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On conformable fractional calculus

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ABSTRACT

Recently, the authors Khalil et al. (2014) introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. In this article we proceed on to develop the definitions there and set the basic concepts in this new simple interesting fractional calculus. The fractional versions of chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace transforms and linear differential systems are proposed and discussed.

1. Introduction

The fractional calculus [1–3] attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially during the last two decades apparently. Many researches started to deal with the discrete versions of this fractional calculus benefitting from the theory of time scales (see [4–8] and the references therein. The main idea behind setting this fractional calculus is summarized into two approaches. The first approach is the Riemann–Liouville which based on iterating the integral operator n times and then replaced it by one integral via the famous Cauchy formula where then $n!$ is changed to the Gamma function and hence the fractional integral of noninteger order is defined. Then integrals were used to define Riemann and Caputo fractional derivatives. The second approach is the Grünwald–Letnikov approach which based on iterating the derivative n times and then fractionalizing by using the Gamma function in the binomial coefficients. The obtained fractional derivatives in this calculus seemed complicated and lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. However, the semigroup properties of these fractional operators behave well in some cases. Recently, the author in [9] define a new well-behaved simple fractional derivative called "the conformable fractional derivative" depending just on the basic limit definition of the derivative. Namely, for a function $f : (0, \infty) \to \mathbb{R}$ the (conformable) fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{1}$$

and the fractional derivative at 0 is defined as $(T_\alpha f)(0) = \lim_{t \to 0^+} (T_\alpha f)(t)$.

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They then define the fractional derivative of higher order (i.e. of order $\alpha > 1$) as we will see below in next sections. They also define the fractional integral of order $0 < \alpha \leq 1$ only. They then proved the product rule, the fractional mean value theorem, and solved some (conformable) fractional differential equations where the fractional exponential function $e^{\frac{t^\alpha}{\alpha}}$ played an important rule. While in case of well-known fractional calculus Mittag-Leffler functions generalized exponential functions. In this article we continue to settle the basic definitions and concepts of this new theory motivated by the fact that there are certain functions which do not have Taylor power series representation or their Laplace transform cannot be calculated and so forth, but will be possible to do so by the help of the theory of this conformable fractional calculus. The article is organized as follows: In Section 2 the left and right (conformable) fractional derivatives and fractional integrals of higher orders are defined, the fractional chain rule and Gronwall inequality are obtained and the action of fractional derivatives and integrals to each other are discussed. The conformable and sequential conformable fractional derivatives of higher orders are discussed at the end points as well. In Section 3, two kinds of fractional integration by parts formulas when $0 < \alpha \leq 1$ are obtained where the usual integration by parts formulas in usual cases are reobtained when $\alpha \to 1$. In Section 4, the fractional power series expansions for certain functions that do not have Taylor power series representation in usual calculus are obtained and the fractional Taylor inequality is proved. Finally, in Section 5 the fractional Laplace transform is defined and used to solve a conformable fractional linear differential equation, where also the fractional Laplace of certain basic functions are calculated.

2. Basic definitions and tools

Definition 2.1. The (left) fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by

$$(T_\alpha^a f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon}. \tag{2}$$

When $a = 0$ we write T_α. If $(T_\alpha f)(t)$ exists on (a, b) then $(T_\alpha^a f)(a) = \lim_{t \to a^+} (T_\alpha^a f)(t)$.

The (right) fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$({}_\alpha^b T f)(t) = -\lim_{\epsilon \to 0} \frac{f(t + \epsilon(b-t)^{1-\alpha}) - f(t)}{\epsilon}. \tag{3}$$

If $({}^bT_\alpha f)(t)$ exists on (a, b) then $({}^bT_\alpha f)(b) = \lim_{t \to b^-} ({}^bT_\alpha f)(t)$.

Note that if f is differentiable then $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t)$ and $({}_\alpha^b T f)(t) = -(b-t)^{1-\alpha} f'(t)$. It is clear that the conformable fractional derivative of the constant function is zero. Conversely, if $T_\alpha f(t) = 0$ on an interval (a, b) then by the help of conformable fractional mean value theorem proved in [9] we can easily show that $f(x) = 0$ for all $x \in (a, b)$. Also, by the help of the fractional mean value theorem there we can show that if the conformable fractional derivative of a function f on an interval (a, b) is positive (negative) then the graph of f is increasing (decreasing) there.

Notation. $(I_\alpha^a f)(t) = \int_a^t f(x) d\alpha(x, a) = \int_a^t (x-a)^{\alpha-1} f(x) dx$. When $a = 0$ we write $d\alpha(x)$. Similarly, in the right case we have $({}^bI_\alpha f)(t) = \int_t^b f(x) d\alpha(b, x) = \int_t^b (b-x)^{\alpha-1} f(x) dx$. The operators I_α^a and ${}^bI_\alpha$ are called conformable left and right fractional integrals of order $0 < \alpha \leq 1$.

In the higher order case we can generalize to the following:

Definition 2.2. Let $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the (left) fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by

$$(\boldsymbol{T}_\alpha^a f)(t) = (T_\beta^a f^{(n)})(t). \tag{4}$$

When $a = 0$ we write $\boldsymbol{T}_\alpha$.

The (right) fractional derivative of order α terminating at b of f is defined by

$$({}_\alpha^b \boldsymbol{T} f)(t) = (-1)^{n+1} ({}_\beta^b T f^{(n)})(t). \tag{5}$$

Note that if $\alpha = n + 1$ then $\beta = 1$ and the fractional derivative of f becomes $f^{(n+1)}(t)$. Also when $n = 0$ (or $\alpha \in (0, 1)$) then $\beta = \alpha$ and the definition coincides with those in Definition 2.1.

Lemma 2.1 ([9]). *Assume that $f : [a, \infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$T_\alpha^a I_\alpha^a f(t) = f(t).$$

In the right case we can similarly prove:

Lemma 2.2. *Assume that $f : (-\infty, b] \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t < b$ we have*

$${}^bT_\alpha {}^bI_\alpha f(t) = f(t).$$

Next we give the definition of left and right fractional integrals of any order $\alpha > 0$.

Definition 2.3. Let $\alpha \in (n, n+1]$ then the left fraction integral of order α starting at a is defined by

$$(I_\alpha^a f)(t) = \boldsymbol{I}_{n+1}^a((t-a)^{\beta-1}f) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx \tag{6}$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_\alpha^a f)(t) = (\mathbf{I}_{n+1}^a f)(t) = \frac{1}{n!}\int_a^t (t-x)^n f(x)dx$, which is by means of Cauchy formula the iterative integral of f, $n+1$ times over $(a, t]$.

Recalling that the left Riemann–Liouville fractional integral of order $\alpha > 0$ starting from a is defined by

$$(\mathbf{I}_\alpha^a f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds, \tag{7}$$

we see that $(I_\alpha^a f)(t) = (\mathbf{I}_\alpha^a f)(t)$ for $\alpha = n+1,\ n = 0, 1, 2, \ldots$.

Example 2.4. Recalling that [1] $(\boldsymbol{I}_\alpha^a (t-a)^{\mu-1})(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}(x-a)^{\alpha+\mu-1},\ \alpha, \mu > 0$, we can calculate the (conformable) fractional integral of $(t-a)^\mu$ of order $\alpha \in (n, n+1]$. Indeed, if $\mu \in \mathbb{R}$ such that $\alpha + \mu - n > 0$ then

$$(I_\alpha^a (t-a)^\mu)(x) = (\boldsymbol{I}_{n+1}^a (t-a)^{\mu+\alpha-n-1})(x) = \frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(x-a)^{\alpha+\mu}. \tag{8}$$

Analogously, we can find the (conformable) right fractional integral of such functions. Namely,

$$({}^b I_\alpha (b-t)^\mu)(x) = ({}^b\boldsymbol{I}_{n+1}(t-a)^{\mu+\alpha-n-1})(x) = \frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(b-x)^{\alpha+\mu}, \tag{9}$$

where $\mu \in \mathbb{R}$ such that $\alpha + \mu - n > 0$.

From the above discussion, we notice that the Riemann fractional integrals and conformable fractional integrals of polynomial functions are different up to a constant multiple and coincide for natural orders.

The following semigroup property relates the composition operator $I_\mu I_\alpha$ and the operator $I_{\alpha+\mu}$.

Proposition 2.3. *Let $f : [a, \infty) \to \mathbb{R}$ be a function and $0 < \alpha,\ \mu \leq 1$ be such that $1 < \alpha + \mu \leq 2$. Then*

$$(I_\alpha I_\mu f)(t) = \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}(I_{\alpha+\mu} f)(t) - \frac{t}{\mu}\int_0^t s^{\alpha+\mu-2} f(s)ds. \tag{10}$$

Proof. Interchanging the order of integrals and noting that

$$(I_{\alpha+\mu} f)(t) = (\mathbf{I}_2 s^{\alpha+\mu-2} f(s))(t) = \int_0^t (t-s)s^{\alpha+\mu-2}ds, \tag{11}$$

we see that

$$\begin{aligned}
(I_\alpha I_\mu f)(t) &= \int_0^t \left(\int_0^{t_1} f(s)s^{\alpha-1}ds\right) t_1^{\mu-1}dt_1 \\
&= \int_0^t f(s)s^{\alpha-1}\left(\int_s^t t_1^{\mu-1}dt_1\right)ds \\
&= \int_0^t f(s)s^{\alpha-1}\left[\frac{t^\mu}{\mu} - \frac{s^\mu}{\mu}\right]ds \\
&= \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}\left[(I_{\alpha+\mu} f)(t) - t\int_0^t s^{\alpha+\mu-2} f(s)ds\right]. \quad \square
\end{aligned} \tag{12}$$

Notice that if in (10) we let $\alpha, \mu \to 1$ we verify that $(I_1 I_1 f)(t) = (I_2 f)(t)$.

Recalling the action of the Q-operator on fractional integrals ($Qf(t) = f(a+b-t),\ f : [a, b] \to \mathbb{R}$) on Riemann left and right fractional integrals, we see that:

$$Q\mathbf{I}_\alpha^a f(t) = {}^b\mathbf{I}_\alpha Qf(t). \tag{13}$$

Indeed, for $\alpha \in (n, n+1]$ we have

$$QI_\alpha^a f(t) = Q\mathbf{I}_{n+1}^a((t-a)^{\alpha-n-1}f(t)) = {}^b\mathbf{I}_{n+1}((b-t)^{\alpha-n-1}f(a+b-t)) = {}^b I_\alpha Qf(t), \tag{14}$$

Now we give the generalized version of Lemma 2.1.

Lemma 2.4. *Assume that $f : [a, \infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$\mathbf{T}_\alpha^a I_\alpha^a f(t) = f(t).$$

Proof. From the definition we have

$$\mathbf{T}_\alpha^a I_\alpha^a f(t) = T_\beta^a \left(\frac{d^n}{dt^n} I_\alpha^a f(t) \right) = T_\beta^a \left(\frac{d^n}{dt^n} I_{n+1}^a ((t-a)^{\beta-1} f(t)) \right) = T_\beta^a (I_1^a ((t-a)^{\beta-1} f(t))) \tag{15}$$

That it is $\mathbf{T}_\alpha^a I_\alpha^a f(t) = T_\beta^a I_\beta^a f(t)$ and hence the result follows by Lemma 2.1. □

Similarly we can generalize Lemma 2.2. Indeed,

Lemma 2.5. *Assume that $f : (-\infty, b] \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t < b$ we have*

$${}^b\mathbf{T}_\alpha {}^b I_\alpha f(t) = f(t).$$

Lemma 2.6. *Let $f, h : [a, \infty) \to \mathbb{R}$ be functions such that T_α^a exists for $t > a$, f is differentiable on (a, ∞) and $T_\alpha^a f(t) = (t-a)^{1-\alpha} h(t)$. Then $h(t) = f'(t)$ for all $t > a$.*

The proof follows by definition and setting $h = \epsilon (t-a)^{1-\alpha}$ so that $h \to 0$ as $\epsilon \to 0$. As a result of Lemma 2.6 we can state

Corollary 2.7. *Let $f : [a, b) \to \mathbb{R}$ be such that $(I_\alpha^a T_\alpha^a) f(t)$ exists for $b > t > a$. Then, $f(t)$ is differentiable on (a, b).*

Lemma 2.8. *Let $f : (a, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a (f)(t) = f(t) - f(a). \tag{16}$$

Proof. Since f is differentiable then by the help of Theorem 2.1 (6) in [9] we have

$$I_\alpha^a T_\alpha^a (f)(t) = \int_a^t (x-a)^{\alpha-1} T_\alpha(f)(x) dx = \int_a^t (x-a)^{\alpha-1} (x-a)^{1-\alpha} f'(x) dx = f(t) - f(a). \quad \square \tag{17}$$

Lemma 2.8 can be generalized for the higher order as follows.

Proposition 2.9. *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times differentiable for $t > a$. Then, for all $t > a$ we have*

$$I_\alpha^a \mathbf{T}_\alpha^a (f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}. \tag{18}$$

Proof. From definition and Theorem 2.1 (6) in [9] we have

$$I_\alpha^a \mathbf{T}_\alpha^a (f)(t) = I_{n+1}^a ((t-a)^{\beta-1} T_\beta^a f^{(n)}(t)) = I_{n+1}^a ((t-a)^{\beta-1} (t-a)^{1-\beta} f^{(n+1)}(t)) = I_{n+1}^a f^{(n+1)}(t). \tag{19}$$

Then integration by parts gives (20). □

Analogously, in the right case we have

Proposition 2.10. *Let $\alpha \in (n, n+1]$ and $f : (-\infty, b] \to \mathbb{R}$ be $(n+1)$ times differentiable for $t < b$. Then, for all $t < b$ we have*

$${}^b I_\alpha {}^b \mathbf{T}_\alpha (f)(t) = f(t) - \sum_{k=0}^{n} \frac{(-1)^k f^{(k)}(b)(b-t)^k}{k!}. \tag{20}$$

In particular, if $n = 0$ or $0 < \alpha \leq 1$, then ${}^b I_\alpha {}^b T_\alpha (f)(t) = f(t) - f(b)$.

Theorem 2.11 (*Chain Rule*)**.** *Assume $f, g : (a, \infty) \to \mathbb{R}$ be (left) α-differentiable functions, where $0 < \alpha \leq 1$. Let $h(t) = f(g(t))$. Then $h(t)$ is (left) α-differentiable and for all t with $t \neq a$ and $g(t) \neq 0$ we have*

$$(T_\alpha^a h)(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}. \tag{21}$$

If $t = a$ we have

$$(T_\alpha^a h)(a) = \lim_{t \to a^+} (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}. \tag{22}$$

Proof. By setting $u = t + \epsilon(t-a)^{1-\alpha}$ in the definition and using continuity of g we see that

$$\begin{aligned} T_\alpha^a h(t) &= = \lim_{u\to t} \frac{f(g(u)) - f(g(t))}{(u-t)} t^{1-\alpha} \\ &= \lim_{u\to t} \frac{f(g(u)) - f(g(t))}{(g(u)-g(t))} \cdot \lim_{u\to t} \frac{g(u)-g(t)}{u-t} t^{1-\alpha} \\ &= \lim_{g(u)\to g(t)} \frac{f(g(u)) - f(g(t))}{(g(u)-g(t))} \cdot g(t)^{1-\alpha} \cdot T_\alpha^a g(t) \cdot g(t)^{\alpha-1} \\ &= (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}. \quad \square \end{aligned} \tag{23}$$

Proposition 2.12. *Let $f : [a,\infty) \to \infty$ be twice differentiable on (a,∞) and $0 < \alpha,\ \beta \le 1$ such that $1 < \alpha + \beta \le 2$. Then*

$$(T_\alpha^a T_\beta^a f)(t) = T_{\alpha+\beta}^a f(t) + (1-\beta)(t-a)^{-\beta} T_\alpha^a f(t). \tag{24}$$

Proof. By the fractional product rule and that f is twice differentiable we have

$$\begin{aligned} (T_\alpha^a T_\beta^a f)(t) &= t^{1-\alpha} \frac{d}{dt}[t^{1-\beta}(t-a)^{-\beta} f'(t)] \\ &= t^{1-\alpha}[t^{1-\beta} f''(t) + (1-\beta)(t-a)^{-\beta} f'(t)] \\ &= T_{\alpha+\beta}^a f(t) + (1-\beta)(t-a)^{-\beta} T_\alpha^a f(t). \quad \square \end{aligned} \tag{25}$$

Note that in (24) if we let $\alpha, \beta \to 1$ then we have $T_\alpha^a T_\beta f(t) = T_2 f(t) = f''(t)$.

Next we prove a fractional version of *Gronwall inequality* which will be useful is studying stability of (conformable) fractional systems.

Theorem 2.13. *Let r be a continuous, nonnegative function on an interval $J = [a,b]$ and δ and k be nonnegative constants such that*

$$r(t) \le \delta + \int_a^t k r(s)(s-a)^{\alpha-1} ds \quad (t \in J).$$

Then for all $t \in J$

$$r(t) \le \delta e^{k\frac{(t-a)^\alpha}{\alpha}}.$$

Proof. Define $R(t) = \delta + \int_a^t kr(s)(s-a)^{\alpha-1}ds = \delta + I_\alpha^a(kr(s))(t)$. Then $R(a) = \delta$ and $R(t) \ge r(t)$, and

$$T_\alpha^a R(t) - kR(t) = kr(t) - kR(t) \le kr(t) - kr(t) = 0. \tag{26}$$

Multiply (26) by $K(t) = e^{-k\frac{(t-a)^\alpha}{\alpha}}$. By the help of chain rule in Theorem 2.11 we see that $T_\alpha^a K(t) = -kK(t)$ and hence by the product rule we conclude that $T_\alpha^a(K(t)R(t)) \le 0$. Since $K(t)R(t)$ is differentiable on (a,b) then Lemma 2.8 implies that

$$I_\alpha^a T_\alpha^a(K(t)R(t)) = K(t)R(t) - K(a)R(a) = K(t)R(t) - \delta \le 0. \tag{27}$$

Hence

$$r(t) \le R(t) \le \frac{\delta}{K(t)} = \delta e^{k\frac{(t-a)^\alpha}{\alpha}} \tag{28}$$

which completes the proof. $\square$

Finally, in this section we discuss the conformable fractional derivative at a in the left case and at b in the right case for some smooth functions. Let $n-1 < \alpha < n$ and assume $f : [a,\infty) \to \mathbb{R}$ be such that $f^{(n)}(t)$ exists and continuous. Then, $(\mathbf{T}_\alpha^a f)(t) = (T_{\alpha+1-n}^a f^{(n-1)})(t) = (t-a)^{n-\alpha} f^{(n)}(t)$ and thus $(\mathbf{T}_\alpha^a f)(a) = \lim_{t\to a^+}(t-a)^{n-\alpha} f^{(n)}(t) = 0$. Similarly, in the right case we have $({}^b\mathbf{T}_\alpha f)(b) = \lim_{t\to b^-}(b-t)^{n-\alpha} f^{(n)}(t) = 0$, for $(-\infty, b] \to \mathbb{R}$ with $f^{(n)}(t)$ exists and continuous. Now, let $0 < \alpha < 1$ and $n \in \{1,2,3,\ldots\}$ then the left (right) sequential conformable fractional derivative of order n is defined by

$${}^{(n)}T_\alpha^a f(t) = \underbrace{T_\alpha^a T_\alpha^a \ldots T_\alpha^a}_{n-times} f(t) \tag{29}$$

and

$$ {}^{b}T_{\alpha}^{(n)}f(t) = \underbrace{{}^{b}T_{\alpha}\,{}^{b}T_{\alpha}\ldots{}^{b}T_{\alpha}}_{n-times} f(t), \tag{30}$$

respectively. If $f : [a,\infty) \to \mathbb{R}$ is second continuously differentiable and $0 < \alpha \leq \frac{1}{2}$ then direct calculations show that

$$ {}^{(2)}T_{\alpha}^{a}(t) = T_{\alpha}^{a}T_{\alpha}^{a}f(t) = \begin{cases} (1-\alpha)(t-a)^{1-2\alpha}f'(t) + (t-a)^{2-2\alpha}f''(t) & \text{if } t > a, \\ 0 & \text{if } t = a. \end{cases}$$

Similarly, in the right case, for $f : (-\infty, b] \to \mathbb{R}$ is second continuously differentiable and $0 < \alpha \leq \frac{1}{2}$ then direct calculations show that

$$ {}^{b}T_{\alpha}^{(2)}(t) = {}^{b}T_{\alpha}\,{}^{b}T_{\alpha}f(t) = \begin{cases} (1-\alpha)(b-t)^{1-2\alpha}f'(t) + (b-t)^{2-2\alpha}f''(t) & \text{if } t < b, \\ 0 & \text{if } t = b. \end{cases}$$

This shows that the second order sequential conformable fractional derivative may not be continuous even f is second continuously differentiable for $\frac{1}{2} < \alpha < 1$. If we proceed inductively, then we can see that if f is n-continuously continuously differentiable and $0 < \alpha \leq \frac{1}{n}$ then the n-th order sequential conformable fractional derivative is continuous and vanishes at the end points (a in the left case and b in the right case).

3. Integration by parts

Theorem 3.1. *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$ \int_a^b f(x)T_{\alpha}^{a}(g)(x)d\alpha(x,a) = fg|_a^b - \int_a^b g(x)T_{\alpha}^{a}(f)(x)d\alpha(x,a) \tag{31}$$

The proof followed by Lemma 2.8 applied to fg and Theorem 2.1 (3) in [9].

The following integration by parts formula is given by means of left and right fractional integrals.

Proposition 3.2. *Let $f, g : [a, b] \to \mathbb{R}$ be functions and $0 < \alpha \leq 1$. Then*

$$ \int_a^b (I_{\alpha}^{a}f)(t)g(t)d_{\alpha}(b,t) = \int_a^b f(t)({}^{b}I_{\alpha}g)(t)d_{\alpha}(t,a). \tag{32}$$

Proof. From definition we get

$$ \int_a^b (I_{\alpha}^{a}f)(t)g(t)d_{\alpha}(t,a) = \int_a^b (\int_a^t (x-a)^{\alpha-1}f(x)dx)g(t)(b-t)^{\alpha-1}dt. \tag{33}$$

Interchanging the order of integrals we reach at

$$ \int_a^b (I_{\alpha}^{a}f)(t)g(t)d_{\alpha}(b,t) = \int_a^b f(x)({}^{b}I_{\alpha}g)(x)d_{\alpha}(x,a)$$

which completes the proof. □

Next we employ Proposition 3.2 to prove an integration by parts formula by means of left and right fractional derivatives.

Theorem 3.3. *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions and $0 < \alpha \leq 1$. Then*

$$ \int_a^b (T_{\alpha}^{a}f)(t)g(t)d_{\alpha}(t,a) = \int_a^b f(t)({}^{b}T_{\alpha}g)(t)d_{\alpha}(b,t) + f(t)g(t)|_a^b. \tag{34}$$

Proof. By Proposition 2.10 and that g is differentiable, we have

$$ \int_a^b (T_{\alpha}^{a}f)(t)g(t)d_{\alpha}(t,a) = \int_a^b (T_{\alpha}^{a}f)(t)\,{}^{b}I_{\alpha}\,{}^{b}T_{\alpha}g(t)d_{\alpha}(t,a) + g(b)\int_a^b (T_{\alpha}^{a}f)(t)d_{\alpha}(t,a). \tag{35}$$

Applying Proposition 3.2 leads to

$$ \int_a^b (T_{\alpha}^{a}f)(t)g(t)d_{\alpha}(t,a) = \int_a^b (I_{\alpha}^{a}T_{\alpha}^{a}f)(t)\,{}^{b}T_{\alpha}g(t)d_{\alpha}(b,t) + g(b)(I_{\alpha}^{a}T_{\alpha}^{a}f)(a). \tag{36}$$

Then the proof is completed by the help of Lemma 2.8 by substituting $(I_{\alpha}^{a}T_{\alpha}^{a}f)(t) = f(t) - f(a)$ using that f is differentiable and by the help of Proposition 2.10 and that g is differentiable by substituting $({}^{b}I_{\alpha}\,{}^{b}T_{\alpha}g)(t) = g(t) - g(b)$. □

Remark 3.1. Notice that if in Theorem 3.1 or Theorem 3.3 we let $\alpha \to 1$ then we obtain the integration by parts formula in usual calculus, where we have to note that $d_\alpha(t,a) \to dt$, $d_\alpha(b,t) \to dt$, $T_\alpha^a f(t) \to f'(t)$ and ${}^bT_\alpha f(t) \to -f'(t)$ as $\alpha \to 1$.

In Theorems 3.1 and 3.3 we needed some differentiability conditions. We next define some function spaces on which the obtained integration by parts formulas are still valid.

Definition 3.1. For $0 < \alpha \le 1$ and an interval $[a,b]$ define

$$I_\alpha([a,b]) = \{f : [a,b] \to \mathbb{R} : f(x) = (I_\alpha^a \psi)(x) + f(a), \quad \texttt{for some } \psi \in L_\alpha(a)\},$$

and

$${}^\alpha I([a,b]) = \{g : [a,b] \to \mathbb{R} : g(x) = ({}^bI_\alpha \varphi)(x) + g(b), \quad \texttt{for some } \varphi \in L_\alpha(b)\},$$

where

$$L_\alpha(a) = \{\psi : [a,b] \to \mathbb{R}\} : (I_\alpha^a \psi)(x) \texttt{ exists for all } x \in [a,b]\},$$

and

$$L_\alpha(b) = \{\varphi : [a,b] \to \mathbb{R}\} : ({}^bI_\alpha \varphi)(x) \texttt{ exists for all } x \in [a,b]\}.$$

Lemma 3.4. *Let $f, g : [a,b] \to \mathbb{R}$ be functions and $0 < \alpha \le 1$. Then*

(a) *If f is left (g is right) α-differentiable then $f \in I_\alpha([a,b])$ ($g \in {}_\alpha I([a,b])$).*
(b) *If $f \in I_\alpha([a,b])$ with $f(x) = (I_\alpha^a \psi)(x) + f(a)$ where ψ is continuous then $\psi(x) = T_\alpha^a f(x)$ and $(I_\alpha^a T_\alpha^a f)(x) = f(x) - f(a)$.*
(c) *If $g \in {}_\alpha I([a,b])$ with $g(x) = ({}^bI_\alpha \varphi)(x) + g(b)$ where φ is continuous then $\varphi(x) = {}^bT_\alpha g(x)$ and $({}^bI_\alpha {}^bT_\alpha g)(x) = g(x) - g(b)$.*

Proof. The proof of (a) follows by Lemma 2.8 and Proposition 2.10 by choosing $\psi(t) = T_\alpha^a f$ and $\varphi(t) = {}^bT_\alpha g$. The proof of (b) follows by Lemma 2.1 and the fact that the left α-derivative of constant function is zero. The proof of (c) follows by Lemma 2.2 and the fact that the right α-derivative of constant function is zero. □

Theorem 3.5. *Let $f, g : [a,b] \to \mathbb{R}$ be functions such that $f \in I_\alpha([a,b])$ with $\psi(t)$ is continuous and $g \in {}_\alpha I([a,b])$ with $\varphi(t)$ is continuous and $0 < \alpha \le 1$. Then*

$$\int_a^b (T_\alpha^a f)(t) g(t) d_\alpha(t,a) = \int_a^b f(t)({}^bT_\alpha g)(t) d_\alpha(b,t) + f(t)g(t)|_a^b. \tag{37}$$

Proof. The proof is similar to that in Theorem 3.3 where we make use of (b) and (c) in Lemma 3.4. □

4. Fractional power series expansions

Certain functions, being not infinitely differentiable at some point, do not have Taylor power series expansion there. In this section we set the fractional power series expansions so that those functions will have fractional power series expansions.

Theorem 4.1. *Assume f is an infinitely α-differentiable function, for some $0 < \alpha \le 1$ at a neighborhood of a point t_0. Then f has the fractional power series expansion:*

$$f(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{1/\alpha},\ R > 0. \tag{38}$$

Here $(T_\alpha^{t_0} f)^{(k)}(t_0)$ means the application of the fractional derivative k times.

Proof. Assume $f(t) = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + c_3(t-t_0)^{3\alpha} + \cdots,\ t_0 < t < t_0 + R^{1/\alpha},\ R > 0.$ □

Then, $f(t_0) = c_0$. Apply $T_\alpha^{t_0}$ to f and evaluate at t_0 we see that $(T_\alpha^{t_0} f)(t_0) = c_1\alpha$ and hence $c_1 = \frac{(T_\alpha^{t_0} f)(t_0)}{\alpha}$. Proceeding inductively and applying $T_\alpha^{t_0}$ to f n-times and evaluating at t_0 we see that $(T_\alpha^{t_0} f)^{(n)}(t_0) = c_n.\alpha(2\alpha)\cdots(n\alpha) = \alpha^n.n!$ and hence

$$c_n = \frac{(T_\alpha^{t_0} f)^{(n)}(t_0)}{\alpha^n.n!}. \tag{39}$$

Hence (38) is obtained and the proof is completed.

Proposition 4.2 (*Fractional Taylor Inequality*). *Assume f is an infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point t_0 has the Taylor power series representation (38) such that $|(T_\alpha^a f)^{n+1}| \leq M, \quad M > 0$ for some $n \in \mathbb{N}$. Then, for all $(t_0, t_0 + R)$*

$$|R_n^\alpha(t)| \leq \frac{M}{\alpha^{n+1}(n+1)!}(t - t_0)^{\alpha(n+1)}, \tag{40}$$

where $R_n^\alpha(t) = \sum_{k=n+1}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!} = f(x) - \sum_{k=0}^{n} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}$.

The proof is similar to that in usual calculus by applying $I_\alpha^{t_0}$ instead of integration.

Example 4.1. Consider the fractional exponential function $f(t) = e^{\frac{(t-t_0)^\alpha}{\alpha}}$, where $0 < \alpha < 1$. The function $f(t)$ is clearly not differentiable at t_0 and thus it does not have Taylor power series representation about t_0. However, $(T_\alpha^{t_0} f)^{(n)}(t_0) = 1$ for all n and hence

$$f(t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^{k\alpha}}{\alpha^k k!}. \tag{41}$$

The ratio test shows that this series is convergent to f on the interval $[t_0, \infty)$.

Example 4.2. The functions $g(t) = \sin \frac{(t-t_0)^\alpha}{\alpha}$ and $h(t) = \sin \frac{(t-t_0)^\alpha}{\alpha}$ do not have Taylor power series expansions about $t = t_0$ for $0 < \alpha < 1$ since they are not differentiable there. However, by the help of Eq. (38) and that $T_\alpha^{t_0} \sin \frac{(t-t_0)^\alpha}{\alpha} = \cos \frac{(t-t_0)^\alpha}{\alpha}$ and $T_\alpha^{t_0} \cos \frac{(t-t_0)^\alpha}{\alpha} = -\sin \frac{(t-t_0)^\alpha}{\alpha}$ we can see that

$$\sin \frac{(t - t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty} (-1)^k \frac{(t - t_0)^{(2k+1)\alpha}}{\alpha^{(2k+1)}(2k+1)!}, \quad t \in [t_0, \infty) \tag{42}$$

and

$$\cos \frac{(t - t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty} (-1)^k \frac{(t - t_0)^{(2k)\alpha}}{\alpha^{(2k)}(2k)!}, \quad t \in [t_0, \infty). \tag{43}$$

Example 4.3. The function $f(x) = \frac{1}{1 - \frac{t^\alpha}{\alpha}}$ does not have Taylor power series representation about $t = 0$ for $0 < \alpha < 1$, since it is not differentiable there. However, by the help of Eq. (38) we can see that

$$\frac{1}{1 - \frac{t^\alpha}{\alpha}} = \sum_{k=0}^{\infty} t^{\alpha k}, \quad t \in [0, 1) \tag{44}$$

or more generally,

$$\frac{1}{1 - \frac{(t-t_0)^\alpha}{\alpha}} = \sum_{k=0}^{\infty} (t - t_0)^{\alpha k}, \quad t \in [t_0, t_0 + 1). \tag{45}$$

Remark 4.1. In case the function f is defined on $(-\infty, a)$ and not differentiable at a then we search for its (conformal) right fractional order derivatives ${}^aT_\alpha$ at a for some $0 < \alpha \leq 1$ and use it for our fractional Taylor series on some $(a - R, a), \ R > 0$. For example the functions $\frac{(a-t)^\alpha}{\alpha}$, $\sin \frac{(a-t)^\alpha}{\alpha}$ and so on.

5. The fractional Laplace transform

In this section we will define the fractional Laplace transform and use it to solve some linear fractional equations to produce the fractional exponential function. Then, we use the method of successive approximation to verify the solution by making use of the fractional power series representation discussed in the above section. Also we shall calculate the Laplace transform for certain (fractional) type functions.

Definition 5.1. Let $t_0 \in \mathbb{R}$, $0 < \alpha \leq 1$ and $f : [t_0, \infty) \to$ be real valued function. Then the fractional Laplace transform of order α starting from a of f is defined by

$$L_\alpha^{t_0}\{f(t)\}(s) = F_\alpha^{t_0}(s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t) d\alpha(t, t_0) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t - t_0)^{\alpha-1} dt. \tag{46}$$

Theorem 5.1. *Let $a \in \mathbb{R}$, $0 < \alpha \leq 1$ and $f : (a, \infty) \to$ be differentiable real valued function. Then*

$$L_\alpha^a\{T_\alpha(f)(t)\}(s) = sF_\alpha(s) - f(a). \tag{47}$$

Proof. The proof followed by definition, Theorem 2.1(6) in [9] and the usual integration by parts. □

Example 5.2. Consider the conformable fractional initial value problem:

$$(T_\alpha^a y)(t) = \lambda y(t), \quad y(a) = y_0, \ t > a, \tag{48}$$

where the solution is assumed to be differentiable on (a, ∞).

Apply the operator I_α^a to the above equation to obtain

$$y(t) = y_0 + \lambda(I_\alpha^a y)(t). \tag{49}$$

Then

$$y_{n+1} = y_0 + \lambda(I_\alpha^a y_n)(t), \quad n = 0, 1, 2, \ldots. \tag{50}$$

For $n = 0$ we see that

$$y_1 = y_0 + \lambda y_0 \frac{(t-a)^\alpha}{\alpha} = y_0\left(1 + \lambda\frac{(t-a)^\alpha}{\alpha}\right). \tag{51}$$

For $n = 1$ we see that

$$y_2 = y_0\left[1 + \lambda\frac{(t-a)^\alpha}{\alpha} + \lambda^2\frac{(t-a)^{2\alpha}}{\alpha(2\alpha)}\right]. \tag{52}$$

If we proceed inductively we conclude that

$$y_n = y_0 \sum_{k=0}^{n} \frac{\lambda^k (t-a)^{k\alpha}}{\alpha^k k!}. \tag{53}$$

Letting $n \to \infty$ we see that

$$y(t) = y_0 \sum_{k=0}^{\infty} \frac{\lambda^k (t-a)^{k\alpha}}{\alpha^k k!}. \tag{54}$$

Which is clearly the fractional Taylor power series representation of the (fractional) exponential function $y_0 e^{\lambda\frac{(t-a)^\alpha}{\alpha}}$.

The following lemma relates the fractional Laplace transform to the usual Laplace transform.

Lemma 5.2. *Let $f : [t_0, \infty) \to \mathbb{R}$ be a function such that $L_\alpha^{t_0}\{f(t)\}(s) = F_\alpha^{t_0}(s)$ exists. Then*

$$F_\alpha^{t_0}(s) = \mathfrak{L}\{f(t_0 + (\alpha t)^{1/\alpha})\}(s), \tag{55}$$

where $\mathfrak{L}\{g(t)\}(s) = \int_0^\infty e^{-st} g(t)dt$.

The proof follows easily by setting $u = \frac{(t-t_0)^\alpha}{\alpha}$.

Example 5.3. In this example we calculate the fractional Laplace for certain functions.

- $L_\alpha^{t_0}\{1\}(s) = \frac{1}{s}, \ s > 0$
- $L_\alpha^{t_0}\{t\}(s) = \mathfrak{L}\{t_0 + (\alpha t)^{1/\alpha}\}(s) = \frac{t_0}{s} + \alpha^{1/\alpha}\frac{\Gamma(1+\frac{1}{\alpha})}{s^{1+\frac{1}{\alpha}}}, \ s > 0.$
- $L_\alpha^0\{t^p\}(s) = \frac{\alpha^{p/\alpha}}{s^{1+p/\alpha}}\Gamma(1 + \frac{1}{\alpha}), \ s > 0.$
- $L_\alpha^0\{e^{\frac{t^\alpha}{\alpha}}\}(s) = \frac{1}{s-1}, \ s > 1.$
- $L_\alpha^0\{\sin\omega\frac{t^\alpha}{\alpha}\}(s) = \mathfrak{L}\{\sin\omega t\}(s) = \frac{1}{\omega^2+s^2}.$
- $L_\alpha^0\{\cos\omega\frac{t^\alpha}{\alpha}\}(s) = \mathfrak{L}\{\cos\omega t\}(s) = \frac{s}{\omega^2+s^2}.$
- $L_\alpha^{t_0}\{e^{-k\frac{(t-t_0)^\alpha}{\alpha}} f(t)\}(s) = \mathfrak{L}\{e^{-kt} f(t_0 + (\alpha t)^{\frac{1}{\alpha}})\}$. For example $L_\alpha^0\{e^{-k\frac{t^\alpha}{\alpha}} \sin\frac{t^\alpha}{\alpha}\}(s) = \mathfrak{L}\{e^{-kt}\sin t\}(s) = \frac{1}{(s+k)^2+1}$ and $L_\alpha^{t_0}\{e^{\lambda\frac{(t-t_0)^\alpha}{\alpha}}\} = \mathfrak{L}\{e^{\lambda t}\} = \frac{1}{s-\lambda}.$

Notice that in the above example there are some functions, with $0 < \alpha < 1$, whose usual Laplace is not easy to be calculated. However, their fractional Laplace can be easily calculated.

Example 5.4. We use the fractional Laplace transform to verify the solution of the conformable fractional initial value problem:

$$(T_\alpha^a y)(t) = \lambda y(t), \quad y(a) = y_0, \ t > a, \tag{56}$$

where the solution is assumed to be differentiable on (a, ∞).

Apply L_α^a and use (47) to conclude that

$$L_\alpha^a\{y(t)\}(s) = \frac{y_0}{s-\lambda}, \tag{57}$$

and hence, $y(t) = y_0 e^{\lambda \frac{(t-a)^\alpha}{\alpha}}$.

Finally, we use the fractional fundamental exponential matrix to express the solution of (conformable) fractional linear systems.

Consider the system

$$T_\alpha^a \mathbf{y}(t) = A\mathbf{y}(t) + \mathbf{f}(t), \quad 0 < \alpha \le 1, \tag{58}$$

where $\mathbf{y}, \mathbf{f} : [a, b) \to \mathbb{R}^n$ are vector functions and A is an $n \times n$ matrix. The general solution of the fractional non-homogeneous system (58) is expressed by

$$\mathbf{y}(t) = e^{A\frac{(t-a)^\alpha}{\alpha}}\mathbf{c} + \int_a^t e^{A\frac{(t-a)^\alpha}{\alpha}} e^{-A\frac{(s-a)^\alpha}{\alpha}} \mathbf{f}(s)(s-a)^{1-\alpha} ds, \tag{59}$$

where $e^{A\frac{(t-a)^\alpha}{\alpha}} = \sum_{k=0}^\infty \frac{A^k (t-a)^{k\alpha}}{\alpha^k k!}$ and $\mathbf{c}$ is a constant vector.

6. Some conclusions and comparisons

1. The conformable fractional derivative behaves well in the product rule and chain rule while complicated formulas appear in case of usual fractional calculus.
2. The conformable fractional derivative of a constant function is zero while it is not the case for Riemann fractional derivatives.
3. Mittag-Leffler functions play important rule in fractional calculus as a generalization to exponential functions while the fractional exponential function $f(t) = e^{\frac{t^\alpha}{\alpha}}$ appears in case of conformable fractional calculus.
4. Conformable fractional derivatives, conformable chain rule, conformable integration by parts, conformable Gronwall's inequality, conformable exponential function, conformable Laplace transform and so forth, all tend to the corresponding ones in usual calculus.
5. In case of usual calculus there some functions that do not have Taylor power series representations about certain points but in the theory of conformable fractional they do have.
6. *Open problem:* Is it hard to fractionalize the conformable fractional calculus, either by iterating the conformable fractional derivative (Grünwald–Letnikov approach) or by iterating the conformable fractional integral of order $0 < \alpha \le 1$ (Riemann approach)? Notice that when $\alpha = 0$ we obtain Hadamard type fractional integrals.

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