 1} \in \mathbb{R}_0^+$ we obtain $0 \leq \sup_{n \geq 1} \phi(z_n) < \frac{1}{2}$;
 - if ϕ is a function of semi-contractive factor, that is, for any strictly decreasing sequence $(z_n)_{n \geq 1} \subset \mathbb{R}_0^+$ we have $0 \leq \sup_{n \geq 1} \phi(z_n) < \frac{1}{2}$ [33];
 - any function defined as $\kappa(y) = \frac{\phi(y)}{2} + \frac{1}{4}$ is also considered as a \mathcal{P} -function.
- $\alpha : \mathbb{R}_0^+ \rightarrow [0, 1)$ is an \mathcal{MT} -function if and only if ϕ is a function of contractive factor, that is, for any strictly decreasing sequence $(z_n)_{n \geq 1} \subset \mathbb{R}_0^+$ we have $0 \leq \sup_{n \geq 1} \alpha(z_n) < 1$.
- If we define $\alpha(y) = 2\phi(y)$ for all $y \in \mathbb{R}_0^+$ then α is truly an \mathcal{MT} -function. For more details about \mathcal{MT} -functions see [35, 36].

Theorem 3.4 ([34]) Let $\alpha : \mathbb{R}_0^+ \rightarrow [0, 1)$ be an \mathcal{MT} -function, and $\Lambda : \Sigma \rightarrow \mathcal{CB}(\Sigma)$ be a multivalued map, where (Σ, ρ) is a complete metric space. Assume that

$$H_\rho(\Lambda w, \Lambda z) \leq \alpha(\rho(w, z))\rho(w, z), \quad \text{for all } w, z \in \Sigma.$$

Then Λ has a fixed point.

Mizoguchi–Takahashi’s fixed point theorem [34] is a positive answer to the conjecture of Reich [37].

Definition 3.5 A function $(w, z) \in AC_\delta^2([1, T], \mathbb{R}) \times AC_\delta^2([1, T], \mathbb{R})$ is called a solution of the coupled system (1.3) if there exist functions $(\sigma, \vartheta) \in L^1([1, T], \mathbb{R}) \times L^1([1, T], \mathbb{R})$ such that $\sigma(y) \in W(y, w(y), z(y))$, and $\vartheta(y) \in Z(y, w(y), z(y))$, a.e. $y \in [1, T]$, and w, z satisfy conditions (1.4) with

$$w(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta} \right)^{\gamma-1} \sigma(\theta) \frac{d\theta}{\theta}, \quad y \in [1, T], \tag{3.2}$$

and

$$z(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \vartheta(\theta) \frac{d\theta}{\theta}, \quad y \in [1, T]. \tag{3.3}$$

We define the operators $\mathcal{N}_1, \mathcal{N}_2 : \Sigma \times \Sigma \rightarrow \mathcal{L}(\Sigma \times \Sigma)$ associated with the problem (1.3)–(1.4) by

$$\mathcal{N}_1(w, z) : \left\{ f_1 \in \Sigma \times \Sigma : f_1(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \sigma(\theta) \frac{d\theta}{\theta}, \sigma \in S_{W,(w,z)} \right\} \tag{3.4}$$

and

$$\mathcal{N}_2(w, z) : \left\{ f_2 \in \Sigma \times \Sigma : f_2(y) = \frac{1}{\Gamma(\beta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\beta-1} \vartheta(\theta) \frac{d\theta}{\theta}, \vartheta \in S_{Z,(w,z)} \right\}. \tag{3.5}$$

Then we define an operator $\mathcal{N} : \Sigma \times \Sigma \rightarrow \mathcal{L}(\Sigma \times \Sigma)$

$$\mathcal{N}(w, z)(y) = \begin{bmatrix} \mathcal{N}_1(w, z)(y) \\ \mathcal{N}_2(w, z)(y) \end{bmatrix}, \tag{3.6}$$

where \mathcal{N}_1 and \mathcal{N}_2 are, respectively, defined by (3.4) and (3.5).

Theorem 3.6 *Let $\phi_1, \phi_2 : \mathbb{R}_0^+ \rightarrow [0, \frac{1}{2})$ be two \mathcal{P} -functions, and define α on \mathbb{R}_0^+ by $\alpha(y) = \phi_1(y) + \phi_2(y)$. Assume that the following hypotheses hold:*

- (H1) $W, Z : [1, T] \times \mathbb{R}^2 \rightarrow \mathcal{K}(\mathbb{R})$ are measurable multi-functions for all $w, z \in \mathbb{R}$.
- (H2) For w, \bar{w}, z , and $\bar{z} \in \mathbb{R}$, we have

$$\begin{aligned} &H_\rho(W(y, w(y), z(y)), W(y, \bar{w}(y), \bar{z}(y))) \\ &\leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|) \end{aligned}$$

and

$$\begin{aligned} &H_\rho(Z(y, w(y), z(y)), Z(y, \bar{w}(y), \bar{z}(y))) \\ &\leq \frac{\Gamma(\zeta + 1)}{(\log(T))^\zeta} \phi_2(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|), \end{aligned}$$

for all $y \in [1, T]$.

If α verifies the Mizoguchi–Takahashi’s condition, the problem (1.3)–(1.4) has at least one solution on $[1, T]$.

Proof We shall show that $\mathcal{N} : \Sigma \times \Sigma \rightarrow \mathcal{L}(\Sigma \times \Sigma)$ given in (3.6) has a fixed point. First we show that \mathcal{N} is a closed subset of $\mathcal{L}(\Sigma \times \Sigma)$ for each $(w, z) \in \Sigma \times \Sigma$.

Let $(f_n, \tilde{f}_n) \in \mathcal{N}(w_n, z_n)$ be a sequence such that $(f_n, \tilde{f}_n) \rightarrow (f, \tilde{f})$ in $\Sigma \times \Sigma$ whenever $n \rightarrow +\infty$. Then there exist a $\sigma_n \in S_{W,(w_n,z_n)}$, and $\vartheta_n \in S_{Z,(w_n,z_n)}$ such that, for each $y \in [1, T]$, we

get

$$f_n(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \sigma_n(\theta) \frac{d\theta}{\theta},$$

$$\bar{f}_n(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \vartheta_n(\theta) \frac{d\theta}{\theta}.$$

By compactness of W and Z , the sequences $(\sigma_n)_{n \geq 1}$ and $(\vartheta_n)_{n \geq 1}$ have sub-sequences, still denoted by $(\sigma_n)_{n \geq 1}$ and $(\vartheta_n)_{n \geq 1}$ which converge strongly to $\sigma \in L^1([1, T], \mathbb{R})$ and $\vartheta \in L^1([1, T], \mathbb{R})$, respectively. Indeed for every $\varpi \in W(y, w(y), z(y))$, we get

$$|\sigma_n(y) - \sigma(y)| \leq |\sigma_n(y) - \varpi| + |\varpi - \sigma|,$$

which implies

$$|\sigma_n(y) - \sigma(y)| \leq H_\rho(W(y, w_n, z_n), W(y, w, z))$$

$$\leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w_n - w| + |z_n - z|)(|w_n - w| + |z_n - z|).$$

Since $\|(w_n - w, z_n - z)\| \rightarrow 0$, we have $\phi_1(\|(w_n - w, z_n - z)\|)(\|(w_n - w, z_n - z)\|) \rightarrow 0$ and hence $\sigma \in S_{W,(w,z)}$. By the same process we show $\vartheta \in S_{Z,(w,z)}$. Thus, for each $y \in J$

$$f_n(y) \rightarrow f(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \sigma(\theta) \frac{d\theta}{\theta}$$

and

$$\bar{f}_n(y) \rightarrow \bar{f}(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \vartheta(\theta) \frac{d\theta}{\theta}.$$

So $(f, \bar{f}) \in \mathcal{N}(w, z)$ and \mathcal{N} is a closed map.

Next for $w, \bar{w}, z, \bar{z} \in \Sigma$, we show

$$H_\rho(\mathcal{N}(w, z), \mathcal{N}(\bar{w}, \bar{z})) \leq \alpha(\|w - \bar{w}\| + \|z - \bar{z}\|)(\|w - \bar{w}\| + \|z - \bar{z}\|).$$

Let $((w, \bar{w}), (z, \bar{z})) \in \Sigma \times \Sigma$, and $(f_1, \bar{f}_2) \in \mathcal{N}(w, z)$. Then there exist $\sigma_1(y) \in S_{W,(w,z)}$ and $\vartheta_1(y) \in S_{Z,(w,z)}$ such that, for each $y \in [1, T]$,

$$f_1(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \sigma_1(\theta) \frac{d\theta}{\theta}$$

and

$$\bar{f}_1(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \vartheta_1(\theta) \frac{d\theta}{\theta}.$$

From (H2) it follows that

$$H_\rho(W(y, w, z), W(y, \bar{w}, \bar{z})) \leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|)$$

and

$$H_\rho(Z(y, w, z), Z(y, \bar{w}, \bar{z})) \leq \frac{\Gamma(\zeta + 1)}{(\log(T))^\zeta} \phi_2(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|),$$

Thus, there exist $\varrho \in W(y, \bar{w}, \bar{z})$ and $\varpi \in Z(y, \bar{w}, \bar{z})$ provided that

$$|\sigma_1(y) - \varrho| \leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|), \quad y \in [1, T],$$

and

$$|\vartheta_1(y) - \varpi| \leq \frac{\Gamma(\zeta + 1)}{(\log(T))^\zeta} \phi_2(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|), \quad y \in [1, T].$$

Define $U_1, U_2 : [1, T] \rightarrow \mathcal{L}(\mathbb{R})$ given by

$$U_1(y) = \left\{ \varrho \in \mathbb{R} : |\sigma_1(y) - \varrho| \leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|) \right\}$$

and

$$U_2(y) = \left\{ \varpi \in \mathbb{R} : |\vartheta_1(y) - \varpi| \leq \frac{\Gamma(\zeta + 1)}{(\log(T))^\zeta} \phi_2(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|) \right\}.$$

Since $U_1(y) \cap W(y, \bar{w}, \bar{z})$ and $U_2(y) \cap Z(y, \bar{w}, \bar{z})$ are two measurable operators [38], we can find a measurable selection $\sigma_2(y)$ for $U_1(y) \cap W(y, \bar{w}, \bar{z})$ and a measurable selection $\vartheta_2(y)$ for $U_2(y) \cap Z(y, \bar{w}, \bar{z})$. Thus $\sigma_2(y) \in W(y, \bar{w}(y), \bar{z}(y))$, $\vartheta_2(y) \in Z(y, \bar{w}(y), \bar{z}(y))$, and for each $y \in [1, T]$, we have

$$|\sigma_1(y) - \sigma_2(y)| \leq \frac{\Gamma(\gamma + 1)}{(\log(T))^\gamma} \phi_1(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|)$$

and

$$|\vartheta_1(y) - \vartheta_2(y)| \leq \frac{\Gamma(\zeta + 1)}{(\log(T))^\zeta} \phi_2(|w - \bar{w}| + |z - \bar{z}|)(|w - \bar{w}| + |z - \bar{z}|).$$

We define $f_2(y)$ for each $y \in [1, T]$, as follows:

$$f_2(y) = \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta} \right)^{\gamma-1} \sigma_2(\theta) \frac{d\theta}{\theta}$$

and

$$\bar{f}_2(y) = \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta} \right)^{\zeta-1} \vartheta_2(\theta) \frac{d\theta}{\theta}.$$

Then for $y \in [1, T]$

$$\begin{aligned} &|f_1(y) - f_2(y)| \\ &\leq \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta} \right)^{\gamma-1} |\sigma_1(\theta) - \sigma_2(\theta)| \frac{d\theta}{\theta} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\gamma)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\gamma-1} \frac{d\theta}{\theta} \frac{\Gamma(\gamma+1)}{(\log(T))^\gamma} \phi_1(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|) \\ &\leq \frac{(\log(T))^\gamma}{\Gamma(\gamma+1)} \frac{\Gamma(\gamma+1)}{(\log(T))^\gamma} \phi_1(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|) \\ &\leq \phi_1(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|) \end{aligned}$$

and

$$\begin{aligned} &|\bar{f}_1(y) - \bar{f}_2(y)| \\ &\leq \frac{1}{\Gamma(\beta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\beta-1} |\vartheta_1(\theta) - \vartheta_2(\theta)| \frac{d\theta}{\theta} \\ &\leq \frac{1}{\Gamma(\zeta)} \int_1^y \left(\log \frac{y}{\theta}\right)^{\zeta-1} \frac{d\theta}{\theta} \frac{\Gamma(\zeta+1)}{(\log(T))^\zeta} \phi_2(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|) \\ &\leq \frac{(\log(T))^\zeta}{\Gamma(\zeta+1)} \frac{\Gamma(\zeta+1)}{(\log(T))^\zeta} \phi_2(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|) \\ &\leq \phi_2(|w-\bar{w}|+|z-\bar{z}|)(|w-\bar{w}|+|z-\bar{z}|). \end{aligned}$$

Therefore,

$$\begin{aligned} \|f_1 - f_2\| &\leq \phi_1(\|w-\bar{w}\| + \|z-\bar{z}\|)(\|w-\bar{w}\| + \|z-\bar{z}\|), \\ \|\bar{f}_1 - \bar{f}_2\| &\leq \phi_2(\|w-\bar{w}\| + \|z-\bar{z}\|)(\|w-\bar{w}\| + \|z-\bar{z}\|). \end{aligned}$$

It follows that

$$\begin{aligned} &H_\rho(\mathcal{N}(w, z), \mathcal{N}(\bar{w}, \bar{z})) \\ &\leq (\phi_1(\|w-\bar{w}\| + \|z-\bar{z}\|) + \phi_2(\|w-\bar{w}\| + \|z-\bar{z}\|))(\|w-\bar{w}\| + \|z-\bar{z}\|) \\ &\leq \alpha(\|w-\bar{w}\| + \|z-\bar{z}\|)(\|w-\bar{w}\| + \|z-\bar{z}\|), \end{aligned}$$

for all $w, \bar{w}, z, \bar{z} \in \Sigma$. By hypothesis, since the function α fulfilled the Mizoguchi–Takahashi’s condition it is an \mathcal{MT} -function, and by Lemma 3.4 \mathcal{N} has a fixed point $(w^*, z^*) \in \Sigma \times \Sigma$ that is a solution to the system (1.3)–(1.4). The proof is now complete. \square

4 Example

Consider the following coupled Caputo–Hadamard fractional differential inclusions with separated boundary conditions:

$$\begin{cases} {}^{Hc}\mathcal{D}^\gamma w(y) \in W(y, w(y), z(y)), & y \in J = [1, 2], 1 < \gamma \leq 2, \\ {}^{Hc}\mathcal{D}^\zeta z(y) \in Z(y, w(y), z(y)), & y \in J = [1, 2], 1 < \zeta \leq 2, \\ w(1) = 0 & \delta w(2) = \delta w(1) = 0, \\ z(1) = 0 & \delta z(2) = \delta z(1) = 0. \end{cases} \tag{4.1}$$

Here $W, Z : [1, 2] \times \mathbb{R}^2 \rightarrow \mathcal{L}(\mathbb{R})$ are multivalued maps given by

$$W(y, w(y), z(y)) = \left[0, \frac{1}{3} \log|z(y)| + \frac{1}{(y+2)} \frac{|w(y)|}{1+|w(y)|} \right]. \tag{4.2}$$

$$Z(y, w(y), z(y)) = \left[0, \frac{1}{6} \arctan^2(|w(y)|) + \frac{\arctan(|z(y)|)}{(2y + 4)(1 + |z(y)|)} \right]. \tag{4.3}$$

Choose \mathcal{P} -functions by

$$\phi_1(u) = \begin{cases} \frac{u}{3} & 0 \leq u < \frac{3}{2}, \\ 0 & u \geq \frac{3}{2}, \end{cases}$$

and $\phi_2(u) = \frac{\phi_1(u)}{2} + \frac{1}{4}$ for all $u \in [0, \infty)$. It is obvious that $\phi_i(u)$, $i = 1, 2$, are \mathcal{P} -functions. Consider a sequence $\{\epsilon_n\} \subset [1, 2] \subset [0, \infty)$ given by

$$\epsilon_n = \begin{cases} \frac{3n}{2} & n < 1, \\ 0 & n \geq 1. \end{cases}$$

We obtain

$$\begin{aligned} H_\rho(W(y, w, z), W(y, \bar{w}, \bar{z})) &\leq \left| \frac{1}{3} (\log(z) - \log(\bar{z})) \right| + \left| \frac{1}{(y + 2)} \frac{w - \bar{w}}{(1 + w)(1 + \bar{w})} \right| \\ &\leq \frac{1}{3} (|z - \bar{z}| + |w - \bar{w}|) \\ &< \frac{\Gamma(\gamma + 1)}{\log(2)} \phi_1(\|w - \bar{w}\| + \|z - \bar{z}\|)(\|w - \bar{w}\| + \|z - \bar{z}\|) \end{aligned}$$

and

$$\begin{aligned} H_\rho(Z(y, w, z), Z(y, \bar{w}, \bar{z})) &\leq \left| \frac{1}{6} (\arctan^2(w) - \arctan^2(\bar{w})) \right| + \left| \frac{1}{2y + 4} \frac{\arctan(z) - \arctan(\bar{z})}{(1 + z)(1 + \bar{z})} \right| \\ &\leq \frac{1}{6} (|w - \bar{w}| + |z - \bar{z}|) \\ &< \frac{\Gamma(\zeta + 1)}{\log(2)} \phi_2(\|w - \bar{w}\| + \|z - \bar{z}\|)(\|w - \bar{w}\| + \|z - \bar{z}\|). \end{aligned}$$

Hence the condition (H2) holds for w, z, \bar{z} and $\bar{w} \in \mathbb{R}$ a.e. $1 < \gamma, \zeta \leq 2$. We see that $(\epsilon_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence; then

$$\begin{aligned} 0 \leq \sup_{n \in \mathbb{N}}(\phi_1(\epsilon_n)) &< \frac{1}{2} \quad \text{and} \quad 0 \leq \sup_{n \in \mathbb{N}}(\phi_2(\epsilon_n)) = \frac{1}{2} \sup_{n \in \mathbb{N}}(\phi_1(\epsilon_n)) + \frac{1}{4} < \frac{1}{2}, \\ \sup_{n \in \mathbb{N}} \alpha(\epsilon_n) &= \sup_{n \in \mathbb{N}}(\phi_1(\epsilon_n) + \phi_2(\epsilon_n)) < \frac{1}{2} + \frac{1}{2} < 1. \end{aligned}$$

It ensures that α is a function of a contractive factor, and thus verifies the Mizoguchi–Takahashi’s condition. We showed that all the hypotheses of Theorem 3.6 are fulfilled, then the system (4.1) with W and Z provided by (4.2) and (4.3) has at least one solution on $[1, 2]$.

5 Conclusions

This paper was focused on the existence theory of solutions for coupled fractional differential inclusions involving Caputo–Hadamard type fractional derivative equipped with

uncoupled boundary conditions. We make use of Mizoguchi–Takahashi’s fixed point theorem for multivalued maps to reach the desired results, which are well illustrated with the aid of an example. The technique developed in the present work can also be used to give results for boundary value problems of coupled fractional differential inclusions consisting of different types of fractional derivatives along with a variety of boundary value conditions.

Acknowledgements

The authors would like to thank the administration of their institutions for their support.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All the authors have equally made contributions in this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, University of Batna 2, 05078 Batna, Algeria. ²Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey. ³Department of Medical Research, China Medical University, Taichung 40402, Taiwan. ⁴Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia. ⁵Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

Publisher’s Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 20 January 2021 Accepted: 28 July 2021 Published online: 11 August 2021

References

1. Benchohra, M., Bouriah, S., Nieto, J.J.: Terminal value problem for differential equations with Hilfer–Katugampola fractional derivative. *Symmetry* **11**(5), 67 (2019)
2. Karapinar, E., Abdeljawad, T., Jarad, F.: Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Equ.* **2019**(1), 421 (2019)
3. Haq, F., Shah, K., Al-Mdallal, Q.M., Jarad, F.: Application of a hybrid method for systems of fractional order partial differential equations arising in the model of the one-dimensional Keller–Segel equation. *Eur. Phys. J. Plus* **134**(9), 461 (2019)
4. Belmor, S., Ravichandran, C., Jarad, F.: Nonlinear generalized fractional differential equations with generalized fractional integral conditions. *J. Taibah Univ. Sci.* (2020). <https://doi.org/10.1080/16583655.2019.1709265>
5. Panda, S.K., Ravichandran, C., Hazarika, B.: Results on system of Atangana–Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems. *Chaos Solitons Fractals* **142**, 110390 (2021)
6. Panda, S.K., Abdeljawad, T., Ravichandran, C.: Novel fixed point approach to Atangana–Baleanu fractional and L^p -Fredholm integral equations. *Alex. Eng. J.* **59**(4), 1959–1970 (2020)
7. Panda, S.K.: Applying fixed point methods and fractional operators in the modelling of novel coronavirus 2019-nCoV/SARS-CoV-2. *Results Phys.* **19**, 103433 (2020)
8. Ardjouni, A., Djoudi, A.: Existence and uniqueness of solutions for nonlinear implicit Caputo–Hadamard fractional differential equations with nonlocal conditions. *Adv. Theory Nonlinear Anal. Appl.* **3**(1), 46–52 (2019)
9. Ardjouni, A., Djoudi, A.: Existence and uniqueness of solutions for nonlinear hybrid implicit Caputo–Hadamard fractional differential equations. *Results Nonlinear Anal.* **2**(3), 136–142 (2019)
10. Karapinar, E., Binh, H.D., Luc, N.H., Can, N.H.: On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems. *Adv. Differ. Equ.* **2021**, 70 (2021)
11. Butzer, P.L., Kilbas, A.A., Trujillo, J.J.: Composition of Hadamard-type fractional integration operators and the semigroup property. *J. Math. Anal. Appl.* **269**(2), 387–400 (2002)
12. Mainardi, F.: *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. World Scientific, Singapore (2010)
13. Kilbas, A., Trujillo, J.: Hadamard-type integrals as G -transforms. *Integral Transforms Spec. Funct.* **14**(5), 413–427 (2003)
14. Podlubny, I., Chechkin, A., Skovranek, T., Chen, Y., Jara, B.M.V.: Matrix approach to discrete fractional calculus II: partial fractional differential equations. *J. Comput. Phys.* **228**(8), 3137–3153 (2009)
15. Jarad, F., Abdeljawad, T., Baleanu, D.: Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, 142 (2012)
16. Gambo, Y.Y., Jarad, F., Baleanu, D., Abdeljawad, T.: On Caputo modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2014**(1), 10 (2014)

17. Mordukhovich, B.S., Wang, L.: Optimal control of neutral functional-differential inclusions. *SIAM J. Control Optim.* **43**(1), 111–136 (2004)
18. Makarova, A.V.: On solvability of stochastic differential inclusions with current velocities II. *Glob. Stoch. Anal.* **2**(1), 101–112 (2012)
19. Jafarian, M.: Robust consensus of unicycles using ternary and hybrid controllers. *Int. J. Robust Nonlinear Control* **27**(17), 4013–4034 (2017)
20. Ben-Naim, E., Knight, J.B., Nowak, E.R., Jaeger, H.M., Nagel, S.R.: Slow relaxation in granular compaction. *Phys. D: Nonlinear Phenom.* **123**(1–4), 380–385 (1998)
21. Sayed, E., Ibrahim, A.G.: Multivalued fractional differential equations of arbitrary orders. *Appl. Math. Comput.* **68**, 15–25 (1995)
22. Yang, D., Bai, C.: Existence of solutions for anti-periodic fractional differential inclusions involving ψ -Riesz–Caputo fractional derivative. *Mathematics* **7**(7), 630 (2019)
23. Ahmad, B., Ntouyas, S.K.: Existence results for fractional differential inclusions arising from real estate asset securitization and HIV models. *Adv. Differ. Equ.* **2013**(1), 216 (2013)
24. Graef, J.R., Guerraiche, N., Hamani, S.: Initial value problems for fractional functional differential inclusions with Hadamard type derivatives in Banach spaces. *Math. Appl.* **13**, 27–40 (2018)
25. Ibrahim, A.G.: Differential equations and inclusions of fractional order with impulse effects in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **43**(1), 69–109 (2020)
26. Zhai, C., Ren, J.: A coupled system of fractional differential equations on the half-line. *Bound. Value Probl.* **2019**(1), 117 (2019)
27. Alsaedi, A., Baleanu, D., Etemad, S., Rezapour, S.: On coupled systems of time-fractional differential problems by using a new fractional derivative. *J. Funct. Spaces* **2016**, Article ID 4626940 (2016)
28. Ali, Z., Zada, A., Shah, K.: On Ulam's stability for a coupled systems of nonlinear implicit fractional differential equations. *Bull. Malays. Math. Sci. Soc.* **42**(5), 2681–2699 (2019)
29. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Coupled systems of fractional differential inclusions with coupled boundary conditions. *Electron. J. Differ. Equ.* **2019**, 69 (2019)
30. Hu, S., Papageorgiou, N.: *Handbook of Multivalued Analysis, Volume I: Theory*. Kluwer Academic, Dordrecht (1997)
31. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
32. Kisielewicz, M.: *Differential Inclusions and Optimal Control*. Kluwer Academic, Dordrecht (1991)
33. Pathak, H.K., Agarwal, R.P., Cho, Y.J.: Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions. *J. Comput. Appl. Math.* **283**, 201–217 (2015)
34. Mizoguchi, N., Takahashi, W.: Fixed point theorems for multivalued mappings on complete metric space. *J. Math. Anal. Appl.* **141**, 177–188 (1989)
35. Du, W.-S.: Some new results and generalizations in metric fixed point theory. *Nonlinear Anal.* **73**, 1439–1446 (2010)
36. Du, W.-S.: Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi–Takahashi's condition in quasiordered metric spaces. *Fixed Point Theory Appl.* **2010**, Article ID 876372 (2010)
37. Reiche, S.: Some fixed point problems. *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* **57**, 194–198 (1974)
38. Castaing, C., Valadier, M.: *Convex Analysis and Measurable Multifunctions*. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
