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ON THE EIGENVALUES OF SECOND-ORDER BOUNDARY-VALUE PROBLEMS

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Abstract In this paper we investigate the properties of eigenvalues of some boundary-value problems generated by second-order Sturm-Liouville equation with distributional potentials and suitable boundary conditions. Moreover, we share a necessary condition for the problem to have an infinitely many eigenvalues. Finally, we introduce some ordinary and Frechet derivatives of the eigenvalues with respect to some elements of the data.

Keywords Boundary-value problems, eigenvalues, Frechet derivative.

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1. Introduction

The problem of existence of eigenvalues of a boundary-value problem has been attacted by the authors extensively for years. The special attempt has been applied for the following problem

$$\begin{aligned}
&-(p(x)y')' + q(x)y = \lambda w(x)y,\ x \in [a,b],\\
&y(a)\cos\alpha - (py')(a)\sin\alpha = 0,\\
&y(b)\cos\beta - (py')(b)\sin\beta = 0,
\end{aligned} \tag{1.1}$$

where p, p', q, w are real-valued and integrable functions, $p > 0$, $w > 0$ and α, β are some real numbers. One of the tools is the Prüfer's transformation. Indeed, with the following new variables

$$y(x) = r(x)\sin\theta(x),\ p(x)y'(x) = r(x)\cos\theta(x),$$

the differential equation in (1.1) is transformed into the equations [1,3,7,8,18]

$$r' = \left(\frac{1}{p} - g\right) r\sin\theta\cos\theta,$$

and

$$\theta' = \frac{1}{p}\cos^2\theta + g\sin^2\theta, \tag{1.2}$$

where

$$g = \lambda w - q.$$

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This method is meaningful provided that y and py' do not vanish simultaneously. Note that Eq. (1.2) has a unique solution θ satisfying the initial condition

$$\theta(a,\lambda)=\alpha.$$

Some certain properties of eigenvalues of the propblem (1.1) can be investigated with the help of the monotone increasing proporty of θ. In fact, one may examine the properties of θ using the following equation

$$\left(\theta_2-\theta_1\right)'=f\left(\theta_2-\theta_1\right)+h,\ \ h\geq 0,$$

where

$$f=\left(g_1-\frac{1}{p_1}\right)\left(\sin\theta_2+\sin\theta_1\right)\left(\frac{\sin\theta_2-\sin\theta_1}{\theta_2-\theta_1}\right),$$

and

$$h=\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\cos^2\theta_2+\left(g_2-g_1\right)\sin^2\theta_2\geq 0,$$

with $p_1>p_2$ and $g_2>g_1$.

Another method belongs to Atkinson [1]. Indeed, for the problem

$$\begin{aligned}
&\left(\frac{1}{r(x)}y'\right)'+(q(x)+\lambda w(x))y=0,\ \ x\in[a,b],\\
&y(a)\cos\alpha-\left(\frac{1}{r}y'\right)(a)\sin\alpha=0,\\
&y(b)\cos\beta-\left(\frac{1}{r}y'\right)(b)\sin\beta=0,
\end{aligned}$$

where r,q,w are real-valued and integrable functions on $[a,b]$ with $r\geq 0$, $w\geq 0$ on $[a,b]$ such that for $x\in(a,b)$

$$\int_a^x w(s)ds>0,\int_x^b w(s)ds>0,\int_a^b r(s)ds>0,$$

and for $c_1,c_2\in[a,b]$

$$\int_{c_1}^{c_2} w(s)ds=0,$$

implies

$$\int_{c_1}^{c_2}\left|q(s)\right|ds=0.$$

He passed to the differential equations generated by y and z instead of the second-order equation such that

$$y'=rz,\ \ z'=-(\lambda w+q)y,$$

where $z=y'/r$ and investigated the properties of eigenvalues of the problem.

In [5], Eckhardt et al have investigated some properties of the eigenvalues and solutions of the following problem

$$\begin{aligned}
&-\left[p(x)\left(y'+s(x)y\right)\right]'+p(x)s(x)\left(y'+s(x)y\right)+q(x)y=\lambda w(x)y,\ \ x\in[a,b],\\
&y(a)\cos\alpha-[p(y'+sy)](a)\sin\alpha=0,\\
&y(b)\cos\beta-[p(y'+sy)](b)\sin\beta=0,
\end{aligned}\tag{1.3}$$

where α, β are some real numbers, p, p', q, s, w are real-valued and integrable functions with $w > 0$ on the given interval. The differential equation appearing in (1.3) is called as differential equation with distributional potentials. This equation and the corresponding problems have also been studied in [2] and [16]. Clearly, this differential equation contains the differential equation in (1.1). However, this one provides a detailed analysis. It is better to note that a weaker version of the differential equation in (1.3) has also been introduced by Savchuk and Shkalikov [13] as

$$-\left[(y' + s(x)y)\right]' + s(x)\left(y' + s(x)y\right) - s^2(x)y = \lambda y,\ x \in [a, b].$$

Clearly for $s \equiv 0$, these equations turn out to be the ordinary Sturm-Liouville equations.

In this paper, we will investigate the properties of eigenvalues of a similar problem with (1.3) with the aid of Atkinson's approach. Moreover, at the end of the paper we will compare this method with Prüfer's transformation. Finally, we will investigate the dependence of the eigenvalues of the problem on some elements of data. This is also a remarkable area for the investigation of the eigenvalues of some Sturm-Liouville boundary value problems and the readers may see the papers [4, 6, 9–12, 14, 15, 17, 19] including recent works on the dependence of the eigenvalues of some ordinary Sturm-Liouville boundary value problems.

2. Boundary-value problem

Let us consider the following system of equations

$$y' + s(x)y = r(x)z,\ z' = (-\lambda w(x) + q(x))y + s(x)z,\ x \in [a, b]. \tag{2.1}$$

Here the basic assumptions are as follows

(i) r, s, q, w are real-valued and integrable functions on $[a, b]$,
(ii) $w \geq 0$, $r \geq 0$ on $[a, b]$,
(iii) $\int_a^x w(t)dt > 0$, $\int_x^b w(t)dt > 0$, $\int_a^b r(t)dt > 0$, $x \in (a, b)$,
(iv) $\int_c^d w(x)dx = 0$ implies $\int_c^d |q(x)|\, dx = \int_c^d |s(x)|\, dx = 0$, $a \leq c < d \leq b$.

We should note that the Eq. (2.1) can also be considered as the following second-order differential equation with distributional potentials

$$-\left(\frac{1}{r(x)}\left[y' + s(x)y\right]\right)' + \frac{1}{r(x)}s(x)\left[y' + s(x)y\right] + q(x)y = \lambda w(x)y,\ x \in [a, b].$$

However, we will continue with the system of Eq.s (2.1).

Lemma 2.1. *If*

$$\int_a^b w(x)\left|y(x)\right|^2 dx = 0,$$

then $y \equiv z \equiv 0$ *on* $[a, b]$, *where* $z = r^{-1}(x)\left[y' + s(x)y\right]$.

Proof. We obtain from the second equation in (2.1) that

$$\mu_a(x)z(x) - z(a) = -\lambda \int_a^x \mu_a(t)w(t)y(t)dt + \int_a^x \mu_a(t)q(t)y(t)dt,$$

where

$$\mu_\delta(x) := \exp\left(-\int_\delta^x s(t)dt\right), a \le \delta < x \le b.$$

Let us consider that

$$\int_a^b w(x)\left|y(x)\right|^2 dx = 0. \tag{2.2}$$

Then the inequality

$$\left|\int_a^x \mu_a(t)w(t)y(t)dt\right| \le \int_a^x \mu_a^2(t)w(t)dt \int_a^x w(t)\left|y(t)\right|^2 dt$$

implies the following

$$\mu_a(x)z(x) - z(a) = \int_a^x \mu_a(t)q(t)y(t)dt. \tag{2.3}$$

Now suppose that we have the following

$$\int_a^b \mu_a(x)\left|q(x)y(x)\right| dx > 0.$$

Then for some $\epsilon > 0$ we get

$$\int_a^b \mu_a(x)\left|q(x)y(x)\right| dx > \epsilon \int_a^b \left|q(x)\right| dx.$$

Therefore on an arbitrarily small interval (c_1, c_2) we have

$$\int_{c_1}^{c_2} \mu_a(x)\left|q(x)y(x)\right| dx > \epsilon \int_{c_1}^{c_2} \left|q(x)\right| dx > 0.$$

Since y is continuous we may consider that $|y(x)| > \epsilon/2$ on (c_1, c_2). Moreover from (iv) we infer that

$$\int_{c_1}^{c_2} w(x)dx > 0.$$

Therefore on (c_1, c_2) one obtains

$$\int_{c_1}^{c_2} w(x)\left|y(x)\right|^2 dx > 0. \tag{2.4}$$

Then (2.2) and (2.4) give a contradiction.

Therefore (2.3) implies that

$$\mu_a(x)z(x) = z(a),$$

or equivalently

$$\mu_a(x)\frac{1}{r(x)}\left[y'(x) + s(x)y(x)\right] = k,$$

where k is a constant. Therefore we get for $k \neq 0$ that

$$\exp\left(\int_a^b \frac{r(x)s(x)}{\mu_a(x)}dx\right) y(b) - y(a) = k\int_a^b \exp\left(\int_a^x \frac{r(t)s(t)}{\mu_a(t)}dx\right)\frac{r(x)}{\mu_a(x)}dx. \tag{2.5}$$

Since the right hand side of (2.5) is positive from (iii), $y(b)$ and $y(a)$ can not be zero at the same time. This implies by the continuity of y that there exists an interval $(a, a+\epsilon)$ or $(b-\epsilon, b)$ such that (2.2) is not satisfied. This completes the proof for $k \neq 0$.

For $k = 0$ one gets

$$y(x) - y(a) = \exp\left(-\int_a^x s(t)dt\right),$$

or

$$y(b) - y(a) = \exp\left(-\int_a^b s(t)dt\right).$$

Consequently, as before, this gives a contradiction with (2.2) and this completes the proof for $k = 0$. □

The boundary conditions for the solutions of (2.1) are considered as follows

$$\begin{aligned} y(a)\cos\gamma - z(a)\sin\gamma = 0, \\ y(b)\cos\varphi - z(b)\sin\varphi = 0, \end{aligned} \tag{2.6}$$

where $0 \leq \gamma < \pi$ and $0 < \varphi \leq \pi$.

The first property of the problem (2.1), (2.6) is the following.

Theorem 2.1. *The eigenvalues of* (2.1), (2.6) *are all real and discrete with the possible limit point at infinity.*

Proof. Consider the equation

$$\frac{\partial}{\partial x}\left(y(x,\lambda)z(x,\overline{\lambda}) - y(x,\overline{\lambda})z(x,\lambda)\right) = (\lambda - \overline{\lambda})w(x)y(x,\lambda)y(x,\overline{\lambda}). \tag{2.7}$$

Integration of both sides of (2.7) and the conditions (2.6) give

$$2i\Im\lambda \int_a^b w(x)\left|y(x)\right|^2 dx = 0.$$

Since y is a nontrivial solution this implies $\Im\lambda = 0$.

The second assertion follows from the conditions (2.6) and the entire property of $y(x,\lambda)$ and $z(x,\lambda)$.

Therefore the proof is completed. □

A direct consequence of (2.1) is the following.

Theorem 2.2. *Following equations are satisfied*

$$\left(\frac{y}{z}\right)' = r + \lambda w\left(\frac{y}{z}\right)^2 - q\left(\frac{y}{z}\right)^2 - 2s\frac{y}{z},\ z \neq 0,$$

and

$$\left(\frac{z}{y}\right)' = -\lambda w + q - r\left(\frac{z}{y}\right)^2 + 2s\frac{z}{y},\ y \neq 0. \tag{2.8}$$

Lemma 2.2. *Following equations are satisfied*

$$\frac{\partial}{\partial\lambda}\frac{y(x,\lambda)}{z(x,\lambda)} = \frac{1}{z(x,\lambda)^2}\int_a^x w(t)y(t,\lambda)^2dt,\ \ z(x,\lambda)\neq 0,$$

and

$$\frac{\partial}{\partial\lambda}\frac{z(x,\lambda)}{y(x,\lambda)}=\frac{-1}{y(x,\lambda)^2}\int_a^x w(t)z(t,\lambda)^2dt,\ \ y(x,\lambda)\neq 0.$$

Proof. Using Eq. (2.7) and (2.6) we get

$$\frac{y(x,\lambda_2)z(x,\lambda_1)-y(x,\lambda_1)z(x,\lambda_1)+y(x,\lambda_1)z(x,\lambda_1)-y(x,\lambda_1)z(x,\lambda_2)}{\lambda_2-\lambda_1}$$
$$=\int_a^x w(t)y(t,\lambda_1)y(t,\lambda_2)dt.$$

Let $\lambda_2\to\lambda_1$. Then one gets

$$z(x,\lambda)\frac{\partial}{\partial\lambda}y(x,\lambda)-y(x,\lambda)\frac{\partial}{\partial\lambda}z(x,\lambda)=\int_a^x w(t)y(t,\lambda)^2dt$$

and the results follow from the last equation. □

Corollary 2.1. *The solutions of* (2.1) *have the following properties*

$$\frac{y(b,\lambda_2)}{z(b,\lambda_2)}>\frac{y(b,\lambda_1)}{z(b,\lambda_1)};\ \ \lambda_2>\lambda_1,$$

and

$$\frac{z(b,\lambda_2)}{y(b,\lambda_2)}<\frac{z(b,\lambda_1)}{y(b,\lambda_1)};\ \ \lambda_2>\lambda_1.$$

Theorem 2.3. *For the nontrivial solutions y and z of* (2.1) *y/z can not be zero at each point on* $[a,b]$.

Proof. We get from (2.8) that

$$\left(\frac{z}{y}\right)'-2s\frac{z}{y}\leq-\lambda w+q,$$

since $r\geq 0$. Therefore we have

$$\exp\left(-2\int_c^x s(t)dt\right)\left(\frac{z}{y}\right)(x)-\left(\frac{z}{y}\right)(c)\leq\int_c^x\exp\left(-2\int_c^t s(t_1)dt_1\right)(-\lambda w(t)+q(t))dt,\tag{2.9}$$

where $a\leq c<x\leq b$ or

$$\exp\left(-2\int_x^c s(t)dt\right)\left(\frac{z}{y}\right)(c)-\left(\frac{z}{y}\right)(x)\leq\int_x^c\exp\left(-2\int_t^c s(t_1)dt_1\right)(-\lambda w(t)+q(t))dt,\tag{2.10}$$

where $a\leq x<c\leq b$. The proof is completed by (2.9) and (2.10) because the right hand sides of (2.9) and (2.10) remain finite on the given intervals. □

3. On the eigenvalues of the problem

In this section we investigate the properties of the problem (2.1), (2.6). For this purpose we shall construct the following function

$$\psi(x,\lambda) = \arctan\left(\frac{y(x,\lambda)}{z(x,\lambda)}\right) = \arg\left[z(x,\lambda) + iy(x,\lambda)\right]. \tag{3.1}$$

Clearly the roots of ψ coincide with the roots of y. Moreover ψ can be considered on $(-\pi/2 + n\pi, \pi/2 + n\pi)$ for the fixed integer n.

One may obtain from (3.1) the following

$$\psi' = r\cos^2\psi + (\lambda w - q)\sin^2\psi - s\sin 2\psi. \tag{3.2}$$

Therefore we may infer that (3.2) has a unique solution satisfying

$$\psi(a,\lambda) = \gamma.$$

Moreover Lemma 2.1 and Lemma 2.2 also imply that $\psi(.,\lambda)$ is increasing in λ.

Theorem 3.1. *Let $c_1 \in (a,b]$ and $y(c_1,\lambda_1) = 0$. If $y(c_2,\lambda_2) = 0$ when $\lambda_2 > \lambda_1$ then $c_2 < c_1$, where $0 < \gamma < \pi$.*

Proof. Suppose that

$$\psi(c_1,\lambda_1) = n\pi,$$

for a fixed n, where $a < c_1 \leq b$ and λ_1 is a real number. Then from Lemma 2.2 we get

$$\psi(c_1,\lambda_2) > n\pi,$$

for $\lambda_2 > \lambda_1$. Since $\psi(a,.) = \gamma < \pi$ we should have

$$\psi(c_2,\lambda_2) = n\pi,$$

at a point c_2, where $a < c_2 < c_1$. This completes the proof. □

Theorem 3.2. *For $\psi(a,.) = 0$ or $r \equiv 0$ the results of Theorem 3.1 may not be true in a right neighborhood of a.*

Proof. Suppose first that $\psi(a,\lambda_*) = 0$. For

$$\int_a^{c_*} w(x)y(x,\lambda_*)^2 dx = 0,$$

we should have z is a constant but nonzero on (a,c_*) because $z(a,\lambda_*) \neq 0$. Therefore ψ can not reach the value $\pi/2$ in (a,c_*) and therefore there is not any root of the equation

$$\psi(\cdot,\lambda) = n\pi,$$

on $(a,c_*]$.

Now let $r \equiv 0$ on an interval (c_1,c_2). Then from (3.2) we get

$$\psi' = (\lambda w - q)\sin^2\psi - s\sin 2\psi. \tag{3.3}$$

At $\psi = n\pi$ for a fixed n, (3.3) implies that $\psi \equiv n\pi$ on (c_1,c_2). This completes the proof. □

Theorem 3.3. *The roots of the equation*

$$\psi(b, \lambda_k) = \varphi + k\pi,$$

where k is the nonnegative integer, are the eigenvalues of (2.1)*,* (2.6)*.*

Proof. Using (3.2) we obtain that

$$\psi(b, \lambda) - \gamma = \int_a^b \left[r\cos^2\psi + \lambda w \sin^2\psi - q\sin^2\psi - s\sin 2\psi \right] dx, \tag{3.4}$$

is uniformly bounded for all real λ and

$$\psi' \leq r + |q| + |s|,$$

for all negative λ. Therefore on (c_1, c_2) we have for $\lambda < 0$ that

$$\psi(c_2, \lambda) - \psi(c_1, \lambda) \leq \int_{c_1}^{c_2} \{r(x) + |q(x)| + |s(x)|\} \, dx. \tag{3.5}$$

Since $\psi(a, 0) < \pi$ we get $\psi(x, 0) < \pi - \epsilon$ for some $\epsilon > 0$ on $[a, b)$. From (3.4) we obtain that

$$|\lambda| \int_a^b w \sin^2\psi dx, \tag{3.6}$$

is uniformly bounded for $\lambda < 0$. Therefore for

$$\int_{c_1}^{c_2} w(x) dx > 0, \tag{3.7}$$

there exists a $x_* \in [c_1, c_2]$ such that

$$|\sin\psi| \leq const. \, |\lambda|^{-1/2}. \tag{3.8}$$

For large negative λ we see that $\sin\psi$ is sufficiently small. Moreover using

$$0 \leq \psi(x, \lambda) \leq \psi(x, 0) < \pi - \epsilon,$$

we obtain that ψ is sufficiently small for large negative λ and so one may infer that

$$\psi(x_*, -\infty) < \psi(x_*, \lambda) < \frac{\pi}{2}.$$

Now suppose that the inequality

$$\int_{d_1}^{d_2} \{r(x) + |q(x)| + |s(x)|\} \, dx < \frac{\pi}{4},$$

holds for some $[d_1, d_2] \subset [a, b]$. Then (3.5) implies

$$\psi(d_2, -\infty) - \psi(d_1, -\infty) < \frac{\pi}{4},$$

and so

$$\psi(d_1, -\infty) > \frac{\pi}{4}.$$

On the other side since $\psi(d_1, -\infty) < \dfrac{\pi}{2}$ for large negative λ we have

$$\psi(d_1, \lambda) < \frac{\pi}{2}. \tag{3.9}$$

Replacing c_1 and c_2 by d_1 and d_2, respectively in (3.5) we see that

$$\psi(x, \lambda) < \frac{3\pi}{4},$$

on $[d_1, d_2]$. Therefore we may write

$$\frac{\pi}{4} < \psi(x, \lambda) < \frac{3\pi}{4},$$

and hence

$$\frac{1}{2} < \sin^2 \psi.$$

Using (3.6)-(3.8) we obtain

$$\int_{d_1}^{d_2} w(x)dx = 0,$$

and by (iv) we get

$$\int_{d_1}^{d_2} |q(x)|\, dx = \int_{d_1}^{d_2} |s(x)|\, dx = 0.$$

So q and s are zero almost everywhere in $[d_1, d_2]$. Now (3.2) shows that

$$\tan\psi(x, \lambda) - \tan\psi(d_1, \lambda) = \int_{d_1}^{x} r(t)dt, \;\; d_1 \leq x < d_2. \tag{3.10}$$

(3.9) and (3.10) imply that

$$\psi(d_2, \lambda) < \frac{\pi}{2}, \;\; \lambda < 0. \tag{3.11}$$

(3.11) particularly shows that

$$0 \leq \psi(b, -\infty) < \frac{\pi}{2}.$$

Now assume that

$$\int_{b_*}^{b} \{r(x) + |q(x)| + |s(x)|\}\, dx < \epsilon_*,$$

where $b_* < b$ and $\epsilon_* > 0$. Therefore we may infer for sufficiently large negative λ from previous calculations on $[b_*, b]$ that

$$\epsilon_* < \psi(x, \lambda) < \frac{\pi}{2} + \epsilon_*, \;\; \lambda < 0.$$

So

$$0 < \sin^2(\epsilon_*) < \sin^2 \psi.$$

Therefore we again infer that

$$\int_{b_*}^{b} w(x)dx = 0,$$

which contradicts with (*iii*). Consequently we should have

$$\psi(b, -\infty) = 0.$$

This completes the proof. $\square$

The following Theorem gives a necessary condition for (2.1), (2.6) to have an infinite number of eigenvalues.

Theorem 3.4. *There are infinitely many eigenvalues of* (2.1), (2.6) *if*

$$\int_{c_{2k}}^{c_{2k+1}} w(x)dx > 0, \ \int_{c_{2k+1}}^{c_{2k+2}} r(x)dx > 0, \ k = 0, 1, \dots .$$

Proof. Let us consider the function $\tan\psi_1$ with the rule

$$\tan\psi_1 = \lambda^{1/2}\frac{y}{z}, \ \lambda > 0,$$

where $\psi_1 = \psi_1(x, \lambda)$ and $|\psi - \psi_1| < \frac{\pi}{2}$. Then we obtain

$$\left(\sec^2\psi_1\right)\psi_1' = \lambda^{1/2} r - 2s\tan\psi_1 + \lambda^{-1/2}\left(\lambda w - q\right)\tan^2\psi_1.$$

Hence for $\lambda > 0$ we obtain

$$\begin{aligned}\psi_1' &= \lambda^{1/2} r\cos^2\psi_1 + \lambda^{-1/2}\left(\lambda w - q\right)\sin^2\psi_1 - s\sin 2\psi_1 \\ &= \lambda^{1/2} r\cos^2\psi_1 + \lambda^{1/2} w\sin^2\psi_1 - \lambda^{-1/2} q\sin^2\psi_1 - s\sin 2\psi_1.\end{aligned}$$

It follows for $\lambda > 0$ that

$$\psi_1' \geq -\lambda^{-1/2}|q| - |s|,$$

and so for $\lambda \geq 1$

$$\psi_1(b, \lambda) - \psi_1(a, \lambda) \geq -\int_a^b |q|\, dx - \int_a^b |s|\, dx.$$

Therefore $\psi_1(b, \lambda) - \psi_1(a, \lambda)$ is bounded from below, uniformly for $\lambda \geq 1$. Now suppose that for $\lambda \geq 1$ there exists a constant l_1 such that

$$\psi_1(b, \lambda) - \psi_1(a, \lambda) \leq l_1,$$

or

$$\lambda^{1/2}\int_a^b \left(r\cos^2\psi_1 + w\sin^2\psi_1\right)dx - \lambda^{-1/2}\int_a^b q\sin^2\psi_1 dx - \int_a^b s\sin 2\psi_1 dx \leq l_1. \tag{3.12}$$

Therefore

$$\int_a^b |\psi_1'|\, dx \leq l_1 + \int_a^b |q|\, dx + \int_a^b |s|\, dx, \ \lambda \geq 1.$$

So ψ_1 is of bounded variation uniformly for $\lambda \geq 1$.

Using (3.12) we obtain

$$\lambda^{1/2} \int_{c_{2k}}^{c_{2k+1}} \left(r \cos^2 \psi_1 + w \sin^2 \psi_1\right) dx \leq l_1 + \int_{c_{2k}}^{c_{2k+1}} |q|\, dx + \int_{c_{2k}}^{c_{2k+1}} |s|\, dx = l_2,$$

and

$$\int_{c_{2k}}^{c_{2k+1}} w \sin^2 \psi_1 dx \leq \lambda^{-1/2} l_2.$$

Then at any rate $x \in [c_{2k}, c_{2k+1}]$ we obtain

$$\sin^2 \psi_1(x, \lambda) \leq \lambda^{-1/2} l_2 \left(\int_{c_{2k}}^{c_{2k+1}} w dx \right)^{-1}.$$

Similarly we obtain

$$\cos^2 \psi_1(x, \lambda) \leq \lambda^{-1/2} l_2 \left(\int_{c_{2k+1}}^{c_{2k+2}} r dx \right)^{-1}.$$

For large values of λ one may find $x \in [c_0, c_1]$ such that ψ_1 is arbitrarily close to a multiple of π, and $x \in [c_1, c_2]$ such that ψ_1 is arbitrarily close to an odd multiple of $\pi/2$. Therefore taking λ large, the variation of $\psi_1(x, \lambda)$ over (a, b) can be made as large as we please and we have a contradiction.

Therefore $\psi_1(b, \lambda) - \psi_1(a, \lambda)$ can be made arbitrarily large and $\psi_1(x, \lambda)$ increases through an arbitrarily large number of multiples of π as x goes from a to b. This completes the proof. $\square$

4. Banach space

In this section we investigate the differentiable property of the eigenvalues of (2.1), (2.6) with respect to some elements of data. For this purpose we shall construct a suitable Banach space as follows

$$\mathbb{B} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times L^1(a', b') \times L^1(a', b') \times L^1(a', b') \times L^1(a', b'),$$

with the norm

$$|l_1| + |l_2| + |l_3| + |l_4| + \int_{a'}^{b'} \{|p_1| + |p_2| + |p_3| + |p_4|\}\, dx,$$

where $l_1, ..., l_4$ are real numbers and $p_1, ..., p_4$ are integrable functions on (a', b').

Now we shall consider the following subspace B_1 of $\mathbb{B}$ consisting of all elements υ_1 such that

$$\upsilon_1 = (a, b, \gamma, \varphi, \widetilde{r}, \widetilde{s}, \widetilde{q}, \widetilde{w}),$$

where the corresponding function $\widetilde{k}$ is defined as follows

$$\widetilde{k} = \begin{cases} k \text{ on } [a, b], \\ 0 \text{ otherwise.} \end{cases}$$

If we define B as the set consisting of all elements υ such that

$$\upsilon = (a, b, \gamma, \varphi, r, s, q, w),$$

then B is not a subset of $\mathbb{B}$. Therefore we identify B with B_1 to inherit the norm from B and the convergence in B.

Lemma 4.1. *Let y and z be the solutions of* (2.1) *satisfying*

$$y(x_*, \lambda) = \xi_1,\ z(x_*, \lambda) = \xi_2,\ \xi_1, \xi_2 \in \mathbb{C},\ x_* \in (a', b').$$

Then $y = y(., x_*, \xi_1, \xi_2, r, s, q, w)$ is continuous of all its variables.

Proof. The proof can be introduced as Theorem 2.7 in [10]. □

Lemma 4.2. *The eigenvalue λ of* (2.1), (2.6) *is a continuous function of υ in the set B.*

Proof. The roots of the function

$$\Phi(\lambda) = y(b, \lambda)\cos\psi - z(b, \lambda)\sin\psi,$$

can be considered as the eigenvalues of the problem (2.1), (2.6) provided that y and z satisfy the certain conditions at a. Since Φ is entire in λ, for the point $\upsilon_0 \in B$, there exists an $\eta > 0$ such that $\Phi(\lambda) \neq 0$ for μ with $|\lambda - \mu| = \eta$. Therefore by the theorem on continuity of roots of an equation as a function of parameters [4] the proof is completed. □

Remark 4.1. We should understand Lemma 4.2 as there exists a continuous eigenvalue branch in B. However, this does not imply that the n th eigenvalue is always continuous in B.Therefore we will consider the eigenvalues in such a continuous eigenvalue branch.

Theorem 4.1. *For the simple eigenvalue λ of* (2.1), (2.6) *belonging to the continuous branch in B there exists a normalized eigenfunction $y_u(x, \lambda)$ such that*

$$\|y_u(x, \lambda(\upsilon)) - y_u(x, \lambda(\upsilon_0))\| \to 0,\ (uniformly)$$

as

$$\|\upsilon - \upsilon_0\| \to 0,\ \upsilon, \upsilon_0 \in B.$$

Proof. For the proof we refer to [10] together with Lemma 4.1 and Lemma 4.2. □

Using Theorem 4.1 we can introduce the following.

Theorem 4.2. *Let λ be an eigenvalue of* (2.1), (2.6). *Then the derivatives of λ with respect to some certain elements of data can be introduced as follows*

$$\frac{d\lambda}{d\gamma} = -\sin^2\gamma\,|y_u(a)|^2 - \cos^2\gamma\,|z_u(a)|^2,$$

$$\frac{d\lambda}{d\varphi} = \sin^2\varphi\,|y_u(b)|^2 + \cos^2\varphi\,|z_u(b)|^2,$$

$$\frac{d\lambda}{dr} = \int_a^b h\,|y_u' + sy_u|^2,\ h \in L^1(a, b),$$

$$\frac{d\lambda}{ds}=2\int_a^b \frac{h}{r}\Re(y_u'\overline{y_u})+2\int_a^b \frac{hs}{r}\left|y_u\right|^2+\int_a^b \frac{h^2}{r}\left|y_u\right|^2, \; h\in L^2(a,b),$$

$$\frac{d\lambda}{dq}=\int_a^b h\left|y_u\right|^2, \; h\in L^1(a,b),$$

$$\frac{d\lambda}{dw}=\lambda\int_a^b h\left|y_u\right|^2, \; h\in L^1(a,b).$$

Proof. The ordinary derivatives of λ with respect to γ and φ and the Frechet derivatives of λ with respect to q and w can be obtained using a similar method in the literature (for example, see [4, 6, 9–12, 14, 15, 17, 19]) . However, the proofs of the derivatives of λ with respect to r and s should be given as they are new in the literature.

We shall consider the Eq. (2.7). Let y_u be a normalized eigenfunction of λ and $y_u = y_u(x,\lambda(1/r))$ and $y_v = y_u(x,\lambda(1/r+h))$, where $h\in L^1(a,b)$. Fixing all the other variables we get

$$\begin{aligned}
&\left(\lambda(\frac{1}{r})-\lambda(\frac{1}{r}+h)\right)\int_a^b w y_u\overline{y_v} \\
&=-\frac{1}{r}(y_u'+sy_u)\overline{y_v}\,\Big|_a^b+\int_a^b \frac{1}{r}(y_u'+sy_u)\overline{y_v'} \\
&\quad+\int_a^b \frac{s}{r}y_u'\overline{y_v}+\int_a^b \frac{s^2}{r}y_u\overline{y_v}+\left(\frac{1}{r}+h\right)y_u(\overline{y_v'}+s\overline{y_v})\,\Big|_a^b-\int_a^b y_u'\left(\frac{1}{r}+h\right)(\overline{y_v'}+s\overline{y_v}) \\
&\quad-\int_a^b\left(\frac{s}{r}+sh\right)y_u\left(\overline{y_v'}+s\overline{y_v}\right) \\
&=-\int_a^b h\left(y_u'+sy_u\right)\left(\overline{y_v'}+s\overline{y_v}\right).
\end{aligned}$$

Therefore the result follows from the last equation.

Now consider that y_u be a normalized eigenfunction of λ and $y_u = y_u(x,\lambda(s))$ and $y_v = y_u(x,\lambda(s+h))$, where $h\in L^2(a,b)$. Fixing all the other variables we obtain

$$(\lambda(s)-\lambda(s+h))\int_a^b w y_u\overline{y_v}=-\int_a^b \frac{h}{r}(y_u'\overline{y_v}+y_u\overline{y_v'})-\int_a^b \frac{hs}{r}2y_u\overline{y_v}-\int_a^b \frac{h^2}{r}y_u\overline{y_v}.$$

Therefore the proof is completed. □

5. Conclusion and remarks

In this paper we investigate some properties of the real eigenvalues of the problem (2.1), (2.6) and we have followed Atkinson's method.

We should note that for the case $s\equiv 0$ on $[a,b]$ the results are well known but in this work there is no need to consider s as identically zero. Therefore the results are new. Furthermore, as can be seen in Theorem 4.2, for the Frechet derivative of the spectral parameter with respect to s we need to consider h as an element of square integrable function space. This construction and result are also new in the literature as well as the results for the derivative of the eigenvalues with respect to the function r.

On the other side, as we have discussed in the introduction, there exists another way following the Prüfer's transformation. It is possible for the problem (2.1), (2.6) to pass to the new variables using Prüfer's transformation and obtain some results for this variables. Indeed, one may consider the following transformations

$$y(x) = \tau(x) \sin \psi(x), \ \ z(x) = \tau(x) \cos \psi(x).$$

These yield

$$\tau' = (r - \lambda w + q)\, \tau \sin \psi \cos \psi + \tau s \cos 2\psi,$$

and

$$\psi' = r \cos^2 \psi + (\lambda w - q) \sin^2 \psi - s \sin 2\psi. \tag{5.1}$$

Then

$$(\psi_2 - \psi_1)' = f\,(\psi_2 - \psi_1) + h,$$

where

$$f = (\lambda w_1 - q_1 - r_1)\,(\sin \psi_2 + \sin \psi_1) \left(\frac{\sin \psi_2 - \sin \psi_1}{\psi_2 - \psi_1} \right) + 2 s_2 \frac{\sin 2\psi_2 - \sin 2\psi_1}{2\psi_2 - 2\psi_1},$$

and

$$h = (r_2 - r_1) \cos^2 \psi_2 + (\lambda w_2 - q_2 - \lambda w_1 + q_1) \sin^2 \psi_2 + (s_1 - s_2) \sin 2\psi_1.$$

For $r_2 > r_1$, $\lambda w_2 - q_2 > \lambda w_1 - q_1$ and $s_1 > s_2$ if ψ is restriced on $(-\pi/2, 0)$ then one may say from (5.1) that ψ is increasing function of ψ. However, on $(0, \pi/2)$ the assumption on s_1 and s_2 should be reversed to infer that ψ is increasing on $(0, \pi/2)$.

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