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On the analysis of an analytical approach for fractional Caudrey-Dodd-Gibbon equations

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Abstract The principal aim of this paper is to study the approximate solution of nonlinear Caudrey-Dodd-Gibbon equation of fractional order by employing an analytical method. The Caudrey-Dodd-Gibbon equation arises in plasma physics and laser optics. The Caputo derivative is applied to model the physical problem. By applying an effective semi-analytical technique, we attain the approximate solutions without linearization. The uniqueness and the convergence analysis for the applied method are shown. The graphical representation of solutions of fractional Caudrey-Dodd-Gibbon equation demonstrates the applied technique is very efficient to obtain the solutions of such type of fractional order mathematical models.

1. Introduction

Since the classical calculus describes the nature of the problem by considering current state but the fractional calculus observes and analyze the behavior of problem by using the past history. Thus, from the several branches of mathematics, fractional calculus is one of the modern and effective branch that deals with the real-world problems very easily and give very impressive results. Fractional order derivatives and integrals are very important to show the behavior of the problems arising in chemistry, biology, engineering, plasma physics, laser optics, mathematical physics etc.[1–6].

To study the behavior of nonlinear partial differential equations occurring in physical sciences are challenging and main issues. But the fractional calculus performs a very crucial part to analyze the nonlinear problems. Consequently, here we are motivated to examine the nature and solution of the Korteweg-deVries (KdV) equation [7,8] in a particular case.

The general form of the fifth order KdV equation is given as follows

$$v_t + \omega v_{yyyyy} + a v v_{yyy} + b v_y v_{yy} + c v^2 v_y = 0 \quad (1)$$

where, $v = v(y,t)$ is a variable, a, b, c and ω are non-zero arbitrary real parameters. Here, we present approximate solutions for a particular case of this equation obtained by setting $a = 30, b = 30, c = 180, \omega = 1$. For these particular values, KdV equation can be expressed as

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$$v_t + v_{yyyyy} + 30vv_{yyy} + 30v_yv_{yy} + 180v^2v_y = 0 \quad (2)$$

Eq. (2) represents the Caudrey-Dodd-Gibbon equation.

Since Caputo fractional derivative [9] narrates the memory of system at past historical stages, because of such property this derivative of arbitrary order is handy to demonstrate new real-world phenomenon of atmospheric physics, earthquake, ocean climate, vibration, dynamical system, polymers etc. Hence, the Caudrey-Dodd-Gibbon equation in Caputo type can be written as

$$D_t^\alpha v(y,t) + v_{yyyyy} + 30vv_{yyy} + 30v_yv_{yy} + 180v^2v_y = 0 \quad (3)$$

where $D_t^\alpha v(y,t)$ represents the fractional order derivatives of $v(y,t)$ in caputo type. For $\alpha = 1$, Eq. (3) represents the classical Caudrey-Dodd-Gibbon equation which was given by Caudrey, Dodd and Gibbon [7]. The fractional model of this nonlinear equation studied as a mathematical model of the internal waves in a shallow density-stratified fluid and surface waves of small amplitude and long wave length on shallow water. It plays a very important role in plasma physics and laser optics. Since in the modelling of nonlinear partial differential equation, crucial aspects i.e. convergence, divergence and efficiency of solutions in numerical evaluation are equally important. So, in order to produce more attainable and more efficient results, various numerical and analytical techniques [10–13] are available for solving the nonlinear partial differential equations. For solving the Caudrey-Dodd-Gibbon equation so many methods are given by numerous mathematicians [14–21]. But these methods have their own drawbacks and limitations for example huge computational work, more computer memory, time and divergent results.

In recent years homotopy techniques are coupled with integral transforms and for handling different kind of mathematical models by several authors [22–23]. In this acrticle, we apply a semi-analytical technique namely homotopy analysis Sumudu transform method [24]. The homotopy analysis Sumudu transform method is amalgamation of two powerful and crucial techniques, the one is homotopy analysis method [25–27] and another is standard Sumudu transform technique [28] with homotopy polynomial. The main advantage of the considered method is that it easily handles the nonlinear terms with high accuracy and control the convergence of solution due to involvement of an auxiliary parameter $\hbar$. This paper is developed as: In Section 2, Sumudu transform as well as arbitrary order derivatives in Riemann-Liouville and Caputo sense are discussed. Section 3 involves the elementary idea of the considered method. Analysis of convergence and uniqueness of the obtained solution by using considered scheme is stated in Section 4. Section5 gives the solution of fractional Caudrey-Dodd-Gibbon model. In Section 6, numerical results are intimated and finally in Section 7, we present concluding remarks.

2. Some primitive definitions

Definition 2.1. *Let a function $h(t) \in C_\beta, \beta \geq -1$, thus the Riemann-Liouville derivative of order $\alpha > 0$ is written in the subsequent manner [1]*

$$G^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} h(\eta) d\eta, (\alpha > 0) \quad (4)$$

Definition 2.2. *The Caputo* derivative *of fractional order $(m-1 < \alpha \leq m)$ of function $h(t)$ is represented as follows [1]*

$$D_t^\alpha h(t) = G^{m-\alpha} D^m h(t)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\eta)^{m-\alpha-1} h^m(\eta) d\eta, \quad (5)$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0$.

Definition 2.3. *Since Fourier, Laplace, Hankel and many more well-known integral transforms are available. In this sequence, Watugala [28] proposed a novel transform namely Sumudu transform, described and interpreted over the set of functions*

$\Xi = \{h(t) : \exists N, \delta_1, \delta_2 > 0, |h(t)| < Ne^{|t|/\delta_j}, \quad \text{if} \quad t \in (-1)^j \times [0,\infty)\}$,
as follows

$$H(u) = S[h(t)] = \int_0^\infty h(ut)e^{-t} dt, u \in (-\delta_1, \delta_2) \quad (6)$$

More properties and other important information are given in so many papers, see [29–31]. The Sumudu transform of $D_t^\alpha h(t)$ is given as follows

$$S\left[D_t^\alpha h(t)\right] = u^{-\alpha} S[h(t)] - \sum_{s=0}^{m-1} u^{-\alpha+s} h^{(s)}(0+), (m-1 < \alpha \leq m). \quad (7)$$

3. Elementary idea of homotopy analysis Sumudu transform method

To discuss and understand the homotopy analysis Sumudu transform method [23–24], here we consider a general fractional order nonlinear partial differential equation as follows

$$D_t^\alpha v(y,t) + Rv(y,t) + Nv(y,t) = \psi(y,t), m-1 < \alpha \leq m \quad (8)$$

here, $v(y,t)$ denotes the function of two variables y and t, D_t^α representsthe Caputo derivative of fractional order α, $m \in \mathbb{N}$, R stands for bounded linear operator in two variables y in addition t, Nindicates the general nonlinear differential operator in y and t and $\psi(y,t)$ is the source term.

Utilizing the Sumudu transform on Eq. (8), we get

$$S[D_t^\alpha v] + S[Rv + Nv] = S[\psi(y,t)] \quad (9)$$

Employing differentiation properties of Sumudu transform, we get the consequent equation

$$u^{-\alpha} S[v] - \sum_{s=0}^{m-1} u^{-\alpha+s} v^{(s)}(y,0) + S[Rv + Nv] = S[\psi(y,t)] \quad (10)$$

On simplifying we have,

$$S[v] - u^\alpha \sum_{s=0}^{m-1} u^{-\alpha+s} v^{(s)}(y,0) + u^\alpha S[Rv + Nv] - u^\alpha S[\psi(y,t)] = 0 \quad (11)$$

By Eq. (11), the nonlinear operator can be expressed as follows

$$N[\xi(y,t;p)] = S[\xi(y,t;p)] - u^{\alpha}\sum_{s=0}^{m-1} u^{-\alpha+s}\xi^{(s)}(y,t;p)(0) + u^{\alpha}[S[R\xi(y,t;p) + N\xi(y,t;p)] - S[\psi(y,t)]] = 0 \tag{12}$$

here $\xi(y,t;p)$ denotes a function of y,t additionally$p$ represents embedding parameter with $0 \le p \le 1$. Now, we represent the homotopy demonstrated by Eq. (13) as

$$(1-p)S[\xi(y,t;p) - v_0] = \hbar N[v(y,t)], \tag{13}$$

where S stands for the Sumudu transform, $\hbar \neq 0$ denotes auxiliary parameter, $v_0(y,t)$ indicatesthe initial approximation of $v(y,t)$ in addition $\xi(y,t;p)$ represents an unknown function. Further, it can be observed that, if we put values of embedding parameter as $p=0$ in addition $p=1$ then, it yields

$$\xi(y,t;0) = v_0(y,t), \xi(y,t;1) = v(y,t), \tag{14}$$

respectively. Hence, solution $\xi(y,t;p)$ changes from initial approximation $v_0(y,t)$ to the solution $v(y,t)$, here p variate from 0 to 1. Expanding the function $\xi(y,t;p)$ in terms of Taylor's series about the parameter$p$, we obtain the subsequent equation

$$\xi(y,t;p) = v_0(y,t) + \sum_{k=1}^{\infty} v_k(y,t)p^k, \tag{15}$$

$$v_k(y,t) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\{\xi(y,t;p)\}|_{p=0}. \tag{16}$$

If the convergence control parameter $\hbar$ and the initial approximation $v_0(y,t)$ are described appropriately, then at $p=1$, Eq. (15) is convergent. Thus, we have

$$v(y,t) = v_0(y,t) + \sum_{k=1}^{\infty} v_k(y,t) \tag{17}$$

The outcome given by Eq. (17) represents the one of the solution of discussed arbitrary order nonlinear differential equation. The governing equation attained by utilizing Eq. (13) and Eq. (17).

We write the vectors in the subsequent manner

$$\vec{v}_k = \{v_0(y,t), v_1(y,t), v_2(y,t), v_3(y,t), \cdots, v_k(y,t)\} \tag{18}$$

Now, differentiating Eq. (13), k – times w. r. t. p, then dividing by $k!$ additionally at the end put $p=0$, we get the following equation

$$S[v_k(y,t) - \chi_k v_{k-1}(y,t)] = \hbar \Im_k(\vec{v}_{k-1}) \tag{19}$$

Now exerting the inverse Sumudu transform operator on the above equation, we get the subsequent result

$$v_k(y,t) = \chi_k v_{k-1}(y,t) + \hbar S^{-1}[\Im_k(\vec{v}_{k-1})] \tag{20}$$

where

$$\chi_k = \begin{cases} 0, k \le 1 \\ 1, k > 1 \end{cases} \tag{21}$$

and we illustrate the value of $\Im_k(\vec{v}_{k-1})$ in an enhanced manner as follows

$$\Im_k(\vec{v}_{k-1}) = S[v_{k-1}(y,t)] - (1-\chi_k)u^{\alpha}(\sum_{s=0}^{m-1} u^{-\alpha+s}v^{(s)}(y,0) + S[\psi(y,t)]) + u^{\alpha}S[Rv_{k-1} + P_{k-1}] \tag{22}$$

In Eq. (22) P_k indicates the homotopy polynomial [32] and described as follows

$$P_k = \frac{1}{\Gamma(k)}\left[\frac{\partial^k}{\partial p^k}N\xi(y,t;p)\right]_{p=0} \tag{23}$$

and

$$\xi(y,t;p) = \xi_0 + p\xi_1 + p^2\xi_2 + \cdots. \tag{24}$$

using Eq. (22) in Eq. (20), we have

$$v_k(y,t) = (\chi_k + \hbar)v_{k-1}(y,t) - \hbar(1-\chi_k)S^{-1}\left(u^{\alpha}\sum_{s=0}^{m-1} u^{-\alpha+s}v^{(s)}(y,0) + u^{\alpha}S[\psi(y,t)]\right) + \hbar S^{-1}[u^{\alpha}S[Rv_{k-1} + P_{k-1}]] \tag{25}$$

Thus, we can find the several components $v_k(y,t)$ for $k \ge 1$ by Eq. (25) and the solution is explored by the subsequent equation as follows

$$v(y,t) = \sum_{k=0}^{\infty} v_k(y,t) \tag{26}$$

4. Analysis of convergence and uniqueness

In this section, we examine the convergence in addition uniqueness of the solution obtained by applied method.

Theorem 1 (*(Uniqueness Theorem)*). The outcome of fractional Caudrey-Dodd-Gibbon equation (3) obtained by HASTM is unique, while, $0 < \rho < 1$, where $\rho = (1+\hbar) + \hbar(\delta_1 + 30(\delta_2 C + D\delta_3) + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7 + 180(\delta_8 C(C+D) + D^2\delta_5))T$.

Proof. Here, we demonstrate the solution of fractional Caudrey-Dodd-Gibbon equation (3)

$$D_t^{\alpha}v + v_{yyyyy} + 30vv_{yyy} + 30v_y v_{yy} + 180v^2 v_y = 0$$

as

$$v(y,t) = \sum\nolimits_{k=0}^{\infty} v_k(y,t), \tag{27}$$

where

$$v_k(y,t) = (\chi_k + \hbar)v_{k-1}(y,t) - \hbar(1-\chi_k)v_0(y,t) + \hbar S^{-1}\left[u^{\alpha}S\left(v_{(k-1)yyyyy} + 30A_{k-1} + 30B_{k-1} + 180C_{k-1}\right)\right] \tag{28}$$

Let, if possible, v as well as v^* be two separate solutions of fractional Caudrey-Dodd-Gibbon equation (3) s.t. $|v| \le C, |v^*| \le D$, then using Eq. (28), we have

$$|v - v^*| = |(1+\hbar)(v - v^*) + \hbar S^{-1}\left[u^{\alpha}S\{\left(v_{yyyyy} - {v^*}_{yyyyy}\right) + 30(vv_{yyy} - v^*{v^*}_{yyy}) + 30\left(v_y v_{yy} - {v^*}_y{v^*}_{yy}\right) + 180(v^2 v_y - {v^*}^2{v^*}_y)\}\right]| \tag{29}$$

Now by applying convolution theorem for Sumudu transform, we get

$$|v - v^*| \le (1+\hbar)|v - v^*| + \hbar\int_0^t \left(|v_{yyyyy} - {v^*}_{yyyyy}| + 30|vv_{yyy} - v^*{v^*}_{yyy}| + 30|v_y v_{yy} - {v^*}_y{v^*}_{yy}| + 180|v^2 v_y - {v^*}^2{v^*}_y|\right)\frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}d\eta$$

$$\leq (1+\hbar)|v-v^*| + \hbar\int_0^t \left(\left|\frac{\partial^5}{\partial y^5}(v-v^*)\right| + 30\left|\frac{\partial^3 v}{\partial y^3}(v-v^*) + v^*\frac{\partial^3}{\partial y^3}(v-v^*)\right| + 30\left|\frac{\partial^2 v}{\partial y^2}\frac{\partial}{\partial y}(v-v^*) + \frac{\partial v^*}{\partial y}\frac{\partial^2}{\partial y^2}(v-v^*)\right| + 180\left|\frac{\partial v}{\partial y}(v-v^*)(v+v^*) + v^{*2}\frac{\partial}{\partial y}(v-v^*)\right|\right)\frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}d\eta$$

$$\leq (1+\hbar)|v-v^*| + \hbar\int_0^t (\delta_1|v-v^*| + 30(\delta_2 C + D\delta_3)|v-v^*| + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7)|v-v^*| + 180(\delta_8 C(C+D) + D^2\delta_5)|v-v^*|)\frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}d\eta \quad (30)$$

Now, applying the mean value theorem [33,34], we get

$$|v-v^*| \leq (1+\hbar)|v-v^*| + \hbar(\delta_1|v-v^*| + 30(\delta_2 C + D\delta_3)|v-v^*| + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7)|v-v^*| + 180(\delta_8 C(C+D) + D^2\delta_5)|v-v^*|)T \quad (31)$$

On simplifying Eq. (31), we get the following relation

$$|v-v^*| \leq \rho|v-v^*| \quad (32)$$

where, $\rho = (1+\hbar) + \hbar(\delta_1 + 30(\delta_2 C + D\delta_3) + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7 + 180(\delta_8 C(C+D) + D^2\delta_5))T$.

It gives $(1-\rho)|v-v^*| \leq 0$. Here, $0 < \rho < 1$, thus $|v-v^*| = 0$ which gives that $v = v^*$. Therefore, the solution is unique.

Theorem 2 (*(Convergence Theorem)*). Let H is a Banach space additionally $G : H \to H$ be a nonlinear mapping also assume that

$$\|G(v) - G(w)\| \leq \|v - w\|, \forall v, w \in H. \quad (33)$$

Then by the fixed-point theory of Banach space, we know that, G has a fixed point. Further, sequence formed by using HASTM solution having an arbitrary solution of $v_0, w_0 \in H$, converges to the fixed point of G and

$$\|v_k - v_r\| \leq \frac{\rho^r}{1-\rho}\|v_1 - v_0\|, \forall v, w \in H \quad (34)$$

Proof. *Let us assume the Banach space* $(C[J], \|.\|)$ *of all the continuous functions on J associated to the norm given as* $\|g(t)\| = \max_{t\in J}|g(t)|$.

Next, we will prove that $\{v_r\}$ is a Cauchy sequence in to the Banach space.

$$\|v_k - v_r\| = \max_{t\in J}|(v_k - v_r)|$$

$$= \max_{t\in J}\Big|(1+\hbar)(v_{k-1} - v_{r-1}) + \hbar S^{-1}\Big[u^\alpha S\Big\{\Big(v_{(k-1)yyyyy} - v_{(r-1)yyyyy}\Big) + 30\big(v_{(k-1)}v_{(k-1)yyy} - v_{(r-1)}v_{(r-1)yyy}\big) + 30\big(v_{(k-1)y}v_{(k-1)yy} - v_{(r-1)y}v_{(r-1)yy}\big) + 180\big(v_{(k-1)}^2 v_{(k-1)y} - v_{(r-1)}^2 v_{(r-1)y}\big)\Big\}\Big]\Big|$$

$$\leq \max_{t\in J}\Big[(1+\hbar)|v_{k-1} - v_{r-1}| + \hbar S^{-1}\Big(u^\alpha S\Big\{\big|v_{(k-1)yyyyy} - v_{(r-1)yyyyy}\big| + 30\big|v_{(k-1)}v_{(k-1)yyy} - v_{(r-1)}v_{(r-1)yyy}\big| + 30\big|v_{(k-1)y}v_{(k-1)yy} - v_{(r-1)y}v_{(r-1)yy}\big| + 180\big|v_{(k-1)}^2 v_{(k-1)y} - v_{(r-1)}^2 v_{(r-1)y}\big|\Big\}\Big)\Big]$$

Implementing the convolution theorem for Sumudu transform, it gives

$$\|v_k - v_r\| \leq \max_{t\in J}\Big[(1+\hbar)|v_{k-1} - v_{r-1}| + \hbar\int_0^t \Big(\big|v_{(k-1)yyyyy} - v_{(r-1)yyyyy}\big| + 30\big|v_{(k-1)}v_{(k-1)yyy} - v_{(r-1)}v_{(r-1)yyy}\big| + 30\big|v_{(k-1)y}v_{(k-1)yy} - v_{(r-1)y}v_{(r-1)yy}\big| + 180\big|v_{(k-1)}^2 v_{(k-1)y} - v_{(r-1)}^2 v_{(r-1)y}\big|\Big)\frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}d\eta\Big]$$

or

$$\|v_k - v_r\| \leq \max_{t\in J}\Big[(1+\hbar)|v_{k-1} - v_{r-1}| + \hbar\int_0^t \Big(\delta_1\big|v_{(k-1)} - v_{(r-1)}\big| + 30(\delta_2 C + D\delta_3)\big|v_{(k-1)} - v_{(r-1)}\big| + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7)\big|v_{(k-1)} - v_{(r-1)}\big| + 180(\delta_8 C(C+D) + D^2\delta_5)|v_{(k-1)} - v_{(r-1)}|\Big) \times \frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}d\eta\Big]$$

Further, applying the mean value theorem [33,34], it yields

$$\|v_k - v_r\| \leq \max_{t\in J}\Big[(1+\hbar)|v_{k-1} - v_{r-1}| + \hbar\Big(\delta_1\big|v_{(k-1)} - v_{(r-1)}\big| + 30(\delta_2 C + D\delta_3)\big|v_{(k-1)} - v_{(r-1)}\big| + 30(\delta_4 C\delta_5 + \delta_6 D\delta_7)\big|v_{(k-1)} - v_{(r-1)}\big| + 180(\delta_8 C(C+D) + D^2\delta_5)|v_{(k-1)} - v_{(r-1)}|\Big)T$$

$$\|v_k - v_r\| \leq \rho\|v_{k-1} - v_{r-1}\|$$

Setting $k = r + 1$, it yields

$$\|v_{r+1} - v_r\| \leq \rho\|v_r - v_{r-1}\| \leq \rho^2\|v_{r-1} - v_{r-2}\| \leq \cdots \leq \rho^r\|v_1 - v_0\|$$

On using triangular inequality, we have

$$\|v_k - v_r\| \leq \|v_{r+1} - v_r\| + \|v_{r+2} - v_{r+1}\| + \cdots + \|v_k - v_{k-1}\|$$

$$\leq \left[\rho^r + \rho^{r+1} + \cdots + \rho^{k-1}\right]|\|v_1 - v_0\||$$

$$\leq \rho^r\left[1 + \rho + \rho^2 + \cdots + \rho^{k-r-1}\right]\|v_1 - v_0\|$$

$$\leq \rho^r\left[\frac{1-\rho^{k-r-1}}{1-\rho}\right]\|v_1 - v_0\|$$

As $0 < \rho < 1$, so $1 - \rho^{k-r-1} < 1$, then we have

$$\|v_k - v_r\| \leq \frac{\rho^r}{1-\rho}\|v_1 - v_0\|$$

Since, $\|v_1 - v_0\| < \infty$, so as $k \to \infty$ then $\|v_k - v_r\| \to 0$.

Thus, the sequence $\{v_r\}$ is convergent as it is a Cauchy sequence in $C[J]$.

5. Solution of fractional Caudrey-Dodd-Gibbon equation

Example 1. *Here, we* evaluate *the fractional Caudrey-Dodd-Gibbon equation* (3) *having the subsequent initial condition*

$$v(y,0) = \frac{15+\sqrt{105}}{30} - tanh^2(y) \tag{35}$$

The exact solution of the standard Caudrey-Dodd-Gibbon equation is attained for the value of $\alpha = 1$, for more details see [14].

Employing ST on both the sides of Eq. (3), using initial condition given by Eq. (35) and on simplification we get the consequent results

$$S[v(y,t)] - \frac{15+\sqrt{105}}{30} + tanh^2(y) + u^\alpha S\left[v_{yyyyy} + 30vv_{yyy} + 30v_y v_{yy} + 180v^2 v_y\right] = 0 \tag{36}$$

Now, we represent a non-linear operator as follows

$$N[\xi(y,t;p)] = S[\xi(y,t;p)] - \frac{15+\sqrt{105}}{30} + tanh^2(y) + u^\alpha S\left[\xi_{yyyyy}(y,t;p) + 30\xi(y,t;p)\xi_{yyy}(y,t;p) + 30\xi_y(y,t;p)\xi_{yy}(y,t;p) + 180\xi^2(y,t;p)\xi_y(y,t;p)\right] = 0 \tag{37}$$

Thus, we define the $\Im_k(\vec{v}_{k-1})$ for given problem in the following manner

$$\Im_k(\vec{v}_{k-1}) = S[v_{k-1}(y,t)] - (1-\chi_k)\left[\frac{15+\sqrt{105}}{30} - tanh^2(y)\right] + u^\alpha S\left[v_{(k-1)yyyyy} + 30A_{(k-1)} + 30B_{(k-1)} + 180C_{(k-1)}\right] \tag{38}$$

The kth order deformation equation is expressed as follows

$$S[v_k(y,t) - \chi_k v_{k-1}(y,t)] = \hbar \Im_k(\vec{v}_{k-1}) \tag{39}$$

Now, on utilizing inverse ST in above equation, we get

$$v_k(y,t) = \chi_k v_{k-1}(y,t) + \hbar S^{-1}\left[\Im_k(\vec{v}_{k-1})\right] \tag{40}$$

On using the recursive formula given by Eq. (25) and initial approximation expressed by Eq. (35), we get the following iteration

$$v_1(y,t) = -4(11-\sqrt{105})\hbar Sech^2(y)tanh(y)\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{41}$$

The remaining components $v_k, k \geq 0$ can be readily obtained by following the same procedure, hence, we obtain the entire solution. Ultimately, the series solution can be expressed as

$$v(y,t) = \lim_{N\to\infty}\sum_{k=0}^{N} v_k(y,t) \tag{42}$$

Example 2. *Here, we analyse the Caudrey-Dodd-Gibbon equation of fractional order* (3) *with the subsequent initial condition*

$$v(y,0) = \mu^2 sech^2(\mu y) \tag{43}$$

The exact solution of the standard Caudrey-Dodd-Gibbon equation is attained for the value of $\alpha = 1$, for more details see [35].

Applying ST on Eq. (3), using initial condition (43) and on simplification, we get the consequent results

$$S[v(y,t)] - \mu^2 sech^2(\mu y) + u^\alpha S\left[v_{yyyyy} + 30vv_{yyy} + 30v_y v_{yy} + 180v^2 v_y\right] = 0 \tag{44}$$

Now, the nonlinear operator can be represented as following way

$$N[\xi(y,t;p)] = S[\xi(y,t;p)] - \mu^2 sech^2(\mu y) + u^\alpha S\left[\xi_{yyyyy}(y,t;p) + 30\xi(y,t;p)\xi_{yyy}(y,t;p) + 30\xi_y(y,t;p)\xi_{yy}(y,t;p) + 180\xi^2(y,t;p)\xi_y(y,t;p)\right] = 0 \tag{45}$$

Hence, we define the $\Im_k(\vec{v}_{k-1})$ for given problem in the following manner

$$\Im_k(\vec{v}_{k-1}) = S[v_{k-1}(y,t)] - (1-\chi_k)\mu^2 sech^2(\mu y) + u^\alpha S\left[v_{(k-1)yyyyy} + 30A_{(k-1)} + 30B_{(k-1)} + 180C_{(k-1)}\right] \tag{46}$$

The kth order deformation equation is given in the following way

$$S[v_k(y,t) - \chi_k v_{k-1}(y,t)] = \hbar \Im_k(\vec{v}_{k-1}) \tag{47}$$

Table 1 Comparative analysis of obtained solution and exact solution for $v(y,t)$ when $\alpha = 1$ and $\hbar = -1$ for the Example 1.

y	t	Exact Solution	Present Solution	Absolute Error
0.5	0.00	0.6280127585	0.6280127585	0.0000000000
	0.01	0.6388936545	0.6389287758	0.0000351211
	0.02	0.6496327491	0.6499137702	0.0002810211
	0.03	0.6602163246	0.6611649272	0.0009486029
	0.04	0.6706305549	0.6728794350	0.0022488807
	0.05	0.6808615377	0.6852544747	0.0043929376
1.0	0.00	0.2615393671	0.2615393671	0.0000000000
	0.01	0.2712439912	0.2712298483	0.0000141429
	0.02	0.2810871794	0.2809741345	0.0001130449
	0.03	0.2910662174	0.2906850246	0.0003811928
	0.04	0.3011780846	0.3002753180	0.0009027666
	0.05	0.3114194433	0.3096578141	0.0017616292

Now, on applying the inverse ST, we get

$$v_k(y,t) = \chi_k v_{k-1}(v,t) + \hbar S^{-1}\left[\Im_k\left(\vec{v}_{k-1}\right)\right] \tag{48}$$

On using the recursive formula represented by Eq. (25) and initial approximation given by Eq. (43), we get the subsequent iteration

$$\begin{aligned} v_1(y,t) = 8\hbar\Big[&-34 - 90tanh(\mu y)^4 + 75sech(\mu y)^2 \\ &- 45sech(\mu y)^4 + 15\Big(8 - 9sech(\mu y)^2\Big)tanh(\mu y)^2\Big] \\ &\times sech(\mu y)^2 \tanh(\mu y)\mu^7 \frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{49}$$

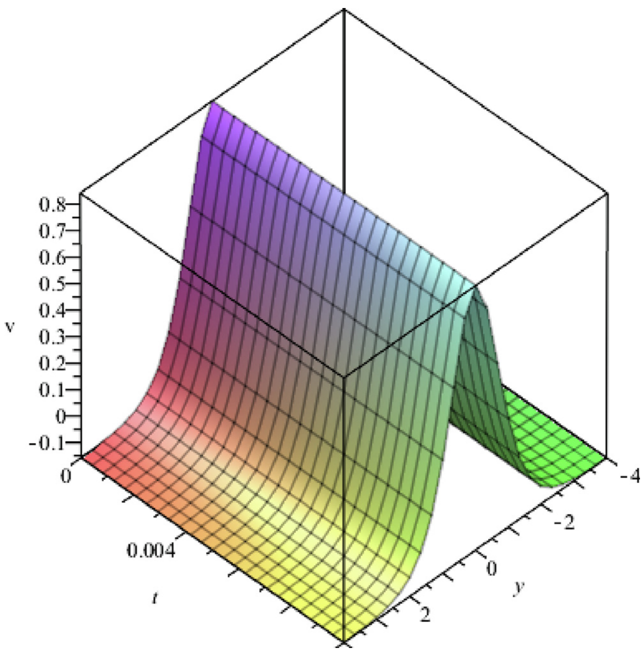

Fig. 1 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 1$.

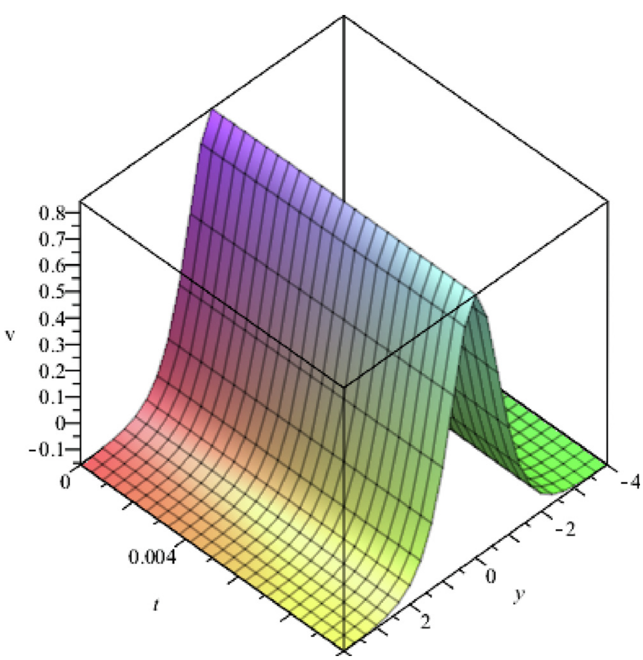

Fig. 2 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.95$.

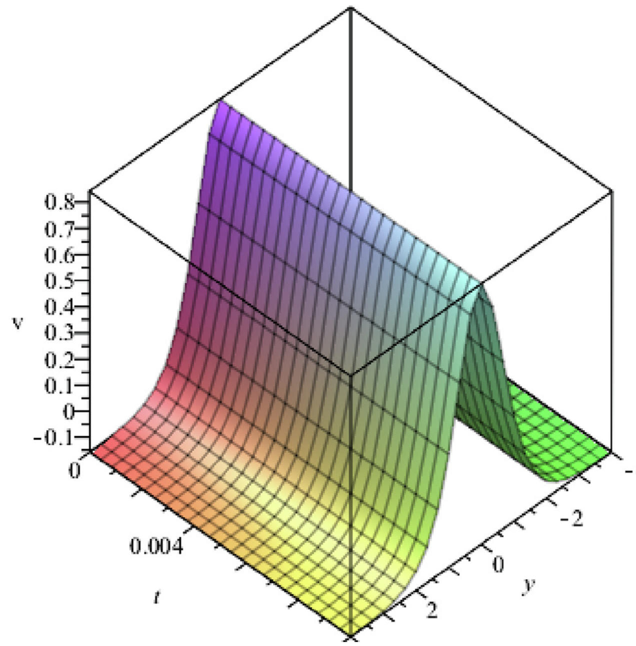

Fig. 3 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.90$.

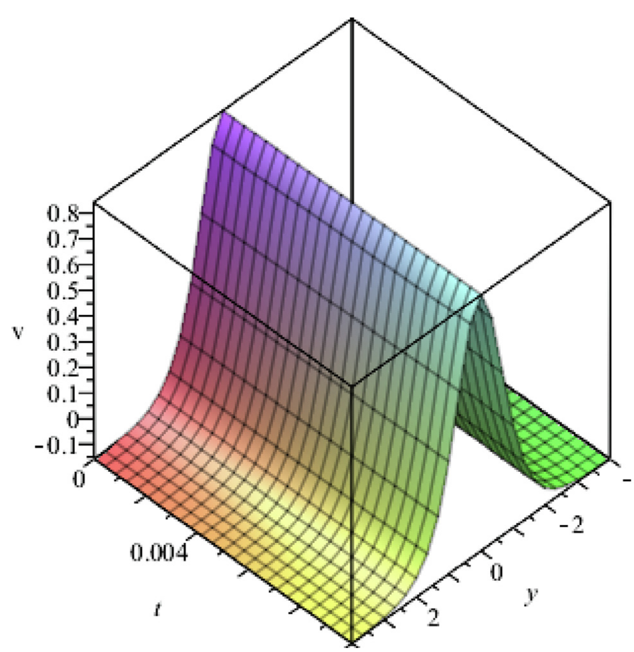

Fig. 4 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.85$.

The remaining components $v_k, k \geq 0$ can be readily obtained by following the same procedure, therefore, we obtain the entire solution. Ultimately, the series solution can be written as

$$v(y,t) = \lim_{N\to\infty} \sum_{k=0}^{N} v_k(y,t) \tag{50}$$

6. Numerical results and discussion

Here, we perform numerical simulation by exerting the suggsted technique for the fractional Caudrey-Dodd-Gibbon equation at fractional Brownian motions $\alpha = 0.95, \alpha = 0.90$ and $\alpha = 0.85$ in addition for the standard motion $\alpha = 1$. The comparative analysis of solution obtained by applied method and

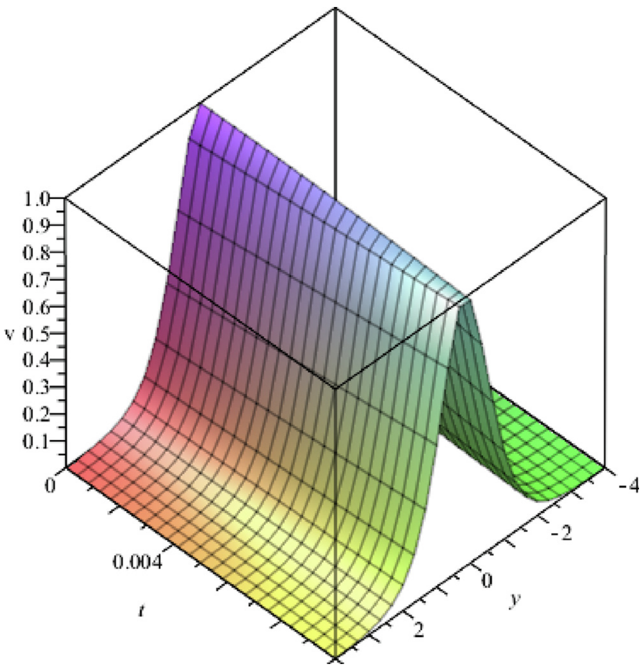

Fig. 5 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 1$.

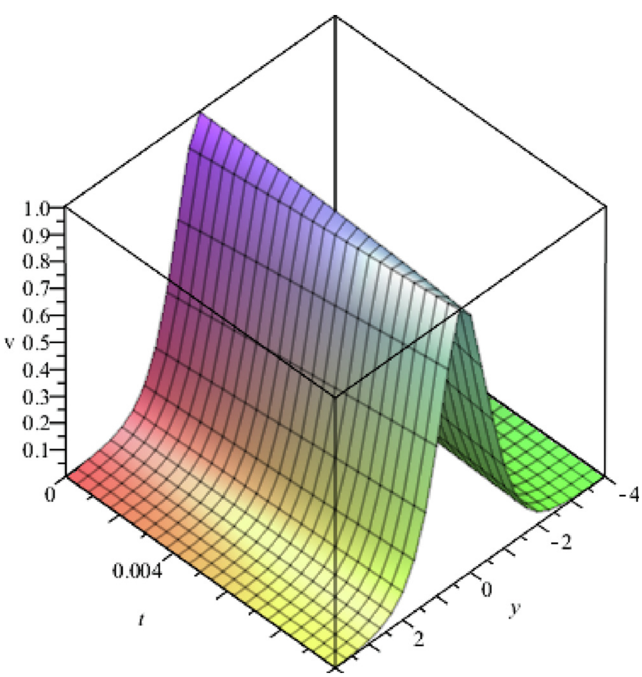

Fig. 6 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.95$.

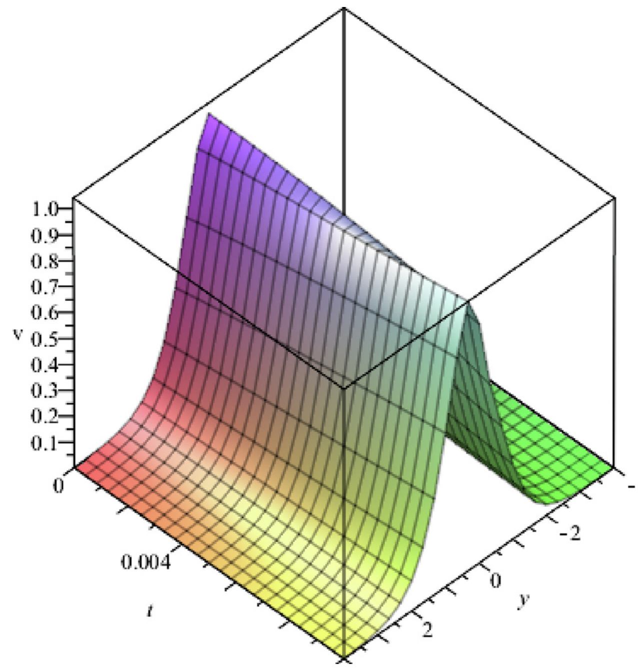

Fig. 7 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.90$.

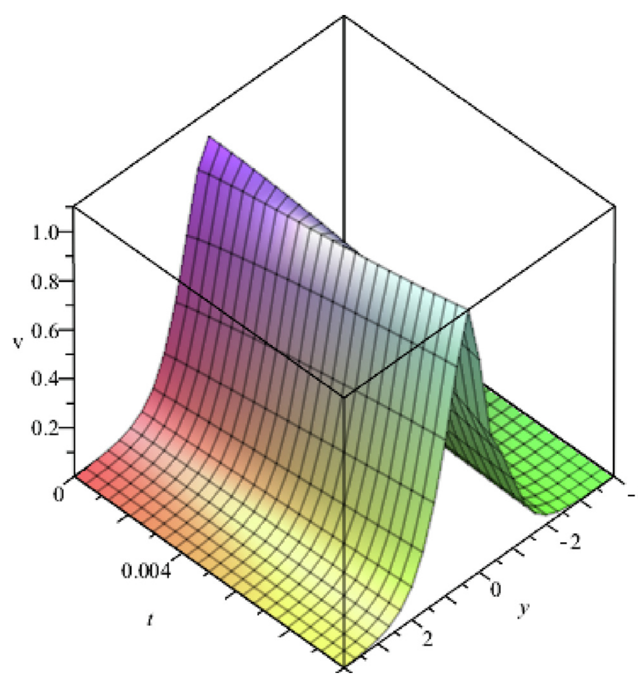

Fig. 8 Surface of $v(y,t)$ w.r.t. y and t for $\alpha = 0.85$.

exact solution for $v(y, t)$ when $\alpha = 1$ and $\hbar = -1$ for the Example 1 is demonstrated in Table 1. The results of this simulation are expressed in the form of Figs. 1-14. Behavior of solutions of standard model is shown in Fig. 1 (for Example 1) and Fig. 5 (for Example 2) respectively. Fig. 2,3,4 (for Example 1) and Fig. 6,7,8 (for Example 2) show the behavior of $v(y, t)$ for the fractional Caudrey-Dodd-Gibbon equation. Fig. 9 (for Example 1) and Fig. 11 (for Example 2) express that the impact of order of fractional derivative on the displacement profile $v(y, t)$. Fig. 10 (for Example 1) and Fig. 12 (for Example 2) reveal the effect of order of fractional derivative on displacement profile $v(y, t)$. Fig. 13 (for Example 1) and Fig. 14 (for Example 2) exhibits $\hbar$-curves for several values of α. In Fig. 13 and Fig. 14, horizontal line segment demonstrates

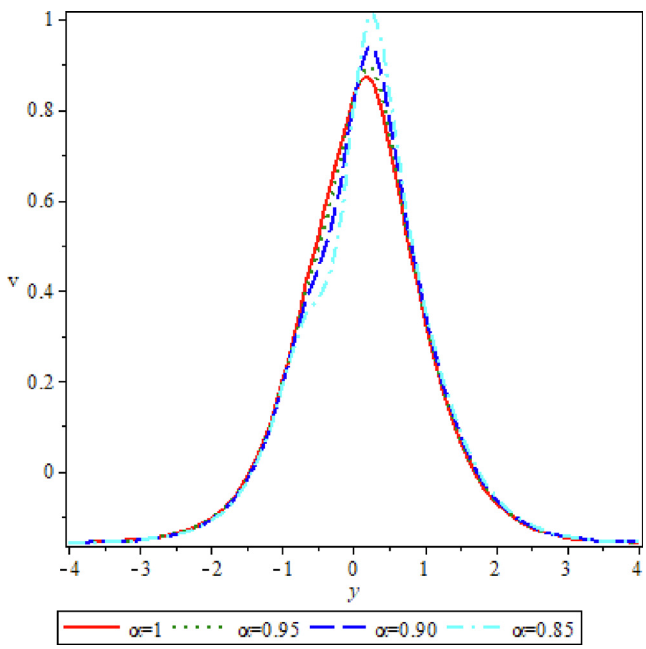

Fig. 9 Response of $v(y, t)$with respect to y for different values of α.

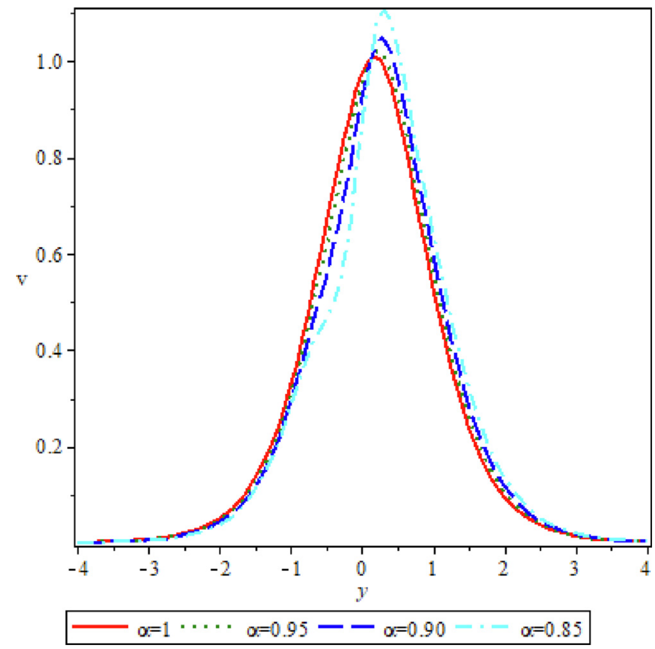

Fig. 11 Response of $v(y, t)$ with respect to y for different values of α.

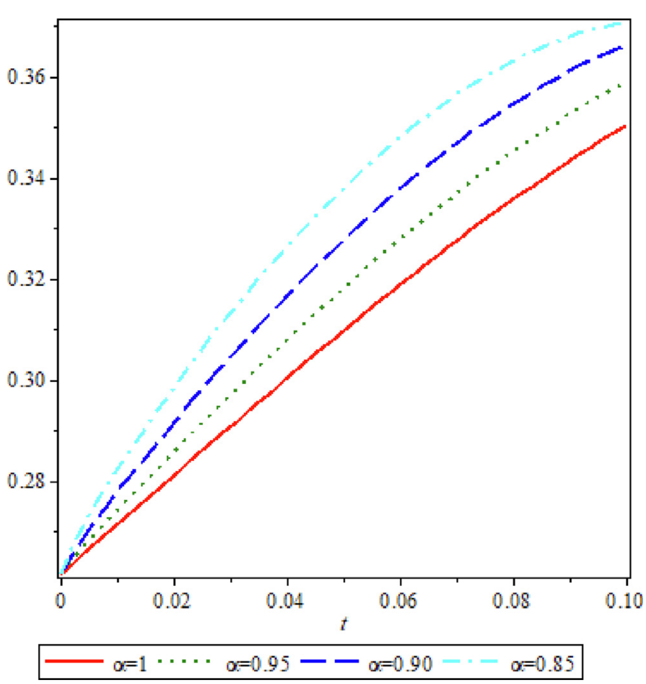

Fig. 10 Characteristic of $v(y, t)$with respect to t for different values of α.

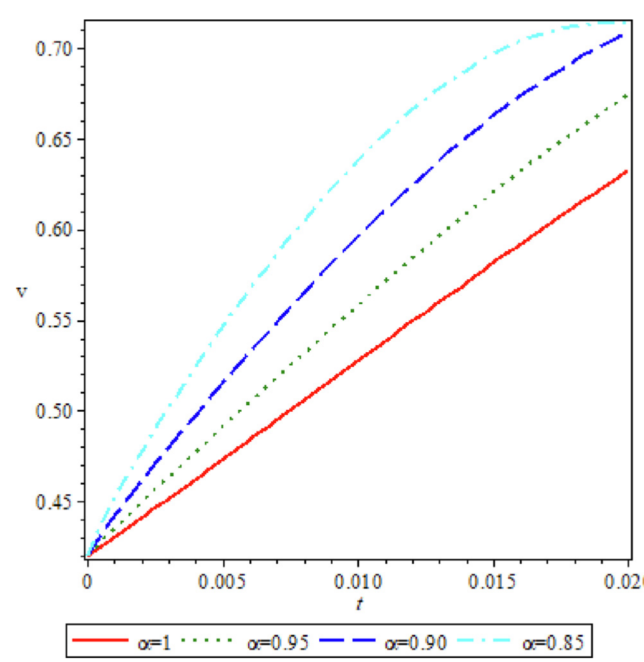

Fig. 12 Characteristic of $v(y, t)$ with respect to t for different values of α.

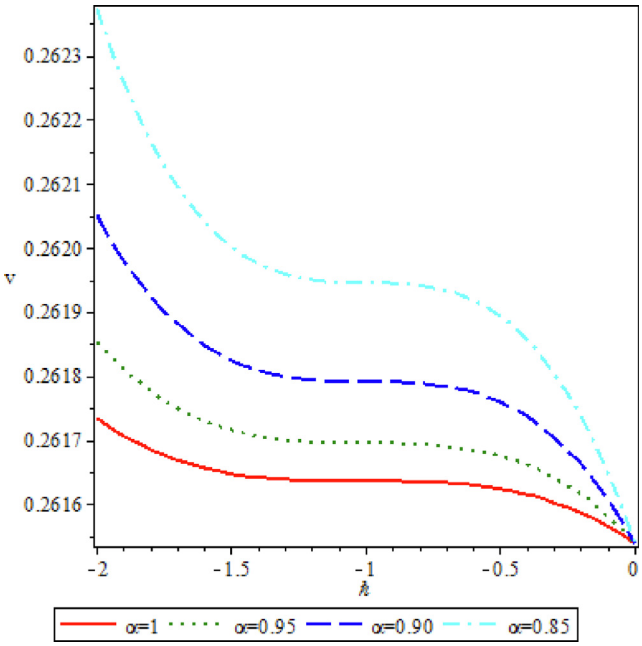

Fig. 13 $\hbar$- curve for several values of α at $t = 0.0001$ for $v(y,t)$.

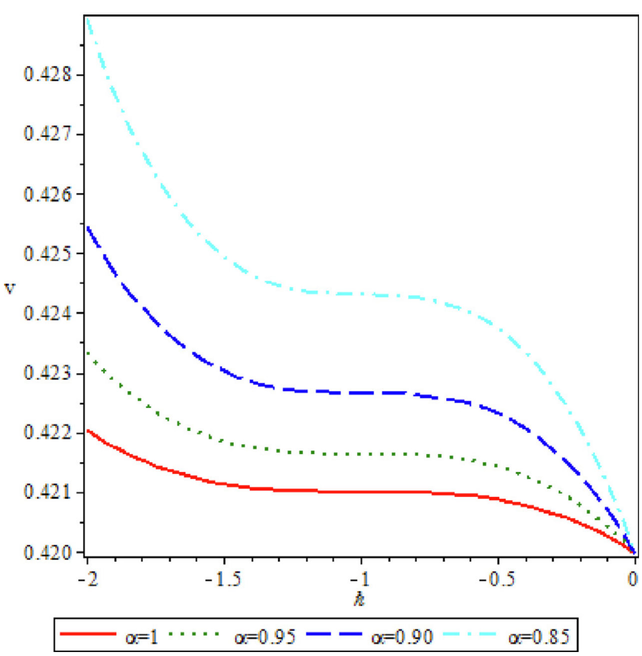

Fig. 14 $\hbar$- curve for several values of α at $t = 0.0001$ for $v(y,t)$.

the range of convergence of the obtained solution. From Figs. 10 and 12, we observe that as we increase the order of fractional derivative than, the displacement profile $v(y,t)$ decreases. From Figs. 13 and 14, we observe that as we increase the order of fractional derivative than, the range of convergence for the obtained solution enhances. Thus, we can observe that there is a significant impact of order of fractional derivative on displacement profile due to well known nonlocal nature of Caputo fractional derivative. The nonlocal nature of Caputo fractional derivative plays a great role in the study of mathematical modelling of physical problems. Therefore, the main advantage of using fractional operators in mathematical modeling of physical systems is that such type of models captures the dynamic behavior of the system in a better manner due to the memory effect of the fractional operators.

7. Conclusions

In this research work, we applied the homotopy based technique to analyze the approximate solution of Caudrey-Dodd-Gibbon equation of fractional order. The results of present work show that utility of applied technique for solution of the considered fractional model is quite worthy. The displacement reveals specific and new attributes for the fractional order model in comparison of the classical model. The employed scheme needs less computational work and gives very precise result. Thus, we observe that the suggested method is very efficient and accurate technique for solving fractional Caudrey-Dodd-Gibbon model with better physical aspects. This work is very useful in the field of sciences and engineering, it opens a new vista in these fields.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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