Miskolc Mathematical Notes
Vol. 21 (2020), No. 1, pp. 127–141
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.2871

ON CONTRACTIONS VIA SIMULATION FUNCTIONS ON EXTENDED b-METRIC SPACES

CRISTIAN CHIFU AND ERDAL KARAPINAR

Received 27 February, 2019

Abstract. In this paper, we introduce the notion of an admissible extended $\mathcal{Z}$-contraction mapping in the setting of extended b-metric spaces. As an application, we consider Ulam stability problems based on our contractions. The presented results cover several existing results in the literature.

2010 *Mathematics Subject Classification:* 46T99; 54H25; 47H10

Keywords: simulation function, $\mathcal{Z}$-contraction, b-metric space, extended b-metric space, fixed point

1. INTRODUCTION AND PRELIMINARIES

The notion of the distance has been discussed from Euclid period. In the beginning of the nineteen century it was axiomatically formulated under the name of metric, by Fréchet and Haussdorff. Since then, the axioms of the metric notions have been relaxed several ways to generalize it. Among all, we mention the concept of a b-metric (see [12, 15]) which was also announced as "quasi-metric" (see [23]).

Definition 1 (Czerwik [15]). Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions:

$(b1)$ $d(x,y) = 0$ if and only if $x = y$;
$(b2)$ $d(x,y) = d(y,x)$ for all $x,y \in X$;
$(b3)$ $d(x,y) \le s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$, where $s \ge 1$.

The function d is called a b-metric and the space (X,d) is called a b-metric space, in short, bMS.

Example 1. Let $X = \{x_i : 1 \le i \le M\}$ for some $M \in \mathbb{N}$ and $s \ge 2$. Define $d : X \times X \to \infty$ as

$$d(x_i,x_j) = \begin{cases} 0 & \text{if } i = j, \\ s & \text{if } |i-j| = 1, \\ 2 & \text{if } |i-j| = 2, \\ 1 & otherwise. \end{cases}$$

Accordingly, we obtain that $d(x_i,x_j) \leq \frac{s}{2}[d(x_i,x_k)+d(x_k,x_j)]$, for all $i,j,k \in \{1,M\}$. As a result, the pair (X,d) forms a b-metric for $s>2$. Note that the standard triangle inequality does not hold.

Example 2 (See e.g. [23]). The space $L^p[0,1]$ (where $0<p<1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x,y) := (\int_0^1 |x(t)-y(t)|^p dt)^{1/p}, \text{ for each } x,y \in L^p[0,1],$$

is a b-metric space. Notice that $s=2^{1/p}$.

Example 3 (See e.g. [23]). Let E be a Banach space and 0_E be the zero vector of E. Let P be a cone in E with $int(P) \neq \varnothing$ and $\preceq$ be a partial ordering with respect to P. Let X be a non-empty set. Suppose the mapping $d: X \times X \to E$ satisfies:

$(M1)$ $0 \preceq d(x,y)$ for all $x,y \in X$;
$(M2)$ $d(x,y)=0$ if and only if $x=y$;
$(M3)$ $d(x,y) \preceq d(x,z)+d(z,y)$, for all $x,y \in X$;
$(M4)$ $d(x,y)=d(y,x)$ for all $x,y \in X$.

Then d is called a cone metric on X, and the pair (X,d) is called a cone metric space (CMS).

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D: X \times X \to [0,\infty)$ be defined by $D(x,y)=||d(x,y)||$, where $d: X \times X \to E$ is a cone metric space. Then (X,D) is a b-metric space with a constant $s:=K \geq 1$.

Czerwik [15] proved the analogue of Banach fixed point theorem. For (common) fixed point results on b-metric spaces, see [1–4, 7, 9–11, 13, 14, 20, 22, 24].

Recently, Kamran [17] introduced a new type of generalized metric spaces and they proved some fixed point theorems on this space.

Definition 2 ([17]). Let X be a non empty set and $\theta: X \times X \to [1,\infty)$. A function $d_\theta: X \times X \to [0,\infty)$ is called an extended $b-$metric if for all $x,y,z \in X$ is satisfies

$(d_\theta 1)$ $d_\theta(x,y)=0$ if and only if $x=y$;
$(d_\theta 2)$ $d_\theta(x,y)=d_\theta(y,x)$;
$(d_\theta 3)$ $d_\theta(x,y) \leq \theta(x,y)\,[d_\theta(x,z)+d_\theta(z,y)]$.

The pair (X,d_θ) is called an extended $b-$metric space, in short extended-bMS.

Remark 1. If $\theta(x,y)=s$, for $s \geq 1$ then we obtain the definition of bMS. Note that neither $b-$metric nor extended $b-$metric is continuous. Throughout the paper, we presume that all considered an extended $b-$metrics are continuous.

Example 4. Let $p \in (0,1)$, $q>1$ and $X=l_p(\mathbb{R}) \cup l_q(\mathbb{R})$ equipped with the metric

$$d(x,y)=\begin{cases} d_p(x,y) & \text{if } x,y \in l_p(\mathbb{R}), \\ d_q(x,y) & \text{if } x,y \in l_q(\mathbb{R}), \\ 0 & otherwise, \end{cases}$$

where

$$l_r(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^r < \infty \right\} \text{ for } r = p,q.$$

and

$$d_r(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^r \right)^{1/r}, \text{ for } r = p,q.$$

It is clear that (X,d) forms an extended b-metric with

$$\theta(x,y) = \begin{cases} 2^{1/p} & \text{if } x,y \in l_p(\mathbb{R}), \\ 2^{1/q} & \text{if } x,y \in l_q(\mathbb{R}), \\ 1 & \text{otherwise.} \end{cases}$$

Example 5 ([17]). Let $X = \{1,2,3\}, \theta : X \times X \to [1,\infty)$ and $d_\theta : X \times X \to [0,\infty)$ as $\theta(x,y) = 1+x+y$ and

$$d_\theta(1,1) = d_\theta(2,2) = d_\theta(3,3) = 0, \quad \text{and} \quad d_\theta(1,2) = d_\theta(2,1) = 80,$$
$$d_\theta(1,3) = d_\theta(3,1) = 1000, \quad \text{and} \quad d_\theta(2,3) = d_\theta(3,2) = 600.$$

Example 6 ([5]). Let $X = [0,1], \theta : X \times X \to [1,\infty)$ and $d_\theta : X \times X \to [0,\infty)$ as $\theta(x,y) = \dfrac{1+x+y}{x+y}$ and

$$d_\theta(x,y) = \frac{1}{xy}, x,y \in (0,1], x \neq y,$$
$$d_\theta(x,y) = 0, x,y \in [0,1], x = y,$$
$$d_\theta(x,0) = d_\theta(0,x) = \frac{1}{x}, x \in (0,1].$$

Some fundamental concepts like convergence, Cauchy sequence and completeness in a extended-bMS are defined as follows [17].

Definition 3 ([17]). Let (X,d_θ) be an extended-bMS.

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x) < \varepsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_m,x_n) < \varepsilon$, for all $m,n \geq N$.

Definition 4. An extended-bMS (X,d_θ) is complete if every Cauchy sequence in X is convergent.

Lemma 1. *Let (X,d_θ) be an complete extended-bMS. If d_θ is continuous, then every convergent sequence has a unique limit.*

Throughout the paper, we presume that all extended-b-metrics are continuous.

The notion of comparison functions is defined by Rus [23] and it has been extensively studied by a number of authors to get more general forms of contractive mappings.

Definition 5 ([23])**.** A function $\phi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\phi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0,\infty)$, where ϕ^n is the n-th iterate of ϕ.

Properties and examples of comparison functions can be found in [23]. An important property of comparison functions is given by the following Lemma.

Lemma 2 ([23])**.** *If $\phi : [0,\infty) \to [0,\infty)$ is a comparison function, then*

(1) *each iterate ϕ^k of ϕ, $k \geq 1$ is also a comparison function;*
(2) *ϕ is continuous at 0;*
(3) *$\phi(t) < t$ for all $t > 0$.*

Definition 6 ([23])**.** Let $s \geq 1$ be a real number. A function $\phi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if ϕ is increasing and

(*) there exist $k_0 \in \mathbb{N}$, $a \in [0,1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq a s^k \phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

The collection of all (b)-comparison functions will be denoted by Ψ. In the literature, a (b)-comparison function is called (c)-comparison functions when $s = 1$. It can be shown that a (c)-comparison function is a comparison function, but the converse is not true in general. Berinde [23] also proved the following important property of (b)-comparison functions.

Lemma 3 ([23])**.** *Let $\phi : [0,\infty) \to [0,\infty)$ be a (b)-comparison function. Then*

(1) *the series $\sum_{k=0}^{\infty} s^k \phi^k(t)$ converges for any $t \in [0,\infty)$;*
(2) *the function $b_s : [0,\infty) \to [0,\infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k \phi^k(t)$ is increasing and is continuous at $t = 0$.*

Remark 2. Any (b)-comparison function ϕ satisfies $\phi(t) < t$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

In order to unify several existing fixed point results in the literature, Khojasteh *et al.* [19] introduced the notion of *simulation functions* and investigate the existence and uniqueness of a fixed point for different types of contractive mappings.

Definition 7. A *simulation function* is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1): $\zeta(t,s) < s-t$ for all $t,s>0$;
(ζ_2): if $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0, \text{ then } \limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

In [19], the condition $\zeta(0,0)=0$ was added, but Argoubi et al. [8] dropped it. Let $\mathcal{Z}$ denote the family of all simulation functions $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$, that is, verifying (ζ_1) and (ζ_2).

Due to the axiom (ζ_1), we have

$$\zeta(t,t) < 0 \text{ for all } t>0.$$

The following example is derived from [6, 16, 19].

Example 7. Let $\phi_i : [0,\infty)\to[0,\infty)$ be continuous functions such that $\phi_i(t)=0$ if and only if, $t=0$. For $i=1,2,3,4,5,6$, we define the mappings $\zeta_i : [0,\infty)\times[0,\infty)\to\mathbb{R}$, as follows

(i): $\zeta_1(t,s)=\phi_1(s)-\phi_2(t)$, for all $t,s\in[0,\infty)$, where $\phi_1,\phi_2 : [0,\infty)\to[0,\infty)$ are two continuous functions such that $\phi_1(t)=\phi_2(t)=0$ if and only if $t=0$ and $\phi_1(t)<t\le\phi_2(t)$ for all $t>0$.
(ii): $\zeta_2(t,s)=s-\dfrac{f(t,s)}{g(t,s)}t$, for all $t,s\in[0,\infty)$, where $f,g : [0,\infty)\times[0,\infty)\to(0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s)>g(t,s)$, for all $t,s>0$.
(iii): $\zeta_3(t,s)=s-\phi_3(s)-t$, for all $t,s\in[0,\infty)$.
(iv): $\zeta_4(t,s)=s\varphi(s)-t$, for all $s,t\in[0,\infty)$, where $\varphi : [0,\infty)\to[0,1)$ is a function such that $\limsup_{t\to r^+}\varphi(t)<1$, for all $r>0$.
(v): $\zeta_5(t,s)=\eta(s)-t$, for all $s,t\in[0,\infty)$, where $\eta : [0,\infty)\to[0,\infty)$ is an upper semi-continuous mapping such that $\eta(t)<t$, for all $t>0$ and $\eta(0)=0$.
(vi): $\zeta_6(t,s)=s-\int_0^t\phi(u)du$, for all $s,t\in[0,\infty)$, where $\phi : [0,\infty)\to[0,\infty)$ is a function such that $\int_0^\varepsilon\phi(u)du$ exists and $\int_0^\varepsilon\phi(u)du>\varepsilon$, for each $\varepsilon>0$.

It is clear that each function ζ_i $(i=1,2,3,4,5,6)$ forms a simulation function.

Suppose (X,d) is a metric space, T is a self-mapping on X and $\zeta\in\mathcal{Z}$. We say that T is a *$\mathcal{Z}$-contraction* with respect to ζ [19], if

$$\zeta(d(T(x),T(y)),d(x,y))\ge 0, \text{ for all } x,y\in X.$$

Theorem 1 ([17]). *Let (X,d_θ) be an extended-bMS such that d_θ is a continuous functional. Let $T : X\to X$ satisfy:*

$$d_\theta(T(x),T(y))\le kd_\theta(x,y), \tag{1.1}$$

for all $x,y\in X$, where $k\in[0,1)$ be such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{k}$, here $x_n=T^n(x_0)$, $n=1,2,\ldots$. Then T has precisely one fixed point u. Moreover, for each $y\in X$, $T^n(y)\to u$.

The notion of α-admissible mappings and the concept of triangular α-admissible mappings [18] are reconsidered and refined by Popescu [21] in the following way:

Definition 8 ([21])**.** Let $\alpha : X \times X \to [0,\infty)$ be a mapping and $X \neq \varnothing$. A self-mapping $T : X \to X$ is said to be an $\alpha-$orbital admissible if for all $s \in X$, we have

$$\alpha(s,T(s)) \geq 1 \Rightarrow \alpha(T(s),T^2(s)) \geq 1.$$

Furthermore, an $\alpha-$orbital admissible mapping T is called triangular α-orbital admissible if

(TO) $\alpha(s,t) \geq 1$ and $\alpha(s,T(t)) \geq 1$ implies that $\alpha(s,T(t)) \geq 1$, for all $s,t \in X$.

In this paper, we investigate the existence of a fixed point of an admissible extended $\mathcal{Z}$-contraction in the context of extended b-metric spaces.

2. Main results

We start this section by introducing nonlinear contractive mappings in the setting of extended b-metric spaces as follows.

Definition 9. Let (X,d) be an extended-bMS and $\theta : X \times X \to [1,\infty)$. A mapping $T : X \to X$ is called an admissible extended $\mathcal{Z}$-contraction mapping if there is a $\zeta \in \mathcal{Z}$ such that

$$\zeta(\alpha(x,y)d_\theta(T(x),T(y)),\phi(M(x,y))) \geq 0, \text{ for all } x,y \in X, \tag{2.1}$$

where $\phi \in \Psi$ and

$$M(x,y) = \max\{d_\theta(x,y), d_\theta(x,T(x)), d_\theta(y,T(y))\}. \tag{2.2}$$

Remark 3. Note that for admissible extended $\mathcal{Z}$-contraction mappings we have

$$\alpha(x,y)d_\theta(Tx,Ty) \leq \phi(M(x,y)) \text{ for all } x,y \in X. \tag{2.3}$$

In what follows we shall express the main theorem of this paper.

Theorem 2. *Let (X,d_θ) be a complete extended b-metric space and $T : X \to X$ a mapping. Suppose that exists a sequence $(q_n)_{n\in\mathbb{N}}$, $q_n > 1$, for all $n \in \mathbb{N}$ such that $\theta(x_n,x_m) < q_n$, for all $m > n$. If T is an admissible extended $\mathcal{Z}$-contraction mapping satisfying*

- (i) *T is triangular $\alpha-$orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq 1$;*
- (iii) *T is continuous,*

then it has a fixed point u. In the case of existence of a fixed point u, we have $T^n(y) \to u$ for each $y \in X$.

Proof. Due to (ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define an recursive sequence $\{x_n\}$ by $x_n = T^n(x_0)$ for $n \in \mathbb{N}$. If for some $n_0 \in \mathbb{N}$, we have $x_{n_0} = x_{n_0+1} =$

$T(x_{n_0})$, then x_{n_0} is a fixed point of T. From now on, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

On the other hand, employing (ii) and regarding that T is $\alpha-$orbital admissible, we find, by iteration, that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.4}$$

Furthermore, by taking (i) into account, we deduce also

$$\alpha(x_n, x_m) \geq 1, \text{ for all } m > n. \tag{2.5}$$

By letting $x = x_n$ and $y = x_{n+1}$ in (2.3) and keeping (2.4) in mind, we derive

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &= d_\theta(T(x_{n-1}), T(x_n)) \\ &\leq \alpha(x_{n-1}, x_n) d_\theta(T(x_{n-1}), T(x_n)) \leq \phi(M(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, T(x_{n-1})), d_\theta(x_n, T(x_n))\} \\ &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \\ &= \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \end{aligned}$$

If for some $n \in \mathbb{N}$, we get

$$M(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1}),$$

then

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),$$

which is a contradiction. Thus, for all $n \geq 1$,

$$M(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_{n-1}, x_n).$$

Hence

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_{n-1}, x_n)) < d_\theta(x_{n-1}, x_n), \quad \forall\, n \geq 1. \tag{2.6}$$

We deduce

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi^n(d_\theta(x_0, x_1)), \quad \forall\, n \geq 0. \tag{2.7}$$

Letting $n \to \infty$ in (2.7), because ϕ is a (b)-comparison function, we get

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

We claim that (x_n) is a Cauchy sequence. For all $m > n$,

$$\begin{aligned} d_\theta(x_n, x_m) &\leq \theta(x_n, x_m)(d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)) \\ &\leq \theta(x_n, x_m) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d_\theta(x_{n+1}, x_{n+2}) \\ &\quad + \ldots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-1}, x_m) d_\theta(x_{m-1}, x_m) \\ &\leq \theta(x_n, x_m)\phi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\phi^{n+1}(d_\theta(x_0, x_1)) \\ &\quad + \ldots + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-1}, x_m)\phi^{m-1}(d_\theta(x_0, x_1)) \\ &\leq \theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\phi^n(d_\theta(x_0, x_1)) \end{aligned}$$

$$+\theta(x_1,x_m)\theta(x_2,x_m)...\theta(x_{n+1},x_m)\phi^{n+1}(d_\theta(x_0,x_1))$$
$$+...+\theta(x_1,x_m)\theta(x_2,x_m)...\theta(x_{m-1},x_m)\phi^{m-1}(d_\theta(x_0,x_1)).$$

Choose for all $n\in\mathbb{N}$

$$S_n=\sum_{j=1}^{n}\phi^j(d_\theta(x_0,x_1))\prod_{i=1}^{j}\theta(x_i,x_m).$$

We deduce that

$$d_\theta(x_n,x_m)\leq S_{m-1}-S_{n-1},\quad \forall\, m>n.$$

Consider the series

$$\sum_{n=1}^{\infty}\phi^n(d_\theta(x_0,x_1))\prod_{i=1}^{n}\theta(x_i,x_m).$$

Let $q=\max\{q_1,...,q_n\}$. We have

$$u_n=\phi^n(d_\theta(x_0,x_1))\prod_{i=1}^{j}\theta(x_i,x_m)\leq\phi^n(d_\theta(x_0,x_1))q^n=v_n.$$

From Lemma 3, we have that the series $\sum\limits_{j=0}^{\infty}\phi^j(d_\theta(x_0,x_1))q^j$ converges. Using comparison criteria for the convergence of series, we obtain that

$$\sum_{n=1}^{\infty}\phi^n(d_\theta(x_0,x_1))\prod_{i=1}^{n}\theta(x_i,x_m)$$

converges, and hence

$$\lim_{n,m\to\infty}d_\theta(x_n,x_m)=0$$

that is, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (X,d) is a complete extended b-metric space, there exists $z\in X$ such that

$$\lim_{n\to\infty}d_\theta(x_n,z)=0.$$

Since T is continuous, we derive that

$$\lim_{n\to\infty}d_\theta(T(x_n),T(z))=0=\lim_{n\to\infty}d_\theta(x_{n+1},T(z))=d_\theta(z,T(z)).$$

Regarding the uniqueness of the limit, we conclude that $Tz=z$. □

Theorem 3. *Let (X,d_θ) be a complete extended b-metric space and $T:X\to X$ a mapping. Suppose that exists a sequence $(q_n)_{n\in\mathbb{N}}$, $q_n>1$, for all $n\in\mathbb{N}$ such that $\theta(x_n,x_m)<q_n$, for all $m>n$. If T is an admissible extended $\mathcal{Z}$-contraction mapping satisfying*

- (i) *T is triangular α-orbital admissible;*
- (ii) *there exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\geq 1$;*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k*.*

then it has a fixed point u*. In the case of existence of a fixed point* u*, we have* $T^n(y) \to u$ *for each* $y \in X$*.*

Proof. Following the proof of Theorem 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ for all $n \geq 0$, converges for some $u \in X$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.1), for all k, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n(k)}, u) d_\theta(T x_{n(k)}, Tu), \phi(M(x_{n(k)}, u))) \\ &= \zeta(\alpha(x_{n(k)}, u) d_\theta(x_{n(k)+1}, Tu), \phi(M(x_{n(k)}, u))) \\ &< M(x_{n(k)}, u) - \alpha(x_{n(k)}, u) d(x_{n(k)+1}, Tu), \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_\theta(x_{n(k)+1}, T(u)) &= d_\theta(T\left(x_{n(k)}\right), T(u)) \\ &\leq \alpha(x_{n(k)}, u) d_\theta(T\left(x_{n(k)}\right), T(u)) \\ &\leq \phi(M(x_{n(k)}, u)) \\ &= \phi(\max\{d_\theta(x_{n(k)}, u), d_\theta(x_{n(k)}, T\left(x_{n(k)}\right)), d_\theta(u, T(u))\}). \end{aligned}$$

Letting $\limsup$ as $k \to \infty$ in the equality above, we have $d_\theta(u, Tu) = 0$, that is, $u = Tu$. □

For the uniqueness of a fixed point of a α-admissible $\mathcal{Z}$*-contraction* with respect to ζ, we shall suggest the following hypothesis.

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 4. *Adding condition* (U) *to the hypotheses of Theorem 1 (resp. Theorem 3), we obtain that* u *is the unique fixed point of* T*.*

Proof. We shall prove that z is unique. Assume that z and w two fixed points of T with $z \neq w$. By (2.1),

$$\begin{aligned} d_\theta(z, w) &= d_\theta(T(z), T(w)) \leq \alpha(z, w) d_\theta(T(z), T(w)) \\ &\leq \phi(M(z, w)) = \phi(\max\{d_\theta(z, w), d_\theta(z, T(z)), d_\theta(w, T(w))\}) \\ &= \phi(d_\theta(z, w)) < d_\theta(z, w), \end{aligned}$$

which is a contradiction. So the fixed point of T is unique. □

3. Ulam stability and well-posedness of the fixed point problem

Definition 10. Let (X,d_θ) be an extended b-metric space and $T:X\to X$ be an operator. The fixed point problem

$$x=T(x),\ x\in X, \tag{3.1}$$

is called generalized Ulam stable if and only if there exists $\psi:\mathbb{R}_+\to\mathbb{R}_+$, increasing, continuous in 0 and $\psi(0)=0$, such that, for each $\varepsilon>0$ and for each solution $y^*\in X$ of the inequality

$$d_\theta(y,T(y))\le\varepsilon, \tag{3.2}$$

there exists a solution x^* of the fixed point problem (3.1) such that

$$d_\theta(x^*,y^*)\le\psi(\varepsilon).$$

Theorem 5. *Let (X,d_θ) be a complete extended b-metric space and $T:X\to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

(i) for any solution $y^*\in X$ of (3.2), we have that $\alpha(x_n,y^*)\ge 1$ and $\lim_{n\to\infty}\theta(x_{n+1},y^*)<q=\max\{q_1,...,q_n\}$, where $x_n=T^n(x_0)$, $n\in\mathbb{N}$;

(ii) $\beta:[0,\infty)\to[0,\infty)$, $\beta(r):=r-q\phi(r)$ is strictly increasing and onto;

(iii) ϕ is continuous.

Then, the fixed point problem (3.1) is generalized Ulam stable.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^*\in X$. Let $y^*\in X$ be a solution of (3.2). We have

$$\begin{aligned} d_\theta(x^*,y^*) &= \lim_{n\to\infty} d_\theta(x_{n+1},y^*)=\lim_{n\to\infty} d_\theta(T(x_n),y^*) \\ &\le \lim_{n\to\infty}\theta(T(x_n),y^*)\left[d_\theta(T(x_n),T(y^*))+d_\theta(y^*,T(y^*))\right] \end{aligned} \tag{3.3}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$\begin{aligned} 0 &\le \zeta(\alpha(x_n,y^*)d_\theta(T(x_n),T(y^*)),\phi(M(x_n,y^*))) \\ &< \phi(M(x_n,y^*))-\alpha(x_n,y^*)d_\theta(T(x_n),T(y^*)) \\ &\le \phi(M(x_n,y^*))-d_\theta(T(x_n),T(y^*)). \end{aligned}$$

Hence, we have

$$d_\theta(T(x_n),T(y^*))\le\phi(M(x_n,y^*)).$$

Accordingly, the inequality (3.3) becomes

$$d_\theta(x^*,y^*)\le\lim_{n\to\infty}\theta(T(x_n),y^*)\left[\phi(M(x_n,y^*))+\varepsilon\right]$$

$$d_\theta(x^*,y^*)\le q\lim_{n\to\infty}\phi(M(x_n,y^*))+q\varepsilon \tag{3.4}$$

Now, we consider

$$M(x_n,y^*)=\max\{d_\theta(x_n,y^*),d_\theta(x_n,T(x_n)),d_\theta(y^*,T(y^*))\}$$

$$M(x_n,y^*) \leq \max\{d_\theta(x_n,y^*), d_\theta(x_n,T(x_n)), \varepsilon\}$$

$$\lim_{n\to\infty} M(x_n,y^*) \leq \max\left\{\lim_{n\to\infty} d_\theta(x_n,y^*), \varepsilon\right\} = \max\{d_\theta(x^*,y^*), \varepsilon\}$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = \varepsilon$, then (3.3) turns into

$$d_\theta(x^*,y^*) \leq q\phi(\varepsilon) + q\varepsilon = \psi(\varepsilon).$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = d_\theta(x^*,y^*)$, then (3.3) becomes

$$d_\theta(x^*,y^*) \leq q\phi(d_\theta(x^*,y^*)) + q\varepsilon$$

That is $\beta(d_\theta(x^*,y^*)) \leq q\varepsilon$. Since β is strictly increasing and onto, we have

$$d_\theta(x^*,y^*) \leq \beta^{-1}(q\varepsilon) = \psi(\varepsilon),$$

and the proof is complete. □

Theorem 6. *Let (X,d_θ) be a complete extended b-metric space and $T: X \to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

(i) for any solution $y^* \in X$ of (3.2), there exists a solution x^* of (3.1) such that $\alpha(x^*,y^*) \geq 1$ and $\theta(x^*,y^*) < q = \max\{q_1,...,q_n\}$;

(ii) $\beta: [0,\infty) \to [0,\infty)$, $\beta(r) := r - q\phi(r)$ is strictly increasing and onto;

Then, the fixed point problem (3.1) is generalized Ulam stable.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^* \in X$. Let $y^* \in X$ be a solution of (3.2). We have

$$\begin{aligned} d_\theta(x^*,y^*) &\leq \theta(x^*,y^*)[d_\theta(T(x^*),T(y^*)) + d_\theta(y^*,T(y^*))] \\ &\leq \theta(x^*,y^*)[d_\theta(T(x^*),T(y^*)) + \varepsilon]. \end{aligned} \tag{3.5}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$d_\theta(T(x^*),T(y^*)) \leq \phi(M(x^*,y^*)).$$

Attendantly, the inequality (3.5) turns into

$$d_\theta(x^*,y^*) \leq q[\phi(M(x^*,y^*)) + \varepsilon] \tag{3.6}$$

Now, we shall consider

$$M(x^*,y^*) = \max\{d_\theta(x^*,y^*), d_\theta(x^*,T(x^*)), d_\theta(y^*,T(y^*))\}$$
$$M(x_n,y^*) \leq \max\{d_\theta(x^*,y^*), \varepsilon\}.$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = \varepsilon$, then (3.6) becomes

$$d_\theta(x^*,y^*) \leq q\phi(\varepsilon) + q\varepsilon = \psi(\varepsilon).$$

If $\max\{d_\theta(x^*,y^*), \varepsilon\} = d_\theta(x^*,y^*)$, then (3.6) turns into

$$d_\theta(x^*,y^*) \leq q\phi(d_\theta(x^*,y^*)) + q\varepsilon$$

That is

$$\beta\left(d_{\theta}\left(x^{*},y^{*}\right)\right)\leq q\varepsilon.$$

Since β is strictly increasing and onto, we have

$$d_{\theta}\left(x^{*},y^{*}\right)\leq\beta^{-1}\left(q\varepsilon\right)=\psi\left(\varepsilon\right),$$

and the proof is complete. ☐

Definition 11. Let (X,d_{θ}) be an extended b-metric space with constant and $T:X\to X$ be an operator. The fixed point problem (19) is well-posed if

- (i) $FixT=\{x^{*}\}$;
- (ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $d_{\theta}\left(x_n,T\left(x_n\right)\right)\to 0$, as $n\to\infty$, then $x_n\to x^{*}$, as $n\to\infty$.

Theorem 7. *Let (X,d_{θ}) be a complete extended b-metric space and $T:X\to X$ a mapping. Suppose that all the hypotheses of Theorem 2 (or Theorem 3) hold. Additionally we suppose that*

- (i) *for any solution $x^{*}\in X$ of (3.1) , we have that $\alpha\left(x_n,x^{*}\right)\geq 1$ and $\lim\limits_{n\to\infty}\theta\left(x_n,x^{*}\right)<q=\max\left\{q_1,...,q_n\right\}$, where $x_n=T^{n}\left(x_0\right)$, $n\in\mathbb{N}$;*
- (ii) *$\beta:[0,\infty)\to[0,\infty)$, $\beta(r):=r-q\phi(r)$ is strictly increasing and onto;*

Then, the fixed point problem (3.1) is well-posed.

Proof. Since the conditions of Theorem 2 hold, the fixed point problem (3.1) has a unique solution $x^{*}\in X$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $d\left(x_n,T\left(x_n\right)\right)\to 0$, as $n\to\infty$.

$$d_{\theta}\left(x_n,x^{*}\right)\leq\theta\left(x_n,x^{*}\right)\left[d_{\theta}\left(x_n,T\left(x_n\right)\right)+d_{\theta}\left(T\left(x_n\right),T\left(x^{*}\right)\right)\right].\tag{3.7}$$

Since T is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$\begin{aligned}0&\leq\zeta(\alpha(x_n,x^{*})d_{\theta}(T\left(x_n\right),T\left(x^{*}\right)),\phi(M(x_n,x^{*})))\\&<\phi(M(x_n,x^{*}))-\alpha(x_n,x^{*})d_{\theta}(T\left(x_n\right),T\left(x^{*}\right))\\&\leq\phi(M(x_n,x^{*}))-d_{\theta}(T\left(x_n\right),T\left(x^{*}\right)).\end{aligned}$$

Hence

$$d_{\theta}(T\left(x_n\right),T\left(x^{*}\right))\leq\phi(M(x_n,x^{*})).$$

(3.7) yields

$$d_{\theta}\left(x_n,x^{*}\right)\leq\theta\left(x_n,x^{*}\right)\left[d_{\theta}\left(x_n,T\left(x_n\right)\right)+\phi(M(x_n,x^{*}))\right]\tag{3.8}$$

Now

$$M(x_n,y^{*})=\max\left\{d_{\theta}\left(x_n,x^{*}\right),d_{\theta}\left(x_n,T\left(x_n\right)\right),d_{\theta}\left(x^{*},T\left(x^{*}\right)\right)\right\}=d_{\theta}\left(x_n,x^{*}\right)$$

Hence, (3.8) implies

$$d_{\theta}\left(x_n,x^{*}\right)\leq\theta\left(x_n,x^{*}\right)\left[d_{\theta}\left(x_n,T\left(x_n\right)\right)+\phi(d_{\theta}(x_n,x^{*}))\right]\tag{3.9}$$

Letting $n \to \infty$ in (3.9) we obtain

$$\lim_{n\to\infty} [d_\theta(x_n, x^*) - q\phi(d_\theta(x_n, x^*))] \le 0$$

which is

$$\lim_{n\to\infty} \beta(d_\theta(x_n, x^*)) = 0.$$

From here

$$\lim_{n\to\infty} d(x_n, x^*) = 0,$$

and the proof is complete. □

In what follows we shall give a data dependence result.

Theorem 8. *Let (X, d_θ) be a complete extended b-metric space and $T_1, T_2 : X \to X$ two mappings such that*

(i) *The conditions of Theorem 2 (or Theorem 3) hold for T_1;*
(ii) *$FixT_2 \neq \varnothing$;*
(iii) *For any $x^* \in FixT_1$ and $y^* \in FixT_2$ we have $\alpha(x^*, y^*) \ge 1$ and $\theta(x^*, y^*) < q = \max\{q_1, ..., q_n\}$;*
(iv) *There exits $\eta > 0$, such that $d_\theta(T_1(x), T_2(y)) \le \eta$, for all $x, y \in X$;*

Then, we have

$$d_\theta(x^*, y^*) \le \sup\{t \in \mathbb{R}_+ | t - q\varphi(t) \le 2q\eta\}.$$

Proof. Since the conditions of Theorem 2 hold for T_1, there exists a unique fixed point x^* of T_1, and because $FixT_2 \neq \varnothing$, there exists y^* a fixed point of T_2. So, we have

$$\begin{aligned} d_\theta(x^*, y^*) &\le \theta(x^*, y^*)[d_\theta(T_1(x^*), T_1(y^*)) + d_\theta(T_1(y^*), T_2(y^*))] \\ &\le \theta(x^*, y^*)[d_\theta(T_1(x^*), T_1(y^*)) + \eta]. \end{aligned} \tag{3.10}$$

Since T_1 is an admissible extended $\mathcal{Z}$-contraction mapping, we have

$$d_\theta(T_1(x^*), T_1(y^*)) \le \phi(M(x^*, y^*)).$$

(3.10) implies that

$$d_\theta(x^*, y^*) \le q[\phi(M(x^*, y^*)) + \eta] \tag{3.11}$$

Consider now

$$\begin{aligned} M(x^*, y^*) &= \max\{d_\theta(x^*, y^*), d_\theta(x^*, T_1(x^*)), d_\theta(y^*, T_1(y^*))\} \\ M(x_n, y^*) &\le \max\{d_\theta(x^*, y^*), \eta\}. \end{aligned}$$

If $\max\{d_\theta(x^*, y^*), \eta\} = \eta$, then (3.11) turns into

$$d_\theta(x^*, y^*) \le q\phi(\eta) + q\eta < 2q\eta. \tag{3.12}$$

If $\max\{d_\theta(x^*, y^*), \eta\} = d_\theta(x^*, y^*)$, then (3.11) becomes

$$d_\theta(x^*, y^*) \le q\phi(d_\theta(x^*, y^*)) + q\eta < q\phi(d_\theta(x^*, y^*)) + 2q\eta. \tag{3.13}$$

From (3.12) or (3.13) we reach the conclusion. □

4. Discussion and Conclusion

It wouldn't be wrong to say that our results yields several consequences most of which were announced in the literature. More precisely, by substituting the mapping ζ in a proper way like in the Example 7, we find a number of corollaries. On the other hand, by proper choice of the auxiliary function α together with the selection of ϕ in Theorem 1-Theorem 4, we are enable to derive some more existing fixed point theorems in the various settings (in the context of *partially ordered set endowed with a metric*, in the setting of *cyclic contraction* etc.). We omit the details since they are straightforward.

References

[1] H. Afshari, M. Atapour, and H. Aydi, "Generalized $\alpha-\psi$-geraghty multivalued mappings on b-metric spaces endowed with a graph," *TWMS Journal of Applied and Engineering Mathematics*, vol. 7, no. 2, pp. 248–260, 2017.

[2] H. Afshari, S. Kalantari, and H. Aydi, "Fixed point results for generalized $\alpha-\psi$-suzuki-contractions in quasi-b-metric-like spaces," *Asian-European Journal of Mathematics*, vol. 11, 2018, doi: 10.1142/S1793557118500122.

[3] U. Aksoy, E. Karapinar, and I. M. Erhan, "Fixed points of generalized α-admissible contractions on b-metric spaces with an application to boundary value problems," *Journal of Nonlinear and Convex Analysis*, vol. 17, no. 6, pp. 1095–1108, 2016.

[4] N. Alharbi, H. Aydi, A. Felhi, C. Ozel, and S. Sahmim, "α-contractive mappings on rectangular b-metric spaces and an application to integral equations," *Journal of Mathematical Analysis*, vol. 9, no. 3, pp. 47–60, 2018.

[5] B. Alqahtani, A. Fulga, and E. Karapinar, "Common fixed point results on extended b-metric space," *Journal of Inequalities and Applications*, vol. 2018:158, 2018, doi: 10.1186/s13660-018-1745-4.

[6] H. H. Alsulami, E. Karapinar, F. Khojasteh, and A. F. R.-L. de Hierro, "A proposal to the study of contractions in quasi-metric spaces," *Discrete Dynamics in Nature and Society*, vol. 2014, p. 10, 2014, doi: 10.1155/2014/269286.

[7] A. H. Ansari, M. A. Barakat, and H. Aydi, "New approach for common fixed point theorems via c-class functions in g_p-metric spaces," *Journal of Functions Spaces*, vol. 4569, p. 9, 2017, doi: 10.1155/2017/2624569.

[8] H. Argoubi, B. Samet, and C. Vetro, "Nonlinear contractions involving simulation functions in a metric space with a partial order," *J. Nonlinear Sci.Appl.*, vol. 8, pp. 1082–1094, 2015.

[9] H. Aydi, R. Banković, I. Mitrović, and M. Nazam, "Nemytzki-edelstein-meir-keeler type results in b-metric spaces," *Discrete Dynamics in Nature and Society*, vol. 2018, p. 7, 2018, doi: 10.1155/2018/4745764.

[10] H. Aydi, M. Bota, E. Karapinar, and S. Mitrović, "A fixed point theorem for set-valued quasi-contractions in b-metric spaces," *Fixed Point Theory Appl.*, vol. 2012:88, 2012, doi: 10.1186/1687-1812-2012-88.

[11] H. Aydi, M. Bota, E. Karapinar, and S. Moradi, "A common fixed point for weak ϕ-contractions in b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[12] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[13] M. F. Bota, C. Chifu, and E. Karapinar, "Fixed point theorems for generalized $(\alpha-\psi)$-Ćirić-type contractive multivalued operators in b-metric spaces," *J. Nonlinear Sci.App.*, vol. 9, pp. 1165–1177, 2016, doi: 10.1155/2013/825293.

[14] M. F. Bota and E. Karapinar, "A note on "some results on multi-valued weakly jungck mappings in b-metric space"," *Cent. Eur. J. Math*, vol. 11, no. 9, pp. 1711–1712, 2013, doi: 10.2478/s11533-013-0272-2.

[15] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Math et Inf. Uni. Ostraviensis*, vol. 1, pp. 5–11, 1993.

[16] A. F. R.-L. de Hierro, E. Karapinar, C. R.-L. de Hierro, and J. Martínez-Moreno, "Coincidence point theorems on metric spaces via simulation functions," *J. Comput. Appl. Math*, vol. 275, pp. 345–355, 2015, doi: 10.1016/j.cam.2014.07.011.

[17] T. Kamran, M. Samreen, and O. U. Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics 5*, 2017, doi: 10.3390/math5020019.

[18] E. Karapinar, P. Kumam, and P. Salimi, "On $\alpha-\psi$-meir-keeler contractive mappings," *Fixed Point Theory Appl.*, vol. 2013, p. 94, 2013, doi: 10.1186/1687-1812-2013-94.

[19] F. Khojasteh, S. Shukla, and S. Radenović, "A new approach to the study of fixed point theorems via simulation functions," *Filomat*, vol. 29, no. 6, pp. 1189–1194, 2015, doi: 10.2298/FIL1506189K.

[20] M. A. Kutbi, E. Karapinar, J. Ahmed, and A. Azam, "Some fixed point results for multi-valued mappings in b-metric spaces," *Journal of Inequalities and Applications*, vol. 2014126, 2014.

[21] O. Popescu, "Some new fixed point theorems for α-geraghty contraction type maps in metric spaces," *Fixed Point Theory Appl.*, vol. 2014190, 2014, doi: 10.1186/1687-1812-2014-190.

[22] H. Qawaqneh, M. Noorani, W. Shatanawi, H. Aydi, and H. Alsamir, "Fixed point results for multi-valued contractions in b-metric spaces and an application," *Mathematics*, vol. 7(132), 2019, doi: 10.3390/math7020132.

[23] I. A. Rus, *Generalized contractions and applications*. Cluj-Napoca: Cluj University Press, 2001.

[24] J. Vujakovic, H. Aydi, S. Radenovic, and A. Mukheimer, "Some remarks and new results in ordered partial b-metric spaces," *Mathematics*, vol. 7(334), 2019, doi: 10.3390/math7040334.

Authors' addresses

Cristian Chifu
Department of Business, Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail address: cristian.chifu@tbs.ubbcluj.ro

Erdal Karapınar
(Correspondig author) Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan
Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey
E-mail address: erdalkarapinar@yahoo.com