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Research article

On fuzzy Volterra-Fredholm integrodifferential equation associated with Hilfer-generalized proportional fractional derivative

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Abstract: This investigation communicates with an initial value problem (IVP) of Hilfer-generalized proportional fractional ($\mathcal{GPF}$) differential equations in the fuzzy framework is deliberated. By means of the Hilfer-$\mathcal{GPF}$ operator, we employ the methodology of successive approximation under the generalized Lipschitz condition. Based on the proposed derivative, the fractional Volterra-Fredholm integrodifferential equations ($\mathcal{FVFIE}s$) via generalized fuzzy Hilfer-$\mathcal{GPF}$ Hukuhara differentiability ($\mathcal{HD}$) having fuzzy initial conditions are investigated. Moreover, the existence of the solution is proposed by employing the fixed-point formulation. The uniqueness of the solution is verified. Furthermore, we derived the equivalent form of fuzzy $\mathcal{FVFIE}s$ which is supposed to demonstrate the convergence of this group of equations. Two appropriate examples are presented for illustrative purposes.

Keywords: Hilfer generalized proportional fractional derivative operator; fuzzy fractional derivative operators; generalized Hukuhara differentiability; fuzzy fractional Volterra-Fredholm intgro-differential equation
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1. Introduction

Recently, fractional calculus has attained assimilated bounteous flow and significant importance due to its rife utility in the areas of technology and applied analysis. Fractional derivative operators have given a new rise to mathematical models such as thermodynamics, fluid flow, mathematical biology, and virology, see [1–3]. Previously, several researchers have explored different concepts related to fractional derivatives, such as Riemann-Liouville, Caputo, Riesz, Antagana-Baleanu, Caputo-Fabrizio, etc. As a result, this investigation has been directed at various assemblies of arbitrary order differential equations framed by numerous analysts, (see [4–10]). It has been perceived that the supreme proficient technique for deliberating such an assortment of diverse operators that attracted incredible presentation in research-oriented fields, for example, quantum mechanics, chaos, thermal conductivity, and image processing, is to manage widespread configurations of fractional operators that include many other operators, see the monograph and research papers [11–22].

In [23], the author proposed a novel idea of fractional operators, which is called $\mathcal{GPF}$ operator, that recaptures the Riemann-Liouville fractional operators into a solitary structure. In [24], the authors analyzed the existence of the $FDEs$ as well as demonstrated the uniqueness of the $\mathcal{GPF}$ derivative by utilizing Kransnoselskii's fixed point hypothesis and also dealt with the equivalency of the mixed type Volterra integral equation.

Fractional calculus can be applied to a wide range of engineering and applied science problems. Physical models of true marvels frequently have some vulnerabilities which can be reflected as originating from various sources. Additionally, fuzzy sets, fuzzy real-valued functions, and fuzzy differential equations seem like a suitable mechanism to display the vulnerabilities marked out by elusiveness and dubiousness in numerous scientific or computer graphics of some deterministic certifiable marvels. Here we broaden it to several research areas where the vulnerability lies in information, for example, ecological, clinical, practical, social, and physical sciences [25–27].

In 1965, Zadeh [28] proposed fuzziness in set theory to examine these issues. The fuzzy structure has been used in different pure and applied mathematical analyses, such as fixed-point theory, control theory, topology, and is also helpful for fuzzy automata and so forth. In [29], authors also broadened the idea of a fuzzy set and presented fuzzy functions. This concept has been additionally evolved and the bulk of the utilization of this hypothesis has been deliberated in [30–35] and the references therein. The concept of $\mathcal{HD}$ has been correlated with fuzzy Riemann-Liouville differentiability by employing the Hausdorff measure of non-compactness in [36, 37].

Numerous researchers paid attention to illustrating the actual verification of certain fuzzy integral equations by employing the appropriate compactness type assumptions. Different methodologies and strategies, in light of $\mathcal{HD}$ or generalized $\mathcal{HD}$ (see [38]) have been deliberated in several credentials in the literature (see for instance [39–49]) and we presently sum up quickly a portion of these outcomes. In [50], the authors proved the existence of solutions to fuzzy $FDEs$ considering Hukuhara fractional Riemann-Liouville differentiability as well as the uniqueness of the aforesaid problem. In [51, 52], the authors investigated the generalized Hukuhara fractional Riemann-Liouville and Caputo differentiability of fuzzy-valued functions. Bede and Stefanini [39] investigated and discovered novel ideas for fuzzy-valued mappings that correlate with generalized differentiability. In [43], Hoa introduced the subsequent fuzzy FDE with order $\vartheta \in (0, 1)$:

$$\begin{cases} ({}_c\mathcal{D}^{\vartheta}_{\sigma_1^+}\Phi)(\zeta) = \mathcal{F}(\zeta, \Phi(\zeta)), \\ \Phi(\sigma_1) = \Phi_0 \in \mathfrak{E}, \end{cases} \tag{1.1}$$

where a fuzzy function is $\mathcal{F} : [\sigma_1, \sigma_2] \times \mathfrak{E} \to \mathfrak{E}$ with a nontrivial fuzzy constant $\Phi_0 \in \mathfrak{E}$. The article addressed certain consequences on clarification of the fractional fuzzy differential equations and showed that the aforesaid equations in both cases (differential/integral) are not comparable in general.

A suitable assumption was provided so that this correspondence would be effective. Hoa et al. [53] proposed the Caputo-Katugampola $FDEs$ fuzzy set having the initial condition:

$$\begin{cases}({}_c\mathcal{D}^{\vartheta,\rho}_{\sigma_1^+}\Phi)(\zeta) = \mathcal{F}(\zeta,\Phi(\zeta)),\\ \Phi(\sigma_1) = \Phi_0,\end{cases} \tag{1.2}$$

where $0 < \sigma_1 < \zeta \leq \sigma_2$, ${}_c\mathcal{D}^{\vartheta,\rho}_{\sigma_1^+}$ denotes the fuzzy Caputo-Katugampola fractional generalized Hukuhara derivative and a fuzzy function is $\mathcal{F} : [\sigma_1,\sigma_2] \times \mathfrak{E} \to \mathfrak{E}$. An approach of continual estimates depending on generalized Lipschitz conditions was employed to discuss the actual as well as the uniqueness of the solution. Owing to the aforementioned phenomena, in this article, we consider a novel fractional derivative (merely identified as Hilfer $\mathcal{GPF}$-derivative). Consequently, in the framework of the proposed derivative, we establish the basic mathematical tools for the investigation of $\mathcal{GPF}$-$\mathcal{FFHD}$ which associates with a fractional order fuzzy derivative. We investigated the actuality and uniqueness consequences of the clarification to a fuzzy fractional IVP by employing $\mathcal{GPF}$ generalized $\mathcal{HD}$ by considering an approach of continual estimates via generalized Lipschitz condition. Moreover, we derived the $\mathcal{FVFIE}$ using a generalized fuzzy $\mathcal{GPF}$ derivative is presented. Finally, we demonstrate the problems of actual and uniqueness of the clarification of this group of equations. The Hilfer-$\mathcal{GPF}$ differential equation is presented as follows:

$$\begin{cases}\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi(\zeta) = \mathcal{F}(\zeta,\Phi(\zeta)), & \zeta \in [\sigma_1,\mathcal{T}],\ 0 \leq \sigma_1 < \mathcal{T}\\ \mathcal{I}^{1-\gamma,\beta}_{\sigma_1}\Phi(\sigma_1) = \sum\limits_{j=1}^{m}\mathcal{R}_j\Phi(\nu_j), & \vartheta \leq \gamma = \vartheta + \mathfrak{q} - \vartheta\mathfrak{q},\ \nu_j \in (\sigma_1,\mathcal{T}],\end{cases} \tag{1.3}$$

where $\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}(.)$ is the Hilfer $\mathcal{GPF}$-derivative of order $\vartheta \in (0,1)$, $\mathcal{I}^{1-\gamma,\beta}_{\sigma_1}(.)$ is the $\mathcal{GPF}$ integral of order $1-\gamma > 0$, $\mathcal{R}_j \in \mathbb{R}$, and a continuous function $\mathcal{F} : [\sigma_1,\mathcal{T}] \times \mathbb{R} \to \mathbb{R}$ with $\nu_j \in [\sigma_1,\mathcal{T}]$ fulfilling $\sigma < \nu_1 < ... < \nu_m < \mathcal{T}$ for $j = 1,...,m$. To the furthest extent that we might actually know, nobody has examined the existence and uniqueness of solution (1.3) regarding $\mathcal{FVFIE}s$ under generalized fuzzy Hilfer-$\mathcal{GPF}$-$\mathcal{HD}$ with fuzzy initial conditions. An illustrative example of fractional-order in the complex domain is proposed and provides the exact solution in terms of the Fox-Wright function.

The following is the paper's summary. Notations, hypotheses, auxiliary functions, and lemmas are presented in Section 2. In Section 3, we establish the main findings of our research concerning the existence and uniqueness of solutions to Problem 1.3 by means of the successive approximation approach. We developed the fuzzy $\mathcal{GPF}$ Volterra-Fredholm integrodifferential equation in Section 4. Section 5 consists of concluding remarks.

2. Preliminaries

Throughout this investigation, $\mathfrak{E}$ represents the space of all fuzzy numbers on $\mathbb{R}$. Assume the space of all Lebsegue measureable functions with complex values $\mathcal{F}$ on a finite interval $[\sigma_1,\sigma_2]$ is identified by $\chi^r_c(\sigma_1,\sigma_2)$ such that

$$\|\mathcal{F}\|_{\chi^r_c} < \infty, \quad c \in \mathbb{R},\ 1 \leq r \leq \infty.$$

Then, the norm

$$\|\mathcal{F}\|_{\chi_c^r} = \Big(\int_{\sigma_1}^{\sigma_2} |\zeta^c \mathcal{F}(\zeta)|^r \frac{d\zeta}{\zeta}\Big)^{1/r} \infty.$$

Definition 2.1. ([53]) A fuzzy number is a fuzzy set $\Phi : \mathbb{R} \to [0,1]$ which fulfills the subsequent assumptions:
(1) Φ is normal, i.e., there exists $\zeta_0 \in \mathbb{R}$ such that $\Phi(\zeta_0) = 1$;
(2) Φ is fuzzy convex in $\mathbb{R}$, i.e, for $\delta \in [0,1]$,

$$\Phi(\delta\zeta_1 + (1-\delta)\zeta_2) \geq \min\{\Phi(\zeta_1), \Phi(\zeta_2)\} \quad for\, any\, \zeta_1, \zeta_2 \in \mathbb{R};$$

(3) Φ is upper semicontinuous on $\mathbb{R}$;
(4) $[z]^0 = cl\{z_1 \in \mathbb{R} \,|\, \Phi(z_1) > 0\}$ is compact.

$C([\sigma_1, \sigma_2], \mathfrak{E})$ indicates the set of all continuous functions and set of all absolutely continuous fuzzy functions signifys by $\mathcal{AC}([\sigma_1, \sigma_2], \mathfrak{E})$ on the interval $[\sigma_1, \sigma_2]$ having values in $\mathfrak{E}$.

Let $\gamma \in (0,1)$, we represent the space of continuous mappings by

$$C_\gamma[\sigma_1, \sigma_2] = \{\mathcal{F} : (\sigma_1, \sigma_2] \to \mathfrak{E} : e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta - \sigma_1)^{1-\gamma}\mathcal{F}(\zeta) \in C[\sigma_1, \sigma_2]\}.$$

Assume that a fuzzy set $\Phi : \mathbb{R} \mapsto [0,1]$ and all fuzzy mappings $\Phi : [\sigma_1, \sigma_2] \to \mathfrak{E}$ defined on $L([\sigma_1, \sigma_2], \mathfrak{E})$ such that the mappings $\zeta \to \bar{\mathcal{D}}_0[\Phi(\zeta), \hat{0}]$ lies in $L_1[\sigma_1, \sigma_2]$.

There is a fuzzy number Φ on $\mathbb{R}$, we write $[\Phi]^{\check{q}} = \{z_1 \in \mathbb{R} \,|\, \Phi(z_1) \geq \check{q}\}$ the $\check{q}$-level of Φ, having $\check{q} \in (0,1]$.

From assertions (1) to (4); it is observed that the $\check{q}$-level set of $\Phi \in \mathfrak{E}$, $[\Phi]^{\check{q}}$ is a nonempty compact interval for any $\check{q} \in (0,1]$. The $\check{q}$-level of a fuzzy number Φ is denoted by $[\underline{\Phi}(\check{q}), \bar{\Phi}(\check{q})]$.

For any $\delta \in \mathbb{R}$ and $\Phi_1, \Phi_2 \in \mathfrak{E}$, then the sum $\Phi_1 + \Phi_2$ and the product $\delta\Phi_1$ are demarcated as: $[\Phi_1 + \Phi_2]^{\check{q}} = [\Phi_1]^{\check{q}} + [\Phi_2]^{\check{q}}$ and $[\delta.\Phi_1]^{\check{q}} = \delta[\Phi_1]^{\check{q}}$, for all $\check{q} \in [0,1]$, where $[\Phi_1]^{\check{q}} + [\Phi_2]^{\check{q}}$ is the usual sum of two intervals of $\mathbb{R}$ and $\delta[\Phi_1]^{\check{q}}$ is the scalar multiplication between δ and the real interval.

For any $\Phi \in \mathfrak{E}$, the diameter of the $\check{q}$-level set of Φ is stated as $diam[\mu]^{\check{q}} = \bar{\mu}(\check{q}) - \underline{\mu}(\check{q})$.

Now we demonstrate the notion of Hukuhara difference of two fuzzy numbers which is mainly due to [54].

Definition 2.2. ([54]) Suppose $\Phi_1, \Phi_2 \in \mathfrak{E}$. If there exists $\Phi_3 \in \mathfrak{E}$ such that $\Phi_1 = \Phi_2 + \Phi_3$, then Φ_3 is known to be the Hukuhara difference of Φ_1 and Φ_2 and it is indicated by $\Phi_1 \ominus \Phi_2$. Observe that $\Phi_1 \ominus \Phi_2 \neq \Phi_1 + (-)\Phi_2$.

Definition 2.3. ([54]) We say that $\bar{\mathcal{D}}_0[\Phi_1, \Phi_2]$ is the distance between two fuzzy numbers if

$$\bar{\mathcal{D}}_0[\Phi_1, \Phi_2] = \sup_{\check{q}\in[0,1]} \mathcal{H}\Big([\Phi_1]^{\check{q}}, [\Phi_2]^{\check{q}}\Big), \quad \forall \Phi_1, \Phi_2 \in \mathfrak{E},$$

where the Hausdroff distance between $[\Phi_1]^{\check{q}}$ and $[\Phi_2]^{\check{q}}$ is defined as

$$\mathcal{H}\Big([\Phi_1]^{\check{q}}, [\Phi_2]^{\check{q}}\Big) = \max\{|\underline{\Phi}(\check{q}) - \bar{\Phi}(\check{q})|, |\bar{\Phi}(\check{q}) - \underline{\Phi}(\check{q})|\}.$$

Fuzzy sets in $\mathfrak{E}$ is also refereed as triangular fuzzy numbers that are identified by an ordered triple $\Phi = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ with $\sigma_1 \leq \sigma_2 \leq \sigma_3$ such that $[\Phi]^{\check{q}} = [\underline{\Phi}(\check{q}), \bar{\Phi}(\check{q})]$ are the endpoints of $\check{q}$-level sets for all $\check{q} \in [0, 1]$, where $\underline{\Phi}(\check{q}) = \sigma_1 + (\sigma_2 - \sigma_1)\check{q}$ and $\bar{\Phi}(\check{q}) = \sigma_3 - (\sigma_3 - \sigma_2)\check{q}$.

Generally, the parametric form of a fuzzy number Φ is a pair $[\Phi]^{\check{q}} = [\underline{\Phi}(\check{q}), \bar{\Phi}(\check{q})]$ of functions $\underline{\Phi}(\check{q}), \bar{\Phi}(\check{q}), \check{q} \in [0, 1]$, which hold the following assumptions:
(1) $\underline{\mu}(\check{q})$ is a monotonically increasing left-continuous function;
(2) $\bar{\mu}(\check{q})$ is a monotonically decreasing left-continuous function;
(3) $\underline{\mu}(\check{q}) \leq \bar{\mu}(\check{q}),\ \check{q} \in [0, 1]$.

Now we mention the generalized Hukuhara difference of two fuzzy numbers which is proposed by [38].

Definition 2.4. ([38]) The generalized Hukuhara difference of two fuzzy numbers $\Phi_1, \Phi_2 \in \mathfrak{E}$ ($\mathfrak{g}H$-difference in short) is stated as follows

$$\Phi_1 \ominus_{\mathfrak{g}H} \Phi_2 = \Phi_3 \quad \Leftrightarrow \quad \Phi_1 = \Phi_2 + \Phi_3 \quad or \quad \Phi_2 = \Phi_1 + (-1)\Phi_3.$$

A function $\Phi : [\sigma_1, \sigma_2] \to \mathfrak{E}$ is said to be $\mathfrak{d}$-increasing ($\mathfrak{d}$-decreasing) on $[\sigma_1, \sigma_2]$ if for every $\check{q} \in [0, 1]$. The function $\zeta \to diam[\Phi(\zeta)]^{\check{q}}$ is nondecreasing (nonincreasing) on $[\sigma_1, \sigma_2]$. If Φ is $\mathfrak{d}$-increasing or $\mathfrak{d}$-decreasing on $[\sigma_1, \sigma_2]$, then we say that Φ is $\mathfrak{d}$-monotone on $[\sigma_1, \sigma_2]$.

Definition 2.5. ([39])The generalized Hukuhara derivative of a fuzzy-valued function $\mathcal{F} : (\sigma_1, \sigma_2) \to \mathfrak{E}$ at ζ_0 is defined as

$$\mathcal{F}'_{\mathfrak{g}H}(\zeta_0) = \lim_{h\to 0} \frac{\mathcal{F}(\zeta_0 + h) \ominus_{\mathfrak{g}H} \mathcal{F}(\zeta_0)}{h},$$

if $(\mathcal{F})'_{\mathfrak{g}H}(\zeta_0) \in \mathfrak{E}$, we say that $\mathcal{F}$ is generalized Hukuhara differentiable ($\mathfrak{g}H$-differentiable) at ζ_0.

Moreover, we say that $\mathcal{F}$ is $[(i) - \mathfrak{g}H]$-differentiable at ζ_0 if

$$\begin{aligned} [\mathcal{F}'_{\mathfrak{g}H}(\zeta_0)]^{\check{q}} &= \left[\left[\lim_{h\to 0} \frac{\underline{\mathcal{F}}(\zeta_0 + h) \ominus_{\mathfrak{g}H} \underline{\mathcal{F}}(\zeta_0)}{h}\right]^{\check{q}}, \left[\lim_{h\to 0} \frac{\bar{\mathcal{F}}(\zeta_0 + h) \ominus_{\mathfrak{g}H} \bar{\mathcal{F}}(\zeta_0)}{h}\right]^{\check{q}}\right] \\ &= [(\underline{\mathcal{F}})'(\check{q}, \zeta_0), (\bar{\mathcal{F}})'(\check{q}, \zeta_0)], \end{aligned} \tag{2.1}$$

and that $\mathcal{F}$ is $[(ii) - \mathfrak{g}H]$-differentiable at ζ_0 if

$$[\mathcal{F}'_{\mathfrak{g}H}(\zeta_0)]^{\check{q}} = [(\bar{\mathcal{F}})'(\check{q}, \zeta_0), (\underline{\mathcal{F}})'(\check{q}, \zeta_0)]. \tag{2.2}$$

Definition 2.6. ([49]) We state that a point $\zeta_0 \in (\sigma_1, \sigma_2)$, is a switching point for the differentiability of $\mathcal{F}$, if in any neighborhood U of ζ_0 there exist points $\zeta_1 < \zeta_0 < \zeta_2$ such that
Type I. at ζ_1 (2.1) holds while (2.2) does not hold and at ζ_2 (2.2) holds and (2.1) does not hold, or
Type II. at ζ_1 (2.2) holds while (2.1) does not hold and at ζ_2 (2.1) holds and (2.2) does not hold.

Definition 2.7. ([23]) For $\beta \in (0, 1]$ and let the left-sided $\mathcal{GPF}$-integral operator of order ϑ of $\mathcal{F}$ is defined as follows

$$\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{F}(\zeta) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)} (\zeta - \nu)^{\vartheta-1} \mathcal{F}(\nu) d\nu, \quad \zeta > \sigma_1, \tag{2.3}$$

where $\beta \in (0, 1], \vartheta \in \mathbb{C}, Re(\vartheta) > 0$ and $\Gamma(.)$ is the Gamma function.

Definition 2.8. ([23]) For $\beta \in (0,1]$ and let the left-sided $\mathcal{GPF}$-derivative operator of order ϑ of $\mathcal{F}$ is defined as follows

$$\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{F}(\zeta) = \frac{\mathcal{D}^{n,\beta}}{\beta^{n-\vartheta}\Gamma(n-\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{n-\vartheta-1}\mathcal{F}(\nu)d\nu, \tag{2.4}$$

where $\beta \in (0,1], \vartheta \in \mathbb{C}$, $Re(\vartheta) > 0,\ n = [\vartheta] + 1$ and $\mathcal{D}^{n,\beta}$ represents the *nth*-derivative with respect to proportionality index β.

Definition 2.9. ([23]) For $\beta \in (0,1]$ and let the left-sided $\mathcal{GPF}$-derivative in the sense of Caputo of order ϑ of $\mathcal{F}$ is defined as follows

$${}^{c}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{F}(\zeta) = \frac{1}{\beta^{n-\vartheta}\Gamma(n-\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{n-\vartheta-1}(\mathcal{D}^{n,\beta}\mathcal{F})(\nu)d\nu, \tag{2.5}$$

where $\beta \in (0,1], \vartheta \in \mathbb{C}$, $Re(\vartheta) > 0$ and $n = [\vartheta] + 1$.

Let $\Phi \in L([\sigma_1, \sigma_2], \mathfrak{E})$, then the $\mathcal{GPF}$ integral of order ϑ of the fuzzy function Φ is stated as:

$$\Phi^{\beta}_{\vartheta}(\zeta) = \left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\Phi(\nu)d\nu, \quad \zeta > \sigma_1. \tag{2.6}$$

Since $[\Phi(\zeta)]^{\check{q}} = [\underline{\Phi}(\check{q},\zeta), \bar{\Phi}(\check{q},\zeta)]$ and $0 < \vartheta < 1$, we can write the fuzzy $\mathcal{GPF}$-integral of the fuzzy mapping Φ depend on lower and upper mappingss, that is,

$$[(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi)(\zeta)]^{\check{q}} = [(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\underline{\Phi})(\check{q},\zeta), (\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\bar{\Phi})(\check{q},\zeta)], \quad \zeta \geq \sigma_1, \tag{2.7}$$

where

$$(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\underline{\Phi})(\check{q},\zeta) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\underline{\Phi}(\check{q},\nu)d\nu, \tag{2.8}$$

and

$$(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\bar{\Phi})(\check{q},\zeta) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\bar{\Phi}(\check{q},\nu)d\nu. \tag{2.9}$$

Definition 2.10. For $n \in \mathbb{N}$, order ϑ and type $\mathfrak{q}$ hold $n-1 < \vartheta \leq n$ with $0 \leq \mathfrak{q} \leq 1$. The left-sided fuzzy Hilfer-proportional $\mathfrak{g}H$-fractional derivative, with respect to ζ having $\beta \in (0,1]$ of a function $\zeta \in C^{\beta}_{1-\gamma}[\sigma_1, \sigma_2]$, is stated as

$$\left(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \left(\mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}(\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\Phi)\right)(\zeta),$$

where $\mathcal{D}^{\beta}\Phi(\nu) = (1-\beta)\Phi(\nu) + \beta\Phi'(\nu)$ and if the $\mathfrak{g}H$-derivative $\Phi'_{(1-\vartheta),\beta}(\zeta)$ exists for $\zeta \in [\sigma_1, \sigma_2]$, where

$$\Phi^{\beta}_{(1-\vartheta)}(\zeta) := \left(\mathcal{I}^{(1-\vartheta),\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \frac{1}{\beta^{1-\vartheta}\Gamma(1-\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta}\Phi(\nu)d\nu, \quad \zeta \geq \sigma_1.$$

Definition 2.11. Let $\Phi' \in L([\sigma_1, \sigma_2], \mathfrak{E})$ and the fractional generalized Hukuhara $\mathcal{GPF}$-derivative of fuzzy-valued function Φ is stated as:

$$({}_{\mathrm{g}H}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+}\Phi)(\zeta) = \mathcal{I}^{1-\vartheta,\beta}_{\sigma_1^+}(\Phi'_{\mathrm{g}H})(\zeta) = \frac{1}{\beta^{1-\vartheta}\Gamma(1-\vartheta)}\int_{\sigma_1}^{\vartheta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta}\Phi'_{\mathrm{g}H}(\nu)d\nu, \ \ \nu \in (\sigma_1, \zeta). \quad (2.10)$$

Furthermore, we say that Φ is ${}^{GPF}[(i) - \mathrm{g}H]$-differentiable at ζ_0 if

$$\begin{aligned}[({}_{\mathrm{g}H}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+})]^{\check{q}} &= \left[\left[\frac{1}{\beta^{1-\vartheta}\Gamma(1-\vartheta)}\int_{\sigma_1}^{\vartheta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta}\underline{\Phi}'_{\mathrm{g}H}(\nu)d\nu\right]^{\check{q}}, \left[\frac{1}{\beta^{1-\vartheta}\Gamma(1-\vartheta)}\int_{\sigma_1}^{\vartheta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta}\bar{\Phi}'_{\mathrm{g}H}(\nu)d\nu\right]^{\check{q}}\right] \\ &= [({}_{\mathrm{g}H}\underline{\mathcal{D}}^{\vartheta,\beta}_{\sigma_1^+})(\check{q},\zeta), ({}_{\mathrm{g}H}\bar{\mathcal{D}}^{\vartheta,\beta}_{\sigma_1^+})(\check{q},\zeta)] \end{aligned} \quad (2.11)$$

and that Φ is ${}^{GPF}[(i) - \mathrm{g}H]$-differentiable at ζ_0 if

$$[({}_{\mathrm{g}H}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+})]^{\check{q}} = [({}_{\mathrm{g}H}\bar{\mathcal{D}}^{\vartheta,\beta}_{\sigma_1^+})(\check{q},\zeta), ({}_{\mathrm{g}H}\underline{\mathcal{D}}^{\vartheta,\beta}_{\sigma_1^+})(\check{q},\zeta)]. \quad (2.12)$$

Definition 2.12. We say that a point $\zeta_0 \in (\sigma_1, \sigma_2)$, is a switching point for the differentiability of $\mathcal{F}$, if in any neighborhood U of ζ_0 there exist points $\zeta_1 < \zeta_0 < \zeta_2$ such that
Type I. at ζ_1 (2.11) holds while (2.12) does not hold and at ζ_2 (2.12) holds and (2.11) does not hold, or
Type II. at ζ_1 (2.12) holds while (2.11) does not hold and at ζ_2 (2.11) holds and (2.12) does not hold.

Proposition 1. *([23]) Let $\vartheta, \varrho \in \mathbb{C}$ such that $Re(\vartheta) > 0$ and $Re(\varrho) > 0$. Then for any $\beta \in (0, 1]$, we have*

$$\begin{aligned}\left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+} e^{\frac{\beta-1}{\beta}}(s-\sigma_1)^{\varrho-1}\right)(\zeta) &= \frac{\Gamma(\varrho)}{\beta^{\vartheta}\Gamma(\varrho+\vartheta)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\varrho+\vartheta-1}, \\ \left(\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+} e^{\frac{\beta-1}{\beta}}(s-\sigma_1)^{\varrho-1}\right)(\zeta) &= \frac{\Gamma(\varrho)}{\beta^{\vartheta}\Gamma(\varrho-\vartheta)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\varrho-\vartheta-1}, \\ \left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+} e^{\frac{\beta-1}{\beta}}(\sigma_2-s)^{\varrho-1}\right)(\zeta) &= \frac{\Gamma(\varrho)}{\beta^{\vartheta}\Gamma(\varrho+\vartheta)} e^{\frac{\beta-1}{\beta}(\sigma_2-s)}(\sigma_2-\zeta)^{\varrho+\vartheta-1}, \\ \left(\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+} e^{\frac{\beta-1}{\beta}}(\sigma_2-s)^{\varrho-1}\right)(\zeta) &= \frac{\Gamma(\varrho)}{\beta^{\vartheta}\Gamma(\varrho-\vartheta)} e^{\frac{\beta-1}{\beta}(\sigma_2-s)}(\sigma_2-s)^{\varrho-\vartheta-1}.\end{aligned}$$

Lemma 2.13. *([24])For $\beta \in (0, 1]$, $\vartheta > 0$, $0 \le \gamma < 1$. If $\Phi \in C_\gamma[\sigma_1, \sigma_2]$ and $\mathcal{I}^{1-\vartheta}_{\sigma_1^+}\Phi \in C^1_\gamma[\sigma_1, \sigma_2]$, then*

$$\left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \Phi(\zeta) - \frac{e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\vartheta-1}}{\beta^{\vartheta-1}\Gamma(\vartheta)}\left(\mathcal{I}^{1-\vartheta,\beta}_{\sigma_1^+}\Phi\right)(\sigma_1).$$

Lemma 2.14. *([24]) Let $\Phi \in L_1(\sigma_1, \sigma_2)$. If $\mathcal{D}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\Phi$ exists on $L_1(\sigma_1, \sigma_2)$, then*

$$\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi = \mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\Phi.$$

Lemma 2.15. *Suppose there is a $\mathfrak{d}$-monotone fuzzy mapping $\Phi \in \mathcal{AC}([\sigma_1, \sigma_2], \mathfrak{E})$, where $[\Phi(\zeta)]^{\check{q}} = [\underline{\Phi}(\check{q},\zeta), \bar{\Phi}(\check{q},\zeta)]$ for $0 \le \check{q} \le 1$, $\sigma_1 \le \zeta \le \sigma_2$, then for $0 < \vartheta < 1$ and $\beta \in (0, 1]$, we have*
(i) $[(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta)]^{\check{q}} = [\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\underline{\Phi}(\check{q},\zeta), \mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\bar{\Phi}(\check{q},\zeta)]$ *for $\zeta \in [\sigma_1, \sigma_2]$, if Φ is $\mathfrak{d}$-increasing;*
(ii) $[(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta)]^{\check{q}} = [\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\bar{\Phi}(\check{q},\zeta), \mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\underline{\Phi}(\check{q},\zeta)]$ *for $\zeta \in [\sigma_1, \sigma_2]$, if Φ is $\mathfrak{d}$-decreasing.*

Proof. It is to be noted that if Φ is $\mathfrak{d}$-increasing, then $[\Phi'(\zeta)]^{\check{q}} = [\frac{d}{d\zeta}\underline{\Phi}(\check{q},\zeta), \frac{d}{d\zeta}\bar{\Phi}(\check{q},\zeta)]$. Taking into account Definition 2.10, we have

$$\begin{aligned}[(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta)]^{\check{q}} &= [\mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}(\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\underline{\Phi})(\check{q},\zeta), \mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}(\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\bar{\Phi})(\check{q},\zeta)] \\ &= [\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\underline{\Phi}(\check{q},\zeta), \mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\bar{\Phi}(\check{q},\zeta)].\end{aligned}$$

If Φ is $\mathfrak{d}$-decreasing, then $[\Phi'(\zeta)]^{\check{q}} = [\frac{d}{d\zeta}\bar{\Phi}(\check{q},\zeta), \frac{d}{d\zeta}\underline{\Phi}(\check{q},\zeta)]$, we have

$$\begin{aligned}[(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta)]^{\check{q}} &= [\mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}(\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\bar{\Phi})(\check{q},\zeta), \mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}(\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\underline{\Phi})(\check{q},\zeta)] \\ &= [\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\bar{\Phi}(\check{q},\zeta), \mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\underline{\Phi}(\check{q},\zeta)].\end{aligned}$$

This completes the proof. □

Lemma 2.16. *For* $\beta \in (0,1]$, $\vartheta \in (0,1)$. *If* $\Phi \in \mathcal{AC}([\sigma_1,\sigma_2],\mathfrak{E})$ *is a* $\mathfrak{d}$*-monotone fuzzy function. We take*

$$z_1(\zeta) := \left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\Phi(\nu)d\nu,$$

and

$$z_1^{(1-\vartheta),\beta} := \left(\mathcal{I}^{(1-\vartheta),\beta}_{\sigma_1^+}\Phi\right)(\zeta) = \frac{1}{\beta^{1-\vartheta}\Gamma(1-\vartheta)}\int_{\sigma_1}^{\vartheta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta}\Phi'_{\mathfrak{g}H}(\nu)d\nu,$$

is $\mathfrak{d}$*-increasing on* $(\sigma_1,\sigma_2]$, *then*

$$(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta) = \Phi(\zeta) \ominus \frac{\sum\limits_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1},$$

and

$$(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi)(\zeta) = \Phi(\zeta).$$

Proof. If $z_1(\zeta)$ is $\mathfrak{d}$-increasing on $[\sigma_1,\sigma_2]$ or $z_1(\zeta)$ is $\mathfrak{d}$-decreasing on $[\sigma_1,\sigma_2]$ and $z_1^{(1-\vartheta),\beta}(\zeta)$ is $\mathfrak{d}$-increasing on $(\sigma_1,\sigma_2]$.

Utilizing the Definitions 2.6, 2.10 and Lemma 2.13 with the initial condition $(\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi)(\sigma_1) = 0$, we have

$$\begin{aligned}(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta) &= \left(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\beta}\mathcal{I}^{(1-\mathfrak{q})(1-\vartheta),\beta}_{\sigma_1^+}\Phi\right)(\zeta) \\ &= \left(\mathcal{I}^{\gamma,\beta}_{\sigma_1^+}\mathcal{D}^{\beta}\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi\right)(\zeta) \\ &= \left(\mathcal{I}^{\gamma,\beta}_{\sigma_1^+}\mathcal{D}^{\gamma,\beta}_{\sigma_1^+}\Phi\right)(\zeta) \\ &= \Phi(\zeta) \ominus \frac{\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi}{\beta^{\gamma-1}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}. \end{aligned} \tag{2.13}$$

Now considering Proposition 1, Lemma 2.13 and Lemma 2.14, we obtain

$$\begin{aligned}(\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi)(\zeta) &= \left(\mathcal{I}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\mathcal{D}^{\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\Phi\right)(\zeta)\\ &= \Phi(\zeta)\ominus\frac{(\mathcal{I}^{1-\mathfrak{q}(1-\vartheta),\beta}_{\sigma_1^+}\Phi)(\sigma_1)e^{\frac{\beta-1}{\beta}}(\zeta-\sigma_1)}{\beta^{\mathfrak{q}(1-\vartheta)}\Gamma(\mathfrak{q}(1-\vartheta))}(\zeta-\sigma_1)^{\mathfrak{q}(1-\vartheta)-1}\\ &= \Phi(\zeta).\end{aligned}$$

On contrast, since $\Phi\in\mathcal{AC}([\sigma_1,\sigma_2],\mathfrak{E})$, there exists a constant $\mathcal{K}$ such that $\mathcal{K}=\sup_{\zeta\in[\sigma_1,\sigma_2]}\bar{\mathcal{D}}_0[\Phi(\zeta),\hat{0}]$. Then

$$\begin{aligned}\bar{\mathcal{D}}_0[\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi(\zeta),\hat{0}] &\le \mathcal{K}\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta}e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}d\nu\\ &\le \mathcal{K}\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|(\zeta-\nu)^{\vartheta-1}d\nu\\ &= \frac{\mathcal{K}}{\beta^{\vartheta}\Gamma(\vartheta+1)}(\zeta-\sigma_1)^{\vartheta},\end{aligned}$$

where we have used the fact $\left|e^{\frac{\beta-1}{\beta}\zeta}\right|<1$ and $\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi(\zeta)=0$ and $\zeta=\sigma_1$.

This completes the proof. □

Lemma 2.17. *Let there be a continuous mapping $\Phi:[\sigma_1,\sigma_2]\to\mathbb{R}^+$ on $[\sigma_1,\sigma_2]$ and hold $\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi(\zeta)\le\mathcal{F}(\xi,\Phi(\xi)),\xi\ge\sigma_1$, where $\mathcal{F}\in C([\sigma_1,\sigma_1]\times\mathbb{R}^+,\mathbb{R}^+)$. Assume that $m(\zeta)=m(\zeta,\sigma_1,\xi_0)$ is the maximal solution of the IVP*

$$\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\xi(\zeta)=\mathcal{F}(\zeta,\xi),\qquad(\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\xi)(\sigma_1)=\xi_0\ge0,\tag{2.14}$$

on $[\sigma_1,\sigma_2]$. Then, if $\Phi(\sigma_1)\le\xi_0$, we have $\Phi(\zeta)\le m(\zeta),\zeta\in[\sigma_1,\sigma_2]$.

Proof. The proof is simple and can be derived as parallel to Theorem 2.2 in [53]. □

Lemma 2.18. *Assume the IVP described as:*

$$\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi(\zeta)=\mathcal{F}(\zeta,\Phi(\zeta)),\qquad(\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi)(\sigma_1)=\Phi_0=0,\quad\zeta\in[\sigma_1,\sigma_2].\tag{2.15}$$

Let $\alpha>0$ be a given constant and $\mathfrak{B}(\Phi_0,\alpha)=\{\Phi\in\mathbb{R}:|\Phi-\Phi_0|\le\alpha\}$. Assume that the real-valued functions $\mathcal{F}:[\sigma_1,\sigma_2]\times[0,\alpha]\to\mathbb{R}^+$ satisfies the following assumptions:
(i) $\mathcal{F}\in C([\sigma_1,\sigma_2]\times[0,\alpha],\mathbb{R}^+)$, $\mathcal{F}(\zeta,0)\equiv0$, $0\le\mathcal{F}(\zeta,\Phi)\le\mathcal{M}_{\mathcal{F}}$ for all $(\zeta,\Phi)\in[\sigma_1,\sigma_2]\times[0,\alpha]$;
(ii) $\mathcal{F}(\zeta,\Phi)$ is nondecreasing in Φ for every $\zeta\in[\sigma_1,\sigma_2]$. Then the problem (2.15) has at least one solution defined on $[\sigma_1,\sigma_2]$ and $\Phi(\zeta)\in\mathfrak{B}(\Phi_0,\alpha)$.

Proof. The proof is simple and can be derived as parallel to Theorem 2.3 in [53]. □

3. Main results and discussion

In this investigation, we find the existence and uniqueness of solution to problem 1.3 by utilizing the successive approximation technique by considering the generalized Lipschitz condition of the right-hand side.

Lemma 3.1. *For $\gamma = \vartheta + \mathfrak{q}(1-\vartheta)$, $\vartheta \in (0,1), \mathfrak{q} \in [0,1]$ with $\beta \in (0,1]$, and let there is a fuzzy function $\mathcal{F} : (\sigma_1, \sigma_2] \times \mathfrak{E} \to \mathfrak{E}$ such that $\zeta \to \mathcal{F}(\zeta, \Phi)$ belongs to $C^{\beta}_{\gamma}([\sigma_1, \sigma_2], \mathfrak{E})$ for any $\Phi \in \mathfrak{E}$. Then a $\mathfrak{d}$-monotone fuzzy function $\Phi \in C([\sigma_1, \sigma_2], \mathfrak{E})$ is a solution of IVP (1.3) if and only if Φ satisfies the integral equation*

$$\Phi(\zeta) \ominus_{gH} \frac{\sum\limits_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}$$
$$= \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu))d\nu, \quad \zeta \in [\sigma_1, \sigma_2], \quad j = 1, 2, ..., m. \tag{3.1}$$

and the fuzzy function $\zeta \to \mathcal{I}^{1-\gamma}_{\sigma_1^+}\mathcal{F}(\zeta, \Phi)$ is $\mathfrak{d}$-increasing on $(\sigma_1, \sigma_2]$.

Proof. Let $\Phi \in C([\sigma_1, \sigma_2], \mathfrak{E})$ be a $\mathfrak{d}$-monotone solution of (1.3), and considering $z_1(\zeta) := \Phi(\zeta) \ominus_{gH} (\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi)(\sigma_1)$, $\zeta \in (\sigma_1, \sigma_2]$. Since Φ is $\mathfrak{d}$-monotone on $[\sigma_1, \sigma_2]$, it follows that $\zeta \to z_1(\zeta)$ is $\mathfrak{d}$-increasing on $[\sigma_1, \sigma_2]$ (see [43]).

From (1.3) and Lemma 2.16, we have

$$(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta) = \Phi(\zeta) \ominus \frac{\sum\limits_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}, \quad \forall \zeta \in [\sigma_1, \sigma_2]. \tag{3.2}$$

Since $\mathcal{F}(\zeta, \Phi) \in C_{\gamma}([\sigma_1, \sigma_2], \mathfrak{E})$ for any $\Phi \in \mathfrak{E}$, and from (1.3), observes that

$$(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\Phi)(\zeta) = \mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\mathcal{F}(\zeta, \Phi(\zeta)) = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int\limits_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu))d\nu, \quad \forall \zeta \in [\sigma_1, \sigma_2]. \tag{3.3}$$

Additionally, since $z_1(\zeta)$ is $\mathfrak{d}$-increasing on $(\sigma_1, \sigma_2]$. Also, we observe that $\zeta \to \mathcal{F}^{\vartheta,\beta}(\zeta, \Phi)$ is also $\mathfrak{d}$-increasing on $(\sigma_1, \sigma_2]$.

Reluctantly, merging (3.2) and (3.3), we get the immediate consequence.

Further, suppose $\Phi \in C([\sigma_1, \sigma_2], \mathfrak{E})$ be a $\mathfrak{d}$-monotone fuzzy function fulfills (3.1) and such that $\zeta \to \mathcal{F}^{\vartheta,\beta}(\zeta, \Phi)$ is $\mathfrak{d}$-increasing on $(\sigma_1, \sigma_2]$. By the continuity of the fuzzy mapping $\mathcal{F}$, the fuzzy mapping $\zeta \to \mathcal{F}^{\vartheta,\beta}(\zeta, \Phi)$ is continuous on $(\sigma_1, \sigma_2]$ with $\mathcal{F}^{\vartheta,\beta}(\sigma_1, \Phi(\sigma_1)) = \lim\limits_{\zeta \to \sigma_1^+} \mathcal{F}^{\vartheta,\beta}(\zeta, \Phi) = 0$. Then

$$\Phi(\zeta) = \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} + (\mathcal{I}_{\sigma_1^+}^{\vartheta,\beta}\mathcal{F}(\zeta,\zeta))(\zeta),$$

$$\mathcal{I}_{\sigma_1^+}^{1-\gamma,\beta}\Phi(\zeta) = \sum_{j=1}^{m} R_j\Phi(\zeta_j) + (\mathcal{I}_{\sigma_1^+}^{1-\mathfrak{q}(1-\vartheta)}\mathcal{F}(\zeta,\Phi(\zeta)))(\zeta),$$

and

$$\mathcal{I}_{\sigma_1^+}^{1-\gamma,\beta}\Phi(0) = \sum_{j=1}^{m} R_j\Phi(\zeta_j).$$

Moreover, since $\zeta \to \mathcal{F}^{\vartheta,\beta}(\zeta,\Phi)$ is $\mathfrak{d}$-increasing on $(\sigma_1,\sigma_2]$. Applying, the operator $\mathcal{D}_{\sigma_1^+}^{\vartheta,\mathfrak{q},\beta}$ on (3.1), yields

$$\begin{aligned}
&\mathcal{D}_{\sigma_1^+}^{\vartheta,\mathfrak{q},\beta}\Big(\Phi(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}\Big)\\
&= \mathcal{D}_{\sigma_1^+}^{\vartheta,\mathfrak{q},\beta}\Big(\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu,\Phi(\nu))d\nu\Big)\\
&= \mathcal{F}(\zeta,\Phi(\zeta)).
\end{aligned}$$

This completes the proof. □

In our next result, we use the following assumption. For a given constant $\hbar > 0$, and let $\mathfrak{B}(\Phi_0,\hbar) = \{\Phi \in \mathfrak{E} : \bar{\mathcal{D}}_0[\Phi,\Phi_0] \leq \hbar\}$.

Theorem 3.2. *Let $\mathcal{F} \in C([\sigma_1,\sigma_2]\times\mathfrak{B}(\Phi_0,\hbar),\mathfrak{E})$ and suppose that the subsequent assumptions hold:*
(i) there exists a positive constant $\mathcal{M}_{\mathcal{F}}$ such that $\bar{\mathcal{D}}_0[\mathcal{F}(\zeta,z_1),\hat{0}] \leq \mathcal{M}_{\mathcal{F}}$, for all $(\zeta,z_1) \in [\sigma_1,\sigma_2]\times\mathfrak{B}(\Phi_0,\hbar)$;
(ii) for every $\zeta \in [\sigma_1,\sigma_2]$ and every $z_1,\omega \in \mathfrak{B}(\Phi_0,\hbar)$,

$$\bar{\mathcal{D}}_0[\mathcal{F}(\zeta,z_1),\mathcal{F}(\zeta,\omega)] \leq \mathfrak{g}(\zeta,\bar{\mathcal{D}}_0[z_1,\omega]), \tag{3.4}$$

where $\mathfrak{g}(\zeta,.) \in C([\sigma_1,\sigma_2]\times[0,\beta],\mathbb{R}^+)$ satisfies the assumption in Lemma 2.18 given that problem (2.15) has only the solution $\phi(\zeta) \equiv 0$ on $[\sigma_1,\sigma_2]$. Then the subsequent successive approximations given by $\Phi^0(\zeta) = \Phi_0$ and for $n = 1,2,...,$

$$\begin{aligned}
&\Phi^n(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}\\
&= \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu,\Phi^{n-1}(\nu))d\nu,
\end{aligned}$$

converges consistently to a fixed point of problem (1.3) on certain interval $[\sigma_1,\mathcal{T}]$ for some $\mathcal{T} \in (\sigma_1,\sigma_2]$ given that the mapping $\zeta \to \mathcal{I}_{\sigma_1^+}^{\vartheta,\beta}\mathcal{F}(\zeta,\Phi^n(\zeta))$ is $\mathfrak{d}$-increasing on $[\sigma_1,\mathcal{T}]$.

Proof. Take $\sigma_1 < \zeta^*$ such that $\zeta^* \leq [\frac{\beta^{\vartheta}\hbar.\Gamma(1+\vartheta)}{\mathcal{M}} + \sigma_1]^{\frac{1}{\vartheta}}$, where $\mathcal{M} = \max\{\mathcal{M}_{\mathfrak{g}}, \mathcal{M}_{\mathcal{F}}\}$ and put $\mathcal{T} := \min\{\zeta^*, \sigma_2\}$. Let $\mathbb{S}$ be a set of continuous fuzzy functions Φ such that $\omega(\sigma_1) = \Phi_0$ and $\omega(\zeta) \in \mathfrak{B}(\Phi_0, \hbar)$ for all $\zeta \in [\sigma_1, \mathcal{T}]$. Further, we suppose the sequence of continuous fuzzy function $\{\Phi^n\}_{n=0}^{\infty}$ given by $\Phi^0(\zeta) = \Phi_0$, $\forall\zeta \in [\sigma_1, \mathcal{T}]$ and for $n = 1, 2, ..,$

$$\Phi^n(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum_{j=1}^{m} R_j \Phi^{n-1}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta - \sigma_1)^{\gamma-1}$$
$$= \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta - \nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi^{n-1}(\nu))d\nu. \tag{3.5}$$

Firstly, we show that $\Phi^n(\zeta) \in C([\sigma_1, \mathcal{T}], \mathfrak{B}(\Phi_0, \hbar))$. For $n \geq 1$ and for any $\zeta_1, \zeta_2 \in [\sigma_1, \mathcal{T}]$ with $\zeta_1 < \zeta_2$, we have

$$\bar{\mathcal{D}}_0\left(\Phi^n(\zeta_1) \ominus_{\mathfrak{g}H} \frac{\sum_{j=1}^{m} R_j \Phi^{n-1}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta - \sigma_1)^{\gamma-1}, \Phi^n(\zeta_2) \ominus_{\mathfrak{g}H} \frac{\sum_{j=1}^{m} R_j \Phi^{n-1}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta - \sigma_1)^{\gamma-1}\right)$$
$$\leq \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta_1} \left[e^{\frac{\beta-1}{\beta}(\zeta_1-\nu)}(\zeta_1 - \nu)^{\vartheta-1} - e^{\frac{\beta-1}{\beta}(\zeta_2-\nu)}(\zeta_2 - \nu)^{\vartheta-1}\right]\bar{\mathcal{D}}_0[\mathcal{F}(\nu, \Phi^{n-1}(\nu)), \hat{0}]d\nu$$
$$+\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\zeta_1}^{\zeta_2} e^{\frac{\beta-1}{\beta}(\zeta_2-\nu)}(\zeta_2 - \nu)^{\vartheta-1}\bar{\mathcal{D}}_0[\mathcal{F}(\nu, \Phi^{n-1}(\nu)), \hat{0}]d\nu.$$

Using the fact that $|e^{\frac{\beta-1}{\beta}\zeta}| < 1$, then, on the right-hand side from the last inequality, the subsequent integral becomes $\frac{1}{\beta^{\vartheta}\Gamma(1+\vartheta)}(\zeta_2 - \zeta_1)^{\vartheta}$. Therefore, with the similar assumption as we did above, the first integral reduces to $\frac{1}{\beta^{\vartheta}\Gamma(1+\vartheta)}[(\zeta_1 - \sigma_1)^{\vartheta} - (\zeta_2 - \sigma_1)^{\vartheta} + (\zeta_2 - \zeta_1)^{\vartheta}]$. Thus, we conclude

$$\begin{aligned}\bar{\mathcal{D}}_0[\Phi^n((\zeta_1), \Phi^n(\zeta_2))] &\leq \frac{\mathcal{M}_{\mathcal{F}}}{\beta^{\vartheta}\Gamma(1+\vartheta)}[(\zeta_1 - \sigma_1)^{\vartheta} - (\zeta_2 - \sigma_1)^{\vartheta} + 2(\zeta_2 - \zeta_1)^{\vartheta}] \\ &\leq \frac{2\mathcal{M}_{\mathcal{F}}}{\beta^{\vartheta}\Gamma(1+\vartheta)}(\zeta_2 - \zeta_1)^{\vartheta}.\end{aligned}$$

In the limiting case as $\zeta_1 \to \zeta_2$, then the last expression of the above inequality tends to 0, which shows Φ^n is a continuous function on $[\sigma_1, \mathcal{T}]$ for all $n \geq 1$.

Moreover, it follows that $\Phi^n \in \mathfrak{B}(\Phi_0, \hbar)$ for all $n \geq 0, \zeta \in [\sigma_1, \mathcal{T}]$ if and only if $\Phi^n(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta - \sigma_1)^{\gamma-1} \in \mathfrak{B}(0, \hbar)$ for all $\zeta \in [\sigma_1, \mathcal{T}]$ and for all $n \geq 0$.

Also, if we assume that $\Phi^{n-1}(\zeta) \in \mathbb{S}$ for all $\zeta \in [\sigma_1, \mathcal{T}]$, $n \geq 2$, then

$$\bar{\mathcal{D}}_0\Big[\Phi^n(\zeta)\ominus_{gH}\frac{\sum\limits_{j=1}^{m}R_j\Phi^{n-1}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)}e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1},\hat{0}\Big]$$
$$\leq \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int\limits_{\sigma_1}^{\zeta}e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\bar{\mathcal{D}}_0[\mathcal{F}(\nu,\Phi^{n-1}(\nu)),\hat{0}]d\nu$$
$$=\frac{\mathcal{M}_{\mathcal{F}}(\zeta-\sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(1+\vartheta)}\leq \hbar.$$

It follows that $\Phi^n(\zeta)\in\mathbb{S}$, $\forall\in[\sigma_1,\mathcal{T}]$.

Henceforth, by mathematical induction, we have $\Phi^n(\zeta)\in\mathbb{S}$, $\forall\zeta\in[\sigma_1,\mathcal{T}]$ and $\forall\, n\geq 1$.

Further, we show that the sequence $\Phi^n(\zeta)$ converges uniformly to a continuous function $\Phi\in C([\sigma_1,\mathcal{T}],\mathfrak{B}(\Phi_0,\hbar))$. By assertion (*ii*) and mathematical induction, we have for $\zeta\in[\sigma_1,\mathcal{T}]$

$$\bar{\mathcal{D}}_0\Big[\Phi^{n+1}(\zeta)\ominus_{gH}\frac{\sum\limits_{j=1}^{m}R_j\Phi^{n}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)}e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1},\Phi^n(\zeta)\ominus_{gH}\frac{\sum\limits_{j=1}^{m}R_j\Phi^{n-1}(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)}e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}\Big]$$
$$\leq\phi^n(\zeta),\qquad n=0,1,2,..., \tag{3.6}$$

where $\phi^n(\zeta)$ is defined as follows:

$$\phi^n(\zeta)=\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int\limits_{\sigma_1}^{\zeta}e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}g(\nu,\phi^{n-1}(\nu))d\nu, \tag{3.7}$$

where we have used the fact that $|e^{\frac{\beta-1}{\beta}\zeta}|<1$ and $\phi^0(\zeta)=\frac{\mathcal{M}(\zeta-\sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(\vartheta+1)}$. Thus, we have, for $\zeta\in[\sigma_1,\mathcal{T}]$ and for $n=0,1,2,...,$

$$\bar{\mathcal{D}}_0[\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n+1}(\zeta),\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n}(\zeta)]$$
$$\leq\bar{\mathcal{D}}_0[\mathcal{F}(\zeta,\Phi^n(\zeta)),\mathcal{F}(\zeta,\Phi^{n-1}(\zeta))]$$
$$\leq g(\zeta,\bar{\mathcal{D}}_0[\Phi^n(\zeta),\Phi^{n-1}(\zeta)])$$
$$\leq g(\zeta,\phi^{n-1}(\zeta)).$$

Let $n\leq m$ and $\zeta\in[\sigma_1,\mathcal{T}]$, then one obtains

$$\begin{aligned}\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\bar{\mathcal{D}}_0[\Phi^n(\zeta),\Phi^m(\zeta)] &\leq\bar{\mathcal{D}}_0[\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n}(\zeta),\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{m}(\zeta)]\\ &\leq\bar{\mathcal{D}}_0[\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n}(\zeta),\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n+1}(\zeta)]+\bar{\mathcal{D}}_0[\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{n+1}(\zeta),\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{m+1}(\zeta)]\\ &\quad+\bar{\mathcal{D}}_0[\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{m+1}(\zeta),\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\Phi^{m}(\zeta)]\\ &\leq 2g(\zeta,\phi^{n-1}(\zeta))+g(\zeta,\bar{\mathcal{D}}_0[\Phi^n(\zeta),\Phi^m(\zeta)]).\end{aligned}$$

From (*ii*), we observe that the solution $\phi(\zeta)=0$ is a unique solution of problem (2.15) and $g(.,\phi^{n-1}):[\sigma_1,\mathcal{T}]\to[0,\mathcal{M}_g]$ uniformly converges to 0, for every $\epsilon>0$, there exists a natural number n_0 such that

$$\mathcal{D}^{\vartheta,q}_{\sigma_1^+}\bar{\mathcal{D}}_0[\Phi^n(\zeta),\Phi^m(\zeta)]\leq g(\zeta,\bar{\mathcal{D}}_0[\Phi^n(\zeta),\Phi^m(\zeta)])+\epsilon,\qquad for\, n_0\leq n\leq m.$$

Using the fact that $\bar{\mathcal{D}}_0[\Phi^n(\sigma_1), \Phi^m(\sigma_1)] = 0 < \epsilon$ and by using Lemma 2.17, we have for $\zeta \in [\sigma_1, \mathcal{T}]$

$$\bar{\mathcal{D}}_0[\Phi^n(\zeta), \Phi^m(\zeta)] \leq \delta_\epsilon(\zeta), \quad n_0 \leq n \leq m, \tag{3.8}$$

where $\delta_\epsilon(\zeta)$ is the maximal solution to the following IVP :

$$(\mathcal{D}^{\vartheta,\mathfrak{q}}_{\sigma_1^+}\delta_\epsilon)(\zeta) = \mathfrak{g}(\zeta, \delta_\epsilon(\zeta)) + \epsilon, \qquad (\mathcal{I}^{1-\gamma}_{\sigma_1^+}\delta_\epsilon) = \epsilon.$$

Taking into account Lemma 2.17, we deduce that $[\phi_\epsilon(., \omega)]$ converges uniformly to the maximal solution $\phi(\zeta) \equiv 0$ of (2.15) on $[\sigma_1, \mathcal{T}]$ as $\epsilon \to 0$.

Therefore, in view of (3.8), we can obtain $n_0 \in \mathbb{N}$ is large enough such that, for $n_0 < n, m$,

$$\sup_{\zeta\in[\sigma_1,\mathcal{T}]} \bar{\mathcal{D}}_0\Bigg[\Phi^n(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum\limits_{j=1}^{m} R_j\Phi^{n-1}(\zeta_j)}{\beta^\gamma\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}, \Phi^m(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum\limits_{j=1}^{m} R_j\Phi^{n-1}(\zeta_j)}{\beta^\gamma\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}\Bigg] \leq \epsilon. \tag{3.9}$$

Since $(\mathfrak{E}, \bar{\mathcal{D}}_0)$ is a complete metric space and (3.9) holds, thus $\{\Phi^n(\zeta)\}$ converges uniformly to $\Phi \in C([\sigma_1, \sigma_2], \mathfrak{B}(\Phi_0, \hbar))$. Hence

$$\begin{aligned}\Phi(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum\limits_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^\gamma\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} &= \lim_{n\to\infty}\left(\Phi^n(\zeta) \ominus_{\mathfrak{g}H} \frac{\sum\limits_{j=1}^{m} R_j\Phi^{n-1}(\zeta_j)}{\beta^\gamma\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}\right)\\ &= \frac{1}{\beta^\vartheta\Gamma(\vartheta)}\int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi^{n-1}(\nu))d\nu. \end{aligned} \tag{3.10}$$

Because of Lemma 3.1, the function $\Phi(\zeta)$ is the solution to (1.3) on $[\sigma_1, \mathcal{T}]$.

In order to find the unique solution, assume that $\Psi : [\sigma_1, \mathcal{T}] \to \mathfrak{E}$ is another solution of problem (1.3) on $[\sigma_1, \mathcal{T}]$. We denote $\kappa(\zeta) = \bar{\mathcal{D}}_0[\Phi(\zeta), \Psi(\zeta)]$. Then $\kappa(\sigma_1) = 0$ and for every $\zeta \in [\sigma_1, \mathcal{T}]$, we have

$$\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}\kappa(\zeta) \leq \bar{\mathcal{D}}_0[\mathcal{F}(\zeta, \Phi(\zeta)), \mathcal{F}(\zeta, \Psi(\zeta))] \leq \mathfrak{g}(\zeta, \kappa(\zeta)). \tag{3.11}$$

Further, using the comaprison Lemma 2.17, we get $\kappa(\zeta) \leq m(\zeta)$, where m is a maximal solution of the IVP $\mathcal{D}^{\vartheta,\mathfrak{q},\beta}_{\sigma_1^+}m(\zeta) \leq \mathfrak{g}(\zeta, m(\zeta))$, $(\mathcal{I}^{1-\gamma}_{\sigma_1^+}m)(\sigma_1) = 0$. By asseration (ii), we have $m(\zeta) = 0$ and hence $\Phi(\zeta) = \Psi(\zeta)$, $\forall \in [\sigma_1, \mathcal{T}]$.

This completes the proof. □

Corollary 1. *For $\beta \in (0, 1]$ and let $C([\sigma_1, \sigma_2], \mathfrak{E})$. Assume that there exist positive constants $\mathcal{L}, \mathcal{M}_{\mathcal{F}}$ such that, for every $z_1, \omega \in \mathfrak{E}$,*

$$\bar{\mathcal{D}}_0[\mathcal{F}(\zeta, z_1), \mathcal{F}(\zeta, \omega)] \leq \mathcal{L}\bar{\mathcal{D}}_0[z_1, \omega], \qquad \bar{\mathcal{D}}_0[\mathcal{F}(\zeta, z_1), \hat{0}] \leq \mathcal{M}_{\mathcal{F}}.$$

Then the subsequent successive approximations given by $\Phi^0(\zeta) = \Phi_0$ *and for* $n = 1, 2, ..$

$$\Phi^n(\zeta) \ominus_{gH} \Phi_0 = \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)} (\zeta - \nu)^{\vartheta-1} \mathcal{F}(\nu, \Phi^{n-1}(\nu)) d\nu,$$

converges consistently to a fixed point of problem (1.3) *on* $[\sigma_1, \mathcal{T}]$ *for certain* $\mathcal{T} \in (\sigma_1, \sigma_2]$ *given that the mapping* $\zeta \to \mathcal{I}^{\vartheta,\beta}_{\sigma_1^+} \mathcal{F}(\zeta, \Phi^n(\zeta))$ *is* $\mathfrak{d}$*-increasing on* $[\sigma_1, \mathcal{T}]$.

Example 3.3. For $\beta \in (0, 1]$, $\gamma = \vartheta + \mathfrak{q}(1 - \vartheta)$, $\vartheta \in (0, 1)$, $\mathfrak{q} \in [0, 1]$ and $\delta \in \mathbb{R}$. Assume that the linear fuzzy $\mathcal{GPF}$-FDE under Hilfer-$\mathcal{GPF}$-derivative and moreover, the subsequent assumptions hold:

$$\begin{cases} (\mathcal{D}^{\vartheta,\mathfrak{q}}_{\sigma_1^+}\Phi)(\zeta) = \delta\Phi(\zeta) + \eta(\zeta), & \zeta \in (\sigma_1, \sigma_2], \\ (\mathcal{I}^{1-\gamma,\beta}_{\sigma_1^+}\Phi)(\sigma_1) = \Phi_0 = \sum_{j=1}^{m} \mathcal{R}_j \Phi(\zeta_j), & \gamma = \vartheta + \mathfrak{q}(1 - \vartheta). \end{cases} \tag{3.12}$$

Applying Lemma 3.1, we have

$$\begin{aligned} &\Phi(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)} (\zeta - \sigma_1)^{\gamma-1} \\ &= \delta \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)} (\zeta - \nu)^{\vartheta-1} \Phi(\nu) d\nu + \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)} \int_{\sigma_1}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)} (\zeta - \nu)^{\vartheta-1} \eta(\nu) d\nu, \quad \zeta \in [\sigma_1, \sigma_2] \\ &= \delta(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi)(\zeta) + (\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\eta)(\zeta), \end{aligned}$$

where $\eta \in C((\sigma_1, \sigma_2], \mathfrak{E})$ and furthermore, assuming the diameter on the right part of the aforementioned equation is increasing. Observing $\mathcal{F}(\zeta, \Phi) := \delta\Phi + \eta$ fulfill the suppositions of Corollary 1.

In order to find the analytical view of (3.12), we utilized the technique of successive approximation. Putting $\Phi^0(\zeta) = \Phi_0$ and

$$\begin{aligned} &\Phi^n(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)} (\zeta - \sigma_1)^{\gamma-1} \\ &= \delta(\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\Phi^{n-1})(\zeta) + (\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\eta)(\zeta), \quad n = 1, 2, ... \end{aligned}$$

Letting $n = 1$, $\delta > 0$, assuming there is a $\mathfrak{d}$-increasing mapping Φ, then we have

$$\begin{aligned} &\Phi^1(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j \Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)} (\zeta - \sigma_1)^{\gamma-1} \\ &= \delta \sum_{j=1}^{m} R_j \Phi(\zeta_j) \frac{(\zeta - \sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(\vartheta + 1)} + (\mathcal{I}^{\vartheta,\beta}_{\sigma_1^+}\eta)(\zeta). \end{aligned}$$

In contrast, if we consider $\delta < 0$ and Φ is $\mathfrak{d}$-decreasing, then we have

$$(-1)\Big(\frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} \ominus_{gH} \Phi^1(\zeta)\Big) = \delta\sum_{j=1}^{m} R_j\Phi(\zeta_j)\frac{(\zeta-\sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(\vartheta+1)} + (\mathcal{I}_{\sigma_1^+}^{\vartheta,\beta}\eta)(\zeta).$$

For $n = 2$, we have

$$\Phi^2(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} = \sum_{j=1}^{m} R_j\Phi(\zeta_j)\Big[\frac{\delta(\zeta-\sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(\vartheta+1)} + \frac{\delta^2(\zeta-\sigma_1)^{2\vartheta}}{\beta^{2\vartheta}\Gamma(2\vartheta+1)}\Big] + (\mathcal{I}_{\sigma_1^+}^{\vartheta,\beta}\eta)(\zeta) + (\mathcal{I}_{\sigma_1^+}^{2\vartheta,\beta}\eta)(\zeta),$$

if $\delta > 0$ and there is $\mathfrak{d}$-increasing mapping Φ, we have

$$(-1)\Big(\frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} \ominus_{gH} \Phi^2(\zeta)\Big) = \sum_{j=1}^{m} R_j\Phi(\zeta_j)\Big[\frac{\delta(\zeta-\sigma_1)^{\vartheta}}{\beta^{\vartheta}\Gamma(\vartheta+1)} + \frac{\delta^2(\zeta-\sigma_1)^{2\vartheta}}{\beta^{2\vartheta}\Gamma(2\vartheta+1)}\Big] + (\mathcal{I}_{\sigma_1^+}^{\vartheta,\beta}\eta)(\zeta) + (\mathcal{I}_{\sigma_1^+}^{2\vartheta,\beta}\eta)(\zeta),$$

and there is $\delta < 0$, *and* $\mathfrak{d}$-increasing mapping Φ. So, continuing inductively and in the limiting case, when $n \to \infty$, we attain the solution

$$\begin{aligned}
&\Phi(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1} \\
&= \sum_{j=1}^{m} R_j\Phi(\zeta_j)\sum_{l=1}^{\infty}\frac{\delta^l(\zeta-\sigma_1)^{l\vartheta}}{\beta^{l\vartheta}\Gamma(l\vartheta+1)} + \int_{\sigma_1}^{\zeta}\sum_{l=1}^{\infty}\frac{\delta^{l-1}(\zeta-\sigma_1)^{l\vartheta}-1}{\beta^{l\vartheta-1}\Gamma(l\vartheta)}\eta(\nu)d\nu \\
&= \sum_{j=1}^{m} R_j\Phi(\zeta_j)\sum_{l=1}^{\infty}\frac{\delta^l(\zeta-\sigma_1)^{l\vartheta}}{\beta^{l\vartheta}\Gamma(l\vartheta+1)} + \int_{\sigma_1}^{\zeta}\sum_{l=0}^{\infty}\frac{\delta^{l}(\zeta-\sigma_1)^{l\vartheta+(\vartheta-1)}}{\beta^{l\vartheta}+(\vartheta-1)\Gamma(l\vartheta+\vartheta)}\eta(\nu)d\nu \\
&= \sum_{j=1}^{m} R_j\Phi(\zeta_j)\sum_{l=1}^{\infty}\frac{\delta^l(\zeta-\sigma_1)^{l\vartheta}}{\beta^{l\vartheta}\Gamma(l\vartheta+1)} + \frac{1}{\beta^{\vartheta-1}}\int_{\sigma_1}^{\zeta}(\zeta-\sigma_1)^{\vartheta-1}\sum_{l=0}^{\infty}\frac{\delta^{l}(\zeta-\sigma_1)^{l\vartheta}}{\beta^{l\vartheta}\Gamma(l\vartheta+\vartheta)}\eta(\nu)d\nu,
\end{aligned}$$

for every $\delta > 0$ and Φ is $\mathfrak{d}$-increasing, or $\delta < 0$ and Φ is $\mathfrak{d}$-decreasing, accordingly. Therefore, by means of Mittag-Leffler function $\mathcal{E}_{\vartheta,\mathfrak{q}}(\Phi) = \sum_{l=1}^{\infty}\frac{\Phi^{\kappa}}{\Gamma(l\vartheta+\mathfrak{q})}$, $\vartheta, \mathfrak{q} > 0$, the solution of problem (3.12) is expressed by

$$\Phi(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}$$
$$= \sum_{j=1}^{m} R_j\Phi(\zeta_j)\mathcal{E}_{\vartheta,1}\Big(\delta(\zeta-\sigma_1)^{\vartheta}\Big) + \frac{1}{\beta^{\vartheta-1}}\int_{\sigma_1}^{\zeta}(\zeta-\sigma_1)^{\vartheta-1}\mathcal{E}_{\vartheta,\vartheta}\Big(\delta(\zeta-\sigma_1)^{\vartheta}\Big)\eta(\nu)d\nu,$$

for every of $\delta > 0$ and Φ is $\mathfrak{d}$-increasing. Alternately, if $\delta < 0$ and Φ is $\mathfrak{d}$-decreasing, then we get the solution of problem (3.12)

$$\Phi(\zeta) \ominus_{gH} \frac{\sum_{j=1}^{m} R_j\Phi(\zeta_j)}{\beta^{\gamma}\Gamma(\gamma)} e^{\frac{\beta-1}{\beta}(\zeta-\sigma_1)}(\zeta-\sigma_1)^{\gamma-1}$$
$$= \sum_{j=1}^{m} R_j\Phi(\zeta_j)\mathcal{E}_{\vartheta,1}\Big(\delta(\zeta-\sigma_1)^{\vartheta}\Big) \ominus (-1)\frac{1}{\beta^{\vartheta-1}}\int_{\sigma_1}^{\zeta}(\zeta-\sigma_1)^{\vartheta-1}\mathcal{E}_{\vartheta,\vartheta}\Big(\delta(\zeta-\sigma_1)^{\vartheta}\Big)\eta(\nu)d\nu.$$

4. Fuzzy generalized proportional $\mathcal{FVFIDE}$

Consider IVP

$$\begin{cases} \big({}_{gH}\mathcal{D}^{\vartheta,\beta}_{\sigma_1^+}\Phi\big)(\zeta) = \mathcal{F}(\zeta, \Phi(\zeta), \mathcal{H}_1\Phi(\zeta), \mathcal{H}_2\Phi(\zeta)), & \zeta \in [\zeta_0, \mathcal{T}] \\ \Phi(\zeta_0) = \Phi_0 \in \mathfrak{E}, \end{cases} \tag{4.1}$$

where $\beta \in (0, 1]$ and $\vartheta \in (0, 1)$ is a real number and the operation ${}_{gH}\mathcal{D}^{\vartheta}_{\sigma_1^+}$ denote the $\mathcal{GPF}$ derivative of order ϑ, $\mathcal{F} : [\zeta_0, \mathcal{T}] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$ is continuous in ζ which fulfills certain supposition that will be determined later, and

$$\mathcal{H}_1\Phi(\zeta) = \int_{\zeta_0}^{\zeta} \mathcal{H}_1(\zeta, s)\Phi(s)ds, \qquad \mathcal{H}_2\Phi(\zeta) = \int_{\zeta_0}^{\mathcal{T}} \mathcal{H}_2(\zeta, s)\Phi(s)ds, \tag{4.2}$$

with $\mathcal{H}_1, \mathcal{H}_2 : [\zeta_0, \mathcal{T}] \times [\zeta_0, \mathcal{T}] \to \mathbb{R}$ such that

$$\mathcal{H}_1^* = \sup_{\zeta\in[\zeta_0,\mathcal{T}]} \int_{\zeta_0}^{\zeta} |\mathcal{H}_1(\zeta, s)|ds, \qquad \mathcal{H}_2^* = \sup_{\zeta\in[\zeta_0,\mathcal{T}]} \int_{\zeta_0}^{\mathcal{T}} |\mathcal{H}_2(\zeta, s)|ds.$$

Now, we investigate the existence and uniqueness of the solution of problem (4.1). To establish the main consequences, we require the following necessary results.

Theorem 4.1. *Let $\mathcal{F} : [\sigma_1, \sigma_2] \to \mathfrak{E}$ be a fuzzy-valued function on $[\sigma_1, \sigma_2]$. Then*
(i) $\mathcal{F}$ is $[(i) - gH]$-differentiable at $c \in [\sigma_1, \sigma_2]$ iff $\mathcal{F}$ is ${}^{GPF}[(i) - gH]$-differentiable at c.
(ii) $\mathcal{F}$ is $[(ii) - gH]$-differentiable at $c \in [\sigma_1, \sigma_2]$ iff $\mathcal{F}$ is ${}^{GPF}[(ii) - gH]$-differentiable at c.

Proof. In view of Definition 2.18 and Definition 2.11, the proof is straightforward. □

Lemma 4.2. *([44]) Let there be a fuzzy valued mapping $\mathcal{F} : [\zeta_0, \mathcal{T}] \to \mathfrak{E}$ such that $\mathcal{F}'_{gH} \in \mathfrak{E} \cap \chi_c^r(\sigma_1, \sigma_2)$, then*

$$\mathcal{I}_{\zeta_0}^{\vartheta,\beta}({}_{gH}\mathcal{D}_{\sigma_1^+}^{\vartheta,\beta}\mathcal{F})(\zeta) = \mathcal{F}(\zeta) \ominus_{gH} \mathcal{F}(\zeta_0). \tag{4.3}$$

Lemma 4.3. *The IVP* (4.1) *is analogous to subsequent equation*

$$\Phi(\zeta) = \Phi_0 + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, \tag{4.4}$$

if $\Phi(\zeta)$ be ${}^{GPF}[(i) - gH]$-differentiable,

$$\Phi(\zeta) = \Phi_0 \ominus \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, \tag{4.5}$$

if $\Phi(\zeta)$ be ${}^{GPF}[(ii) - gH]$-differentiable, and

$$\Phi(\zeta) = \begin{cases} \Phi_0 + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, & \zeta \in [\sigma_1, \sigma_3], \\ \Phi_0 \ominus \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, & \zeta \in [\sigma_3, \sigma_2], \end{cases} \tag{4.6}$$

if there exists a point $\sigma_3 \in (\sigma_1, \sigma_2)$ such that $\Phi(\zeta)$ is ${}^{GPF}[(i) - gH]$-differentiable on $[\sigma_1, \sigma_3]$ and ${}^{GPF}[(ii) - gH]$-differentiable on $[\sigma_3, \sigma_2]$ and $\mathcal{F}(\sigma_3, \Phi(\sigma_3, \Phi(\sigma_3), \mathcal{H}_1\Phi(\sigma_3)) \in \mathbb{R}$.

Proof. By means of the integral operator (2.6) on both sides of (4.1), yields

$$\mathcal{I}_{\zeta_0}^{\vartheta,\beta}({}_{gH}\mathcal{D}_{\sigma_1^+}^{\vartheta,\beta}\Phi(\zeta)) = \mathcal{I}_{\zeta_0}^{\vartheta,\beta}(\mathcal{F}(\zeta, \Phi(\zeta), \mathcal{H}_1\Phi(\zeta), \mathcal{H}_2\Phi(\zeta)). \tag{4.7}$$

Utilizing Lemma 4.2 and Definition 2.6, we gat

$$\Phi(\zeta) \ominus_{gH} \Phi_0 = \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu. \tag{4.8}$$

In view of Defnition 2.17 and Theorem 4.1, if $\Phi(\zeta)$ be ${}^{GPF}[(i) - gH]$-differentiable,

$$\Phi(\zeta) = \Phi_0 + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu \tag{4.9}$$

and if $\Phi(\zeta)$ be ${}^{GPF}[(ii) - gH]$-differentiable

$$\Phi(\zeta) = \Phi_0 \ominus \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu. \tag{4.10}$$

In addition, when we have a switchpoint $\sigma_3 \in (\sigma_1, \sigma_2)$ of type (I) the ${}^{GPF}[\mathfrak{g}H]$-differentiability changes from type (I) to type (II) at $\zeta = \sigma_3$. Then by (4.9) and (4.10) and Definition 2.12, The proof is easy to comprehend. □

Also, we proceed with the following assumptions:
$(\mathbb{A}_1)$. $\mathcal{F} : [\zeta_0, \mathcal{T}] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$ is continuous and there exist positive real functions $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ such that

$$\begin{aligned}&\bar{\mathcal{D}}_0\Big(\mathcal{F}(\zeta, \Phi(\zeta), \mathcal{H}_1\Phi(\zeta), \mathcal{H}_2\Phi(\zeta)), \mathcal{F}(\zeta, \Psi(\zeta), \mathcal{H}_1\Psi(\zeta), \mathcal{H}_2\Psi(\zeta))\Big)\\ &\leq \mathcal{L}_1(\zeta)\bar{\mathcal{D}}_0(\Phi, \Psi) + \mathcal{L}_2(\zeta)\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi, \mathcal{H}_1\Psi) + \mathcal{L}_3(\zeta)\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi, \mathcal{H}_2\Psi).\end{aligned}$$

$(\mathbb{A}_2)$. There exist a number ϵ such that $\delta \leq \epsilon < 1$, $\zeta \in [\zeta_0, \mathcal{T}]$

$$\delta = \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{P}(1 + \mathcal{H}_1^* + \mathcal{H}_2^*)$$

and

$$\mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{P} = \sup_{\zeta\in[0,\mathcal{T}]} \{\mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_1, \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_2, \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_3\}.$$

Theorem 4.4. *Let* $\mathcal{F} : [\zeta_0, \mathcal{T}] \times \mathfrak{E} \times \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$ *be a bounded continuous functions and holds* $(\mathbb{A}_1)$. *Then the IVP* (4.1) *has a unique solution which is* ${}^{GPF}[(i) - \mathfrak{g}H]$*-differentiable on* $[\zeta_0, \mathcal{T}]$, *given that* $\delta < 1$, *where* δ *is given in* $(\mathbb{A}_2)$.

Proof. Assuming $\Phi(\zeta)$ is ${}^{GPF}[(i) - \mathfrak{g}H]$-differentiability and $\Phi_0 \in \mathfrak{E}$ be fixed. Propose a mapping $\mathfrak{F} : C([\zeta_0, \mathcal{T}], \mathfrak{E}) \to C([\zeta_0, \mathcal{T}], \mathfrak{E})$ by

$$(\mathfrak{F}\Phi)(\zeta) = \Phi_0 + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta - \nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, \quad for\, all\, \zeta \in [\zeta_0, \mathcal{T}]. \tag{4.11}$$

Next we prove that $\mathfrak{F}$ is contraction. For $\Phi, \Psi \in C([\zeta_0, \mathcal{T}], \mathfrak{E})$ by considering of $(\mathbb{A}_1)$ and by distance properties (2.3), one has

$$\begin{aligned}&\bar{\mathcal{D}}_0(\mathfrak{F}\Phi(\zeta), \mathfrak{F}\Psi(\zeta))\\ &\leq \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta - \nu)^{\vartheta-1}\right|\bar{\mathcal{D}}_0(\mathcal{F}(\zeta, \Phi(\zeta), \mathcal{H}_1\Phi(\zeta), \mathcal{H}_2\Phi(\zeta)), \mathcal{F}(\zeta, \Psi(\zeta), \mathcal{H}_1\Psi(\zeta), \mathcal{H}_2\Psi(\zeta))\Big)d\nu\\ &\leq \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta - \nu)^{\vartheta-1}\right|[\mathcal{L}_1\bar{\mathcal{D}}_0(\Phi, \Psi) + \mathcal{L}_2\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi, \mathcal{H}_1\Psi) + \mathcal{L}_3\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi, \mathcal{H}_2\Psi)]d\nu\\ &\leq \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta - \nu)^{\vartheta-1}\right|\mathcal{L}_1\bar{\mathcal{D}}_0(\Phi, \Psi)d\nu + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta - \nu)^{\vartheta-1}\right|\mathcal{L}_2\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi, \mathcal{H}_1\Psi)d\nu\\ &\quad + \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})} \int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta - \nu)^{\vartheta-1}\right|\mathcal{L}_3\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi, \mathcal{H}_2\Psi)d\nu.\end{aligned} \tag{4.12}$$

Now, we find that

$$\frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_2\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi,\mathcal{H}_1\Psi)d\nu$$
$$\leq \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta}\Big(\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_2\bar{\mathcal{D}}_0(\Phi,\Psi)\int_{\zeta_0}^{\nu}|\mathcal{H}_1(\nu,x)|dx\Big)d\nu$$
$$\leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_2\mathcal{H}_1^*.\bar{\mathcal{D}}_0(\Phi,\Psi). \tag{4.13}$$

Analogously,

$$\frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_3\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi,\mathcal{H}_2\Psi)d\nu \leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_3\mathcal{H}_1^*.\bar{\mathcal{D}}_0(\Phi,\Psi),$$
$$\frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_1\bar{\mathcal{D}}_0(\Phi,\Psi)d\nu = \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_1\bar{\mathcal{D}}_0(\Phi,\Psi). \tag{4.14}$$

Then we have

$$\begin{aligned}\bar{\mathcal{D}}_0(\mathfrak{F}\Phi,\mathfrak{F}\Psi) &\leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_1\bar{\mathcal{D}}_0(\Phi,\Psi)+\mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_2\mathcal{H}_1^*.\bar{\mathcal{D}}_0(\Phi,\Psi)+\mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{L}_3\mathcal{H}_2^*.\bar{\mathcal{D}}_0(\Phi,\Psi)\\ &\leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{P}(1+\mathcal{H}_1^*+\mathcal{H}_2^*)\bar{\mathcal{D}}_0(\Phi,\Psi)\\ &< \bar{\mathcal{D}}_0(\Phi,\Psi).\end{aligned} \tag{4.15}$$

Consequently, $\mathfrak{F}$ is a contraction mapping on $C([\zeta_0,\mathcal{T}],\mathfrak{E})$ having a fixed point $\mathfrak{F}\Phi(\zeta)=\Phi(\zeta)$. Henceforth, the IVP (4.1) has unique solution. □

Theorem 4.5. *For $\beta\in(0,1]$ and let $\mathcal{F}:[\zeta_0,\mathcal{T}]\times\mathfrak{E}\times\mathfrak{E}\times\mathfrak{E}\to\mathfrak{E}$ be a bounded continuous functions and satisfies $(\mathbb{A}_1)$. Let the sequence $\Phi_n:[\zeta_0,\mathcal{T}]\to\mathfrak{E}$ is given by*

$$\begin{aligned}\Phi_{n+1}(\zeta) &= \Phi_0\ominus\frac{-1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu,\Phi_n(\nu),\mathcal{H}_1\Phi_n(\nu),\mathcal{H}_2\Phi_n(\nu))d\nu,\\ \Phi_0(\zeta) &= \Phi_0,\end{aligned} \tag{4.16}$$

is described for any $n\in\mathbb{N}$. Then the sequence $\{\Phi_n\}$ converges to fixed point of problem (4.1) *which is ${}^{GPF}[(ii)-\mathfrak{g}H]$-differentiable on $[\zeta_0,\mathcal{T}]$, given that $\delta<1$, where δ is defined in $(\mathbb{A}_2)$.*

Proof. We now prove that the sequence $\{\Phi_n\}$, given in (4.16), is a Cauchy sequence in $C([\zeta_0,\mathcal{T}],\mathfrak{E})$. To do just that, we'll require

$$\begin{aligned}\bar{\mathcal{D}}_0(\Phi_1,\Phi_0) &= \bar{\mathcal{D}}_0\Big(\Phi_0\ominus\frac{-1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta}e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu,\Phi_0(\nu),\mathcal{H}_1\Phi_0(\nu),\mathcal{H}_2\Phi_0(\nu))d\nu,\Phi_0\Big)\\ &\leq \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\bar{\mathcal{D}}_0\Big(\mathcal{F}(\nu,\Phi_0(\nu),\mathcal{H}_1\Phi_0(\nu),\mathcal{H}_2\Phi_0(\nu)),\hat{0}\Big)d\nu\\ &\leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{M},\end{aligned} \tag{4.17}$$

where $\mathcal{M} = \sup_{\zeta\in[\zeta_0,\mathcal{T}]} \bar{\mathcal{D}}_0(\mathcal{F}(\zeta, \Phi, \mathcal{H}_1\Phi, \mathcal{H}_2\Phi), \hat{0})$.

Since $\mathcal{F}$ is Lipschitz continuous, In view of Definition (2.3), we show that

$$\begin{aligned}
&\bar{\mathcal{D}}_0(\Phi_{n+1}, \Phi_n)\\
&\leq \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\bar{\mathcal{D}}_0(\mathcal{F}(\nu, \Phi_n(\nu), \mathcal{H}_1\Phi_n(\nu), \mathcal{H}_2\Phi_n(\nu)), \mathcal{F}(\nu, \Phi_{n-1}(\nu), \mathcal{H}_1\Phi_{n-1}(\nu), \mathcal{H}_2\Phi_{n-1}(\nu))\Big)d\nu\\
&\leq \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_1.\bar{\mathcal{D}}_0(\Phi_n, \Phi_{n-1})d\nu\\
&\quad+\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_2.\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi_n, \mathcal{H}_1\Phi_{n-1})d\nu\\
&\quad+\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_3.\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi_n, \mathcal{H}_2\Phi_{n-1})d\nu\\
&\leq \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{P}(1+\mathcal{H}_1^*+\mathcal{H}_2^*)\bar{\mathcal{D}}_0(\Phi_n, \Phi_{n-1}) \leq \delta\bar{\mathcal{D}}_0(\Phi_n, \Phi_{n-1}) \leq \delta^n\bar{\mathcal{D}}_0(\Phi_1, \Phi_0) \leq \delta^n\mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{M}. \qquad (4.18)
\end{aligned}$$

Since $\delta < 1$ promises that the sequence $\{\Phi_n\}$ is a Cauchy sequence in $C([\zeta_0, \mathcal{T}], \mathfrak{E})$. Consequently, there exist $\Phi \in C([\zeta_0, \mathcal{T}], \mathfrak{E})$ such that $\{\Phi_n\}$ converges to Φ. Thus, we need to illustrate that Φ is a solution of the problem (4.1).

$$\begin{aligned}
&\bar{\mathcal{D}}_0\Big(\Phi(\zeta) + \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, \Phi_0\Big)\\
&= \bar{\mathcal{D}}_0\Big(\Phi(\zeta) + \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi(\nu), \mathcal{H}_1\Phi(\nu), \mathcal{H}_2\Phi(\nu))d\nu, \Phi_{n+1}\\
&\quad+\frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu, \Phi_n(\nu), \mathcal{H}_1\Phi_n(\nu), \mathcal{H}_2\Phi_n(\nu))d\nu\Big)\\
&\leq \bar{\mathcal{D}}_0(\Phi(\zeta), \Phi_{n+1}) + \frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_1.\bar{\mathcal{D}}_0(\Phi(\nu), \Phi_n)d\nu\\
&\quad+\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_2.\bar{\mathcal{D}}_0(\mathcal{H}_1\Phi(\nu), \mathcal{H}_1\Phi_n)d\nu\\
&\quad+\frac{1}{\beta^{\vartheta}\Gamma(\vartheta)}\int_{\zeta_0}^{\zeta} \left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\mathcal{L}_3.\bar{\mathcal{D}}_0(\mathcal{H}_2\Phi(\nu), \mathcal{H}_2\Phi_n)d\nu\\
&\leq \bar{\mathcal{D}}_0(\Phi(\zeta), \Phi_{n+1}) + \mathcal{I}_{\zeta_0}^{\vartheta,\beta}\mathcal{P}(1+\mathcal{H}_1^*+\mathcal{H}_2^*)\bar{\mathcal{D}}_0(\Phi(\zeta), \Phi_n). \qquad (4.19)
\end{aligned}$$

In the limiting case, when $n \to \infty$. Thus we have

$$\Phi(\zeta) + \frac{-1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta} e^{\frac{\beta-1}{\beta}(\zeta-\nu)}(\zeta-\nu)^{\vartheta-1}\mathcal{F}(\nu,\Phi(\nu),\mathcal{H}_1\Phi(\nu),\mathcal{H}_2\Phi(\nu))d\nu = \Phi_0. \quad (4.20)$$

By Lemma 4.3, we prove that Φ is a solution of the problem (4.1). In order to prove the uniqness of $\Phi(\zeta)$, let $\Psi(\zeta)$ be another solution of problem (4.1) on $[\zeta_0,\mathcal{T}]$. Utilizing Lemma 4.3, gets

$$\bar{\mathcal{D}}_0(\Phi,\Psi) \le \frac{1}{\beta^{\vartheta}\Gamma(\mathfrak{q})}\int_{\zeta_0}^{\zeta}\left|e^{\frac{\beta-1}{\beta}(\zeta-\nu)}\right|\left|(\zeta-\nu)^{\vartheta-1}\right|\bar{\mathcal{D}}_0\Big(\mathcal{F}(\nu,\Phi(\nu),\mathcal{H}_1\Phi(\nu),\mathcal{H}_2\Phi(\nu),\mathcal{F}(\nu,\Psi(\nu),\mathcal{H}_1\Psi(\nu),\mathcal{H}_2\Psi(\nu))\Big)d\nu.$$

Analogously, by employing the distance properties $\bar{\mathcal{D}}_0$ and Lipschitiz continuity of $\mathcal{F}$, consequently, we deduce that $(1-\delta)\bar{\mathcal{D}}_0(\Phi,\Psi) \le 0$, since $\delta < 1$, we have $\Phi(\zeta) = \Psi(\zeta)$ for all $\zeta \in [\zeta_0,\mathcal{T}]$. Hence, the proof is completed. □

Example 4.6. Suppose the Cauchy problem by means of differential operator (2.4)

$$\mathcal{D}_z^{\vartheta,\beta}\Phi(z) = \mathcal{F}(z,\Phi(z)), \quad (4.21)$$

where $\mathcal{F}(z,\Phi(z))$ is analytic in Φ and $\Phi(z)$ is analytic in the unit disk. Therefore, $\mathcal{F}$ can be written as

$$\mathcal{F}(z,\Phi) = \varphi\Phi(z).$$

Consider $\mathcal{Z} = z^{\vartheta}$. Then the solution can be formulated as follows:

$$\Phi(\mathcal{Z}) = \sum_{j=0}^{\infty}\Phi_j\mathcal{Z}^j, \quad (4.22)$$

where Φ_j are constants. Putting (4.22) in (4.21), yields

$$\frac{\partial}{\partial z}\sum_{j=0}^{\infty}\Upsilon_{\vartheta,\beta,j}\Phi_j\mathcal{Z}^j - \varphi\sum_{j=0}^{\infty}\Phi_j\mathcal{Z}^j = 0.$$

Since

$$\Upsilon_{\vartheta,\beta,j} = \frac{\beta^{\vartheta}\Gamma(\frac{j\vartheta}{\beta}+1)}{j\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)},$$

then the simple computations gives the expression

$$\frac{\beta^{\vartheta}\Gamma(\frac{j\vartheta}{\beta}+1)}{\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)}\Phi_j - \varphi\Phi_{j-1} = 0.$$

Consequently, we get

$$\Phi_j = \Big(\frac{\varphi}{\beta^{\vartheta}}\Big)^j\frac{\Gamma(\frac{(j-1)\vartheta}{\beta}+1-\vartheta)\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)}{\Gamma(\frac{(j-1)\vartheta}{\beta}+1)\Gamma(\frac{j\vartheta}{\beta}+1)}.$$

Therefore, we have the subsequent solution

$$\Phi(\mathcal{Z}) = \sum_{j=0}^{\infty}\Big(\frac{\varphi}{\beta^{\vartheta}}\Big)^{j}\frac{\Gamma(\frac{(j-1)\vartheta}{\beta}+1-\vartheta)\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)}{\Gamma(\frac{(j-1)\vartheta}{\beta}+1)\Gamma(\frac{j\vartheta}{\beta}+1)}\mathcal{Z}^{j},$$

or equivalently

$$\Phi(\mathcal{Z}) = \sum_{j=0}^{\infty}\Big(\frac{\varphi}{\beta^{\vartheta}}\Big)^{j}\frac{\Gamma(j+1)\Gamma(\frac{(j-1)\vartheta}{\beta}+1-\vartheta)\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)}{\Gamma(\frac{(j-1)\vartheta}{\beta}+1)\Gamma(\frac{j\vartheta}{\beta}+1)}\frac{\mathcal{Z}^{j}}{j!},$$

where φ is assumed to be arbitrary constant, we take

$$\varphi := \beta^{\vartheta}.$$

Therefore, for appropriate ϑ, we have

$$\begin{aligned}\Phi(\mathcal{Z}) &= \sum_{j=0}^{\infty}\Big(\frac{\varphi}{\beta^{\vartheta}}\Big)^{j}\frac{\Gamma(j+1)\Gamma(\frac{(j-1)\vartheta}{\beta}+1-\vartheta)\Gamma(\frac{j\vartheta}{\beta}+1-\vartheta)}{\Gamma(\frac{(j-1)\vartheta}{\beta}+1)\Gamma(\frac{j\vartheta}{\beta}+1)}\frac{\mathcal{Z}^{j}}{j!}\\ &= {}_{3}\Psi_{2}\left[\begin{matrix}(1,1),\Big(1-\vartheta-\frac{\vartheta}{\beta},\frac{\vartheta}{\beta}\Big),\Big(1-\vartheta,\frac{\vartheta}{\beta}\Big);\\ \Big(1-\frac{\vartheta}{\beta},\frac{\vartheta}{\beta},\Big),\Big(1,\frac{\vartheta}{\beta}\Big);\end{matrix}\quad \mathcal{Z}\right]\\ &= {}_{3}\Psi_{2}\left[\begin{matrix}(1,1),\Big(1-\vartheta-\frac{\vartheta}{\beta},\frac{\vartheta}{\beta}\Big),\Big(1-\vartheta,\frac{\vartheta}{\beta}\Big);\\ \Big(1-\frac{\vartheta}{\beta},\frac{\vartheta}{\beta},\Big),\Big(1,\frac{\vartheta}{\beta}\Big);\end{matrix}\quad z^{\vartheta\beta}\right],\end{aligned}$$

where $|z| < 1$.

5. Conclusions

The present investigation deal with an IVP for $\mathcal{GPF}$ fuzzy $FDEs$ and we employ a new scheme of successive approximations under generalized Lipschitz condition to obtain the existence and uniqueness consequences of the solution to the specified problem. Furthermore, another method to discover exact solutions of $\mathcal{GPF}$ fuzzy $FDEs$ by utilizing the solutions of integer order differential equations is considered. Additionally, the existence consequences for $\mathcal{FVFIDE}s$ under $\mathcal{GPF}$-$\mathcal{HD}$ with fuzzy initial conditions are proposed. Also, the uniqueness of the so-called integrodifferential equations is verified. Meanwhile, we derived the equivalent integral forms of the original fuzzy $\mathcal{FVFIDE}s$ whichis utilized to examine the convergence of these arrangements of conditions. Two examples enlightened the efficacy and preciseness of the fractional-order $\mathcal{HD}$ and the other one presents the exact solution by means of the Fox-Wright function. For forthcoming mechanisms, we will relate the numerical strategies for the estimated solution of nonlinear fuzzy $FDEs$.

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Conflict of interest

The authors declare that they have no competing interests.

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