ON FRACTIONAL KdV-BURGERS AND POTENTIAL KdV EQUATIONS
Existence and Uniqueness Results

by

Mir Sajjad HASHEMI[a], Mustafa INC[b], and Dumitru BALEANU[c,d,e*]

[a] Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab, Iran
[b] Department of Mathematics, Science Faculty, Firat University, Elazig, Turkey
[c] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara, Turkey
[d] Institute of Space Sciences, Magurele-Bucharest, Romania
[e] Department of Mathematics and Statistics, Faculty of Science, Tshwane University of Technology, Pretoria, South Africa

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Recently a new kind of derivatives, namely the conformable derivative is introduced which have not many drawbacks of other fractional derivatives. Two types of KdV equations with conformable derivative are investigated in this paper. Existence and uniqueness of two different equations of KdV class with conformable derivatives are investigated. It is also shown that the invariant subspace method can be extended to find the exact solutions of these equations.

Key words: *conformable fractional derivative, existence and uniqueness, invariant subspace method, KdV-Burgers equation, potential KdV equation*

Introduction

It is well-known that finding exact solutions for fractional differential equations and dealing with to properties of fractional derivatives are very difficult than the classical ones. There are only few analytical methods which can obtain the solutions of fractional differential equations [1-24].

Against the ordinary and partial differential equation, there introduced different types of fractional derivatives, such as Riemann-Liouville, Grunwald-Letnikov, Caputo [25], Caputo-Fabrizio [26], Atangana-Baleanu [27] and more recently one, the conformable fractional derivative [28]. The chain rule is not valid for most of fractional derivatives whereas it is valid for later one.

One of the most important filed in the analysis of differential equations in existence and uniqueness investigation. There are numerous published papers for the existence and uniqueness of the solutions for fractional differential equations with different derivative types. Some researchers analysed this issue for different differential equations with Riemann-Liouville derivative [29-31], Caputo derivative [32-34], and Caputo-Fabrizio derivative [35, 36]. To the best of the author knowledge, existence and uniqueness of solutions for conformable

* Corresponding author, e-mail: dumitru@cankaya.edu.tr

fractional differential equations has not been studied in literature. There are some interesting papers which have investigated differential equations with fractional derivatives by analytical methods [37-41].

Some of the differential equations have many applications in various branches of science and technology. The Korteweg-de Vries (KdV) equation and its variants are the most important ones which play an essential rule in the physics and engineering, specially in the modelling of waves on shallow water surfaces. Investigation on the solutions of these equations with fractional derivatives is the major interest of many researchers in literature [42-46].

Conformable fractional derivative

Khalil *et al*. in [28], introduced a new kind of fractional derivatives which have not many drawbacks of other fractional derivatives. This derivatives is called as the conformable fractional derivative.

Definition 2.1 Let f is a real valued function defined on $[a, b] \times (0, \infty)$, then:

$$ {}_t\mathbb{T}_\alpha(f)(x,t) = \lim_{\grave{o}\to 0} \frac{f\left(x, t+\epsilon t^{1-\alpha}\right) - f(x,t)}{\epsilon}, \quad \alpha \in (0,1], \quad t>0 \tag{1} $$

is called the conformable fractional derivative of f.

Any real function in previous definition which corresponding limit exists, is well-known as the α-differentiable function. There are some properties of conformable fractional derivatives which we list:

Theorem 2.1 [28] For any real constants a, b and $\alpha \in (0, 1]$ we have:

$$ {}_t\mathbb{T}_\alpha(au+bv) = a\,{}_t\mathbb{T}_\alpha(u) + b\,{}_t\mathbb{T}_\alpha(v) $$

$$ {}_t\mathbb{T}_\alpha(t^\lambda) = \lambda t^{\lambda-\alpha}, \quad \lambda \in \mathbb{R} $$

$$ {}_t\mathbb{T}_\alpha(uv) = u\ {}_t\mathbb{T}_\alpha(v) + v\ {}_t\mathbb{T}_\alpha(u) $$

$$ {}_t\mathbb{T}_\alpha\left(\frac{u}{v}\right) = \frac{u\ {}_t\mathbb{T}_\alpha(v) - v\ {}_t\mathbb{T}_\alpha(u)}{v^2} $$

$$ {}_t\mathbb{T}_\alpha(u)(t) = t^{1-\alpha}u'(t), \ \ u \in C^1 $$

Corresponding integral of the conformable fractional derivative is introduced as [28]:

$$ {}_t\mathbb{I}_\alpha(f)(x,t) = \int_0^t f(x,s)\mathrm{d}\alpha(s) = \int_0^t s^{\alpha-1} f(x,s)\mathrm{d}s \tag{2} $$

Interactions between conformable integral and derivative operators are provided by two following lemmas:

Lemma 2.1 [47] Suppose f: $[a, b] \times (0, \infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then:

$$ {}_t\mathbb{T}_\alpha\ {}_t\mathbb{I}_\alpha(f)(x,t) = f(x,t) $$

Lemma 2.2 [28] Let f: $[a, b] \times (0, \infty) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then:

$$ {}_t\mathbb{I}_\alpha\ {}_t\mathbb{T}_\alpha(f)(x,t) = f(x,t) - f(x,0) $$

The important point to note here is the validation of chain rule conformable fractional derivatives demonstrated by Abdeljawad [47]:

Theorem 2.2 Suppose f is an α – the differentiable, and g – the differentiable functions. Then:

$$ {}_t\mathbb{T}_\alpha(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)) $$

Existence and uniqueness

In this section, we want to consider the existence and uniqueness of the conformable fractional Korteweg-deVries-Burgers (KdVB) equation:

$$ {}_t\mathbb{T}_\alpha(u_1)(x,t) = \nu\frac{\partial^2 u_1}{\partial x^2} - 2u_1\frac{\partial u_1}{\partial x} - \mu\frac{\partial^3 u_1}{\partial x^3} \tag{3} $$

and conformable fractional potential KdV (PKdV) equation:

$$ {}_t\mathbb{T}_\alpha(u_2)(x,t) = \frac{\mu}{2}\left(\frac{\partial u_2}{\partial x}\right)^2 + \frac{\partial^3 u_2}{\partial x^3} \tag{4} $$

Imposing the conformable integral operator on both sides of eqs. (3) and (4):

$$ u_1(x,t) - u_1(x,0) = {}_t\mathbb{I}_\alpha\left(\nu\frac{\partial^2 u_1}{\partial x^2} - 2u_1\frac{\partial u_1}{\partial x} - \mu\frac{\partial^3 u_1}{\partial x^3}\right) \tag{5} $$

and

$$ u_2(x,t) - u_2(x,0) = {}_t\mathbb{I}_\alpha\left[\frac{\mu}{2}\left(\frac{\partial u_2}{\partial x}\right)^2 + \frac{\partial^3 u_2}{\partial x^3}\right] \tag{6} $$

respectively.

With the notations:

$$ \mathcal{K}_1(x,t,u_1) = \nu\frac{\partial^2 u_1}{\partial x^2} - 2u_1\frac{\partial u_1}{\partial x} - \mu\frac{\partial^3 u_1}{\partial x^3},\ \mathcal{K}_2(x,t,u_2) = \frac{\mu}{2}\left(\frac{\partial u_2}{\partial x}\right)^2 + \frac{\partial^3 u_2}{\partial x^3} \tag{7} $$

Eqs. (5)-(6) become:

$$ u_m(x,t) - u_m(x,0) = {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_m)\right],\ m = 1,2 \tag{8} $$

for conformable fractional KdVB and PKdV equations, respectively. Now, it is required to show the Lipschitz condition for the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ with respect to the third variable *i. e.*:

$$ \| \mathcal{K}_m(x,t,u_m) - \mathcal{K}_m(x,t,v_m) \| \le \mathcal{H}_m \| u_m - v_m \|,\ m = 1,2 \tag{9} $$

Here the used norm is defined:

$$ \| u_m(x,t) \| = \max_{(x,t)\in[a,b]\times[0,\infty)} | u_m(x,t) |,\ m = 1,2 $$

The main part is finding the Lipschitz constants $\mathcal{H}_1$ and $\mathcal{H}_2$. Let us firstly consider the related operator of conformable fractional KdVB equation:

$$ \begin{aligned} &\| \mathcal{K}_1(x,t,u_1) - \mathcal{K}_1(x,t,v_1) \| \\ &= \| \nu\frac{\partial^2 u_1}{\partial x^2} - 2u_1\frac{\partial u_1}{\partial x} - \mu\frac{\partial^3 u_1}{\partial x^3} - \left(\nu\frac{\partial^2 v_1}{\partial x^2} - 2v_1\frac{\partial v_1}{\partial x} - \mu\frac{\partial^3 v_1}{\partial x^3}\right) \| \\ &= \| \nu\left(\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2}\right) - 2\left(u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial v_1}{\partial x}\right) - \mu\left(\frac{\partial^3 u_1}{\partial x^3} - \frac{\partial^3 v_1}{\partial x^3}\right) \| \\ &\le \nu \| \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2} \| + 2 \| u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial v_1}{\partial x} \| + \mu \| \frac{\partial^3 u_1}{\partial x^3} - \frac{\partial^3 v_1}{\partial x^3} \| \\ &\le \nu \| \frac{\partial^2 (u_1 - v_1)}{\partial x^2} \| + \| \frac{\partial (u_1^2 - v_1^2)}{\partial x} \| + \mu \| \frac{\partial^3 (u_1 - v_1)}{\partial x^3} \| \end{aligned} \tag{10} $$

Let us to assume u_1 and v_1 are bounded, *i. e.* there is a positive constant $\kappa_1 > 0$ such that $\max\{\| u_1\|, \| v_1 \|\} \leq \kappa_1$. Then, their first, second, and third order derivative functions satisfy the Lipschitz condition and so, there is a constant $\vartheta_1 \geq 0$.

$$\| \mathcal{K}_1(x,t,u_1) - \mathcal{K}_1(x,t,v_1) \|$$

$$\leq \nu\, \vartheta_1^2 \| u_1 - v_1 \| + \vartheta_1 \| u_1^2 - v_1^2 \| + \mu\, \vartheta_1^3 \| u_1 - v_1 \|$$

$$\leq \nu\, \vartheta_1^2 \| u_1 - v_1 \| + \vartheta_1 \| u_1 - v_1 \| \| u_1 + v_1 \| + \mu\, \vartheta_1^3 \| u_1 - v_1 \|$$

$$\leq \nu\, \vartheta_1^2 \| u_1 - v_1 \| + \vartheta_1 \left(\| u_1 \| + \| v_1 \| \right) \| u_1 - v_1 \| + \mu\, \vartheta_1^3 \| u_1 - v_1 \|$$

$$\leq \left(\nu\, \vartheta_1^2 + 2\kappa_1 \vartheta_1 + \mu\, \vartheta_1^3 \right) \| u_1 - v_1 \|$$

Therefore, we obtain the Lipschitz condition eq. (9) for eq. (3), provided that u_1, v_1 are bounded:

$$\mathcal{H}_1 = \nu \vartheta_1^2 + 2\kappa_1 \vartheta_1 + \mu \vartheta_1^3 \tag{11}$$

Likewise, we can consider the Lipschitz condition for the operator $\mathcal{K}_2$ corresponding to conformable fractional PKdV equation:

$$\| \mathcal{K}_2(x,t,u_2) - \mathcal{K}_2(x,t,v_2) \|$$

$$= \| \frac{\mu}{2}\left(\frac{\partial u_2}{\partial x}\right)^2 + \frac{\partial^3 u_2}{\partial x^3} - \left[\frac{\mu}{2}\left(\frac{\partial v_2}{\partial x}\right)^2 + \frac{\partial^3 v_2}{\partial x^3} \right] \|$$

$$= \| \frac{\mu}{2}\left[\left(\frac{\partial u_2}{\partial x}\right)^2 - \left(\frac{\partial v_2}{\partial x}\right)^2 \right] + \left(\frac{\partial^3 u_2}{\partial x^3} - \frac{\partial^3 v_2}{\partial x^3} \right) \|$$

$$\leq \frac{\mu}{2} \| \left(\frac{\partial u_2}{\partial x}\right)^2 - \left(\frac{\partial v_2}{\partial x}\right)^2 \| + \| \frac{\partial^3 u_2}{\partial x^3} - \frac{\partial^3 v_2}{\partial x^3} \|$$

$$\leq \frac{\mu}{2} \| \frac{\partial (u_2 - v_2)}{\partial x} \| \| \frac{\partial (u_2 + v_2)}{\partial x} \| + \| \frac{\partial^3 (u_2 - v_2)}{\partial x^3} \|$$

$$\leq \frac{\mu}{2} \left(\| \frac{\partial u_2}{\partial x} \| + \| \frac{\partial v_2}{\partial x} \| \right) \| \frac{\partial (u_2 - v_2)}{\partial x} \| + \| \frac{\partial^3 (u_2 - v_2)}{\partial x^3} \| \tag{12}$$

Without loss of generality, we can assume u_2, v_2, $\partial u_2/\partial x$, and $\partial v_2/\partial x$ are bounded, *i. e.* there is a positive constant $\kappa_2 > 0$ such that $\max\{\| u_2 \|, \| v_2 \|, \| \partial u_2/\partial x \|, \|v_2/\partial x \|\} \leq \kappa_2$. Then, there is a constant $\vartheta_2 \geq 0$ such that:

$$\| \mathcal{K}_2(x,t,u_2) - \mathcal{K}_2(x,t,v_2) \|$$

$$\leq \mu\kappa_2 \| \frac{\partial (u_2 - v_2)}{\partial x} \| + \| \frac{\partial^3 (u_2 - v_2)}{\partial x^3} \|$$

$$\leq \left(\mu\kappa_2 \vartheta^2 + \vartheta^3 \right) \| u_2 - v_2 \| \tag{13}$$

Therefore, we obtain eq. (9) provided that u_2, v_2, $\partial u_2/\partial x$, and $\partial v_2/\partial x$ are bounded:

$$\mathcal{H}_2 = \mu\kappa_2 \vartheta^2 + \vartheta^3 \tag{14}$$

Existence of a special solution

Here, we will use the notion of iterative formula to prove the existence of special solutions for conformable fractional KdVB and PKdV equations. An iterative formula can be immediately concluded from eq. (8):

$$\begin{cases} u_{n+1,m}(x,t) = {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_{n,m})\right] \\ u_{0,m}(x,t) = u_m(x,0), \qquad m=1,2 \end{cases} \tag{15}$$

Additionally, we consider the notation:

$$\mathcal{G}_n^m(x,t) = u_{n,m}(x,t) - u_{n-1,m}(x,t), \qquad m=1,2 \tag{16}$$

for both of eqs. (3) and (4). We emphasize:

$$u_{n,m}(x,t) = \sum_{i=0}^{n}\mathcal{G}_i^m(x,t), \qquad m=1,2 \tag{17}$$

From eqs. (15) and (16) we can deduce:

$$\begin{aligned} \mathcal{G}_n^m(x,t) &= {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_{n-1,m})\right] - {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_{n-2,m})\right] \\ &= {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_{n-1,m}) - \mathcal{K}_m(x,t,u_{n-2,m})\right] \\ &= \int_0^t s^{\alpha-1}[\mathcal{K}_m(x,s,u_{n-1,m}) - \mathcal{K}_m(x,s,u_{n-2,m})]\mathrm{d}s, \qquad m=1,2 \end{aligned} \tag{18}$$

Therefore:

$$\begin{aligned} \|\mathcal{G}_n^m(x,t)\| &= \left\|\int_0^t s^{\alpha-1}[\mathcal{K}_m(x,s,u_{n-1,m}) - \mathcal{K}_m(x,s,u_{n-2,m})]\mathrm{d}s\right\| \\ &\le \int_0^t s^{\alpha-1}\|\mathcal{K}_m(x,s,u_{n-1,m}) - \mathcal{K}_m(x,s,u_{n-2,m})\|\,\mathrm{d}s \\ &\le \int_0^t s^{\alpha-1}\mathcal{H}_m\|u_{n-1,m} - u_{n-2,m}\|\,\mathrm{d}s = \mathcal{H}_m\int_0^t s^{\alpha-1}\|\mathcal{G}_{n-1}^m(x,s)\|\,\mathrm{d}s \\ &\le \frac{\mathcal{H}_m t^\alpha}{\alpha}\|\mathcal{G}_{n-1}^m(x,t)\|, \qquad m=1,2 \end{aligned} \tag{19}$$

Theorem 3.1 The KdVB and PKdV equations with time conformable fractional derivatives have unique continuous solutions under the condition that we can find $\hat{t}$ satisfying:

$$\mathcal{H}_m < \alpha\hat{t}^{-\alpha}, \qquad m=1,2 \tag{20}$$

Proof: We can write:

$$\|\mathcal{G}_n^m(x,t)\| \le \left(\frac{\mathcal{H}_m t^\alpha}{\alpha}\right)^n u_m(x,0), \qquad m=1,2 \tag{21}$$

If eq. (20) is hold, then $[(\mathcal{H}_m\hat{t}^\alpha)/\alpha] < 1$ and therefore:

$$\lim_{n\to\infty}\|\mathcal{G}_n^m(x,\hat{t})\| = 0, \qquad m=1,2 \tag{22}$$

This fact shows:

$$u_m(x,t) = \sum_{i=0}^{\infty}\mathcal{G}_i^m(x,t), \qquad m=1,2 \tag{23}$$

exist and are smooth functions for both of conformable fractional KdVB and PKdV equations. Now, we want to show that, obtained $u_1(x, t)$ and $u_2(x, t)$ are the solutions of conformable fractional KdVB and PKdV equations, respectively:

$$\mathcal{R}_n^m(x,t) = u_m(x,t) - u_{n,m}(x,t), \qquad m = 1,2 \tag{24}$$

where $u_m(x, t)$ are obtained from eq. (23). It follows from eqs. (15) and (24):

$$\begin{aligned}\mathcal{R}_{n+1}^m(x,t) &= {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_m)\right] - {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_{n,m})\right] \\ &= {}_t\mathbb{I}_\alpha\left[\mathcal{K}_m(x,t,u_m) - \mathcal{K}_m(x,t,u_{n,m})\right] \\ &= \int_0^t s^{\alpha-1}[\mathcal{K}_m(x,s,u_m) - \mathcal{K}_m(x,s,u_{n,m})]\mathrm{d}s, \qquad m = 1,2\end{aligned} \tag{25}$$

Hence:

$$\begin{aligned}\| \mathcal{R}_{n+1}^m(x,t) \| &= \| \int_0^t s^{\alpha-1}[\mathcal{K}_m(x,s,u_m) - \mathcal{K}_m(x,s,u_{n,m})]\mathrm{d}s \| \\ &\le \int_0^t s^{\alpha-1} \| \mathcal{K}_m(x,s,u_m) - \mathcal{K}_m(x,s,u_{n,m}) \| \,\mathrm{d}s \\ &\le \int_0^t s^{\alpha-1}\mathcal{H}_m \| u_m - u_{n,m} \| \,\mathrm{d}s = \mathcal{H}_m \int_0^t s^{\alpha-1} \| \mathcal{R}_n^m(x,s) \| \,\mathrm{d}s \\ &\le \frac{\mathcal{H}_m t^\alpha}{\alpha} \| \mathcal{R}_n^m(x,t) \|, \qquad m = 1,2\end{aligned} \tag{26}$$

Repeating this process recursively, yields:

$$\| \mathcal{R}_{n+1}^m(x,t) \| \le \left(\frac{\mathcal{H}_m t^\alpha}{\alpha}\right)^{n+1} \| u_m(x,0) \|, \qquad m = 1,2 \tag{27}$$

Then applying the infinity limit on both sides of eq. (27) and from eq. (20):

$$\lim_{n\to\infty} \| \mathcal{R}_n^m(x,t) \| = 0, \qquad m = 1,2 \tag{28}$$

This completes the proof.

Uniqueness of a special solution

We can now proceed analogously, to show that the solutions of conformable fractional KdVB and PKdV equations are unique. To do this, we suppose that u_m, v_m, $m = 1, 2$, are two different solutions for eqs. (3) and (4). Under the condition of *Theorem 3.1*:

$$\begin{aligned}\| u_m - v_m \| &= \| \int_0^t s^{\alpha-1}[\mathcal{K}_m(x,s,u_m) - \mathcal{K}_m(x,s,v_m)]\mathrm{d}s \| \\ &\le \int_0^t s^{\alpha-1} \| \mathcal{K}_m(x,s,u_m) - \mathcal{K}_m(x,s,v_m) \| \,\mathrm{d}s \\ &\le \int_0^t s^{\alpha-1}\mathcal{H}_m \| u_m - v_m \| \,\mathrm{d}s \le \\ &\le \frac{\mathcal{H}_m t^\alpha}{\alpha} \| u_m - v_m \|, \qquad m = 1,2\end{aligned} \tag{29}$$

Therefore:

$$\| u_m - v_m \| \left(\frac{\mathcal{H}_m t^\alpha}{\alpha} - 1 \right) \geq 0, \qquad m = 1, 2$$

From *Theorem 3.1*, we have $(\mathcal{H}_m t^\alpha/\alpha) - 1 < 0$, so $\|u_m - v_m\| = 0$, or equivalently $u_m = v_m$, $m = 1, 2$.

Invariant subspace method

In this section we introduce some preliminaries of the invariant subspace method and then we develop it to the conformable time fractional differential equations.

Invariant subspace method for conformable time fractional differential equations

We consider an equation of the form:

$$(u)(x,t) = \Xi[u], \qquad \in (0,1] \tag{30}$$

where x is independent variable and $\Xi[u]$ is differential operator w.r.t. the dependent variable u.

Definition 4.1 The linear space $W_n = \text{span}\{\omega_1(x), \omega_2(x),..., \omega_n(x)\}$ is called the invariant subspace w.r.t. (30), whenever $\Xi[W_n] \subseteq W_n$.

Theorem 4.1 Suppose that eq. (30) admits the invariant subspace $W_n = \text{span}\{\omega_1(x), \omega_2(x),..., \omega_n(x)\}$. Then, there exist $\psi_1, \psi_2,..., \psi_\text{n}$ in such a way:

$$\Xi[\sum_{i=1}^{n} \lambda_i \omega_i(x)] = \sum_{i=1}^{n} \psi_i(\lambda_1, \lambda_2, \ldots, \lambda_n)\omega_i(x), \qquad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n \tag{31}$$

Moreover when the coefficients $\lambda_i(t)$ satisfy:

$${}_t\mathbb{T}_\alpha(\lambda_i)(t) = \psi_i[\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)], \quad i = 1, \ldots, n \tag{32}$$

then

$$u(x,t) = \sum_{i=1}^{n} \lambda_i(t)\omega_i(x) \tag{33}$$

is the exact solution of eq. (30).

Proof: Eqs. (30) and (33) yields:

$${}_t\mathbb{T}_\alpha(u)(x,t) = \sum_{i=1}^{n} {}_t\mathbb{T}_\alpha(\lambda_i)(t)\omega_i(x) \tag{34}$$

Making use of eq. (31) concludes:

$$\Xi[\sum_{i=1}^{n} u(x,t)] = \sum_{i=1}^{n} \psi_i[\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)]\omega_i(x), \qquad i = 1, \ldots, n \tag{35}$$

Comparison of eqs. (35) and (34) with eq. (30):

$$\sum_{i=1}^{n} [{}_t\mathbb{T}_\alpha(\lambda_i)(t) - \psi_i(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))]\omega_i(x) = 0 \tag{36}$$

Finally, linear independence of $\{\omega_1(x), \omega_2(x),..., \omega_n(x)\}$results system eq. (32).

Following theorem gives a manner to find a invariant subspace W_n [16, 48].

Theorem 4.2 Let functions $\omega_1(x),..., \omega_n(x)$ form a set of solutions:

$$\mathfrak{L}[y] \equiv y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{37}$$

Then $W_n = \text{span}\{\omega_1(x), \omega_2(x),..., \omega_n(x)\}$ is invariant w.r.t. E if and only if:

$$\mathfrak{L}(\Xi[u])|_{\mathfrak{L}[u]=0} = 0 \tag{38}$$

Applications

First we consider time conformable fractional KdVB equation:

$$_t\mathbb{T}_\alpha(u)(x,t) = \nu\frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} - \mu\frac{\partial^3 u}{\partial x^3} \tag{39}$$

Here we have:

$$\Xi[u] = \nu\frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} - \mu\frac{\partial^3 u}{\partial x^3} \tag{40}$$

Regarding to the *Definition 4.1*, we obtain $W_2 = \text{span}\{1, x\}$, which produces a solution of eq. (39):

$$u(x,t) = \lambda_0(t) + \lambda_1(t)x \tag{41}$$

where $\lambda_0(t)$ and $\lambda_1(t)$ are unknown coefficients to be determined. Substituting eq. (41) into eq. (39) concludes:

$$\begin{cases} _t\mathbb{T}_\alpha(\lambda_0)(t) = -2\lambda_0(t)\lambda_1(t) \\ _t\mathbb{T}_\alpha(\lambda_1)(t) = -2\lambda_1^2(t) \end{cases} \tag{42}$$

We conclude from the second equation of eq. (42):

$$\lambda_1(t) = \frac{\alpha}{2}t^{-\alpha} \tag{43}$$

Therefore, from the first equation of eq. (42) we get:

$$\lambda_0(t) = t^{-\alpha} \tag{44}$$

Hence, regarding to eq. (41), we find the final exact solution:

$$u(x,t) = t^{-\alpha}\left(1 + \frac{\alpha}{2}x\right)$$

Now let us consider, time conformable fractional PKdV equation:

$$_t\mathbb{T}_\alpha(u)(x,t) = \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial x^3} \tag{45}$$

Let us assumed:

$$\Xi[u] = \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial x^3} \tag{46}$$

According to the *Definition 4.1*, $W_3 = \text{span}\{1, x, x^2\}$and therefore:

$$u(x,t) = \lambda_0(t) + \lambda_1(t)x + \lambda_2(t)x^2 \tag{47}$$

where $\lambda_0(t)$, $\lambda_1(t)$, and $\lambda_2(t)$ have to be determined. Substituting eq. (47) into eq. (45) yields:

$$\begin{cases} {}_t\mathbb{T}_\alpha(\lambda_0)(t) = \dfrac{\mu}{2}\lambda_1^2(t) \\ {}_t\mathbb{T}_\alpha(\lambda_1)(t) = 2\mu\lambda_1(t)\lambda_2(t) \\ {}_t\mathbb{T}_\alpha(\lambda_2)(t) = 2\mu\lambda_2^2(t) \end{cases} \tag{48}$$

We conclude from the third equation of eq. (48):

$$\lambda_2(t) = -\frac{\alpha}{2\mu}t^{-\alpha} \tag{49}$$

On substituting $\lambda_2(t)$ into the second equation of eq. (48) we get:

$$\lambda_1(t) = t^{-\alpha} \tag{50}$$

In the same fashion:

$$\lambda_0(t) = -\frac{\mu}{2\alpha}t^{-\alpha} \tag{51}$$

Therefore, according to eq. (47), we find an exact solution of time conformable fractional PKdV equation:

$$u(x,t) = -\frac{\mu}{2\alpha}t^{-\alpha} + t^{-\alpha}x - \frac{\alpha}{2\mu}t^{-\alpha}x^2$$

Conclusion

We discussed about the uniqueness and existence results of solutions for conformable fractional KdVB and PKdV equations. To the best of our knowledge, this paper is firstly investigates the uniqueness and existence results for conformable fractional differential equations. The invariant subspace method is extended for these equations in order to find the exact solutions, as well.

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