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Article

On Hybrid Contractions in the Context of Quasi-Metric Spaces

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Abstract: In this manuscript, we will investigate the existence of fixed points for mappings that satisfy some hybrid type contraction conditions in the setting of quasi-metric spaces. We provide examples to assure the validity of the given results. The results of this paper generalize several known theorems in the recent literature.

Keywords: contractions; hybrid contractions; quasi-metric spaces; metric spaces

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1. Introduction and Preliminaries

Roughly speaking, a quasi-metric is a distance function that is not symmetry but satisfies both the triangle inequality and self-distance property. The notion of quasi-metric was first introduced by Wilson in 1930s [1]. This is a subject of intensive research not only in the setting of topology [2–4] and functional analysis, but also several qualitative sciences, such as theoretical computer science [5–8], biology [9], and many other qualitative disciplines. In particular, as it is mentioned in [10], the notion of quasi-metric plays crucial roles in several distinct branches of mathematics, such as in the existence and uniqueness of iterated function systems' attractor (fractal), in the existence and uniqueness of Hamilton-Jacobi equations, and so on.

Another crucial notion that has no metric counterpart is that of an engaged partial order. Each partial order can be associated with a quasi-metric, and vice versa. Consequently, quasi-metric not only generalizes the concept of the metric, but also partial orders. This is a crucial fact for both the theoretical computer science applications and also has significance in the framework of biology [9].

For the sake of the completeness, we shall give the formal definition of quasi-metric. Throughout the paper, $\mathcal{X}$ is a nonempty set A distance function $q:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ is called a quasi-metric on $\mathcal{X}$ if

(q_1) $q(u,v)=0 \Leftrightarrow u=v$;

(q_2) $q(u,w)\leq q(u,v)+q(v,w)$, for all $u,v,w\in\mathcal{X}$.

In addition, the pair $(\mathcal{X},q)$ is called a quasi-metric space.

In what follows, we indicate the close relation between a standard metric and a quasi-metric. Given q be a quasi-metric on X, it is clear that the function $q_* : X \times X \to [0, \infty)$ defined by $q_*(u, v) = q(v, u)$ forms also a quasi-metric and it is also called the dual (conjugate) of q. The functions $d_1, d_2 : X \times X \to [0, \infty)$, where

$$\begin{aligned} d_1(v,u) &= q(u,v) + q_*(u,v), \\ d_2(v,u) &= \max\{q(u,v), q_*(u,v)\} \end{aligned}$$

form standard metrics on X.

We will provide an overview of quasi-metric spaces, presenting the notions of convergence, completeness, and continuity.

Let $\{u_n\}$ be a sequence in X, and $u \in X$, where (X, q) a quasi-metric space. We say that:

1. $\{u_n\}$ converges to u if and only if
$$\lim_{n\to\infty} q(u_n, u) = \lim_{n\to\infty} q(u, u_n) = 0. \tag{1}$$
2. $\{u_n\}$ is left-Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $k = k(\epsilon)$ such that $q(u_n, u_m) < \epsilon$ for all $n \geq m > k$.
3. $\{u_n\}$ is right-Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $k = k(\epsilon)$ such that $q(u_n, u_m) < \epsilon$ for all $m \geq n > k$.
4. $\{u_n\}$ is Cauchy if and only if it is left-Cauchy and right-Cauchy.

We would remark here that, in a quasi-metric space (X, q), the limit for a convergent sequence is unique. Indeed, if $u_n \to u$, for all $v \in X$, we have

$$\lim_{n\to\infty} q(u_n, v) = q(u, v) \text{ and } \lim_{n\to\infty} q(v, u_n) = q(v, u).$$

A quasi-metric space (X, q) is said to be: complete (respectively, left-complete or right-complete) if and only if each Cauchy sequence (respectively, left-Cauchy sequence or right-Cauchy sequence) in X is convergent. Notice, in this context, that "right completeness" is equivalent to "Smyth completeness" [11]. See also [12].

A mapping $T : X \to X$ is continuous provided that, for any sequence $\{u_n\}$ in X such that $u_n \to u \in X$, the sequence $\{Tu_n\}$ converges to Tu, that is,

$$\lim_{n\to\infty} q(Tu_n, Tu) = \lim_{n\to\infty} q(Tu, Tu_n) = 0 \tag{2}$$

If $T : X \to X$, then the fixed point set of T is $\mathcal{F}_T(X) := \{x \in X : Tx = x\}$.

A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called an *extended simulation function* if the following axioms are fulfilled:

(z_d) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(z_0) $\zeta(t, 0) \leq 0$ for every $t \geq 0$ and $\zeta(t, 0) = 0 \Leftrightarrow t = 0$.

Notice that the axiom (z_d) implies that $\zeta(t, t) < 0$ for all $t > 0$. Let us denote by $\mathcal{Z}$ the family of all extended simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$.

A function $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* [13] if:

(c_1) φ is increasing;
(c_2) $\lim_{n\to\infty} \varphi^n(t) = 0$, for $t \in [0, \infty)$.

Proposition 1. *If φ is a comparison function, then:*

(i) *each φ^k is also a comparison function, for all $k \in \mathbb{N}$;*
(ii) *φ is continuous at 0;*
(iii) *$\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.*

A function $\psi : [0, \infty) \to [0, \infty)$ is called a *c-comparison function* [13,14] if:

(cc_1) ψ is increasing;
(cc_2) $\sum_{n=0}^{\infty} \psi^n(t) < \infty$, for all $t \in (0, \infty)$.

We denote by Ψ the family of c-comparison functions. In some papers, instead of a c-comparison function, the term of strong comparison function is used. See [13].

Remark 1. *Any c-comparison function is a comparison function.*

Let $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function. We say that a mapping $T : \mathcal{X} \to \mathcal{X}$ is α-orbital admissible [15] if for each $u \in \mathcal{X}$ we have

$$\alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, T^2u) \geq 1.$$

Lemma 1. *Let $T : \mathcal{X} \to \mathcal{X}$ be an α-orbital admissible function. If there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then the sequence $(u_n)_{n\in\mathbb{N}}$, defined by $u_n = Tu_{n-1}$, $n \in \mathbb{N}$ satisfies the following relations:*

$$\alpha(u_n, u_{n+1}) \geq 1 \text{ and } \alpha(u_{n+1}, u_n) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

We say that the set $\mathcal{X}$ is *regular* with respect to mapping $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ if the following condition is satisfied: if $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\alpha(u_{n+1}, u_n) \geq 1$ and $\alpha(u_n, u_{n+1}) \geq 1$ for any $n \in \mathbb{N}$ and $u_n \to u \in \mathcal{X}$ as $n \to \infty$, then there exists a subsequence $\left\{u_{n(i)}\right\}$ of $\{u_n\}$ such that

$$\alpha(u_{n(i)}, u) \geq 1 \text{ and } \alpha(u, u_{n(i)}) \geq 1,$$

for each i.

In this manuscript, we will investigate the existence of fixed points for mappings that satisfy some hybrid type contraction conditions in the setting of quasi-metric spaces. We provide examples to assure the validity of the given results. The results of this paper generalize several known theorems in the recent literature, see [13,14,16–25].

2. Main Results

We start with the formal definition of hybrid almost contraction of type $\mathbb{I}$.

Definition 1. *Let $(\mathcal{X}, q)$ be a quasi-metric space. We say that the mapping $T : \mathcal{X} \to \mathcal{X}$ is a* hybrid almost contraction of type $\mathbb{I}$, *if there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$, $p \geq 0$, $L \geq 0$ and $a_1, a_2, a_3 \in [0, 1]$ with $a_1 + a_2 > 0$, $a_1 + a_2 + a_3 = 1$, such that, for all distinct $u, v \in \mathcal{X}$, we have*

$$\begin{aligned} &\tfrac{1}{2}\min\{q(u, Tu), q(v, Tv)q(Tv, v)\} \leq q(u, v) \text{ implies} \\ &\qquad \zeta(\alpha(u, v)q(Tu, Tv), \psi(I_p(u, v) + L\mathcal{N}(u, v))) \geq 0, \end{aligned} \tag{3}$$

where

$$I_p(u,v) = \begin{cases} [a_1(q(u,v))^p + a_2(q(u,Tu))^p + a_3(q(v,Tv))^p]^{1/p}, & \text{for } p > 0, \\ (q(u,v))^{a_1} \cdot (q(u,Tu))^{a_2} \cdot (q(v,Tv))^{a_3} & \text{for } p = 0 \end{cases}$$

and

$$\mathcal{N}(u,v) = \min\{q(u,Tv), q(v,Tu)\}.$$

Theorem 1. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a mapping such that:*

(i) $u = Tu$ *implies* $\alpha(u,v) > 0$ *for every* $v \in \mathcal{X}$*;*
(ii) $v = Tv$ *implies* $\alpha(u,v) > 0$ *for every* $u \in \mathcal{X}$*.*

Suppose that $T : \mathcal{X} \to \mathcal{X}$ is an hybrid almost contraction of type $\mathbb{I}$ and

(C_1) *T is α-orbital admissible;*
(C_2) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$;*
(C_3) *T is continuous.*

Then, T has a fixed point.

Proof. Let the sequence $\{u_n\}$ in $\mathcal{X}$ be defined by

$$u_1 = Tu_0,\ u_2 = Tu_1,\ \ldots,\ u_n = Tu_{n-1} = T^{n-1}u_0$$

where $u_0 \in \mathcal{X}$ is the point such that, from (C_2), $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. Indubitably, for all $n \in \mathbb{N}$, we have $u_{n+1} \neq u_n$. As a matter of fact, if we suppose that there is $N_0 \in \mathbb{N}$ such that $u_{N_0} = u_{N_0+1}$, from the manner in which the sequence $\{u_n\}$ was defined, we get

$$u_{N_0} = Tu_{N_0} = u_{N_0+1}$$

so that the fixed point of T is u_{N_0} and the proof is completed. Thus, choosing $u = u_{n-1}$ respectively $v = u_n$ and since $\frac{1}{2}\min\{q(u_{n-1}, Tu_{n-1}), q(u_n, Tu_n), q(Tu_n, u_n)\} \leq \frac{1}{2}q(u_{n-1}, Tu_{n-1}) < q(u_{n-1}, u_n)$ holds for any $n \in \mathbb{N}$, by (3), we get

$$\zeta(\alpha(u_{n-1}, u_n)q(Tu_{n-1}, Tu_n), \psi(I_p(u_{n-1}, u_n) + L\mathcal{N}(u_{n-1}, u_n))) \geq 0. \tag{4}$$

In other words, taking into account (z_d),

$$0 \leq \psi(I_p(u_{n-1}, u_n) + L\mathcal{N}(u_{n-1}, u_n)) - \alpha(u_{n-1}, u_n)q(Tu_{n-1}, Tu_n). \tag{5}$$

However, T is an α-orbital admissible and, on the strength of Lemma 1, the above inequality yields

$$q(Tu_{n-1}, Tu_n) \leq \alpha(u_{n-1}, u_n)q(Tu_{n-1}, Tu_n) \leq \psi(I_p(u_{n-1}, u_n) + L\mathcal{N}(u_{n-1}, u_n)). \tag{6}$$

Since

$$\begin{aligned} \mathcal{N}(u_{n-1}, u_n) &= \min\{q(u_{n-1}, Tu_n), q(u_n, Tu_{n-1})\} \\ &= \min\{q(u_{n-1}, u_n), q(u_n, u_n)\} = 0, \end{aligned} \tag{7}$$

the inequality (6) becomes

$$q(Tu_{n-1}, Tu_n) \leq \psi(I_p(u_{n-1}, u_n)). \tag{8}$$

In addition, by taking $u = u_n$, respectively, $v = u_{n-1}$, we have

$$\begin{aligned}\frac{1}{2}\min\{q(u_n, Tu_n), q(u_{n-1}, Tu_{n-1}), q(Tu_{n-1}, u_{n-1})\} &\leq \frac{1}{2}\min\{q(u_n, u_{n+1}), q(u_{n-1}, u_n), q(u_n, u_{n-1})\}\\ &< q(u_n, u_{n-1}).\end{aligned}$$

As a consequence, (3) becomes

$$\zeta(\alpha(u_n, u_{n-1})q(Tu_n, Tu_{n-1}), \psi(I_p(u_n, u_{n-1}) + L\mathcal{N}(u_n, u_{n-1}))) \geq 0, \tag{9}$$

or, taking into account (z_d),

$$0 \leq \psi(I_p(u_n, u_{n-1}) + L\mathcal{N}(u_n, u_{n-1})) - \alpha(u_n, u_{n-1})q(Tu_n, Tu_{n-1}).$$

By Lemma 1, the above inequality yields

$$\begin{aligned}q(u_{n+1}, u_n) &= q(Tu_n, Tu_{n-1}) \leq \alpha(u_n, u_{n-1})q(Tu_n, Tu_{n-1})\\ &\leq \psi(I_p(u_n, u_{n-1}) + L\mathcal{N}(u_n, u_{n-1})).\end{aligned} \tag{10}$$

However,

$$\begin{aligned}\mathcal{N}(u_n, u_{n-1}) &= \min\{q(u_n, Tu_{n-1}), q(u_{n-1}, Tu_n)\}\\ &= \min\{q(u_n, u_n), q(u_{n-1}, u_{n+1})\} = 0,\end{aligned} \tag{11}$$

and then we get

$$q(Tu_n, Tu_{n-1}) \leq \psi(I_p(u_n, u_{n-1})). \tag{12}$$

From this point of the proof, we will consider the two cases separately: $p > 0$ and $p = 0$.

Case 1. For the case $p > 0$,

$$\begin{aligned}I_p(u_{n-1}, u_n) &= [a_1(q(u_{n-1}, u_n))^p + a_2(q(u_{n-1}, Tu_{n-1}))^p + a_3(q(u_n, Tu_n))^p]^{1/p}\\ &= [a_1(q(u_{n-1}, u_n))^p + a_2(q(u_{n-1}, u_n))^p + a_3(q(u_n, u_{n+1}))^p]^{1/p}\\ &= [(a_1 + a_2)(q(u_{n-1}, u_n))^p + a_3(q(u_n, u_{n+1}))^p]^{1/p}\end{aligned}$$

and the inequality (6) becomes

$$q(u_n, u_{n+1}) = q(Tu_{n-1}, Tu_n) \leq \psi([(a_1 + a_2)(q(u_{n-1}, u_n))^p + a_3(q(u_n, u_{n+1}))^p]^{1/p}). \tag{13}$$

Onward, being a c-comparison function, ψ satisfies (iii) by Proposition 1 that is $\psi(t) < t$ for any $t > 0$, we obtain

$$\begin{aligned}q(u_n, u_{n+1}) &\leq \psi([(a_1 + a_2)(q(u_{n-1}, u_n))^p + a_3(q(u_n, u_{n+1}))^p]^{1/p})\\ &< [(a_1 + a_2)(q(u_{n-1}, u_n))^p + (1 - a_1 - a_2)(q(u_n, u_{n+1}))^p]^{1/p},\end{aligned}$$

which is equivalent with

$$(a_1 + a_2)[q(u_n, u_{n+1})]^p < (a_1 + a_2)[q(u_{n-1}, u_n)]^p,$$

or (since $a_1 + a_2 > 0$)

$$q(u_n, u_{n+1}) < q(u_{n-1}, u_n). \tag{14}$$

Using the fact that $\psi \in \Psi$ is increasing, by (13), we have

$$q(u_n, u_{n+1}) < \psi(q(u_{n-1}, u_n)) < \psi^2(q(u_{n-2}, u_{n-1})) < ... < \psi^n(q(u_0, u_1)) \tag{15}$$

Let now $l \geq 1$. By using (15) and the triangle inequality, we get

$$\begin{aligned} q(u_n, u_{n+l}) &\leq q(u_n, u_{n+1}) + ... + q(u_{n+l-1}, u_{n+l}) \\ &\leq \sum_{j=n}^{n+l-1} \psi^j(q(u_0, u_1)) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)). \end{aligned} \tag{16}$$

Letting $n \to \infty$ in the above inequality, we derive that $\sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0$. Hence, $q(u_n, u_{n+l}) \to 0$ as $n \to \infty$. Thus, $\{u_n\}$ is a right-Cauchy sequence in $(\mathcal{X}, q)$.

Similarly, since

$$\begin{aligned} I_p(u_n, u_{n-1}) &= [a_1(q(u_n, u_{n-1}))^p + a_2(q(u_n, Tu_n))^p + a_3(q(u_{n-1}, Tu_{n-1}))^p]^{1/p} \\ &= [a_1(q(u_n, u_{n-1}))^p + a_2(q(u_n, u_{n+1}))^p + a_3(q(u_{n-1}, u_n))^p]^{1/p}, \end{aligned}$$

the inequality (12) becomes

$$\begin{aligned} q(u_{n+1}, u_n) &\leq \psi(I_p(u_n, u_{n-1})) < I_p(u_n, u_{n-1}) \\ &= [a_1(q(u_n, u_{n-1}))^p + a_2(q(u_n, u_{n+1}))^p + a_3(q(u_{n-1}, u_n))^p]^{1/p}. \end{aligned} \tag{17}$$

Taking into account (14), we get

$$\begin{aligned} (q(u_{n+1}, u_n))^p &< a_1(q(u_n, u_{n-1}))^p + a_2(q(u_n, u_{n+1}))^p + a_3(q(u_{n-1}, u_n))^p \\ &= a_1(q(u_n, u_{n-1}))^p + a_2(q(u_n, u_{n+1}))^p + (1 - a_1 - a_2)(q(u_{n-1}, u_n))^p \\ &< a_1(q(u_n, u_{n-1}))^p + (1 - a_1)(q(u_{n-1}, u_n))^p, \text{ for any } n \in \mathbb{N}. \end{aligned}$$

We are able to examine it with the following cases.

a. If $q(u_n, u_{n-1}) < q(u_{n-1}, u_n)$ for any $n \in \mathbb{N}$, the above inequality becomes

$$(q(u_{n+1}, u_n))^p < (q(u_{n-1}, u_n))^p,$$

and then, together with (15),

$$q(u_{n+1}, u_n) < q(u_{n-1}, u_n) < \psi^{n-1}(u_0, u_1), \forall n \geq 1. \tag{18}$$

From the triangle inequality and (18), for all $l \geq 1$, we get that

$$\begin{aligned} q(u_{n+l}, u_n) &\leq q(u_{n+l}, u_{n+l-1}) + ... + q(u_{n+1}, u_n) \\ &\leq \sum_{j=n}^{n+l-1} \psi^j(q(u_0, u_1)) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

b. If, for any $n \in \mathbb{N}$, $q(u_n, u_{n-1}) \leq q(u_{n-1}, u_n)$, we have

$$q(u_{n+1}, u_n) < q(u_n, u_{n-1})$$

and, from (17), regarding $\psi \in \Psi$, we get that

$$q(u_{n+1}, u_n) < \psi(q(u_n, u_{n-1})) < ... < \psi^n(q(u_1, u_0)). \tag{19}$$

Again, by triangle inequality,

$$\begin{aligned} q(u_{n+l}, u_n) &\leq q(u_{n+l}, u_{n+l-1}) + ... + q(u_{n+1}, u_n) \\ &\leq \sum_{j=n}^{n+l-1} \psi^j(q(u_1, u_0)) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_1, u_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

c. If $q(u_i, u_{i-1}) \leq q(u_{i-1}, u_i)$ for some $i \in \mathbb{N}$ and $q(u_k, u_{k-1}) > q(u_{k-1}, u_k)$ for some $k \in \mathbb{N}$, then we have for $l \in \mathbb{N}$

$$\begin{aligned} q(u_{n+l}, u_n) &\leq q(u_{n+l}, u_{n+l-1}) + ... + q(u_{n+1}, u_n) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_1, u_0)) + \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, we proved that $\{u_n\}$ is a left-Cauchy in (X, q).

Thus, being left and right Cauchy, the sequence $\{u_n\}$ is a Cauchy in complete quasi-metric space (X, q), which implies that there is $u^* \in X$ such that

$$\lim_{n\to\infty} q(u_n, u^*) = \lim_{n\to\infty} q(u^*, u_n) = 0. \tag{20}$$

Using the continuity of T and $(q1)$, we have

$$\lim_{n\to\infty} q(u_n, Tu^*) = \lim_{n\to\infty} q(Tu_{n-1}, Tu^*) = 0,$$

$$\lim_{n\to\infty} q(Tu^*, u_n) = \lim_{n\to\infty} q(Tu^*, Tu_{n-1}) = 0$$

and so

$$\lim_{n\to\infty} q(u_n, Tu^*) = \lim_{n\to\infty} q(Tu^*, u_n) = 0. \tag{21}$$

It follows from (20) and (21) that $Tu^* = u^*$, that is, u^* is a fixed point of T.

Case 2. In the case $p = 0$, we have

$$\begin{aligned} I_p(u_{n-1}, u_n) &= (q(u_{n-1}, u_n))^{a_1} \cdot (q(u_{n-1}, Tu_{n-1}))^{a_2} \cdot (q(u_n, Tu_n))^{a_3} \\ &= (q(u_{n-1}, u_n))^{a_1} \cdot (q(u_{n-1}, u_n))^{a_2} \cdot (q(u_n, u_{n+1}))^{a_3}. \end{aligned}$$

Replacing in (6) and taking into account (7), we get

$$\begin{aligned} q(u_n, u_{n+1}) &= q(Tu_{n-1}, Tu_n) \leq \alpha(u_{n-1}, u_n) q(Tu_{n-1}, Tu_n) \leq \psi(I_p(u_{n-1}, u_n)) \\ &< I_p(u_{n-1}, u_n) = (q(u_{n-1}, u_n))^{a_1+a_2} \cdot (q(u_n, u_{n+1}))^{a_3} \end{aligned} \tag{22}$$

and we deduce that

$$(q(u_n, u_{n+1}))^{a_1+a_2} < (q(u_{n-1}, u_n))^{a_1+a_2}.$$

Thus, taking into account $a_1 + a_2 > 0$, we have

$$q(u_n, u_{n+1}) < q(u_{n-1}, u_n) \tag{23}$$

and, from (22), since $\psi \in \Psi$ we are able to say that, for any $n \in \mathbb{N}$,

$$q(u_n, u_{n+1}) \leq \psi(q(u_{n-1}, u_n)) < ... < \psi^n(q(u_0, u_1)). \tag{24}$$

Following the above lines and using the triangle inequality, we obtain that the sequence u_n is right Cauchy. Likewise, because

$$\begin{aligned} I_p(u_n, u_{n-1}) &= (q(u_n, u_{n-1}))^{a_1} \cdot (q(u_n, Tu_n))^{a_2} \cdot (q(u_{n-1}, Tu_{n-1}))^{a_3}] \\ &= (q(u_n, u_{n-1}))^{a_1} \cdot (q(u_n, u_{n+1}))^{a_2} \cdot (q(u_{n-1}, u_n))^{a_3}, \end{aligned}$$

taking into account (11) and (23), we have

$$\begin{aligned} q(u_{n+1}, u_n) &= q(Tu_n, Tu_{n-1}) \leq \alpha(u_n, u_{n-1}) q(Tu_n, Tu_{n-1}) \leq \psi(I_p(u_n, u_{n-1})) \\ &< I_p(u_n, u_{n-1}) = (q(u_n, u_{n-1}))^{a_1} \cdot (q(u_n, u_{n+1}))^{a_2} \cdot (q(u_{n-1}, u_n))^{a_3} \\ &< (q(u_n, u_{n-1}))^{a_1} \cdot (q(u_{n-1}, u_n))^{a_2+a_3} \\ &\leq (\max\{q(u_n, u_{n-1}), q(u_{n-1}, u_n)\})^{a_1+a_2+a_3} \\ &= \max\{q(u_n, u_{n-1}), q(u_{n-1}, u_n)\}. \end{aligned}$$

We must examine two cases.

If $\max\{q(u_n, u_{n-1}), q(u_{n-1}, u_n)\} = q(u_{n-1}, u_n)$, then since $q(u_{n-1}, u_n) > 0$, we get that

$$q(u_{n+1}, u_n) \leq \psi(q(u_{n-1}, u_n)),$$

and recursively

$$q(u_{n+1}, u_n) \leq \psi^n(q(u_0, u_1)). \tag{25}$$

If $\max\{q(u_n, u_{n-1}), q(u_{n-1}, u_n)\} = q(u_n, u_{n-1})$, we have

$$q(u_{n+1}, u_n) \leq \psi(q(u_n, u_{n-1})) < ... < \psi^n(q(u_1, u_0)). \tag{26}$$

Therefore, by combining (25) with (26), we derive (due to (c_2)) that

$$\lim_{n\to\infty} q(u_{n+1}, u_n) = \lim_{n\to\infty} \max\{\psi^n(q(u_0, u_1)), \psi^n(q(u_1, u_0))\} = 0.$$

Again, using the triangle inequality, and the above inequalities for all $l \geq 1$, we get

$$\begin{aligned} q(u_{n+l}, u_n) &\leq q(u_{n+l}, u_{n+l-1}) + ... + q(u_{n+1}, u_n) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_1, u_0)) + \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

that is, the sequence $\{u_n\}$ is left Cauchy, so that is a Cauchy sequence in a complete quasi-metric space $(\mathcal{X}, q)$. Thus, there is $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} q(u^*, u_n) = 0 = \lim_{n\to\infty} q(u^*, u_n). \tag{27}$$

Of course, using (q_1) and the continuity of T, we have $Tu^* = u^*$. □

Corollary 1. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space, a function $\alpha : X \times X \to [0, \infty)$ and a mapping $T : \mathcal{X} \to \mathcal{X}$ such that there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for $p \geq 0$, $L \geq 0$ and $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 > 0$ and $a_1 + a_2 + a_3 = 1$, we have*

$$\zeta(\alpha(u, v)q(Tu, Tv), \psi(I_p(u, v) + L\mathcal{N}(u, v))) \geq 0, \text{ for all distinct } u, v \in \mathcal{X}. \tag{28}$$

Suppose also that the following assumptions hold:

(i) $u = Tu$ *implies* $\alpha(u, v) > 0$ *for every* $v \in \mathcal{X}$*;*
(ii) $v = Tv$ *implies* $\alpha(u, v) > 0$ *for every* $u \in \mathcal{X}$*;*
(i) *T is α-orbital admissible;*
(ii) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$;*
(iv) *T is continuous.*

Then, T has a fixed point.

Remark 2. *Of course, in particular letting $L = 0$ in the above Corollary, we find Theorem 2.1. in [16].*

Corollary 2. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and a mapping $T : \mathcal{X} \to \mathcal{X}$ such that there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for $p \geq 0$, $L \geq 0$ and $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 > 0$ and $a_1 + a_2 + a_3 = 1$, we have*

$$\zeta(q(Tu, Tv), \psi(I_p(u, v) + L\mathcal{N}(u, v))) \geq 0, \text{ for all distinct } u, v \in \mathcal{X}. \tag{29}$$

Then, T has a fixed point.

Proof. Let $\alpha(u, v) = 1$ in Corollary 1. □

Corollary 3. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space, a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and a continuous mapping $T : \mathcal{X} \to \mathcal{X}$ such that there exist $\psi \in \Psi$ such that, for $p \geq 0$ and $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 > 0$ and $a_1 + a_2 + a_3 = 1$, we have*

$$\alpha(u, v)q(Tu, Tv) \leq \psi(I_p(u, v)), \text{ for all distinct } u, v \in \mathcal{X}. \tag{30}$$

Suppose that there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. Then, T has a fixed point.

Proof. Let $\zeta(t, s) = \psi(s) - t$ in Corollary 1. □

Moreover, it easy to see that Theorem 1 is a generalization of Theorem 2.1 in [18] in the context of quasi-metric space. Indeed, if we take $L = 0$ and $p = 0$ in Corollary 3, we find:

Corollary 4. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space, a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, and a continuous mapping $T : \mathcal{X} \to \mathcal{X}$ such that there exists $\psi \in \Psi$ such that, for $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 > 0$ and $a_1 + a_2 + a_3 = 1$, we have*

$$\alpha(u, v) q(Tu, Tv) \leq \psi((q(u, v))^{a_1} \cdot (q(u, Tu))^{a_2} \cdot (q(v, Tv))^{a_3}), \text{ for all distinct } u, v \in \mathcal{X}. \tag{31}$$

Suppose that there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. Then, T has a fixed point.

Inspired by the example of [10], we consider the following:

Example 1. *Let the set $\mathcal{X} = [1, \infty)$ and the quasi-metric $q : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ given by*

$$q(u, v) = \begin{cases} \ln v - \ln u, & \text{if } u \leq v \\ \frac{1}{3}(\ln u - \ln v), & \text{if } u > v \end{cases}.$$

(see Example 4.1 in [10].) Let the mapping $T : \mathcal{X} \to \mathcal{X}$, be defined by

$$Tu = \begin{cases} 1, & \text{if } u \in [1, 2] \\ e^{u-2}, & \text{if } u \in (2, \infty) \end{cases}$$

and the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by

$$\alpha(u, v) = \begin{cases} 2, & \text{if } u, v \in [1, 2) \\ 3, & \text{if } u = 1, v = 2 \text{ or } u = 2, v = 1 \\ 6, & \text{if } u = 3, v = 2 \\ 0, & \text{otherwise} \end{cases}.$$

Since the mapping T is continuous and for $u = 2$, $\alpha(2, T2) = \alpha(2, 1) = 3$ and $\alpha(T2, 2) = \alpha(1, 2) = 3$, we have that the assumptions $(C_2), (C_3)$ are satisfied. Moreover, for any $u \in [1, 2)$, we have

$$\alpha(u, Tu) = \alpha(u, 1) = 2 \Rightarrow \alpha(T1, T^2 1) = \alpha(1, 1) = 2$$

and

$$\alpha(2, T2) = \alpha(2, 1) = 3 \Rightarrow \alpha(T2, T^2 2) = \alpha(1, 1) = 2,$$

so that T is α-orbital admissible.

Choosing $\psi(t) = \frac{1}{3}t$, $p = 2$, $a_1 = a_2 = a_3 = \frac{1}{3}$ and $L = 24$, we have the following cases:

***Case 1.** If $u, v \in [1, 2]$, then $q(u, v) = q(1, 1) = 0$ and (3) holds for every $\zeta \in \mathcal{Z}$.*

***Case 2.** If $u = 3$, $v = 2$, then we have*

$$\begin{aligned} &q(3, T3) = q(3, e) = \tfrac{1}{3} \ln \tfrac{3}{e}, \quad q(2, T2) = q(2, 1) = \tfrac{1}{3} \ln 2, q(T2, 2) = q(1, 2) = \ln 2, \\ &q(3, 2) = \tfrac{1}{3} \ln \tfrac{3}{2}, \quad q(T3, T2) = q(e, 1) = \tfrac{1}{3}, \\ &q(3, T2) = q(3, 1) = \tfrac{1}{3} \ln 3, \quad q(2, T3) = q(2, e) = \ln \tfrac{e}{2}. \end{aligned}$$

Thus, we have

$$\frac{1}{2} \min \{q(3, T3), q(2, T2), q(T2, 2)\} = \frac{1}{6} \ln \frac{3}{e} < \frac{1}{3} \ln \frac{3}{2} = q(3, 2)$$

and

$$\alpha(3,2)q(T3,T2) = \frac{6}{3} < \frac{1}{3}\left[\sqrt{\frac{1}{3}}\left((\frac{1}{3}\ln\frac{3}{2})^2 + (\frac{1}{3}\ln\frac{3}{e})^2 + (\frac{1}{3}\ln 2)^2\right)^{1/2} + 24\ln\frac{e}{2}\right] = \psi(I_p(3,2) + L\mathcal{N}(3,2)),$$

so that T is a hybrid almost contraction for any $\zeta \in \mathcal{Z}$.

The other cases are not interesting, while $\alpha(u,v) = 0$. (Consequently, the mapping T has two fixed points, $u_1 = 1$ and $u_2 \in (3,4)$.)

On the other hand, since

$$\begin{aligned}\alpha(3,2)q(T3,T2) &= 2 > (\tfrac{1}{3}\ln\tfrac{3}{2})^{\gamma}(\tfrac{1}{3}\ln\tfrac{3}{e})^{\beta}(\tfrac{1}{3}\ln 2)^{1-\gamma-\beta}\\ &> \psi\left((q(3,2))^{\gamma}(q(3,T3))^{\beta},(q(2,T2))^{1-\gamma-\beta}\right)\end{aligned}$$

for every $\gamma,\beta \in (0,1)$ and $\psi \in \Psi$, Theorem 2.1 in [18] can not be applied.

In particular, for the case $p = 0$, the continuity condition of T can be replaced with the regularity condition of the space X.

Theorem 2. *Let (X,q) be a complete quasi-metric space, a function $\alpha : X \times X \to [0,\infty)$ and a mapping $T : X \to X$ such that there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$, $L \geq 0$ and $a_1, a_2, a_3 \in [0,1]$ with $a_1 + a_2 + a_3 = 1$, such that, for all distinct $u, v \in X$, we have*

$$\begin{aligned}&\tfrac{1}{2}\min\{q(u,Tu), q(v,Tv), q(Tv,v)\} \leq q(u,v) \text{ implies}\\ &\qquad \zeta(\alpha(u,v)q(Tu,Tv), \psi((q(u,v))^{a_1}\cdot(q(u,Tu))^{a_2}\cdot(q(v,Tv))^{a_3} + L\mathcal{N}(u,v))) \geq 0.\end{aligned} \tag{32}$$

Suppose also that

- *(i)* *$u = Tu$ implies $\alpha(u,v) > 0$ for every $v \in X$;*
- *(ii)* *$v = Tv$ implies $\alpha(u,v) > 0$ for every $u \in X$;*
- (C_1) *T is α-orbital admissible;*
- (C_2) *there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$;*
- (C_3) *X is regular with respect to the mapping α.*

Then, T has a fixed point.

Proof. From the above theorem, there exists $u^* \in X$ such that (27) hold. In what follows, we claim that

$$\begin{aligned}&\tfrac{1}{2}\min\left\{q(u^*,Tu^*), q(u_{n(i)},Tu_{n(i)}), q(Tu_{n(i)},u_{n(i)})\right\} \leq q(u^*,u_{n(i)}) \quad \text{or}\\ &\tfrac{1}{2}\min\left\{q(u_{n(i)-1},Tu_{n(i)-1}), q(u^*,Tu^*), q(Tu^*,u^*)\right\} \leq q(u_{n(i)-1},u^*).\end{aligned} \tag{33}$$

Indeed, using the method of Reductio ad Absurdum, we assume that that there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}&\tfrac{1}{2}\min\{q(u^*,Tu^*), q(u_k,Tu_k), q(Tu_k,u_k)\} > q(u^*,u_k) \quad \text{and}\\ &\tfrac{1}{2}\min\{q(u_{k-1},Tu_{k-1}), q(u^*,Tu^*), q(Tu^*,u^*)\} > q(u_{k-1},u^*)\end{aligned}$$

Therefore, we have

$$\begin{aligned} q(u_{k-1}, u_k) &\leq q(u_{k-1}, u^*) + q(u^*, u_k) \\ &< \tfrac{1}{2} \min\{q(u_{k-1}, Tu_{k-1}), q(u^*, Tu^*), q(Tu^*, u^*)\} + \tfrac{1}{2} \min\{q(u^*, Tu^*), q(u_{k-1}, Tu_{k-1}), q(Tu_{k-1}, u_{k-1})\} \\ &< \tfrac{1}{2}[\min\{q(u_{k-1}, u_k), q(u^*, Tu^*), q(Tu^*, u^*)\} + \min\{q(u^*, Tu^*), q(u_{k-1}, u_k), q(u_k, u_{k-1})\}] \\ &\leq \tfrac{1}{2}[q(u_{k-1}, u_k) + q(u_{k-1}, u_k)] \\ &= q(u_{k-1}, u_k), \end{aligned}$$

which is a contradiction.

In the alternative hypothesis, if the space X is regular with respect to mapping α, we have $\alpha(u^*, u_{n(i)}) \geq 1$, where$\{u_{n(i)}\}$ is a sub-sequence of $\{u_n\}$, for $i \in \mathbb{N}$. We will suppose by *reductio ad absurdum* that $u^* \neq Tu^*$. Then, for $u = u^*$ and $v = u_{n(i)}$ in (3), we get

$$\zeta(\alpha(u^*, u_{n(i)})q(Tu^*, Tu_{ni)}), \psi(\mathcal{A}_p(u^*, u_{n(i)}))) \geq 0.$$

Taking into account the properties of function ζ, ψ, and α, the above relation becomes

$$\begin{aligned} q(Tu^*, u^*) &\leq q(Tu^*, Tu_n) + q(Tu_n, u^*) \leq \alpha(u^*, u_n)q(Tu^*, Tu_{n(i)}) + q(u_{n(i)+1}, u^*) \\ &\leq \psi((q(u^*, u_{n(i)}))^{a_1} \cdot (q(u^*, Tu^*))^{a_2} \cdot (q(u_{n(i)}, Tu_{n(i)}))^{a_3} + \mathcal{N}(u^*, u_{n(i)})) + q(u_{n(i)+1}, u^*), \end{aligned}$$

Letting $i \to \infty$, we have

$$0 < q(Tu^*, u^*) < \lim_{i \to \infty} \psi((q(u^*, u_{n(i)}))^{a_1} \cdot (q(u^*, Tu^*))^{a_2} \cdot (q(u_{n(i)}, Tu_{n(i)}))^{a_3} + \mathcal{N}(u^*, u_{n(i)})) + q(u_{n(i)+1}, u^*)$$

and, since ψ is continuous in 0, $\psi(0) = 0$, we get $q(Tu^*, u^*) = 0$. □

Corollary 5. *Let (X, q) be a complete quasi-metric space and $T : X \to X$ be a given mapping. Assume that there exist $L \geq 0$, $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for all distinct $u, v \in X$, we have*

$$\tfrac{1}{2} \min\{q(u, Tu), q(v, Tv)q(Tv, v)\} \leq q(u, v) \text{ implies } \zeta(q(Tu, Tv), \psi(I_p(u, v) + L\mathcal{N}(u, v))) \geq 0,$$

for all distinct $u, v \in X$. Then, T has a fixed point.

Proof. It is sufficient to take $\alpha(u, v) = 1$ for $u, v \in X$ in Theorem 5. □

Corollary 6. *Let (X, q) be a complete quasi-metric space and $T : X \to X$ be a given mapping. Assume that there exist $L \geq 0$, $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for all distinct $u, v \in X$, we have*

$$\tfrac{1}{2} \min\{q(u, Tu), q(v, Tv)q(Tv, v)\} \leq q(u, v) \text{ implies } q(Tu, Tv) \leq kI_p(u, v)$$

for all distinct $u, v \in X$. Then, T has a fixed point.

Proof. It is sufficient to take $L = 0$, $\zeta(t, s) = k_1 s - t$, $\psi(u) = k_2 u$ with $k_1, k_2 \in (0, 1)$ and $k = k_1 k_2$ in Corollary 5. □

Corollary 7. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and $T : \mathcal{X} \to \mathcal{X}$ a continuous mapping such that*

$$\begin{aligned} &\tfrac{1}{2}\min\{q(u,Tu), q(v,Tv)q(Tv,v)\} \leq q(u,v) \text{ implies} \\ &q(Tu,Tv) \leq \tfrac{\Bbbk}{\sqrt{3}} \cdot \sqrt{(q(u,v))^2 + (q(u,Tu))^2 + (q(v,Tv))^2} \end{aligned} \tag{34}$$

for all distinct $u, v \in \mathcal{X}$ and some $\Bbbk \in (0,1)$. Then, T has a fixed point in $\mathcal{X}$.

Proof. Let $p = 2$ and $a_1 = a_2 = a_3 = \frac{1}{3}$ in Corollary 6. □

In the next theorem, we involve a Jaggi type expression with the hybrid contractions.

Definition 2. *Let $(\mathcal{X}, q)$ be a quasi-metric space. A mapping $T : \mathcal{X} \to \mathcal{X}$ is called a* hybrid almost contraction of type $\mathbb{J}$, *if there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for $p \geq 0$, $L \geq 0$ and $a_1, a_2 > 0$ with $a_1 + a_2 < 1$, we have*

$$\begin{aligned} &\tfrac{1}{2}\min\{q(u,Tu), q(v,Tv)q(Tv,v)\} \leq q(u,v) \text{ implies} \\ &\qquad \zeta(\alpha(u,v)q(Tu,Tv), \psi(\mathcal{J}_p(u,v) + L\mathcal{N}(u,v))) \geq 0, \end{aligned} \tag{35}$$

for all distinct $u, v \in \mathcal{X}$, where

$$\mathcal{J}_p(u,v) = \begin{cases} [a_1(q(u,v))^p + a_2(\frac{q(u,Tu)\cdot(q(v,Tv)}{q(u,v)})^p]^{1/p}, & \text{for } p > 0 \\ (q(u,v))^{a_1} \cdot (q(u,Tu))^{a_1} \cdot (q(v,Tv))^{1-a_1-a_2}, & \text{for } p = 0 \end{cases}$$

and

$$\mathcal{N}(u,v) = \min\{q(u,Tv), q(v,Tu)\}.$$

Theorem 3. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping such that:*

- (i) *$u = Tu$ implies $\alpha(u,v) > 0$ for every $v \in \mathcal{X}$;*
- (ii) *$v = Tv$ implies $\alpha(u,v) > 0$ for every $u \in \mathcal{X}$.*

Suppose that $T : \mathcal{X} \to \mathcal{X}$ is a hybrid almost contraction of type $\mathbb{J}$ such that the following assumptions hold:

- (i) *T is α-orbital admissible;*
- (ii) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$;*
- (iii) *there exists $\Delta > 0$ such that, $(a_1 + a_2\Delta^{2p})^{1/p} \leq 1$ (where $p > 0$) and*

$$\frac{1}{\Delta}q(u,v) \leq q(v,u) \leq \Delta q(u,v), \text{ for all } u,v \in \mathcal{X};$$

- (iv) *T is continuous.*

Then, T has a fixed point.

Proof. We will consider only the case $p > 0$ because, for $p = 0$, the expression is similar to the one in Theorem 1. By verbatim of the first lines in the proof of Theorem 1, starting from a point u_0, we are able to build a sequence $\{u_n\} \subset \mathcal{X}$. Onward, as in the proof of Theorem 1, we suppose that $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}$ and from (35), we have $\frac{1}{2}\min\{q(u_{n-1}, Tu_{n-1}), q(u_n, Tu_n), q(Tu_n, u_n)\} \leq q(u_{n-1}, u_n)$, which implies

$$\zeta(\alpha(u_{n-1}, u_n)q(Tu_{n-1}, Tu_n), \psi(\mathcal{J}_p(u_{n-1}, u_n) + L\mathcal{N}(u_{n-1}, u_n))) \geq 0.$$

By the axiom (z_d), Lemma 1 and taking into account (7), this inequality becomes

$$\begin{aligned} q(u_n, u_{n+1}) &\leq \alpha(u_{n-1}, u_n) q(Tu_{n-1}, Tu_n) \leq \psi(\mathcal{I}_p(u_{n-1}, u_n) + L\mathcal{N}(u_{n-1}, u_n)) < \mathcal{I}_p(u_{n-1}, u_n) \\ &= [a_1(q(u_{n-1}, u_n))^p + a_2(\tfrac{q(u_{n-1}, Tu_{n-1}) \cdot q(u_n, Tu_n)}{q(u_{n-1}, u_n)})^p]^{1/p} \\ &= [a_1(q(u_{n-1}, u_n))^p + a_2(\tfrac{q(u_{n-1}, u_n) \cdot (qu_n, u_{n+1})}{q(u_{n-1}, u_n)})^p]^{1/p} \\ &= [a_1(q(u_{n-1}, u_n))^p + a_2(q(u_n, u_{n+1}))^p]^{1/p}. \end{aligned} \tag{36}$$

Thereupon,

$$q(u_n, u_{n+1}) < \left(\frac{a_1}{1 - a_2}\right)^{1/p} q(u_{n-1}, u_n) < q(u_{n-1}, u_n)$$

and then, from (36), we have $q(u_n, u_{n+1}) < \psi(q(u_{n-1}, u_n))$. Since $\psi \in \Psi$, recursively, we get

$$q(u_n, u_{n+1}) < \psi(q(u_{n-1}, u_n)) < \ldots < \psi^n(q(u_0, u_1)). \tag{37}$$

In order to prove that $\{u_n\}$ is a right-Cauchy sequence, let $l \in \mathbb{N}$. From (37) and the triangle inequality, we get that

$$\begin{aligned} q(u_n, u_{n+l}) &\leq q(u_n, u_{n+1}) + \ldots + q(u_{n+l-1}, u_{n+l}) \\ &\leq \sum_{j=n}^{n+l-1} \psi^j(q(u_0, u_1)) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0, \text{ as } n \to \infty. \end{aligned}$$

We conclude that $\{q_n\}$ is a right-Cauchy sequence in (X, q).

Substituting in (35) $u = u_n$ and $v = u_{n-1}$ and since $\frac{1}{2}\min\{q(u_n, Tu_n, q(u_{n-1}, Tu_{n-1}), q(Tu_{n-1}, u_{n-1}))\} \leq q(u_n, u_{n-1})$, we have (taking into account (11))

$$\begin{aligned} q(u_{n+1}, u_n) &\leq \alpha(u_n, u_{n-1}) q(Tu_n, Tu_{n-1}) \leq \psi(\mathcal{I}_p(u_n, u_{n-1}) + L\mathcal{N}(u_n, u_{n-1})) < \mathcal{I}_p(u_n, u_{n-1}) \\ &= [a_1(q(u_n, u_{n-1}))^p + a_2(\tfrac{q(u_n, u_{n+1})) \cdot (q(u_{n-1}, u_n)}{q(u_n, u_{n-1}})^p]^{1/p} \end{aligned}$$

i.e.,

$$(q(u_{n+1}, u_n))^p < a_1(q(u_n, u_{n-1}))^p + a_2(\frac{q(u_n, u_{n+1})) \cdot (q(u_{n-1}, u_n)}{q(u_n, u_{n-1}})^p.$$

On one hand, we have already proved that $q(u_n, u_{n+1}) < q(u_{n-1}, u_n)$. On the other hand, by (iii), there exists a positive constant Δ such that $q(u_{n-1}, u_n) \leq \Delta q(u_n, u_{n-1})$ for $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} (q(u_{n+1}, u_n))^p &< a_1(q(u_n, u_{n-1}))^p + a_2(\tfrac{(q(u_{n-1}, u_n))^2}{q(u_n, u_{n-1}}))^p \\ &< a_1(q(u_n, u_{n-1}))^p + a_2(\tfrac{(\Delta \cdot q(u_n, u_{n-1}))^2}{q(u_n, u_{n-1})})^p \\ &= (a_1 + a_2\Delta^{2p}) \cdot (q(u_n, u_{n-1}))^p, \end{aligned}$$

which is equivalent to the next inequality

$$q(u_{n+1}, u_n) < (a_1 + a_2\Delta^{2p})^{1/p} q(u_n, u_{n-1}) \leq q(u_n, u_{n-1}).$$

Thus,

$$q(u_{n+1}, u_n) < \psi(q(u_n, u_{n-1})) < ... < \psi^n(q(u_1, u_0)) \tag{38}$$

Again, considering triangle inequality, together with (38), for $l \in \mathbb{N}$, we have

$$\begin{aligned} q(u_{n+l}, u_n) &\leq q(u_{n+l}, u_{n+l-1}) + ... + q(u_{n-1}, u_n) \\ &\leq \sum_{j=n}^{\infty} \psi^j(q(u_0, u_1)) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Analogously, we deduce that $\{u_n\}$ is left-Cauchy, so that it is a Cauchy sequence in complete quasi-metric space.

Thus, there exists $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} q(u_n, u^*) = \lim_{n\to\infty} q(u^*, u_n) = 0. \tag{39}$$

Under the assumption (iv), from the continuity of T and (q_1), we have

$$\lim_{n\to\infty} q(u_n, Tu^*) = \lim_{n\to\infty} q(Tu_{n-1}, Tu^*) = 0,$$

$$\lim_{n\to\infty} q(Tu^*, u_n) = \lim_{n\to\infty} q(Tu^*, Tu_{n-1}) = 0$$

so that

$$\lim_{n\to\infty} q(u_n, Tu^*) = \lim_{n\to\infty} q(Tu^*, u_n) = 0. \tag{40}$$

Hence, $Tu^* = u^*$ that is, u^* is a fixed point of T. □

The following is a special case for $p = 0$.

Corollary 8. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space, a function $\alpha : X \times X \to [0, \infty)$ and a mapping $T : \mathcal{X} \to \mathcal{X}$ such that there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that, for $p \geq 0$, $L \geq 0$ and $a_1, a_2, \in [0, 1)$ with $a_1 + a_2 < 1$, we have*

$$\zeta(\alpha(u, v)q(Tu, Tv), \psi(\mathcal{I}_p(u, v) + L\mathcal{N}(u, v))) \geq 0, \text{ for all distinct } u, v \in \mathcal{X}. \tag{41}$$

Suppose also that the following assumptions hold:

- *(i)* *$u = Tu$ implies $\alpha(u, v) > 0$ for every $v \in \mathcal{X}$;*
- *(ii)* *$v = Tv$ implies $\alpha(u, v) > 0$ for every $u \in \mathcal{X}$;*
- (i) *T is α-orbital admissible;*
- (ii) *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$;*
- (iii) *there exists $\Delta > 0$ such that, $(a_1 + a_2\Delta^{2p})^{1/p} \leq 1$ (where $p > 0$) and*

$$\frac{1}{\Delta} q(u, v) \leq q(v, u) \leq \Delta q(u, v), \text{ for all } u, v \in \mathcal{X};$$

- (iv) *T is continuous.*

Then, T has a fixed point.

Example 2. *Let $\mathcal{X} = [0,1]$ and the function*

$$q(u,v) = \begin{cases} u - v, & \text{for } u \geq v \\ 2(v-u), & \text{for } u < v \end{cases}$$

It is easy to see that the pair $(\mathcal{X}, q)$ forms a quasi-metric space.

Let the map $T : \mathcal{X} \to \mathcal{X}$ defined by

$$Tu = \begin{cases} \frac{1}{8}, & \text{for } u \in [0, \frac{1}{2}] \\ \frac{u}{4}, & \text{for } u \in [\frac{1}{2}, 1] \end{cases}$$

and choose $\zeta(u,v) = \frac{1}{2}v - u$ and $\psi(t) = \frac{1}{2}t$. For $p = 2$, $L = 0$, $\Delta = 2$, $a_1 = \frac{1}{4}$ and $a_2 = \frac{1}{32}$ because $(a_1 + a_2 \cdot \Delta^{2p})^{1/p} = \frac{1}{4} + \frac{1}{32} \cdot 2^4 = \frac{3}{4} \leq 1$, the assumption (iii) is satisfied. In this case, (41) becomes

$$\alpha(u,v)q(u,v) \leq \mathcal{I}_p(u,v) = \frac{1}{4}\sqrt{\frac{1}{4}(q(u,v))^2 + \frac{1}{32}(\frac{q(u,Tu)q(v,Tv)}{q(u,v)})^2}. \tag{42}$$

Define $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that

$$\alpha(u,v) = \begin{cases} 3, & \text{for } u,v \in [0, \frac{1}{2}) \\ 1, & \text{for } u = 1, v = 0 \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that T is α-admissible. Indeed, we have

$$\alpha(u,v) = 3 \Rightarrow \alpha(Tu, Tv) = \alpha(1/8, 1/8) = 3, \text{ for } u,v \in [0, \frac{1}{2})$$

and

$$\alpha(1,0) = 1 \Rightarrow \alpha(T1, T0) = \alpha(1/4, 1/8) = 3.$$

On the other hand, for $q_0 = 0$,

$$\alpha(0, T0) = \alpha(T0, 0) = \alpha(0,0) = 3,$$

so that the presumptions (i), (ii), and (iv) are satisfied. Of course, if $u,v \in [0, \frac{1}{2})$, we have $q(Tu, Tv) = q(\frac{1}{8}, \frac{1}{8}) = 0$ and (41) is verified. For $u = 1$ and $v = 0$, we have $q(T1, T0) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$, $q(0, T0) = q(0, 1/8) = 2(1/8 - 0) = 1/4$, $q(1, T1) = q(1, 1/4) = 3/4$ and

$$\begin{aligned} \alpha(1,0)q(T1,T0) &= \tfrac{1}{8} \leq \tfrac{1}{4}\sqrt{\tfrac{1}{4} + \tfrac{1}{32}(\tfrac{3}{16})^2} \\ &= \tfrac{1}{4}\sqrt{\tfrac{1}{4}(q(1,0))^2 + \tfrac{1}{32}(\tfrac{q(0,T0)q(1,T1)}{q(1,0)})^2} \end{aligned} \tag{43}$$

The other cases are not interesting since $\alpha(u,v) = 0$ and the condition (42) is fulfilled trivially. Thus, the presumptions of Theorem 8 are provided and $u = \frac{1}{8}$ is the fixed point of T.

Corollary 9. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and T be a continuous self-mapping on $\mathcal{X}$. Suppose that there exist $\zeta \in \mathcal{Z}$, $\psi \in \Psi$ such that*

$$\zeta(q(Tu, Tv), \psi(\mathcal{I}_p(u,v))) \geq 0,$$

for each distinct $u, v \in \mathcal{X}$. If there exists $\Delta > 0$ such that $(a_1 + a_2 \cdot \Delta^{2p})^{1/p} \leq 1$ for $p > 0$, and $\frac{1}{\Delta} q(u,v) \leq q(v,u) \leq \Delta q(u,v)$ for all $u, v \in \mathcal{X}$, then T has a fixed point.

Proof. It is sufficient to take $L = 0$ and $\alpha(u,v) = 1$ for $u, v \in \mathcal{X}$ in Corollary 8. □

Corollary 10. *Let $(\mathcal{X}, q)$ be a complete quasi-metric space and T be a self-mapping on $\mathcal{X}$. Suppose that there exists $\Delta > 0$ such that $(a_1 + a_2 \cdot \Delta^{2p})^{1/p} \leq 1$ for $p > 0$, and $\frac{1}{\Delta} q(u,v) \leq q(v,u) \leq \Delta q(u,v)$ for all $u, v \in \mathcal{X}$. The mapping T has a fixed point provided that*

$$q(Tu, Tv) \leq c \cdot \mathcal{I}_p(u,v)$$

for each distinct $u, v \in \mathcal{X}$ and some $c \in (0,1)$.

Proof. We set $\zeta(t,s) = c_1 s - t$, $\psi(u) = c_2 u$ with $c_1, c_2 \in [0,1)$ and $c = c_1 + c_2$ in Corollary 9. □

Remark 3. *Letting $p = 0$ in Corollary 10, we find Theorem 2.2. in [20].*

Example 3. *Let $(\mathcal{X}, q)$ be the quasi-metric space, where $\mathcal{X} = [1, \infty)$ and*

$$q(u,v) = \begin{cases} u - v, & \text{for } u \geq v \\ 2(v-u), & \text{for } u < v \end{cases}$$

Let

$$Tu = \begin{cases} u^3 - 8u^2 + 19u - 9, & \text{for } u \in [1,5] \\ ln(u^2 - 24) + u + 6, & \text{for } u \in (5, \infty). \end{cases}$$

Consider the function ζ be arbitrary in $\mathcal{Z}$, $\psi \in \Psi$ with $\psi(t) = \frac{t}{\sqrt{3}}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that

$$\alpha(u,v) = \begin{cases} u^2 + 1, & \text{for } (u,v) \in \{(3,3), (3,4), (4,3), (3,1), (1,3)\} \\ 1, & \text{for } (u,v) = (2,1) \\ 0, & \text{otherwise}. \end{cases}$$

It is easily verified that T is α-orbital admissible. Whereas $T1 = T3 = T4 = 3$, taking into account the definition of function α, we have that the inequality (41) holds for every pair $(u,v) \in \mathcal{X}^2 \setminus \{(2,1)\}$. For the case $u = 2$ and $v = 1$, choosing $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{48}$ and $p = 2$, we find that axiom (iii) holds. On the other hand,

$$\begin{aligned} \mathcal{I}_p(2,1) &= \left[\tfrac{1}{2} (q(2,1))^2 + \tfrac{1}{48} \left(\tfrac{q(2,T2) \cdot q(1,T1)}{q(2,1)} \right)^2 \right]^{1/2} \\ &= \sqrt{\tfrac{1}{2} + \tfrac{1}{48} \cdot \left(\tfrac{q(2,5) \cdot q(1,3)}{q(2,1)} \right)^2} = \sqrt{\tfrac{25}{2}} \end{aligned}$$

and

$$\alpha(2,1) q(T2, T1) = q(5,3) = 2 < \sqrt{\frac{25}{6}} = \psi(\mathcal{I}_p(2,1)).$$

Consequently, by Theorem 8, we have that the mapping T has a fixed point in $\mathcal{X}$.

On the other hand, we can observed that, for $u = 1$ *and* $v = 5$,

$$q(T1, T(4.5)) = q(2, 5.625) = 7.25, \; q(1, T1) = q(1, 2) = 2, \; q(4.5, T(4.5)) = q(4.5, 5.625) = 1.125,$$

so that, since

$$q(T1, T(4.5)) > \lambda(q(1, T1))^{\alpha}(q(4.5, T(4.5)))^{1-\alpha}$$

for any $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$, *Theorem 2.2 in [20] can not be applied.*

Corollary 11. *Let* (X, q) *be a complete quasi-metric space and* $T : X \to X$ *a continuous mapping. Then, T has a fixed point provided that*

$$q(Tu, Tv) \leq k_1 \cdot q(u, v) + k_2 \cdot \frac{q(u, Tu)q(v, Tv)}{q(u, v)} \tag{44}$$

for each $u, v \in X$ *and* $k_1, k_2 \in (0, 1)$ *with* $k_1 + k_2 < 1$

Proof. Let $p = 1$ and $k_i = c \cdot a_i$, for $i \in \{1, 2\}$ in Corollary 10. □

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