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Research article

On ψ-Hilfer generalized proportional fractional operators

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Abstract: In this paper, we introduce a generalized fractional operator in the setting of Hilfer fractional derivatives, the ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function. The proposed operator can be viewed as an interpolator between the Riemann-Liouville and Caputo generalized proportional fractional operators. The properties of the proposed operator are established under some classical and standard assumptions. As an application, we formulate a nonlinear fractional differential equation with a nonlocal initial condition and investigate its equivalence with Volterra integral equations, existence, and uniqueness of solutions. Finally, illustrative examples are given to demonstrate the theoretical results.

Keywords: Hilfer fractional derivative; generalized proportional fractional derivative; existence and uniqueness; weighed space; fixed point theorems
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1. Introduction

The fractional calculus, which analyses the integrals and derivatives of arbitrary order, attracted the attention of many researchers in the last century and continues to do so even in the present century, as it is perceived as one of the most solid and powerful mathematical tools both in theory and applications [1–9]. One of the pivotal efficacy of fractional calculus is that there are copious types of fractional operators that appear from different aspects. The most illustrious ones are the Riemann-Liouville and Caputo's fractional operators that were effectively applied in developing models of long-term memory processes and the problems that came to the fore in many areas of science and technology [10–18]. Being incapacitated to model all the veracious problems of the world with the operators in the traditional calculus, and for the sake of enhanced understanding and modeling the real-world problems more accurately, researchers continuously observe the need to develop and discover new types of fractional operators for the said cause that were confined to Riemann-Liouville fractional derivatives and Caputo's fractional derivatives before the hit of this century.

Katugampola [19, 20] in 2011, came up with a new type of fractional operators called generalized fractional operators to combine Riemann-Liouville and Hadamard fractional operators. Later Jarad et al. in [21] modified these operators to include Caputo and Caputo-Hadamard fractional derivatives. In [22], the authors established a new derivative and named it as a Conformable fractional derivative. But this derivative faulted that it does not give the original function if the order tends to 0 and that is a dearth. However, it is mandatory for any derivative that it should give the original function when the order is zero and if the order is 1, it provides the first-order derivative of the function. To evade this issue in conformable derivative, the authors in [23] provided a modification generated from the former definition of the conformable derivative. Also, some more generalizations of these operators are mentioned in [24]. Following the same trend, very recently some authors constructed new types of derivative operators by replacing the singular kernel of Riemann-Liouville and Caputo with non-singular (bounded) kernels. Although these operators suffer from various drawbacks which makes it hard to use them, still discrete authors [25–28] and many more became enthusiastic in working with these operators as they hold profits of Riemann-Liouville and Caputo derivative operators.

Jarad et al. in [29] developed a new class of generalized fractional operators in the sense of Riemann-Liouville and Caputo involving a special case of proportional derivatives. After that, the authors in [30] generalize the work done in [29] by using the concept of proportional derivatives of a function with respect to another function. Moreover, in [31], Idris et al., constructed a new operator called Hilfer generalized proportional fractional derivative that merges the operators defined in [29]. They also provide some fundamental properties and important lemmas.

Motivated by [31, 32], we study a proportional derivatives and provide a generalization of the operator defined in [31] and named it as ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function which acts as a connection between proportional fractional derivatives in Riemann-Liouville and Caputo sense as defined in [30].

The paper is organized as follows: In Section 2, we mention some preliminary definitions, lemmas, and theorems that are used in other sections of the paper. In Section 3, we define the new operator, the ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function, together with some of its properties and important results. Furthermore, we discuss the equivalence between a generalized Cauchy problem and the Volterra integral equation with this operator. The existence and uniqueness results for the proposed problem are also conferred. In Section 4, three illustrative examples were given, which show the theoretical analysis. In Section 5, we discuss conclusions obtained from the analysis.

2. Preliminaries

In this section, we recall some definitions, theorems, lemmas, corollaries and propositions which we use later in this paper [21, 30, 32–34].
Let $\Omega = [a, b]$ $(0 \le a < b < \infty)$ be a finite interval and γ be a parameter such that $n-1 \le \gamma < n$. The space of continuous functions f on Ω is denoted by $C[a, b]$ and the associated norm is defined by [1,34]

$$\|f\|_{C[a,b]} = \max_{y \in [a,b]} |f(y)|,$$

and

$$AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{R};\ f^{(n-1)} \in AC[a, b] \right\},$$

be the space of n times absolutely continuous differentiable functions.
The weighted space $C_{\gamma,\psi}[a, b]$ of functions f on $(a, b]$ is defined by

$$C_{\gamma,\psi}[a, b] = \left\{ f : (a, b] \to \mathbb{R}; (\psi(y) - \psi(a))^{\gamma} f(y) \in C[a, b] \right\},$$

having norm

$$\|f\|_{C_{\gamma,\psi}[a,b]} = \left\| (\psi(y) - \psi(a))^{\gamma} f(y) \right\|_{C[a,b]} = \max_{y \in [a,b]} \left| (\psi(y) - \psi(a))^{\gamma} f(y) \right|.$$

The weighted space $C^n_{\gamma,\psi}[a, b]$ of functions f on $(a, b]$ is defined by

$$C^n_{\gamma,\psi}[a, b] = \left\{ f : [a, b] \to \mathbb{R};\ f(y) \in C^{n-1}[a, b];\ f^{(n)}(y) \in C_{\gamma,\psi}[a, b] \right\},$$

along the with norm

$$\|f\|_{C^n_{\gamma,\psi}[a,b]} = \sum_{k=0}^{n-1} \left\| f^{(k)} \right\|_{C[a,b]} + \left\| f^{(n)} \right\|_{C_{\gamma,\psi}[a,b]}.$$

The above spaces satisfy the following properties:

(i) $C^0_{\gamma,\psi}[a, b] = C_{\gamma,\psi}[a, b]$, for $n = 0$.

(ii) For $n - 1 \le \gamma_1 < \gamma_2 < n$, $C_{\gamma_1,\psi}[a, b] \subset C_{\gamma_2,\psi}[a, b]$.

Definition 2.1. ([30, 32]) Let φ_0, $\varphi_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be two continuous functions such that for all $y \in \mathbb{R}$ and for $\vartheta \in [0, 1]$, we have

$$\lim_{\vartheta \to 0^+} \varphi_0(\vartheta, y) = 0, \quad \lim_{\vartheta \to 0^+} \varphi_1(\vartheta, y) = 1, \quad \lim_{\vartheta \to 1^-} \varphi_0(\vartheta, y) = 1, \quad \lim_{\vartheta \to 1^-} \varphi_1(\vartheta, y) = 0,$$

and $\varphi_0(\vartheta, y) \neq 0$, $\vartheta \in (0, 1]$; $\varphi_1(\vartheta, y) \neq 0$ for $\vartheta \in [0, 1)$. Also let $\psi(y)$ be a strictly positive increasing continuous function. Then,

$$\mathcal{D}^{\vartheta,\psi} f(y) = \varphi_1(\vartheta, y) f(y) + \varphi_0(\vartheta, y) \frac{f'(y)}{\psi'(y)}, \tag{2.1}$$

gives the proportional differential operator of order ϑ with respect to function $\psi(y)$ of a function $f(y)$.
In particular, when $\varphi_0(\vartheta, y) = \vartheta$ and $\varphi_1(\vartheta, y) = 1 - \vartheta$. Then, the operator $\mathcal{D}^{\vartheta,\psi}$ (2.1) becomes

$$\mathcal{D}^{\vartheta,\psi} f(y) = (1 - \vartheta) f(y) + \vartheta \frac{f'(y)}{\psi'(y)}, \tag{2.2}$$

and the integral corresponding to proportional derivative (2.2) is given as

$$\mathcal{J}_a^{1,\vartheta,\psi} f(y) = \frac{1}{\vartheta} \int_a^y e^{\frac{\vartheta-1}{\vartheta}\left(\psi(y)-\psi(s)\right)} f(s)\psi'(s)ds, \tag{2.3}$$

where we assume that $\mathcal{J}_a^{0,\vartheta,\psi} f(y) = f(y)$.

The generalized proportional integral of order n corresponding to proportional derivative $\mathcal{D}^{n,\vartheta,\psi} f(y)$, is given as follows

$$\left(\mathcal{J}_a^{n,\vartheta,\psi} f\right)(y) = \frac{1}{\vartheta^n \Gamma(n)} \int_a^y e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))} (\psi(y) - \psi(s))^{n-1} \psi'(s) f(s) ds, \tag{2.4}$$

where, $\mathcal{D}^{n,\vartheta,\psi} = \underbrace{\mathcal{D}^{\vartheta,\psi} \cdot \mathcal{D}^{\vartheta,\psi} \cdot \mathcal{D}^{\vartheta,\psi} \cdots \mathcal{D}^{\vartheta,\psi}}_{n-times}$.

Now the general proportional fractional integral based on (2.4) is defined as;

Definition 2.2. ([30, 32]) If $\vartheta \in (0, 1]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Then the fractional integral

$$\left(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f\right)(y) = \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_{a^+}^y e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))} (\psi(y) - \psi(s))^{\alpha-1} \psi'(s) f(s) ds, \quad y > a, \tag{2.5}$$

is called the left-sided generalized fractional proportional integral of order α of the function f with respect to another function ψ.

Definition 2.3. ([30, 32]) For $\vartheta \in (0, 1]$, $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $\psi \in C[a, b]$, where $\psi'(s) > 0$, the generalized left proportional fractional derivative of order α of the function f with respect to ψ is defined as

$$\left(\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi} f\right)(y) = \frac{\mathcal{D}_y^{n,\vartheta,\psi}}{\vartheta^{n-\alpha} \Gamma(n-\alpha)} \int_{a^+}^y e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))} (\psi(y) - \psi(s))^{n-\alpha-1} \psi'(s) f(s) ds,$$

where $\Gamma(\cdot)$ is the gamma function and $n = [Re(\alpha)] + 1$.

Proposition 2.1. ([30,32]) If $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha) \geq 0$ and $Re(\beta) > 0$, then for any $\vartheta > 0$, we have

$$(i) \left(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))} (\psi(s) - \psi(a))^{\beta-1}\right)(y) = \frac{\Gamma(\beta)}{\vartheta^\alpha \Gamma(\alpha+\beta)} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} (\psi(y) - \psi(a))^{\alpha+\beta-1},$$

$$(ii) \left(\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi} e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))} (\psi(s) - \psi(a))^{\beta-1}\right)(y) = \frac{\vartheta^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} (\psi(y) - \psi(a))^{\beta-\alpha-1}.$$

Theorem 2.1. *([30, 32, 33]) Suppose $\vartheta \in (0, 1]$, $Re(\alpha) > 0$ and $Re(\beta) > 0$. Then, if f is continuous and defined for $y \geq a$, we have*

$$\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} (\mathcal{J}_{a^+}^{\beta,\vartheta,\psi} f)(y) = \mathcal{J}_{a^+}^{\beta,\vartheta,\psi} (\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f)(y) = (\mathcal{J}_{a^+}^{\alpha+\beta,\vartheta,\psi} f)(y).$$

Theorem 2.2. *([30, 32]) Suppose $\vartheta \in (0, 1]$, $0 \leq n < [Re(\alpha)] + 1$ with $n \in \mathbb{N}$. If $f \in L_1(a, b)$, then*

$$\mathcal{D}_{a^+}^{n,\vartheta,\psi} (\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f)(y) = (\mathcal{J}_{a^+}^{\alpha-n,\vartheta,\psi} f)(y). \tag{2.6}$$

In particular, for $n = 1$, by using the Leibnitz rule , we have

$$\mathcal{D}_{a^+}^{1,\vartheta,\psi} (\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f)(y) = \frac{\alpha - 1}{\vartheta^{\alpha-1} \Gamma(\alpha)} \int_{a^+}^y e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))} (\psi(y) - \psi(s))^{\alpha-2} \psi'(s) f(s) ds. \tag{2.7}$$

Corollary 2.1. ([30,32]) If $0 < Re(\beta) < Re(\alpha)$ and $n-1 < Re(\beta) \leq n$, $n \in \mathbb{N}$. then, we have

$$\mathcal{D}_{a^+}^{\beta,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(y) = \mathcal{J}_{a^+}^{\alpha-\beta,\vartheta,\psi}f(y). \tag{2.8}$$

Theorem 2.3. *([30,32]) Suppose $f \in L_1(a,b)$ and $Re(\alpha) > 0$, $\vartheta \in (0,1]$, $n = [Re(\alpha)] + 1$. Then, the following equality holds*

$$\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(y) = f(y), \quad y \geq a. \tag{2.9}$$

Lemma 2.4. *([32]) If $\alpha > n$, $\vartheta \in (0,1]$ and n is positive integer, then we have*

$$\left(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{n,\vartheta,\psi}f\right)(y) = \left(\mathcal{D}_{a^+}^{n,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f\right)(y) - \sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\big(\psi(y)-\psi(a)\big)^{\alpha-n+k}}{\vartheta^{\alpha-n+k}\Gamma(\alpha+k-n+1)}(\mathcal{D}^{k,\vartheta,\psi}f)(a). \tag{2.10}$$

Theorem 2.5. *([32]) Assume that $Re(\alpha) > 0$, $n = -[-Re(\alpha)]$, $f \in L_1(a,b)$, and $(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f)(y) \in AC^n[a,b]$. Then,*

$$\left(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}f\right)(y) = f(y) - \sum_{j=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\big(\psi(y)-\psi(a)\big)^{\alpha-j}}{\vartheta^{\alpha-j}\Gamma(\alpha-j+1)}(\mathcal{J}_{a^+}^{j-\alpha,\vartheta,\psi}f)(a). \tag{2.11}$$

Definition 2.4. ([30,32]) If $\vartheta \in (0,1]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$, then the generalized left Caputo proportional fractional derivative of function f with respect to function ψ is defined as

$$\begin{aligned}\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}f\right)(y) &= \mathcal{J}_{a^+}^{n-\alpha,\vartheta,\psi}(\mathcal{D}^{n,\vartheta,\psi}f)(y)\\ &= \frac{1}{\vartheta^{n-\alpha}\Gamma(n-\alpha)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}\big(\psi(y)-\psi(s)\big)^{n-\alpha-1}\psi'(s)(\mathcal{D}_{s}^{n,\vartheta,\psi}f)(s)ds,\end{aligned} \tag{2.12}$$

where, $n = [Re(\alpha)] + 1$.

Corollary 2.2. [32] Let $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$ and $\vartheta \in (0,]$, $n = [Re(\alpha)] + 1$. If $f \in C^n[a,b]$ then

$$\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}f\right)(y) = \mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}\left[f(y) - \sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\big(\psi(y)-\psi(a)\big)^{k}}{\vartheta^{k}\cdot k!}(\mathcal{D}_{a^+}^{k,\vartheta,\psi}f)(a)\right]. \tag{2.13}$$

Proposition 2.2. ([30,32]) If $\alpha,\beta \in \mathbb{C}$ with $Re(\alpha) > 0$ and $Re(\beta) > 0$, then for any $\vartheta > 0$ and $n = [Re(\alpha)] + 1$, we obtain as follows

$$\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))}\big(\psi(s)-\psi(a)\big)^{\beta-1}\right)(y) = \frac{\vartheta^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\big(\psi(y)-\psi(a)\big)^{\beta-\alpha-1}.$$

For $k = 0,1,2,\ldots,n-1$, we have

$$\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))}\big(\psi(s)-\psi(a)\big)^{k}\right)(y) = 0.$$

In particular, $\left({}^{C}\mathcal{D}_{a^+}^{\alpha,\vartheta,\psi}e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))}\right)(y) = 0$.

3. Main results

Definition 3.1. Let $I = [a,b]$, where $-\infty \leq a < b \leq \infty$ be an interval and $f,\ \psi \in C^n[a,b]$ be two functions such that ψ is positive, strictly increasing and $\psi'(y) \neq 0$, for all $y \in I$. The ψ-Hilfer generalized proportional fractional derivatives (left-sided/right-sided) of order α and type β of f with respect to another function ψ are defined by

$$\left(\mathcal{D}_{a^{\pm}}^{\alpha,\beta,\vartheta,\psi} f\right)(y) = \left(\mathcal{J}_{a^{\pm}}^{\beta(n-\alpha),\vartheta,\psi}(\mathcal{D}^{n,\vartheta,\psi})\mathcal{J}_{a^{\pm}}^{(1-\beta)(n-\alpha),\vartheta,\psi} f\right)(y), \tag{3.1}$$

where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$ with $n \in \mathbb{N}$ and $\vartheta \in (0,1]$. Also, $\mathcal{D}^{\vartheta,\psi} f(y) = (1-\vartheta)f(y) + \vartheta\frac{f'(y)}{\psi'(y)}$ and $\mathcal{J}$ is the generalized proportional fractional derivative defined in (2.5).

In particular, if $n = 1$, then $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, so (3.1) becomes,

$$\left(\mathcal{D}_{a^{\pm}}^{\alpha,\beta,\vartheta,\psi} f\right)(y) = \left(\mathcal{J}_{a^{\pm}}^{\beta(1-\alpha),\vartheta,\psi}(\mathcal{D}^{1,\vartheta,\psi})\mathcal{J}_{a^{\pm}}^{(1-\beta)(1-\alpha),\vartheta,\psi} f\right)(y).$$

Remark 3.1. From the Definition 3.1, we can view the operator $\mathcal{D}_{a^{\pm}}^{\alpha,\beta,\vartheta,\psi}$ as an interpolator between the Riemann–Liouville and Caputo generalized proportional fractional derivatives, respectively, since

$$\mathcal{D}_{a^{\pm}}^{\alpha,\beta,\vartheta,\psi} f = \begin{cases} \mathcal{D}^{n,\vartheta,\psi}\mathcal{J}_{a^{\pm}}^{n-\alpha,\vartheta,\psi} f, & \text{if } \beta = 0, \\ \mathcal{J}_{a^{\pm}}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}^{n,\vartheta,\psi} f, & \text{if } \beta = 1. \end{cases}$$

Remark 3.2. In this paper we discuss our results involving ψ-Hilfer generalized proportional fractional derivatives using only one sided (left) operator. The similar procedure can be developed for the right-sided operator.

The operator $\mathcal{D}_{a^{+}}^{\alpha,\beta,\vartheta,\psi}$ can be expressed in terms of the operators given in Definition 2.2 and Definition 2.3. This is given by the following property:

Property 3.1. The ψ-Hilfer generalized proportional fractional derivatives $\mathcal{D}_{a^{+}}^{\alpha,\beta,\vartheta,\psi}$ is equivalent to

$$\left(\mathcal{D}_{a^{+}}^{\alpha,\beta,\vartheta,\psi} f\right)(y) = \left(\mathcal{J}_{a^{+}}^{\beta(n-\alpha),\vartheta,\psi}(\mathcal{D}^{n,\vartheta,\psi})\mathcal{J}_{a^{+}}^{(1-\beta)(n-\alpha),\vartheta,\psi} f\right)(y) = \left(\mathcal{J}_{a^{+}}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^{+}}^{\gamma,\vartheta,\psi} f\right)(y),$$

where $\gamma = \alpha + \beta(n-\alpha)$.

Proof. By the Definition 3.1 of $\mathcal{D}_{a^{+}}^{\alpha,\beta,\vartheta,\psi}$ and using (2.2), (2.5) , we have

$$\begin{aligned}\left(\mathcal{D}_{a^{+}}^{\alpha,\beta,\vartheta,\psi} f\right)(y) &= \left(\mathcal{J}_{a^{+}}^{\beta(n-\alpha),\vartheta,\psi}(\mathcal{D}^{n,\vartheta,\psi})\mathcal{J}_{a^{+}}^{(1-\beta)(n-\alpha),\vartheta,\psi} f\right)(y) \\ &= \mathcal{J}_{a^{+}}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}^{n-1,\vartheta,\psi}\left\{\frac{\mathcal{D}^{1,\vartheta,\psi}}{\vartheta^{n-\gamma}\Gamma(n-\gamma)}\right. \\ &\quad \left.\times\int_{a^{+}}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{n-\gamma-1}\psi'(s)f(s)ds\right\},\end{aligned}$$

this gives by using (2.7)

$$\begin{aligned}&= \mathcal{J}_{a^{+}}^{\beta(n-\alpha),\vartheta,\psi}(\mathcal{D}^{n-1,\vartheta,\psi})\left\{\frac{1}{\vartheta^{n-\gamma-1}\Gamma(n-\gamma-1)}\right. \\ &\quad \left.\times\int_{a^{+}}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{n-\gamma-2}\psi'(s)f(s)ds\right\}.\end{aligned}$$

Now repeating the above process $(n-1)$ times and using (2.7), we obtain the required result. □

Theorem 3.1. *Let $n-1<\alpha<n$, with $n\in\mathbb{N}$, $0\le\beta\le 1$, $\vartheta\in(0,1]$ and $\gamma=\alpha+\beta(n-\alpha)$. For $\eta\in\mathbb{R}$ such that $\eta>n$, then the image of the function $f(y)=e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\eta-1}$ under the operator $\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}$ is given as*

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)=\frac{\vartheta^{\alpha}\Gamma(\eta)}{\Gamma(\eta-\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\eta-\alpha-1}.$$

Proof. From Proposition 2.1. and (3.1), we obtain

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)&=\mathcal{J}_{a^+}^{\gamma-\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(y)\\&=\mathcal{J}_{a^+}^{\gamma-\alpha,\vartheta,\psi}\left(\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\eta-1}\right)\\&=\frac{\vartheta^{\gamma}\Gamma(\eta)}{\Gamma(\eta-\gamma)}\mathcal{J}_{a^+}^{\gamma-\alpha,\vartheta,\psi}\left(e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\eta-\gamma-1}\right)\\&=\frac{\vartheta^{\alpha}\Gamma(\eta)}{\Gamma(\eta-\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\eta-\alpha-1}.\end{aligned}$$

□

Lemma 3.2. *Let $n-1<\alpha<n$, with $n\in\mathbb{N}$ $0\le\beta\le 1$, $\vartheta\in(0,1]$ and $\gamma=\alpha+\beta(n-\alpha)$. For $\theta>0$, consider the function $f(y)=e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}E_{\alpha}(\theta(\psi(y)-\psi(a))^{\alpha})$, where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function with one parameter. Then,*

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)=\theta\vartheta^{\alpha}f(y).$$

Proof. Using the definition of Mittag-Leffler function and the Theorem 3.1. , we have

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)&=\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\left\{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}E_{\alpha}(\theta(\psi(y)-\psi(a))^{\alpha})\right\}\\&=\sum_{j=0}^{\infty}\frac{\theta^{j}}{\Gamma(\alpha j+1)}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\alpha j}\\&=\theta\vartheta^{\alpha}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\sum_{j=1}^{\infty}\frac{\theta^{j-1}(\psi(y)-\psi(a))^{\alpha j-\alpha}}{\Gamma(\alpha(j-1)+1)}\\&=\theta\vartheta^{\alpha}f(y).\end{aligned}$$

□

Property 3.2. Assume that the parameters α,β,γ satisfying the relations as

$$\gamma=\alpha+\beta(n-\alpha),\ n-1<\alpha,\ \gamma\le n,\ 0\le\beta\le 1,$$

and

$$\gamma\ge\alpha,\ \gamma>\beta,\ n-\gamma<n-\beta(n-\alpha).$$

Therefore, we consider the following weighted spaces of continuous functions on $(a,b]$ as follows:

$$C_{n-\gamma,\psi}^{\alpha,\beta}[a,b]=\left\{f\in C_{n-\gamma,\psi}[a,b],\ \mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f\in C_{\gamma,\psi}[a,b]\right\},$$

and

$$C_{n-\gamma,\psi}^{\gamma}[a,b]=\left\{f\in C_{n-\gamma,\psi}[a,b],\ \mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f\in C_{n-\gamma,\psi}[a,b]\right\}.$$

Since $\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} f = \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi} \mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}$, it follows that

$$C_{n-\gamma,\psi}^{\gamma}[a,b] \subset C_{n-\gamma,\psi}^{\alpha,\beta}[a,b].$$

Lemma 3.3. *Let* $n-1 \leq \gamma < n$, $n-1 < \alpha < n$ *with* $n \in \mathbb{N}$ *and* $\vartheta \in (0,1]$. *If* $f \in C_\gamma[a,b]$ *then*

$$\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f(a) = \lim_{y\to a^+} \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f(y) = 0, \quad n-1 \leq \gamma < \alpha. \tag{3.2}$$

Proof. Since $f \in C_\gamma[a,b]$, then $(\psi(y)-\psi(a))^\gamma f(y)$ is continuous on [a,b] and hence

$$\left|(\psi(y)-\psi(a))^\gamma f(y)\right| < \mathbf{N},$$

where $y \in [a,b]$ and $\mathbf{N} > 0$ is a constant. Therefore,

$$\left|\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} f(y)\right| < \mathbf{N}\Big[\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} e^{\frac{\vartheta-1}{\vartheta}(\psi(s)-\psi(a))} (\psi(s)-\psi(a))^{-\gamma}\Big](y),$$

and by Proposition 2.1, we can write

$$\left|\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} f(y)\right| < \mathbf{N}\left[\frac{\Gamma(n-\gamma)}{\vartheta^\alpha \Gamma(\alpha-\gamma+n)} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} (\psi(y)-\psi(a))^{\alpha-\gamma}\right].$$

As $\alpha > \gamma$, the RHS of above equation approaches to 0 as $y \to a^+$. □

Lemma 3.4. *Let* $n-1 < \alpha < n$, $\vartheta \in (0,1]$, $0 \leq \beta \leq 1$, *with* $n \in \mathbb{N}$ *and* $\gamma = \alpha + \beta(n-\alpha)$. *If* $f \in C_{n-\gamma}^{\gamma}[a,b]$ *then*

$$\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi} \mathcal{D}_{a^+}^{\gamma,\vartheta,\psi} f = \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} \mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} f,$$

and

$$\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi} \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f = \mathcal{D}_{a^+}^{\beta(n-\alpha),\vartheta,\psi} f.$$

Proof. Using Theorem 2.1 and Property 3.1, we can write

$$\begin{aligned}
\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi} \mathcal{D}_{a^+}^{\gamma,\vartheta,\psi} f &= \mathcal{J}_{a^+}^{\gamma,\vartheta,\psi}\left(\mathcal{J}_{a^+}^{-\beta(n-\alpha),\vartheta,\psi} \mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} f\right) \\
&= \mathcal{J}_{a^+}^{\alpha+\beta(n-\alpha),\vartheta,\psi} \mathcal{J}_{a^+}^{-\beta(n-\alpha),\vartheta,\psi} \mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} f \\
&= \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} \mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} f.
\end{aligned}$$

Again using Definition 2.3 and Theorem 2.1, we obtain

$$\begin{aligned}
\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi} \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f &= \mathcal{D}^{n,\vartheta,\psi} \mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi} \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f \\
&= \mathcal{D}^{n,\vartheta,\psi} \mathcal{J}_{a^+}^{n-\beta(n-\alpha),\vartheta,\psi} f = \mathcal{D}_{a^+}^{\beta(n-\alpha),\vartheta,\psi} f.
\end{aligned}$$

□

Lemma 3.5. *Let* $f \in L_1(a,b)$. *If* $\mathcal{D}^{\beta(n-\alpha),\vartheta,\psi} f$ *exists in* $L_1(a,b)$, *then*

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi} \mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f = \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi} \mathcal{D}_{a^+}^{\beta(n-\alpha),\vartheta,\psi} f.$$

Proof. From Definition 2.3., Theorem 2.1. and (3.1), we have

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f &= \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f = \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}(\mathcal{D}_{a^+}^{n,\vartheta,\psi}\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi})\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f\\ &= \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{n,\vartheta,\psi}\mathcal{J}_{a^+}^{n-\beta(n-\alpha),\vartheta,\psi}f\\ &= \mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}f.\end{aligned}$$

□

Lemma 3.6. *Assume $n-1<\alpha<n$ for $n\in\mathbb{N}$; $\vartheta\in(0,1]$, $0\le\beta\le 1$, and $\gamma=\alpha+\beta(n-\alpha)$. If $f\in C_{n-\gamma}^{\gamma}[a,b]$ and $\mathcal{J}_{a^+}^{n-\beta(n-\alpha),\vartheta,\psi}f\in C_{n-\gamma,\psi}^{n}[a,b]$, then $\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f$ exists in $(a,b]$ and*

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(y)=f(y),\qquad y\in(a,b].$$

Proof. With the help of Theorem 2.5, Lemma 3.3 and Lemma 3.5, we get as follows

$$\begin{aligned}\left(\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f\right)(y) &= \left(\mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}f\right)(y)\\ &= f(y)-\sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\left(\psi(y)-\psi(a)\right)^{\beta(n-\alpha)-k}}{\vartheta^{\beta(n-\alpha)-k}\Gamma(\beta(n-\alpha)-k+1)}(\mathcal{J}_{a^+}^{k-\beta(n-\alpha),\vartheta,\psi})(a)\\ &= f(y).\end{aligned}$$

□

Theorem 3.7. *For $n-1<\alpha<n$, with $n\in\mathbb{N}$, $\vartheta\in(0,1]$, and $0\le\beta\le1$. If $f\in C^{n}[a,b]$, then*

$$\left(\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)=\mathcal{D}_{a^+}^{n-\beta(n-\alpha),\vartheta,\psi}\left[\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f(y)-\sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\left(\psi(y)-\psi(a)\right)^{k}}{\vartheta^{k}k!}(\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f)(a)\right],$$

where $\gamma=\alpha+\beta(k-\alpha)$.

Proof. Suppose that $g(y)=\mathcal{J}_{a^+}^{(1-\beta)(n-\alpha),\vartheta,\psi}f(y)$ and $\eta=n-\beta(n-\alpha)$, then, by using Definition 2.4 and Corollary 2.2, we get from (3.1) as

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y) &= {}^{C}\mathcal{D}_{a^+}^{\eta,\vartheta,\psi}g(y)\\ &= \mathcal{D}_{a^+}^{\eta,\vartheta,\psi}\left[g(y)-\sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\left(\psi(y)-\psi(a)\right)^{k}}{\vartheta^{k}k!}(\mathcal{D}_{a^+}^{k,\vartheta,\psi}g)(a)\right]\\ &= \mathcal{D}_{a^+}^{\eta,\vartheta,\psi}\left[\mathcal{J}_{a^+}^{(1-\beta)(n-\alpha),\vartheta,\psi}f(y)-\sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\left(\psi(y)-\psi(a)\right)^{k}}{\vartheta^{k}k!}\right.\\ &\qquad\left.\times\left\{\mathcal{D}_{a^+}^{k,\vartheta,\psi}(\mathcal{J}_{a^+}^{(1-\beta)(k-\alpha),\vartheta,\psi}f)(a)\right\}\right]\\ &= \mathcal{D}_{a^+}^{\eta,\vartheta,\psi}\left[\mathcal{J}_{a^+}^{(1-\beta)(n-\alpha),\vartheta,\psi}f(y)-\sum_{k=0}^{n-1}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\left(\psi(y)-\psi(a)\right)^{k}}{\vartheta^{k}k!}(\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f)(a)\right].\end{aligned}$$

□

Lemma 3.8. *Let $n-1<\alpha<n$ where $n\in\mathbb{N}$, $\vartheta\in(0,1]$, $0\leq\beta\leq 1$, with $\gamma=\alpha+\beta(n-\alpha)$ such that $n-1<\gamma<n$. If $f\in C_{\gamma}[a,b]$ and $\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\in C_{\gamma,\psi}^{n}[a,b]$, then*

$$\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)=f(y)-\sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-k}}{\vartheta^{\gamma-k}\Gamma(\gamma-k+1)}(\mathcal{J}_{a^+}^{k-\gamma,\vartheta,\psi})(a). \tag{3.3}$$

Proof. Using Theorem 2.1 and Property 3.1, we get as

$$\begin{aligned}\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}f(y)&=\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{\beta(n-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f\Big)(y)\\&=\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(y).\end{aligned}$$

Now,

$$\begin{aligned}&\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(y)\\&=\frac{1}{\vartheta^{\gamma}\Gamma(\gamma)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\gamma-1}\psi'(s)\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(s)ds\\&=\frac{1}{\vartheta^{\gamma}\Gamma(\gamma)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\gamma-1}\psi'(s)\Big\{\mathcal{D}_{a^+}^{n,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\Big)(s)\Big\}ds\\&=\frac{1}{\vartheta^{\gamma}\Gamma(\gamma)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\gamma-1}\psi'(s)\mathcal{D}_{a^+}^{1,\vartheta,\psi}\Big\{\mathcal{D}_{a^+}^{n-1,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\Big)(s)\Big\}ds.\end{aligned}$$

Using (2.2) and then integrating by parts, we obtain

$$\begin{aligned}&\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(y)\\&=-\frac{1}{\vartheta^{\gamma-1}\Gamma(\gamma)}\Big[e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1}\{\mathcal{D}_{a^+}^{n-1,\vartheta,\psi}(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f)(a)\}\Big]\\&\quad+\frac{1}{\vartheta^{\gamma-1}\Gamma(\gamma-1)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\gamma-2}\psi'(s)\Big\{\mathcal{D}_{a^+}^{n-1,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\Big)(s)\Big\}ds.\end{aligned}$$

Now, continue the above process $(n-1)$ times, we get

$$\begin{aligned}&\mathcal{J}_{a^+}^{\gamma,\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}f(y)\\&=\frac{1}{\vartheta^{\gamma-n}\Gamma(\gamma-n)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\gamma-(n-1)}\psi'(s)\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f(s)ds\\&\quad-\sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-k}}{\vartheta^{\gamma-k}\Gamma(\gamma-k+1)}\Big\{\mathcal{D}_{a^+}^{n-k,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\Big)(a)\Big\}\\&=\mathcal{J}_{a^+}^{\gamma-n,\vartheta,\psi}\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f(y)-\sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-k}}{\vartheta^{\gamma-k}\Gamma(\gamma-k+1)}\Big\{\mathcal{D}_{a^+}^{n-k,\vartheta,\psi}\Big(\mathcal{J}_{a^+}^{n-\gamma,\vartheta,\psi}f\Big)(a)\Big\}.\end{aligned}$$

Therefore, by using Theorem 2.1 and Theorem 2.2, we arrive at (3.3). □

3.1. Equivalence between the generalized Cauchy problem and the Volterra integral equation

We consider the following nonlinear ψ-Hilfer generalized proportional fractional differential equation:

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\phi(y)=f(y,\phi(y)),\quad y\in I=[a,b],\ b>a\geq 0, \tag{3.4}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function subject to the following nonlocal initial condition

$$\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a) = \sum_{i=1}^{m} \mu_i \phi(\tau_i), \quad \gamma = \alpha + \beta(1-\alpha),\ \tau_i \in (a,b) \text{ and } \mu_i \in \mathbb{R}. \tag{3.5}$$

Now, to shows the equivalence between the Cauchy problem (3.4)–(3.5) and the Volterra integral equation

$$\begin{aligned}\phi(y) = & \frac{\Lambda}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1} \\ & \times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s) f(s,\phi(s))ds \\ & + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_{a^+}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\alpha-1}\psi'(s) f(s,\phi(s))ds,\end{aligned} \tag{3.6}$$

where

$$\Lambda = \frac{1}{\vartheta^{\gamma-1}\Gamma(\gamma) - \sum_{i=1}^{m}\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}(\psi(\tau_i)-\psi(a))^{\gamma-1}}.$$

We state and prove the following lemma.

Lemma 3.9. *Let* $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta(1-\alpha)$, *and assume that* $f(\cdot,\phi(\cdot)) \in C_{1-\gamma}[a,b]$ *for any* $\phi \in C_{1-\gamma}[a,b]$ *where* $f : (a,b)\times\mathbb{R} \to \mathbb{R}$ *be a function. If* $\phi \in C_{1-\gamma}^{\gamma}[a,b]$, *then* ϕ *satisfies (3.4)–(3.5) if and only if* ϕ *satisfies (3.6).*

Proof. Assume that $\phi \in C_{1-\gamma}^{\gamma}[a,b]$ be a solution of (3.4)–(3.5). We prove that ϕ is also solution of (3.6). From the Lemma (3.8) with $n = 1$, we have

$$\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\phi(y) = \phi(y) - \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1}}{\vartheta^{\gamma-1}\Gamma(\gamma)}\left(\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi\right)(a),$$

which implies that

$$\begin{aligned}\phi(y) = & \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1}}{\vartheta^{\gamma-1}\Gamma(\gamma)}\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a) \\ & + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_{a^+}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\alpha-1}\psi'(s) f(s,\phi(s))ds.\end{aligned} \tag{3.7}$$

Next, taking $y = \tau_i$ and then multiplying on both side by μ_i in (3.7), we obtain

$$\mu_i\phi(\tau_i) = \frac{(\psi(\tau_i)-\psi(a))^{\gamma-1}}{\vartheta^{\gamma-1}\Gamma(\gamma)}\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a) + \mu_i\left(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi} f(s,\phi(s))\right)(\tau_i),$$

this implies that

$$\begin{aligned}\sum_{i=1}^{m}\mu_i\phi(\tau_i) = & \frac{1}{\vartheta^{\gamma-1}\Gamma(\gamma)}\sum_{i=1}^{m}\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}(\psi(\tau_i)-\psi(a))^{\gamma-1}\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a) \\ & + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s) f(s,\phi(s))ds, \quad \tau_i > a.\end{aligned} \tag{3.8}$$

From the initial condition (3.5, we obtain

$$\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a) = \frac{\vartheta^{\gamma-1}\Gamma(\gamma)}{\vartheta^{\alpha}\Gamma(\alpha)}\Lambda\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi^{'}(s)f(s,\phi(s))ds. \tag{3.9}$$

Thus, the required result is obtained by replacing (3.9) in (3.7), which shows that $\phi(y)$ satisfies (3.6). Conversely, suppose that $\phi \in C_{1-\gamma}^{\gamma}[a,b]$ satisfies (3.6), we show that ϕ also satisfies (3.4)–(3.5). Now by applying the operator $\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}$ on both sides of (3.6) and then using Proposition 2.1 and Lemma 3.4, yields

$$\begin{aligned}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\phi(y) &= \mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\bigg(\frac{\Lambda}{\vartheta^{\alpha}\Gamma(\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1}\\ &\quad\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi^{'}(s)f(s,\phi(s))ds\bigg)\\ &\quad+\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\bigg(\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_{a^+}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\alpha-1}\psi^{'}(s)f(s,\phi(s))ds\bigg)\\ &= \mathcal{D}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}f(s,\phi(s))(y).\end{aligned} \tag{3.10}$$

Since by hypothesis $\phi \in C_{1-\gamma}^{\gamma}[a,b]$ and by the definition of $C_{1-\gamma}^{\gamma}[a,b]$ we have $\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\phi \in C_{1-\gamma}[a,b]$; so from (3.10) , we have

$$\mathcal{D}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}f = \mathcal{D}^{1,\vartheta,\psi}\mathcal{J}^{1-\beta(1-\alpha),\vartheta,\psi}f \in C_{1-\gamma,\psi}[a,b].$$

Also for $f(\cdot,\phi(\cdot)) \in C_{1-\gamma}[a,b]$ and from Theorem 2.3, it follows that

$$\mathcal{J}_{a^+}^{1-\beta(1-\alpha),\vartheta,\psi}f \in C_{1-\gamma,\psi}[a,b],$$

this implies from the definition of $C_{1-\gamma,\psi}^{n}[a,b]$, that

$$\mathcal{J}_{a^+}^{1-\beta(1-\alpha),\vartheta,\psi}f \in C_{1-\gamma,\psi}^{1}[a,b].$$

Now, applying the operator $\mathcal{J}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}$ on both sides of (3.10) and with the help of Theorem 2.5 and Lemma 3.3, we obtain

$$\begin{aligned}\mathcal{J}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\gamma,\vartheta,\psi}\phi(y) &= \mathcal{J}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}\mathcal{D}_{a^+}^{\beta(1-\alpha),\vartheta,\psi}f(s,\phi(s))(y)\\ &= f(y,\phi(y)) - \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\mathcal{J}_{a^+}^{1-\beta(1-\alpha),\vartheta,\psi}f)(a)}{\vartheta^{\beta(1-\alpha)-1}\Gamma(\beta(1-\alpha))}(\psi(y)-\psi(a))^{\beta(1-\alpha)-1}\\ &= f(y,\phi(y)).\end{aligned} \tag{3.11}$$

Hence,

$$\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}\phi(y) = f(y,\phi(y)), \quad y \in [a,b].$$

□

Next, we prove that the initial condition of (3.4) also holds. To prove this, applying $\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}$ to both sides of (3.6) and then using Proposition 2.1 and Theorem 2.1, we get

$$\begin{aligned}\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(y) &= \mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\bigg(\frac{\Lambda}{\vartheta^\alpha\Gamma(\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1}\\ &\quad\times\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds\bigg)\\ &\quad+\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\bigg(\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds\bigg)\\ &= \frac{\vartheta^{\gamma-1}\Gamma(\gamma)}{\vartheta^\alpha\Gamma(\alpha)}\Lambda e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds\\ &\quad+\mathcal{J}_{a^+}^{1-\beta(1-\alpha),\vartheta,\psi}f(s,\phi(s))(y).\end{aligned}\tag{3.12}$$

Since $1-\gamma<\beta(1-\alpha)$, so taking the limit as $y\to a^+$ and using Lemma 3.3 in (3.12), we get

$$\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a^+)=\frac{\vartheta^{\gamma-1}\Gamma(\gamma)}{\vartheta^\alpha\Gamma(\alpha)}\Lambda\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds.\tag{3.13}$$

Now, substituting $y=\tau_i$ and then multiplying through out by μ_i in (3.6),

$$\begin{aligned}\mu_i\phi(\tau_i) &= \frac{\Lambda}{\vartheta^\alpha\Gamma(\alpha)}\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}(\psi(\tau_i)-\psi(a))^{\gamma-1}\\ &\quad\times\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds\\ &\quad+\frac{\mu_i}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds,\end{aligned}$$

this implies that

$$\begin{aligned}\sum_{i=1}^m\mu_i\phi(\tau_i) &= \Lambda\sum_{i=1}^m\mu_i\Big(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(s,\phi(s))\Big)(\tau_i)\sum_{i=1}^m\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}(\psi(\tau_i)-\psi(a))^{\gamma-1}\\ &\quad+\sum_{i=1}^m\mu_i\Big(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(s,\phi(s))\Big)(\tau_i)\\ &= \sum_{i=1}^m\mu_i\Big(\mathcal{J}_{a^+}^{\alpha,\vartheta,\psi}f(s,\phi(s))\Big)(\tau_i)\bigg(1+\lambda\sum_{i=1}^m\mu_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(a))}(\psi(\tau_i)-\psi(a))^{\gamma-1}\bigg).\end{aligned}$$

Thus,

$$\sum_{i=1}^m\mu_i\phi(\tau_i)=\frac{\vartheta^{\gamma-1}\Gamma(\gamma)}{\vartheta^\alpha\Gamma(\alpha)}\Lambda\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds.\tag{3.14}$$

Hence, from (3.13) and (3.14), we have

$$\mathcal{J}_{a^+}^{1-\gamma,\vartheta,\psi}\phi(a^+)=\sum_{i=1}^m\mu_i\phi(\tau_i),\tag{3.15}$$

and this completes the proof.

3.2. Existence of solution

Utilizing the concepts of krasnoselskii's fixed point theorem, in this subsection, we state and prove the existence of at least one solution of problem (3.4)–(3.5) in the weighted space $C^{\alpha,\beta}_{1-\gamma,\psi}[a,b]$.

Theorem 3.10. ***(Krasnoselskii's fixed point theorem)*** *Let B be a nonempty bounded closed convex subset of a Banach space $\mathcal{X}$. Let $N, M : B \to \mathcal{X}$ be two continuous operators satisfying:*

(i) $Nx + My \in B$ whenever $x, y \in B$;

(ii) N is compact and continuous;

(iii) M is contraction mapping;

then, there exist $u \in B$ such that $u = Nu + Mu$.

For that firstly we called of the following assumptions:

$(C1)$. Let $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C^{\beta(1-\alpha)}_{1-\gamma,\psi}[a,b]$, for any $\phi \in C^{\gamma}_{1-\gamma,\psi}[a,b]$.

$(C2)$. There exists a constant $\mathbf{K} > 0$ such that

$$\left|f(y,\omega) - f(y,\overline{\omega})\right| \leq \mathbf{K}|\omega - \overline{\omega}|, \text{ for all } \omega, \overline{\omega} \in \mathbb{R} \text{ and } y \in I.$$

$(C3)$. Assume that

$$\mathbf{K}\Psi < 1,$$

where

$$\Psi = \frac{\mathcal{B}(\gamma,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}\left(|\Lambda| \sum_{i=1}^{m} \mu_i(\psi(\tau_i) - \psi(a))^{\alpha+\gamma-1} + (\psi(b) - \psi(a))^{\alpha}\right), \tag{3.16}$$

and

$$\mathcal{B}(\gamma,\alpha) = \int_0^1 y^{\gamma-1}(1-y)^{\alpha-1}dy, \quad Re(\gamma),\ Re(\alpha) > 0,$$

is the beta function.

$(C4)$. Also let

$$\mathbf{K}\triangle < 1,$$

where

$$\triangle = \frac{\mathcal{B}(\gamma,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}|\Lambda| \sum_{i=1}^{m} \mu_i(\psi(\tau_i) - \psi(a))^{\alpha+\gamma-1}. \tag{3.17}$$

Now, the following theorem yields the existence of at least one solution for the problem (3.4).

Theorem 3.11. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1-\alpha)$. Suppose that the assumptions $(C1), (C2)$, and $(C4)$ holds. Then the problem (3.4)–(3.5) has at least one solution in the space $C^{\gamma}_{1-\gamma}[a,b]$.*

Proof. Given that $\|\chi\|_{C_{1-\gamma,\psi}[a,b]} = \sup_{y\in J}\left|(\psi(y) - \psi(a))^{1-\gamma}\chi(y)\right|$ and choose $\varepsilon \geq \mathscr{M}\|\chi\|_{C_{1-\gamma,\psi}[a,b]}$, where

$$\mathscr{M} = \frac{\mathcal{B}(\gamma,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}\left(|\Lambda| \sum_{i=1}^{m} \mu_i(\psi(\tau_i) - \psi(a))^{\alpha+\gamma-1}(\psi(b) - \psi(a))^{\alpha}\right), \tag{3.18}$$

also consider $\mathbf{B}_\varepsilon = \{\phi \in C[a,b] : \|\phi\|_{C_{1-\gamma}[a,b]} \leq \varepsilon\}$. Thus, for all $y \in [a,b]$ consider the operators N and M defined on $\mathbf{B}_\varepsilon$ by

$$\begin{aligned}
(N\phi)(y) &= \frac{1}{\vartheta^\alpha \Gamma(\alpha)} \int_{a^+}^{y} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))} (\psi(y)-\psi(s))^{\alpha-1} \psi'(s) f(s,\phi(s)) ds, \\
(M\phi)(y) &= \frac{\Lambda}{\vartheta^\alpha \Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} (\psi(y)-\psi(a))^{\gamma-1} \\
&\quad \times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))} (\psi(\tau_i)-\psi(s))^{\alpha-1} \psi'(s) f(s,\phi(s)) ds.
\end{aligned}$$

Step 1. For all $\phi, \bar{\phi} \in \mathbf{B}_\varepsilon$, yields

$$\begin{aligned}
&\left| (N\phi(y) + M\bar{\phi}(y))(\psi(y)-\psi(a))^{1-\gamma} \right| \\
&\leq \frac{(\psi(y)-\psi(a))^{1-\gamma}}{\vartheta^\alpha \Gamma(\alpha)} \int_{a^+}^{y} (\psi(y)-\psi(s))^{\alpha-1} (\psi(s)-\psi(a))^{\gamma-1} \psi'(s) \left| f(s,\phi(s))(\psi(s)-\psi(a))^{1-\gamma} \right| ds \\
&\quad + \frac{|\Lambda|}{\vartheta^\alpha \Gamma(\alpha)} \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} (\psi(\tau_i)-\psi(s))^{\alpha-1} (\psi(\tau_i)-\psi(a))^{\gamma-1} \left| f(s,\bar{\phi}(s))(\psi(\tau_i)-\psi(a))^{1-\gamma} \right| ds \\
&\leq \|\chi\| \left[\frac{\mathcal{B}(\gamma,\alpha)}{\vartheta^\alpha \Gamma(\alpha)} |\Lambda| \sum_{i=1}^{m} \mu_i (\psi(\tau_i)-\psi(a))^{\alpha+\gamma-1} + \frac{(\gamma,\alpha)}{\vartheta^\alpha \Gamma(\alpha)} (\psi(b)-\psi(a))^{\alpha} \right] \\
&\leq \|\chi\| \mathscr{M} \\
&\leq \vartheta < \infty,
\end{aligned}$$

this implies that $N\phi + M\bar{\phi} \in \mathbf{B}_\varepsilon$.

Step 2. We show that M is a contraction. Let $\phi, \bar{\phi} \in C_{1-\gamma}[a,b]$ and $y \in I$, then

$$\begin{aligned}
&\left| (M\phi(y) - M\bar{\phi}(y))(\psi(y)-\psi(a))^{1-\gamma} \right| \\
&\quad = \left| \Lambda e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))} \sum_{i=1}^{m} \mu_i \mathcal{J}_{a^+}^{1-\beta(1-\alpha),\vartheta,\psi} (f(s,\phi(s)) - f(s,\bar{\phi}(s)))(\tau_i) \right| \\
&\quad \leq \frac{\mathbf{K}|\Lambda|}{\vartheta^\alpha \Gamma(\alpha)} \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} (\psi(\tau_i)-\psi(s))^{\alpha-1} (\psi(s)-\psi(a))^{\gamma-1} \psi'(s) |\phi(s) - \bar{\phi}(s)| ds \\
&\quad \leq \left[\frac{\mathbf{K}|\Lambda|}{\vartheta^\alpha \Gamma(\alpha)} \mathcal{B}(\gamma,\alpha) \sum_{i=1}^{m} \mu_i (\psi(\tau_i)-\psi(a))^{\alpha+\gamma-1} \right] \|\phi - \bar{\phi}\|_{C_{1-\gamma,\psi}[a,b]} \\
&\quad \leq \mathbf{K} \triangle \|\phi - \bar{\phi}\|_{C_{1-\gamma,\psi}[a,b]}
\end{aligned} \tag{3.19}$$

Hence, it follows from $(C4)$ that M is a contraction.

Step 3. Now we verify that the operator N is continuous and compact.
Since the function f is continuous, so the operator N is also continuous ,
Hence, for any $\phi \in C_{1-\gamma}[a,b]$, we obtain

$$\|N\phi\| \leq \|\chi\| \frac{\mathcal{B}(\gamma,\alpha)}{\vartheta^\alpha \Gamma(\alpha)} (\psi(b)-\psi(a))^{\alpha} < \infty.$$

This shows that N is uniformly bounded on $\mathbf{B}_\varepsilon$. So, it remains to prove that the operator N is compact. Denoting $\sup_{(y,\phi)\in I\times\mathbf{B}_\varepsilon}|f(y,\phi(y))| = \delta < \infty$ and for any $a < \tau_1 < \tau_2 < b$,

$$\begin{aligned}
&\Big|(\psi(\tau_2)-\psi(a))^{1-\gamma}(N\phi(\tau_2)) + (\psi(\tau_1)-\psi(a))^{1-\gamma}(N\phi(\tau_1))\Big| \\
&= \Bigg|\frac{(\psi(\tau_2)-\psi(a))^{1-\gamma}}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{\tau_2} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_2)-\psi(s))}(\psi(\tau_2)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds \\
&\quad + \frac{(\psi(\tau_1)-\psi(a))^{1-\gamma}}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{\tau_1} e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_1)-\psi(s))}(\psi(\tau_1)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds\Bigg| \\
&\leq \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{\tau_2}\Big[(\psi(\tau_2)-\psi(a))^{1-\gamma}(\psi(\tau_2)-\psi(s))^{\alpha-1}(\psi(\tau_1)-\psi(a))^{1-\gamma}(\psi(\tau_1)-\psi(s))^{\alpha-1}\Big] \qquad (3.20)\\
&\quad\times\psi'(s)\Big|f(s,\phi(s))\Big|ds \\
&\quad + \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi(\tau_1)-\psi(a))^{1-\gamma}(\psi(\tau_1)-\psi(s))^{\alpha-1}\psi'(s)\Big|f(s,\phi(s))\Big|ds \\
&\longrightarrow 0 \ \text{ as } \tau_2\to\tau_1. \qquad (3.21)
\end{aligned}$$

As a consequence of Arzela-Ascoli theorem N is compact on $\mathbf{B}_\varepsilon$. Thus, as a result of Theorem 3.10, problem (3.4)–(3.5) has at least one solution. □

3.3. Uniqueness of solution

In this subsection, we state and prove the uniqueness of solutions of problem (3.4)−(3.5) via Banach contraction principle,.

Theorem 3.12. *(**Contraction mapping principle**) Let $\mathcal{X}$ be a Banach space, $\mathcal{S}\subset\mathcal{X}$ be closed and $\mathbb{T}:\mathcal{S}\to\mathcal{S}$ a contraction mapping i.e*

$$\|\mathbb{T}z-\mathbb{T}\bar{z}\| \leq k\|z-\bar{z}\|, \text{ for all } z,\bar{z}\in\mathcal{S}, \text{ and some } k\in(0,1).$$

Then $\mathcal{S}$ has a unique fixed point.

Theorem 3.13. *Let $0<\alpha<1$, $0\leq\beta\leq1$ and $\gamma=\alpha+\beta(1-\alpha)$. Suppose that the assumptions $(C2)-(C3)$ holds, then the problem (3.4)–(3.5) has a unique solution in the space $C^\gamma_{1-\gamma,\psi}[a,b]$.*

Proof. Consider the fractional operator $\mathbb{T}: C_{1-\gamma,\psi}[a,b]\to C_{1-\gamma,\psi}[a,b]$ defined by:

$$\begin{aligned}
(\mathbb{T}\phi)(y) = {}& \frac{\Lambda}{\vartheta^\alpha\Gamma(\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(a))}(\psi(y)-\psi(a))^{\gamma-1} \\
&\times\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(s))}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds \\
&+\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_{a^+}^{y}e^{\frac{\vartheta-1}{\vartheta}(\psi(y)-\psi(s))}(\psi(y)-\psi(s))^{\alpha-1}\psi'(s)f(s,\phi(s))ds. \qquad (3.22)
\end{aligned}$$

Clearly the operator $\mathbb{T}$ is well defined. Now for any $\phi_1,\phi_2\in C_{1-\gamma}[a,b]$, $y\in I$ and $\Big|e^{\frac{\vartheta-1}{\vartheta}\psi(y)}\Big|<1$, gives

$$\Big|((\mathbb{T}\phi_1)(y)-(\mathbb{T}\phi_2)(y))(\psi(y)-\psi(a))^{1-\gamma}\Big|$$

$$\leq \frac{|\Lambda|}{\vartheta^{\alpha}\Gamma(\alpha)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}(\psi(\tau_i)-\psi(s))^{\alpha-1}\psi'(s)\Big|f(s,\phi_1(s))-f(s,\phi_2(s))\Big|ds$$
$$+\frac{(\psi(y)-\psi(a))^{1-\gamma}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_{a^+}^{y}(\psi(y)-\psi(s))^{\alpha-1}\psi'(s)\Big|f(s,\phi_1(s))-f(s,\phi_2(s))\Big|ds$$
$$\leq \frac{\mathbf{K}|\Lambda|}{\vartheta^{\alpha}\Gamma(\alpha)}\left(\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}(\psi(\tau_i)-\psi(s))^{\alpha-1}(\psi(s)-\psi(a))^{\gamma-1}\psi'(s)ds\right)\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]}$$
$$+\frac{\mathbf{K}}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(y)-\psi(a))^{1-\gamma}\left(\int_{a^+}^{y}(\psi(y)-\psi(s))^{\alpha-1}(\psi(s)-\psi(a))^{\gamma-1}\psi'(s)ds\right)\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]}$$
$$\leq \frac{\mathbf{K}|\Lambda|}{\vartheta^{\alpha}\Gamma(\alpha)}\mathcal{B}(\gamma,\alpha)\sum_{i=1}^{m}\mu_i(\psi(\tau_i)-\psi(s))^{\alpha+\gamma-1}\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]}$$
$$+\frac{\mathbf{K}}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha}\mathcal{B}(\gamma,\alpha)\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]} \tag{3.23}$$

Hence,

$$\|(\mathbb{T}\phi_1)-(\mathbb{T}\phi_2)\|_{C_{1-\gamma,\psi}[a,b]}$$
$$\leq \frac{\mathbf{K}}{\vartheta^{\alpha}\Gamma(\alpha)}\mathcal{B}(\gamma,\alpha)\left(|\Lambda|\sum_{i=1}^{m}\mu_i(\psi(\tau_i)-\psi(s))^{\alpha+\gamma-1}+(\psi(b)-\psi(a))^{\alpha}\right)\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]}$$
$$\leq \mathbf{K}\Psi\|\phi_1-\phi_2\|_{C_{1-\gamma,\psi}[a,b]} \tag{3.24}$$

Thus, from (C3) it follows that $\mathbb{T}$ is a contraction map. So, in view of the Theorem 3.12, there exists a unique solution of problem (3.4)–(3.5). □

4. Illustrative examples

Example 4.1. Consider the following fractional differential equation with generalized Hilfer's proportional fractional derivative as:

$$\begin{cases}\mathcal{D}_{0^+}^{\frac{1}{2},\frac{2}{3},\frac{4}{5},\psi}\phi(y)=\frac{\cos x}{4}+\frac{1}{90}\frac{|\phi(y)|}{1+|\phi(y)|}, & y\in I=[0,1],\\ \mathcal{J}_{0^+}^{1-\gamma,\frac{4}{5},\psi}\phi(0)=5\phi\left(\frac{1}{3}\right)+\sqrt{3}\phi(\frac{3}{5}).\end{cases} \tag{4.1}$$

On comparing (3.4)–(3.5) with (4.1), we obtain the values as follows

$$\alpha=\frac{1}{2},\ \beta=\frac{2}{3},\ \vartheta=\frac{4}{5},\ \gamma=\frac{5}{6},\ a=0,\ b=1,\ \mu_1=5,\ \mu_2=\sqrt{3},\ \text{as } m=2,\ \text{so } \tau_1=\frac{1}{3},\ \tau_2=\frac{3}{5}\in I.$$

Also $f: I\times\mathbb{R}\to\mathbb{R}$ is a function defined by

$$f(y,\phi(y))=\frac{\cos y}{4}+\frac{1}{90}\frac{|\phi(y)|}{1+|\phi(y)|},\qquad y\in I.$$

Clearly, f is continuous function and

$$\Big|f(y,\phi_1)-f(y,\phi_2)\Big|\leq\frac{1}{90}|\phi_1-\phi_2|.$$

It follows that conditions $(C1)$ and $(C2)$ holds with $\mathbf{K} = \frac{1}{90}$. Now, choose $\psi(y) = y^2 + 1$, then it implies that $\psi(y)$ is positive increasing and continuous function in $[0, 1]$ and $\psi'(y) \neq 0$ for all $y \in I$. Substituting these values and after simple calculation, yields

$$|\Lambda| = \left| \frac{1}{(\frac{4}{5})^{\frac{5}{6}-1}\Gamma(\frac{5}{6}) - \left(5e^{\left(\frac{-1}{36}\right)}(\frac{1}{9}+1-1)^{\frac{5}{6}-1} + \sqrt{3}e^{\left(\frac{-9}{100}\right)}(\frac{9}{25}+1-1)^{\frac{5}{6}-1}\right)} \right| \approx 0.12,$$

and

$$\Psi = \frac{(\frac{5}{6}, \frac{1}{2})}{(\frac{4}{5})^{\frac{1}{2}}\Gamma(\frac{1}{2})} \left\{ |\Lambda| \left(5\Big(\frac{1}{9}+1-1\Big)^{\frac{1}{2}+\frac{5}{6}-1} + \sqrt{3}\Big(\frac{9}{25}+1-1\Big)^{\frac{1}{2}+\frac{5}{6}-1} \right) + (2-1)^{\frac{1}{2}} \right\} \approx 2.04,$$

this implies that $\mathbf{K}\Psi < 1$, which is $(C3)$.

Furthermore, $\triangle \approx 0.61 > 0$ and $\mathbf{K}\triangle < 1$, which means that the assumption $(C4)$ is also satisfied. Hence, by Theorem 3.11 and Theorem 3.13, problem (4.1) has at least one solution and hence is unique on the interval I.

Example 4.2. Consider the ψ-Hilfer generalized proportional fractional differential equation of the form

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{7}, \frac{3}{4}, \frac{1}{2}, \psi}\phi(y) = \frac{y}{3} + \frac{2}{35}\frac{|\phi(y)|}{1+|\phi(y)|}, & y \in I = [0, 2], \\ \mathcal{J}_{0^+}^{1-\gamma, \frac{1}{2}, \psi}\phi(0) = -3\phi\left(\frac{5}{7}\right) + \frac{5}{2}\phi\left(\frac{11}{13}\right) + 3\sqrt{2}\phi\left(\frac{9}{10}\right), & \tau_1,\ \tau_2,\ \tau_3 \in (0, 2). \end{cases} \tag{4.2}$$

After doing the same steps as in Example 1 above with $\psi(y) = 2y^3 + 3y^2 + 1$, we obtain the values as $|\Lambda| \approx 0.73$, $\Psi \approx 14.43$ and $\triangle \approx 7.23$. Therefore,

$$\mathbf{K}\Psi \approx 0.82 < 1,$$

and

$$\mathbf{K}\triangle \approx 0.57 < 1,$$

where, $\mathbf{K} = \frac{2}{35}$. So, again in view of Theorem 3.11 and Theorem 3.13, the problem (4.2) has atleast one solution and hence a unique solution on I.

Example 4.3. Let

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{3}, \frac{7}{10}, 1, \psi}\phi(y) = \frac{\sin 2y}{40e^{2y}}\left(\frac{|\phi(y)|}{2+|\phi(y)|}\right), & y \in I = [0, 4], \\ \mathcal{J}_{0^+}^{1-\gamma, 1, \psi}\phi(0) = -\phi\left(\frac{1}{2}\right) + 2\phi\left(\frac{3}{2}\right) + 2\sqrt{3}\phi\left(\frac{5}{2}\right) + 4\sqrt{2}\phi\left(\frac{7}{2}\right), & \tau_1,\ \tau_2,\ \tau_3,\ \tau_4 \in (0, 4). \end{cases} \tag{4.3}$$

be the ψ-Hilfer generalized proportional fractional differential equation.

On comparing (4.3) with (3.4), (3.5), we have the values of parameters as follows: $\alpha = \frac{1}{3}$, $\beta = \frac{7}{10}$, $\vartheta = 1$, $\gamma = \frac{4}{5}$, $a = 0$, $b = 4$, $\mu_1 = -1$, $\mu_2 = 2$, $\mu_3 = 2\sqrt{3}$, $\mu_4 = 4\sqrt{2}$ as $m = 4$, so $\tau_1 = \frac{1}{2}$, $\tau_2 = \frac{3}{2}$, $\tau_3 = \frac{5}{2}$, $\tau_4 = \frac{7}{2} \in I$. In addition, let $\psi(y) = y^3 + 2y + 1$. Now, after performing simple computations, we obtain the estimated values as $|\Lambda| \approx 0.26$, $\Psi \approx 14.06$ and $\triangle \approx 6.29$. Since,

$$\mathbf{K}\Psi \approx 0.70 < 1,$$

and

$$\mathbf{K}\triangle \approx 0.31 < 1,$$

where, $\mathbf{K} = \frac{1}{20}$. Thus, with the help of Theorem 3.11 and Theorem 3.13, the problem (4.3) has atleast one solution and hence a unique solution on I.

5. Conclusions

The main aim of this paper is to propose a generalized fractional derivative $\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}$ with three parameters α, β and ϑ of a function with respect to another function ψ, in the setting of Hilfer generalized proportional fractional derivative. We derived some important properties of the proposed derivative and we investigated conditions for which the semigroup properties are valid. Considering the nonlinear fractional differential equations in sense of the proposed derivative, we established the relationship between the Volterra integral equations and investigated its existence and uniqueness of solutions using Krasnoselskii's and Banach fixed point theorems. Furthermore, some examples are illustrated to support the theoretical analysis. In addition, this paper improves the preceding ones as it unifies two different derivatives which has many applications in science and engineering. Besides, its of great important to note that:

- Setting $\psi(y) = y$ in problem (3.4)–(3.5), the formulation reduces to Hifer generalized proportional fractional derivative studied by Idris et al. [31].
- Setting $\vartheta = 1$, then the derivative operator $\mathcal{D}_{a^+}^{\alpha,\beta,\vartheta,\psi}$ reduces to the ψ-Hilfer fractional derivative $\mathcal{D}_{a^+}^{\alpha,\beta,\psi}$ studied by J.Vanterler et al. [34].

Finally, we conclude that the results obtained are new and generalized the existence ones in the literature and this achievement can be regarded towards the improvement of qualitative aspect of fractional calculus.

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Conflicts of interest

The authors declare no conflict of interest.

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