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On the weighted fractional integral inequalities for Chebyshev functionals

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Abstract

The goal of this present paper is to study some new inequalities for a class of differentiable functions connected with Chebyshev's functionals by utilizing a fractional generalized weighted fractional integral involving another function $\mathcal{G}$ in the kernel. Also, we present weighted fractional integral inequalities for the weighted and extended Chebyshev's functionals. One can easily investigate some new inequalities involving all other type weighted fractional integrals associated with Chebyshev's functionals with certain choices of $\omega(\theta)$ and $\mathcal{G}(\theta)$ as discussed in the literature. Furthermore, the obtained weighted fractional integral inequalities will cover the inequalities for all other type fractional integrals such as Katugampola fractional integrals, generalized Riemann–Liouville fractional integrals, conformable fractional integrals and Hadamard fractional integrals associated with Chebyshev's functionals with certain choices of $\omega(\theta)$ and $\mathcal{G}(\theta)$.

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1 Introduction

The Chebyshev functional [1] for two integrable functions $\mathcal{U}$ and $\mathcal{V}$ on $[\nu_1, \nu_2]$ is defined by

$$\begin{aligned} T(\mathcal{U},\mathcal{V}) = {} & \frac{1}{\nu_1-\nu_2}\int_{\nu_1}^{\nu_2}\mathcal{U}(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta \\ & - \frac{1}{\nu_1-\nu_2}\left(\int_{\nu_1}^{\nu_2}\mathcal{U}(\vartheta)\,d\vartheta\right)\frac{1}{\nu_1-\nu_2}\left(\int_{\nu_1}^{\nu_2}\mathcal{V}(\vartheta)\,d\vartheta\right). \end{aligned} \tag{1.1}$$

The weighted Chebyshev functional (WCF in short) [1] for two integrable functions $\mathcal{U}$ and $\mathcal{V}$ on $[\nu_1, \nu_2]$ is defined by

$$\begin{aligned} T(\mathcal{U},\mathcal{V},\hbar) = {} & \int_{\nu_1}^{\nu_2}\hbar(\vartheta)\,d\vartheta \int_{\nu_1}^{\nu_2}\hbar(\vartheta)(\vartheta)(\vartheta)\,d\vartheta \\ & - \int_{\nu_1}^{\nu_2}\hbar(\vartheta)(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta, \end{aligned} \tag{1.2}$$

where the function $\hbar$ is positive and integrable on $[\nu_1,\nu_2]$. Applications of (1.2) can be found in the field of probability and statistical problems. Also, applications of functional (1.2) can be found in the field of differential and integral equations. The reader may consult [2–4].

Dragomir [5] defined the following inequality for two differentiable functions $\mathcal{U}$ and $\mathcal{V}$:

$$\left|T(\mathcal{U},\mathcal{V},\hbar)\right| \le \|\mathcal{U}'\|\,\|\mathcal{V}'\|\left[\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\vartheta^2\hbar(\vartheta)\,d\vartheta-\left(\int_{\nu_1}^{\nu_2}\vartheta\,\hbar(\vartheta)\,d\vartheta\right)^2\right],$$

where $\mathcal{U}',\mathcal{V}'\in L_\infty(\nu_1,\nu_2)$ and the function $\hbar$ is positive and integrable on $[\nu_1,\nu_2]$. The researchers have studied the functionals (1.1) and (1.2) and established certain remarkable inequalities by employing different techniques. We refer the reader to [5–21]. The existence and uniqueness of a miscible flow equation through porous media with a nonsingular fractional derivative, unified integral inequalities comprising pathway operators, certain results comprising the weighted Chebyshev functional using pathway fractional integrals and integral inequalities associated with Gauss hypergeometric function fractional integral operator can be found in [22–25].

Elezovic et al. [26] proved the following inequality for WCF:

$$\begin{aligned}\left|T(\mathcal{U},\mathcal{V},\hbar)\right| &\le \frac{1}{2}\left(\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}\hbar(\xi)\hbar(\zeta)|\xi-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_\zeta^\xi\left|\mathcal{U}'(\vartheta)\right|^p d\vartheta\right|^{\frac{r}{p}}d\xi\,d\zeta\right)^{\frac{1}{r}}\\ &\quad\times\left(\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}h(\xi)h(\zeta)|\xi-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_\zeta^\xi\left|\mathcal{V}'(\vartheta)\right|^q d\vartheta\right|^{\frac{r}{q}}d\xi\,d\zeta\right)^{\frac{1}{r'}}\\ &\le\frac{1}{2}\|\mathcal{U}'\|\,\|\mathcal{V}'\|\left(\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}\hbar(\xi)h(\zeta)|\xi-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\xi\,d\zeta\right),\end{aligned}\tag{1.3}$$

where $\mathcal{U}'\in L^p([\nu_1,\nu_2])$, $\mathcal{V}'\in L^q([\nu_1,\nu_2])$, $p,q,r>1$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$.

In [9], the authors established the following fractional integral inequality for Chebyshev functional (1.2):

$$\begin{aligned}&2\left|\mathcal{I}^\alpha\hbar(\tau)\mathcal{I}^\alpha\hbar\mathcal{U}\mathcal{V}(\theta)-\mathcal{I}^\alpha\hbar\mathcal{U}(\theta)\mathcal{I}^\alpha\hbar\mathcal{V}(\theta)\right|\\ &\quad\le\frac{\|\mathcal{U}'\|_p\,\|\mathcal{V}'\|_q}{\Gamma^2(\alpha)}\int_0^\theta\int_0^\theta(\theta-\vartheta)^{\alpha-1}(\theta-\zeta)^{\alpha-1}|\vartheta-\zeta|\hbar(\vartheta)\hbar(\zeta)\,d\vartheta\,d\zeta,\end{aligned}$$

where $\mathcal{U}'\in L^p([0,\infty[)$, $\mathcal{V}'\in L^q([0,\infty[)$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The extended Chebyshev functional [27, 28] is defined by

$$\begin{aligned}\tilde{T}(\mathcal{U},\mathcal{V},\hbar,\hbar') &= \int_{\nu_1}^{\nu_2}\hbar'(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\mathcal{U}(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta\\ &\quad+\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\hbar'(\vartheta)\mathcal{U}(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta\\ &\quad-\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\mathcal{U}(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\hbar'(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta\\ &\quad-\int_{\nu_1}^{\nu_2}\hbar'(\vartheta)\mathcal{U}(\vartheta)\,d\vartheta\int_{\nu_1}^{\nu_2}\hbar(\vartheta)\mathcal{V}(\vartheta)\,d\vartheta.\end{aligned}\tag{1.4}$$

Recently the researchers [29–38] presented certain remarkable inequalities by considering certain type of fractional integrals.

This paper is designed as follows.

In Sect. 2, we present some well-known definitions. Section 3 is devoted to the weighted fractional integral inequalities associated with Chebyshev's functionals (1.1) and (1.2) concerning another function $\mathcal{G}$ in the kernel. In Sect. 4, we give some new weighted fractional integral inequalities associated with weighted and extended Chebyshev's functionals (1.3) and (1.4) with respect to another function $\mathcal{G}$ in the kernel. Finally, we discuss concluding remarks in Sect. 5.

2 Preliminaries

In this section, we present the preliminaries and definitions.

Definition 2.1 ([39, 40]) The function $\mathcal{U}$ is said to be in the space $L_{p,r}[0,\infty[$ if

$$L_{p,r}[0,\infty[=\left\{\mathcal{U}:\|\mathcal{U}\|_{L_{p,r}[0,\infty[}=\left(\int_r^s\left|\mathcal{U}(\vartheta)\right|^p\vartheta^r\,d\vartheta\right)^{\frac{1}{p}}<\infty, 1\leq p<\infty, r\geq 0\right\}. \tag{2.1}$$

Applying $r=0$ in (2.1) gives

$$L_{p}[0,\infty[=\left\{\mathcal{U}:\|\mathcal{U}\|_{L_{p}[0,\infty[}=\left(\int_r^s\left|\mathcal{U}(\vartheta)\right|^p\,d\vartheta\right)^{\frac{1}{p}}<\infty, 1\leq p<\infty\right\}.$$

Definition 2.2 ([41]) Let the function $\mathcal{U}\in V_1[0,\infty[$ and suppose that the function $\mathcal{G}$ is positive, increasing and monotone on $[0,\infty[$ and having continuous derivative $\mathcal{G}'$ on $[0,\infty[$ with $\mathcal{G}(0)=0$. Then the function $\hbar_1$ defined on $[0,\infty[$ (Lebesgue real-valued measurable function) will be in the space $X_{\mathcal{G}}^p(0,\infty)$, $(1\leq p<\infty)$ if

$$\|\mathcal{U}\|_{X_{\mathcal{G}}^p}=\left(\int_r^s\left|\mathcal{U}(\vartheta)\right|^p\mathcal{G}'(\vartheta)\,d\vartheta\right)^{\frac{1}{p}}<\infty,\quad 1\leq p<\infty.$$

When $p=\infty$, then

$$\|\mathcal{U}\|_{X_{\mathcal{G}}^\infty}=\mathbf{ess}\sup_{0\leq\vartheta<\infty}\left[\mathcal{G}'(\vartheta)\mathcal{U}(\vartheta)\right].$$

Remark 2.1 Note that:

i. if we set $\mathcal{G}(\vartheta)=\vartheta$ for $1\leq p<\infty$, then the space $X_{\mathcal{G}}^p(0,\infty)$ will coincide with the space $L_p[0,\infty[$,

ii. if $\mathcal{G}(\vartheta)=\ln\vartheta$ for $1\leq p<\infty$, then the space $X_{\mathcal{G}}^p(0,\infty)$ will coincide with the space $L_{p,r}[1,\infty[$.

Definition 2.3 The Riemann–Liouville (R-L) fractional integrals (left- and right-sided) $\mathfrak{I}_{\nu_1}^\kappa$ and $\mathfrak{I}_{\nu_2}^\kappa$ of order $\kappa>0$, for a function $\mathcal{U}(\theta)$ are, respectively, given by

$$\mathfrak{I}_{\nu_1}^\kappa\mathcal{U}(\theta)=\frac{1}{\Gamma(\kappa)}\int_{\nu_1}^\theta(\theta-\vartheta)^{\kappa-1}\mathcal{U}(\vartheta)\,d\vartheta,\quad \nu_1<\theta, \tag{2.2}$$

and

$$\Im_{\nu_2}^{\kappa}\mathcal{U}(\theta)=\frac{1}{\Gamma(\kappa)}\int_{\theta}^{\nu_2}(\vartheta-\theta)^{\kappa-1}\mathcal{U}(\vartheta)\,d\vartheta,\quad \nu_2>\theta, \tag{2.3}$$

where Γ is denoted by the well-known gamma function [42].

Definition 2.4 The one-sided R-L fractional integral $\Im_0^{\kappa}$ of order $\kappa>0$, for a function $\mathcal{U}(\theta)$ is given by

$$\Im_0^{\kappa}\mathcal{U}(\theta)=\frac{1}{\Gamma(\kappa)}\int_{0}^{\theta}(\theta-\vartheta)^{\kappa-1}\mathcal{U}(\vartheta)\,d\vartheta,\quad \nu_1<\theta. \tag{2.4}$$

Definition 2.5 ([43, 44]) Let the function $\mathcal{U}$ be an integrable in $X_{\Psi}^{p}(0,\infty)$ and suppose the function $\mathcal{G}$ is positive, increasing and monotone on $[0,\infty)$ and having continuous derivative on $[0,\infty)$ such that $\mathcal{G}(0)=0$. Then the generalized R-L left- and right-sided fractional integrals of a function $\mathcal{U}$ concerning another function $\mathcal{G}$ are, respectively, defined by

$$\left({}^{\mathcal{G}}\Im {\nu_1}^{\kappa}\mathcal{U}\right)(\theta)=\frac{1}{\Gamma(\kappa)}\int_{\nu_1}^{\theta}\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\mathcal{G}'(\vartheta)\hbar_1(\vartheta)\,d\vartheta,\quad \nu_1<\theta, \tag{2.5}$$

and

$$\left({}^{\mathcal{G}}\Im_{\nu_2}^{\kappa}\mathcal{U}\right)(\theta)=\frac{1}{\Gamma(\kappa)}\int_{\theta}^{\nu_2}\left(\mathcal{G}(\vartheta)-\mathcal{G}(\theta)\right)^{\kappa-1}\mathcal{G}'(\vartheta)\hbar(\vartheta)\,d\vartheta,\quad \theta<\nu_2, \tag{2.6}$$

where $\kappa\in\mathbb{C}$ with $\Re(\kappa)>0$.

Definition 2.6 ([45]) Let the function $\mathcal{U}$ be an integrable in $X_{\Psi}^{p}(0,\infty)$ and suppose the function $\mathcal{G}$ is positive, increasing and monotone on $[0,\infty)$ and having continuous derivative on $[0,\infty)$ such that $\mathcal{G}(0)=0$. Then the left-sided weighted fractional integral of a function $\mathcal{U}$ concerning another function $\mathcal{G}$ is defined by

$$\left({}_{\nu_1}^{\mathcal{G}}\mathcal{I}_{\omega}^{\kappa}\mathcal{U}\right)(\theta)=\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_{\nu_1}^{\theta}\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\omega(\vartheta)\mathcal{G}'(\vartheta)\mathcal{U}(\vartheta)\,d\vartheta,\quad \nu_1<\theta, \tag{2.7}$$

where $\kappa,\in\mathbb{C}$ with $\Re(\kappa)>0$.

Remark 2.2 The following new weighted fractional integrals can be easily obtained:

i. setting $\mathcal{G}(\theta)=\theta$ in Definition 2.6, we get the following weighted R-L fractional integral:

$$\left({}_{\nu_1}\mathcal{I}_{\omega}^{\kappa}\mathcal{U}\right)(\theta)=\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_{\nu_1}^{\theta}(\theta-\vartheta)^{\kappa-1}\omega(\vartheta)\mathcal{U}(\vartheta)\,d\vartheta,\quad \nu_1<\theta.$$

ii. setting $\mathcal{G}(\theta)=\ln\theta$ in Definition 2.6, then we get the following weighted Hadamard fractional integral operator:

$$\left({}_{x_1}\mathcal{I}_{\omega}^{\kappa}\mathcal{U}\right)(\theta)=\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_{\nu_1}^{\theta}(\ln\theta-\ln\vartheta)^{\kappa-1}\omega(\vartheta)\mathcal{U}(\vartheta)\,\frac{d\vartheta}{\vartheta},\quad \nu_1<\theta.$$

iii. setting $\mathcal{G}(\theta) = \frac{\theta^\eta}{\eta}$, $\eta > 0$ in Definition 2.6, then we obtain the following weighted Katugampola fractional integral:

$$\big({}_{\nu_1}\mathcal{I}^\kappa_\omega \mathcal{U}\big)(\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_{\nu_1}^{\theta} \bigg(\frac{\theta^\eta - \vartheta^\eta}{\eta}\bigg)^{\kappa-1} \omega(\vartheta)\mathcal{U}(\vartheta)\,\frac{d\vartheta}{\vartheta^{1-\eta}}, \quad \nu_1 < \theta.$$

Similarly, one can obtain other type of weighted fractional integrals.

Remark 2.3 The following new weighted fractional integrals can be easily obtained:

i. setting $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \theta$ in Definition 2.6, we get (2.2),

ii.setting $\omega(\theta) = 1$ in Definition 2.6, then it will reduce to the left-sided generalized R-L fractional integral operator (2.5),

iii. setting $\omega(\theta) = \theta^u$ and $\mathcal{G}(\theta) = \ln\theta$ in Definition 2.6, then it will reduce to the left-sided Hadamard integral operator [43, 44],

iv. setting $\mathcal{G}(\theta) = \frac{\theta^\eta}{\eta}$, $\eta > 0$ and $\omega(\theta) = 1$ in Definition 2.6, then it will reduce to the left-sided Katugampola [40] fractional integral,

v. setting $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \frac{\theta^{\alpha+s}}{\alpha+s}$ (where $\alpha \in (0,1]$, $s \in \mathbb{R}$ and $\alpha + s \neq 0$) in Definition 2.6, then it reduces to the left-sided generalized fractional conformable integral given by [46],

vi. setting $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \frac{(\theta-x_1)^\alpha}{\alpha}$, $\alpha > 0$ in Definition 2.6, then it reduces to the fractional conformable integral defined by Jarad et al. [47].

In this paper, we consider the following one-sided generalized weighted fractional integral.

Definition 2.7 Let the function $\mathcal{U}$ be an integrable in the space $X^p_{\mathcal{U}}(0,\infty)$ and suppose the function $\mathcal{G}$ is positive, increasing and monotone on $[0,\infty)$ and having continuous derivative on $[0,\infty)$ such that $\mathcal{G}(0) = 0$. Then the one-sided generalized weighted fractional integral of the function $\hbar_1$ concerning another function $\mathcal{G}$ in the kernel is defined by

$$\big({}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0 \mathcal{U}\big)(\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_0^\theta \big(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\big)^{\kappa-1} \omega(\vartheta)\mathcal{G}'(\vartheta)\mathcal{U}(\vartheta)\,d\vartheta. \tag{2.8}$$

3 Weighted fractional integral inequalities associated with Chebyshev's functional

In this section, we present weighted fractional integral inequalities for a class of differentiable functions connected with Chebyshev's functional (1.1).

Theorem 1 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ such that $\mathcal{U}', \mathcal{V}' \in L_\infty([0,\infty[)$ and assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0) = 0$. Then, for all $\theta > 0$, $\kappa > 0$, the following weighted fractional integral inequality holds*:

$$\begin{aligned} &\big|{}^{\mathcal{G}}\mathcal{I}^\kappa_0\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0\mathcal{V}(\theta)\big| \\ &\quad\leq \|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\big[{}^{\mathcal{G}}\mathcal{I}^\kappa_0\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0\theta^2 - \big({}^{\mathcal{G}}_{\omega}\mathcal{I}^\kappa_0\theta\big)^2\big], \end{aligned} \tag{3.1}$$

where ${}^{\mathcal{G}}\mathcal{I}^\kappa_0\omega(\theta)$ is defined by

$${}^{\mathcal{G}}\mathcal{I}^\kappa_0\omega(\theta) = \frac{\omega^{-1}(\theta)}{\Gamma(\kappa)} \int_0^\theta \big(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\big)^{\kappa-1} \omega(\vartheta)\mathcal{G}'(\vartheta)\,d\vartheta.$$

Proof Let us define

$$H(\vartheta,\zeta)=\big(\mathcal{U}(\vartheta)-\mathcal{U}(\zeta)\big)\big(\mathcal{V}(\vartheta)-\mathcal{V}(\zeta)\big);\quad \vartheta,\zeta\in(0,\theta). \tag{3.2}$$

Multiplying (3.2) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\vartheta))^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)$ and then integrating with respect to ϑ over $(0,\theta)$ and applying (2.7), we have

$$\begin{aligned}&\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)H(\vartheta,\zeta)\,d\vartheta\\&\quad={}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{U}\mathcal{V}(\theta)-\mathcal{U}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{V}(\theta)-\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{U}(\theta)+\mathcal{U}(\zeta)\mathcal{V}(\zeta)\,{}^{\mathcal{G}}\mathcal{I}^{\kappa}_{0}\omega(\theta).\end{aligned} \tag{3.3}$$

Again, multiplying (3.3) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\zeta))^{\kappa-1}\mathcal{G}'(\zeta)\omega(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$, we have

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta\\&\quad=2\big({}^{\mathcal{G}}\mathcal{I}^{\kappa}_{0}\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\mathcal{V}(\theta)\big).\end{aligned} \tag{3.4}$$

Also, on the other hand, we have

$$H(\vartheta,\zeta)=\int_\vartheta^\zeta\int_\vartheta^\zeta\mathcal{U}'(x)\mathcal{V}'(y)\,dx\,dy. \tag{3.5}$$

Since the functions $\mathcal{U}'(x),\mathcal{V}'(y)\in L_\infty([0,\infty[)$, we have

$$\big|H(\vartheta,\zeta)\big|\le\left|\int_\vartheta^\zeta\mathcal{U}'(x)\,dx\right|\left|\int_\vartheta^\zeta\mathcal{V}'(y)\,dy\right|\le\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty(\vartheta-\zeta)^2. \tag{3.6}$$

Thus, we can write

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta\\&\quad\le\frac{\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\\&\qquad\times\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\\&\qquad\times\big(\vartheta^2-2\vartheta\zeta+\zeta^2\big)\,d\vartheta\,d\zeta.\end{aligned} \tag{3.7}$$

From (3.7), we estimate the following inequality:

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta\\&\quad\le2\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\big[{}^{\mathcal{G}}\mathcal{I}^{\kappa}_{0}\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\theta^2-\big({}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\theta\big)^2\big].\end{aligned} \tag{3.8}$$

Hence, from (3.4) and (3.8), we get the desired proof. □

Corollary 1 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ such that $\mathcal{U}',\mathcal{V}' \in L_\infty([0,\infty[)$ and assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$. Then, for all $\theta>0$, $\kappa>0$, the following fractional integral inequality holds*:

$$\begin{aligned}&\left|\frac{(\mathcal{G}(\theta))^\kappa}{\Gamma(\kappa+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{V}(\theta)\right|\\&\quad\le\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\left[\frac{(\mathcal{G}(\theta))^\kappa}{\Gamma(\kappa+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\theta^2-\left({}^{\mathcal{G}}\mathcal{I}_0^\kappa\theta\right)^2\right],\end{aligned}$$

where ${}^{\mathcal{G}}\mathcal{I}_0^\kappa(1)$ is defined by

$$\begin{aligned}&{}^{\mathcal{G}}\mathcal{I}_0^\kappa(1)=\frac{1}{\Gamma(\kappa)}\int_0^\theta\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\mathcal{G}'(\vartheta)\,d\vartheta,\\&\frac{(\mathcal{G}(\theta))^\kappa}{\Gamma(\kappa+1)},\quad\left(\mathcal{G}(0)=0\right).\end{aligned}$$

Theorem 2 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ such that $\mathcal{U}',\mathcal{V}' \in L_\infty([0,\infty[)$ and suppose that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$. Then, for all $\theta>0$, κ, $\mu>0$, the following weighted fractional integral inequality holds*:

$$\begin{aligned}&\left|{}^{\mathcal{G}}\mathcal{I}_0^\mu\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}\mathcal{V}(\theta)+{}^{\mathcal{G}}\mathcal{I}_0^\kappa\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}\mathcal{V}(\theta)\right.\\&\quad\left.-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\mu}\mathcal{I}_0^\kappa\mathcal{V}(\theta)\right|\\&\quad\le\|\mathcal{U}'\|_\infty\,\|\mathcal{V}'\|_\infty\left[{}^{\mathcal{G}}\mathcal{I}_0^\kappa\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\theta^2-2\left({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\theta\right)\left({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\theta\right)+{}^{\mathcal{G}}\mathcal{I}_0^\mu\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\theta^2\right].\end{aligned}\tag{3.9}$$

Proof Multiplying (3.3) by $\frac{\omega^{-1}(\theta)}{\Gamma(\mu)}(\mathcal{G}(\theta)-\mathcal{G}(\zeta))^{\mu-1}\mathcal{G}'(\zeta)\omega(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$ and using (2.7), we have

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\left(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\right)^{\mu-1}\\&\quad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta\\&\quad={}^{\mathcal{G}}\mathcal{I}_0^\mu\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}\mathcal{V}(\theta)+{}^{\mathcal{G}}\mathcal{I}_0^\kappa\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}\mathcal{V}(\theta)\\&\quad-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\mu}\mathcal{I}_0^\kappa\mathcal{V}(\theta).\end{aligned}\tag{3.10}$$

Using (3.6), we get

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\left(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\right)^{\mu-1}\\&\quad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\left|H(\vartheta,\zeta)\right|d\vartheta\,d\zeta\\&\quad\le\frac{\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta\left(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\right)^{\kappa-1}\left(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\right)^{\mu-1}\\&\quad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\left(\vartheta^2-2\vartheta\zeta+\zeta^2\right)d\vartheta\,d\zeta.\end{aligned}\tag{3.11}$$

From (3.10) and (3.11), we get the desired proof. □

Corollary 2 *Let the two functions $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ such that $\mathcal{U}',\mathcal{V}' \in L_\infty([0,\infty[)$ and assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$. Then, for all $\theta>0$, κ, $\mu>0$, the following fractional integral inequality holds*:

$$\begin{aligned}&\bigg|\frac{(\mathcal{G}(\theta))^\mu}{\Gamma(\mu+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{U}\mathcal{V}(\theta)+\frac{(\mathcal{G}(\theta))^\kappa}{\Gamma(\kappa+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\mu\mathcal{U}\mathcal{V}(\theta)\\&\quad-{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}_0^\mu\mathcal{V}(\theta)-{}^{\mathcal{G}}\mathcal{I}_0^\mu\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\mathcal{V}(\theta)\bigg|\\&\quad\le\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\bigg[\frac{(\mathcal{G}(\theta))^\mu}{\Gamma(\mu+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\theta^2-2\big({}^{\mathcal{G}}\mathcal{I}_0^\kappa\theta\,{}^{\mathcal{G}}\mathcal{I}_0^\mu\theta\big)+\frac{(\mathcal{G}(\theta))^\kappa}{\Gamma(\kappa+1)}\,{}^{\mathcal{G}}\mathcal{I}_0^\mu\theta^2\bigg].\end{aligned}$$

Remark 3.1 If we put $\kappa=\mu$ in Theorem 2, then we obtain Theorem 1.

Theorem 3 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ with $\mathcal{V}'(\tau)\neq 0$ and assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$ and let there exist $M>0$ such that $\frac{\mathcal{U}'(\theta)}{\mathcal{V}'(\theta)}\le M$. Then, for all $\theta>0$, κ, $\mu>0$, the following weighted fractional integral inequality holds*:

$$\begin{aligned}&\big|{}^{\mathcal{G}}\mathcal{I}_0^\mu\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}\mathcal{V}(\theta)+{}^{\mathcal{G}}\mathcal{I}_0^\kappa\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}\mathcal{V}(\theta)\\&\quad-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{V}(\theta)\big|\\&\quad\le M\big[{}^{\mathcal{G}}\mathcal{I}_0^\kappa\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{V}^2(\theta)-2\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu\mathcal{V}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{V}(\theta)+{}^{\mathcal{G}}\mathcal{I}_0^\mu\omega(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\mathcal{V}^2(\theta)\big].\end{aligned}\tag{3.12}$$

Proof Suppose that the functions $\mathcal{U}$ and $\mathcal{V}$ satisfy the hypothesis of Theorem 3. Then, for every $\vartheta,\zeta\in[0,\theta]$; $u\neq v$, $\theta>0$, there exists a constant c between ϑ and ζ such that

$$\frac{\mathcal{U}(\vartheta)-\mathcal{U}(\zeta)}{\mathcal{V}(\vartheta)-\mathcal{V}(\zeta)}=\frac{\mathcal{U}'(c)}{\mathcal{V}'(c)}.$$

Thus, for every $\vartheta,\zeta\in[0,\theta]$, we have

$$\big|\mathcal{U}(\vartheta)-\mathcal{U}(\zeta)\big|\le M\big|\mathcal{V}(\vartheta)-\mathcal{V}(\zeta)\big|.$$

It follows that

$$\big|H(\vartheta,\zeta)\big|\le M\big(\mathcal{V}(\vartheta)-\mathcal{V}(\zeta)\big)^2.$$

Therefore, we get

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\mu-1}\\&\qquad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta\\&\quad\le\frac{\omega^{-2}(\theta)M}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\mu-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\\&\qquad\times\big(\mathcal{V}^2(\vartheta)-2\mathcal{V}(\vartheta)\mathcal{V}(\zeta)+\mathcal{V}^2(\zeta)\big)\,d\vartheta\,d\zeta.\end{aligned}\tag{3.13}$$

Hence from (3.13), we get the desired inequality. □

Setting $\omega = 1$ in Theorem 4, then we led to the following new result in terms of generalized fractional integral concerning another function $\mathcal{G}$ in the kernel.

Corollary 3 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ with $\mathcal{V}'(\tau) \neq 0$ and assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0) = 0$ and let there exist $M > 0$ such that $\frac{\mathcal{U}'(\theta)}{\mathcal{V}'(\theta)} \leq M$. Then, for all $\theta > 0$, κ, $\mu > 0$, the following fractional integral inequality holds*:

$$\begin{aligned}
&\left| \frac{(\mathcal{G}(\theta))^{\mu}}{\Gamma(\mu+1)} {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\mathcal{U}\mathcal{V}(\theta) + \frac{(\mathcal{G}(\theta))^{\kappa}}{\Gamma(\kappa+1)} {}^{\mathcal{G}}\mathcal{I}_0^{\mu}\mathcal{U}\mathcal{V}(\theta) \right. \\
&\qquad \left. - {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\mathcal{U}(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^{\mu}\mathcal{V}(\theta) - {}^{\mathcal{G}}\mathcal{I}_0^{\mu}\mathcal{U}(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\mathcal{V}(\theta) \right| \\
&\qquad \leq M\left[\frac{(\mathcal{G}(\theta))^{\mu}}{\Gamma(\mu+1)} {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\mathcal{V}^2(\theta) - 2\, {}^{\mathcal{G}}\mathcal{I}_0^{\mu}\mathcal{V}(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\mathcal{V}(\theta) + \frac{(\mathcal{G}(\theta))^{\kappa}}{\Gamma(\kappa+1)} {}^{\mathcal{G}}\mathcal{I}_0^{\mu}\mathcal{V}^2(\theta) \right].
\end{aligned}$$

Corollary 4 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ with $\mathcal{V}'(\tau) \neq 0$ and assume that the function $\mathcal{G}$ is positive, monotone and increasing on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0) = 0$ and suppose there exists $M > 0$ such that $\frac{\mathcal{U}'(\theta)}{\mathcal{V}'(\theta)} \leq M$. Then, for all $\theta > 0$, $\kappa > 0$, the following weighted fractional integral inequality holds*:

$$\begin{aligned}
&\left| {}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\omega(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\mathcal{V}(\theta) \right| \\
&\qquad \leq M\left[{}^{\mathcal{G}}\mathcal{I}_0^{\kappa}\omega(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\mathcal{V}^2(\theta) - \left({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\mathcal{V}(\theta) \right)^2 \right].
\end{aligned} \tag{3.14}$$

Proof By considering $\kappa = \mu$ in Theorem 3, we get the desired corollary. □

Remark 3.2 If we consider $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \theta$ in Theorems 1–3, then we get the results proved earlier by Dahmani [48].

Now, we present the generalization of Theorems 1 and 2.

Theorem 4 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable and having same sense of variations on $[0,\infty)$ and the function $\hbar$ be a positive on $[0,\infty)$. Assume that the function $\mathcal{G}$ is positive, monotone and increasing on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ with $\mathcal{G}(0) = 0$. If $\mathcal{U}', \mathcal{V}' \in L_{\infty}([0,\infty[)$, then the following weighted fractional inequality holds for all $\theta > 0$, $\kappa > 0$*;

$$\begin{aligned}
0 &\leq {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\hbar\mathcal{U}(\tau)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\hbar\mathcal{V}(\theta) \\
&\leq \left\|\mathcal{U}'\right\|_{\infty} \left\|\mathcal{V}'\right\|_{\infty} \left[{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\theta^2\hbar(\tau) - \left({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^{\kappa}\theta\hbar(\theta) \right)^2 \right].
\end{aligned} \tag{3.15}$$

Proof Define

$$\begin{aligned}
H(\vartheta,\zeta) &= \left(\mathcal{U}(\vartheta) - \mathcal{U}(\zeta)\right)\left(\mathcal{V}(\zeta) - \mathcal{V}(\zeta)\right); \vartheta,\zeta \in (0,\theta), \theta > 0 \\
&= \mathcal{U}(\vartheta)\mathcal{V}(\vartheta) - \mathcal{U}(\vartheta)\mathcal{V}(\zeta) - \mathcal{U}(\zeta)\mathcal{V}(\vartheta) + \mathcal{U}(\zeta)\mathcal{V}(\zeta).
\end{aligned} \tag{3.16}$$

Since the functions $\mathcal{U}$ and $\mathcal{V}$ satisfy the hypothesis of Theorem 4, we have

$$H(\vartheta,\zeta)\geq 0.$$

Multiplying both sides of (3.16) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\vartheta))^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)$ and then integrating with respect to ϑ over $(0,\theta)$ and using (2.7), we have

$$\begin{aligned}&\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)H(\vartheta,\zeta)\,d\vartheta\\&\quad={}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}\mathcal{V}(\theta)-\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}(\theta)-\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{V}(\theta)\\&\qquad+\mathcal{U}(\zeta)\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta)\geq 0.\end{aligned}\tag{3.17}$$

Again, multiplying (3.17) $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\zeta))^{\kappa-1}\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$ and using (2.7), we get

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{2\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\\&\qquad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta\\&\quad={}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{V}(\theta)\geq 0.\end{aligned}\tag{3.18}$$

From (3.6), it follows that

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{2\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\\&\qquad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta\\&\quad\leq\frac{\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\omega^{-2}(\theta)}{2\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\\&\qquad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)\big(\vartheta^2-2\vartheta\zeta+\zeta^2\big)\,d\vartheta\,d\zeta.\end{aligned}\tag{3.19}$$

Consequently, it follows

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{2\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta\big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\\&\qquad\times\mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta\\&\quad\leq 2\big\|\mathcal{U}'\big\|_\infty\big\|\mathcal{V}'\big\|_\infty\big[{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\theta^2\hbar(\theta)-\big({}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\theta\hbar(\theta)\big)^2\big].\end{aligned}\tag{3.20}$$

According to (3.18) and (3.20), we get the desired proof. □

Applying Theorem 4 for $\omega=1$, we obtain the following new result in terms of generalized fractional integral with respect to another function $\mathcal{G}$ in the kernel.

Corollary 5 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable and having same sense of variations on $[0,\infty)$ and the function $\hbar$ be a positive on $[0,\infty)$. Assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$*

such that $\mathcal{G}(0) = 0$. *If* $\mathcal{U}', \mathcal{V}' \in L_\infty([0,\infty[)$, *then the following fractional inequality holds for all* $\theta > 0$, $\kappa > 0$:

$$\begin{aligned}0 &\leq {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{U}(\tau)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{V}(\theta)\\ &\leq \|\mathcal{U}'\|_\infty \|\mathcal{V}'\|_\infty \bigl[{}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \theta^2\hbar(\tau) - \bigl({}^{\mathcal{G}}\mathcal{I}_0^\kappa \theta\hbar(\theta)\bigr)^2\bigr].\end{aligned}$$

Remark 3.3 By considering $\hbar(\theta) = 1$ in Theorem 4, we get Theorem 1. Similarly, taking $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \theta$, we obtain the result of Dahmani [49].

Theorem 5 *Let the two function* $\mathcal{U}$ *and* $\mathcal{V}$ *be differentiable and having same sense of variations on* $[0,\infty)$ *and the function* $\hbar$ *be a positive on* $[0,\infty)$. *Suppose that the function* $\mathcal{G}$ *is increasing, positive and monotone on* $[0,\infty[$ *and having continuous derivative on* $[0,\infty[$ *such that* $\mathcal{G}(0) = 0$. *If* $\mathcal{U}', \mathcal{V}' \in L_\infty([0,\infty[)$, *then the following weighted fractional inequality holds for all* $\theta > 0$, $\kappa, \mu > 0$:

$$\begin{aligned}0 &\leq {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{U}\mathcal{V}(\theta) + {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}\mathcal{V}(\theta)\\ &\quad - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{V}(\theta)\\ &\leq \|\mathcal{U}'\|_\infty \|\mathcal{V}'\|_\infty \bigl[{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \theta^2\hbar(\theta)\\ &\quad - 2\bigl({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \theta\hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \theta\hbar(\theta)\bigr) + {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \theta^2\hbar(\theta)\bigr].\end{aligned} \tag{3.21}$$

Proof Multiplying both sides of (3.18) by $\frac{\omega^{-1}(\theta)}{\Gamma(\mu)}(\mathcal{G}(\theta) - \mathcal{G}(\zeta))^{\mu-1}\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$ and using (2.7), we get

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta \bigl(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\bigr)^{\mu-1}\\ &\quad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta\\ &\quad = {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{U}\mathcal{V}(\theta) + {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}\mathcal{V}(\theta)\\ &\quad - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{V}(\theta) \geq 0.\end{aligned} \tag{3.22}$$

From Eqs. (3.5) and (3.6), we have

$$\begin{aligned}&\frac{\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta \bigl(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\bigr)^{\mu-1}\\ &\quad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)\bigl|H(\vartheta,\zeta)\bigr|\,d\vartheta\,d\zeta\\ &\quad \leq \frac{\|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\omega^{-2}(\theta)}{\Gamma(\kappa)\Gamma(\mu)}\int_0^\theta\int_0^\theta \bigl(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\bigr)^{\mu-1}\\ &\quad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)\bigl(\vartheta^2 - 2\vartheta\zeta + \zeta^2\bigr)\,d\vartheta\,d\zeta\\ &\quad = \|\mathcal{U}'\|_\infty\|\mathcal{V}'\|_\infty\bigl[{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \theta^2\hbar(\theta) - 2\bigl({}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \theta\hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \theta\hbar(\theta)\bigr)\\ &\quad + {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \theta^2\hbar(\theta)\bigr].\end{aligned} \tag{3.23}$$

Hence from (3.22) and (3.23), we get the desired proof. □

Applying Theorem 5 for $\omega = 1$, we obtain the following new result in terms of generalized fractional integral with respect to another function $\mathcal{G}$ in the kernel.

Corollary 6 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable and having same sense of variations on $[0,\infty)$ and the function $\hbar$ be a positive on $[0,\infty)$. Assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ with $\mathcal{G}(0) = 0$. If $\mathcal{U}', \mathcal{V}' \in L_\infty([0,\infty[)$, then the following fractional inequality holds for all $\theta > 0$, $\kappa, \mu > 0$:*

$$\begin{aligned}
0 &\le {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\mu \hbar\mathcal{U}\mathcal{V}(\theta) + {}^{\mathcal{G}}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{U}\mathcal{V}(\theta) \\
&\quad - {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\mu \hbar\mathcal{V}(\theta) - {}^{\mathcal{G}}\mathcal{I}_0^\mu \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar\mathcal{V}(\theta) \\
&\le \|\mathcal{U}'\|_\infty \|\mathcal{V}'\|_\infty \big[{}^{\mathcal{G}}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\mu \theta^2\hbar(\theta) - 2\big({}^{\mathcal{G}}\mathcal{I}_0^\kappa \theta\hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\mu \theta\hbar(\theta)\big) \\
&\quad + {}^{\mathcal{G}}\mathcal{I}_0^\mu \hbar(\theta)\, {}^{\mathcal{G}}\mathcal{I}_0^\kappa \theta^2\hbar(\theta)\big].
\end{aligned}$$

Remark 3.4 Taking $\kappa = \mu$ in Theorem 5, we get Theorem 4. Similarly, taking $h(\theta) = 1$ in Theorem 5, we get Theorem 2.

4 Weighted fractional integral inequalities associated with the weighted and the extended Chebyshev functionals

The following results presented in this section related to extended, weighted Chebyshev functionals.

Theorem 6 *Let the two function $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ and the function $\hbar$ be positive and integrable on $[0,\infty)$. Assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0) = 0$. If $\mathcal{U}' \in L^p([0,\infty[)$, $\mathcal{V}' \in L^q([0,\infty[)$, $p,q,r > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, then the following weighted fractional integral inequality holds for all $\tau > 0$, $\alpha, \beta > 0$;*

$$\begin{aligned}
&2\big| {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{U}(\theta)\, {}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa \hbar\mathcal{V}(\theta)\big| \\
&\quad\le \bigg(\frac{\|\mathcal{U}'\|_p^r \omega^{-r}(\theta)}{\Gamma^r(\kappa)} \int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\big)^{\kappa-1} \big(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\big)^{\kappa-1} \\
&\qquad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)|\vartheta - \zeta|^{\frac{1}{p'}+\frac{1}{q'}}\, d\vartheta\, d\zeta \bigg)^{\frac{1}{r}} \\
&\qquad \times \bigg(\frac{\|\mathcal{V}'\|_q^{r'} \omega^{-r'}(\theta)}{\Gamma^{r'}(\kappa)} \int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\big)^{\kappa-1} \big(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\big)^{\kappa-1} \\
&\qquad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)|\vartheta - \zeta|^{\frac{1}{p'}+\frac{1}{q'}}\, d\vartheta\, d\zeta \bigg)^{\frac{1}{r'}} \\
&\quad\le \frac{\|\mathcal{U}'\|_p \|\mathcal{V}'\|_q \omega^{-2}(\theta)}{\Gamma^2(\kappa)} \bigg(\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\big)^{\kappa-1} \big(\mathcal{G}(\theta) - \mathcal{G}(\zeta)\big)^{\kappa-1} \\
&\qquad \times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)|\vartheta - \zeta|^{\frac{1}{p'}+\frac{1}{q'}}\, d\vartheta\, d\zeta \bigg). \qquad (4.1)
\end{aligned}$$

Proof Let us define

$$H(\vartheta,\zeta) = \big(\mathcal{U}(\vartheta)-\mathcal{U}(\zeta)\big)\big(\mathcal{V}(\vartheta)-\mathcal{U}(\zeta)\big); \vartheta,\zeta \in (0,\theta)$$
$$= \mathcal{U}(\vartheta)\mathcal{V}(\vartheta)-\mathcal{U}(\vartheta)\mathcal{V}(\zeta)-\mathcal{U}(\zeta)\mathcal{V}(\vartheta)+\mathcal{U}(\zeta)\mathcal{V}(\zeta). \tag{4.2}$$

Multiplying (4.2) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\vartheta))^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)$ and then integrating with respect to ϑ over $(0,\theta)$ and using (2.7), we get

$$\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}\int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)H(\vartheta,\zeta)\,d\vartheta$$
$$= {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}\mathcal{V}(\theta)-\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}(\theta)-\mathcal{U}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{V}(\theta)+\mathcal{U}(\zeta)\mathcal{V}(\zeta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta). \tag{4.3}$$

Again, multiplying (4.3) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\zeta))^{\kappa-1}\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$ and using (2.7), we get

$$\frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}$$
$$\times \big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta$$
$$= 2\big({}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{V}(\theta)\big). \tag{4.4}$$

Also, on the other hand, we have

$$H(\vartheta,\zeta) = \int_\zeta^\vartheta\int_\zeta^\vartheta \mathcal{U}'(u)\mathcal{V}'(v)\,du\,dv. \tag{4.5}$$

By employing the Hölder inequality, we have

$$\big|\mathcal{U}(\vartheta)-\mathcal{V}(\zeta)\big| \le |\vartheta-\zeta|^{\frac{1}{p'}}\left|\int_\zeta^\vartheta \big|\mathcal{U}'(u)\big|^p\,du\right|^{\frac{1}{p}} \tag{4.6}$$

and

$$\big|\mathcal{V}(\vartheta)-\mathcal{V}(\zeta)\big| \le |\vartheta-\zeta|^{\frac{1}{q'}}\left|\int_\zeta^\vartheta \big|\mathcal{V}'(v)\big|^q\,dv\right|^{\frac{1}{q}}. \tag{4.7}$$

Then H can be calculated as

$$\big|H(\vartheta,\zeta)\big| \le |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_\zeta^\vartheta \big|\mathcal{U}'(u)\big|^p\,du\right|^{\frac{1}{p}}\left|\int_\zeta^\vartheta \big|\mathcal{V}'(v)\big|^q\,dv\right|^{\frac{1}{q}}. \tag{4.8}$$

Therefore, from (4.4) and (4.8), we can write

$$2\big|{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_{0}\hbar\mathcal{V}(\theta)\big|$$
$$= \frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta\int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}$$
$$\times \big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)\big|H(\vartheta,\zeta)\big|\,d\vartheta\,d\zeta$$

$$\leq \frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)} \int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$$
$$\times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}} \left|\int_\zeta^\vartheta |\mathcal{U}'(u)|^p\,du\right|^{\frac{1}{p}} \left|\int_\zeta^\vartheta |\mathcal{V}'(v)|^q\,dv\right|^{\frac{1}{q}} d\vartheta\,d\zeta. \tag{4.9}$$

Applying the Hölder inequality for a double integral to (4.9), we obtain

$$2\big|{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{V}(\theta)\big|$$
$$\leq \frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\bigg(\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$$
$$\times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}} \left|\int_\zeta^\vartheta |\mathcal{U}'(u)|^p\,du\right|^{\frac{r}{p}} d\vartheta\,d\zeta\bigg)^{\frac{1}{r}}$$
$$\times \bigg(\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)$$
$$\times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}} \left|\int_\zeta^\vartheta |\mathcal{V}'(v)|^q\,dv\right|^{\frac{r'}{q}} d\vartheta\,d\zeta\bigg)^{\frac{1}{r'}}. \tag{4.10}$$

Now, using the following properties

$$\left|\int_\zeta^\vartheta |\mathcal{U}'(u)|^p\,du\right| \leq \|\mathcal{U}'\|_p^p \quad\text{and}\quad \left|\int_\zeta^\vartheta |\mathcal{V}'(v)|^q\,dv\right| \leq \|\mathcal{V}'\|_q^q, \tag{4.11}$$

then (4.10) can be written as

$$2\big|{}^{\beta}\mathfrak{I}^{\alpha}h(\tau)\,{}^{\beta}\mathfrak{I}^{\alpha}hfg(\tau)-{}^{\beta}\mathfrak{I}^{\alpha}hf(\tau)\,{}^{\beta}\mathfrak{I}^{\alpha}hg(\tau)\big|$$
$$\leq \bigg(\frac{\|\mathcal{U}'\|_p^r\omega^{-r}(\theta)}{\Gamma^r(\kappa)}\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)$$
$$\times \mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\bigg)^{\frac{1}{r}}\bigg(\frac{\|\mathcal{V}'\|_p^{r'}\omega^{-r'}(\theta)}{\Gamma^{r'}(\kappa)}\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}$$
$$\times \big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\bigg)^{\frac{1}{r'}}. \tag{4.12}$$

From (4.12), we get

$$2\big|{}^{\beta}\mathfrak{I}^{\alpha}h(\tau)\,{}^{\beta}\mathfrak{I}^{\alpha}hfg(\tau)-{}^{\beta}\mathfrak{I}^{\alpha}hf(\tau)\,{}^{\beta}\mathfrak{I}^{\alpha}hg(\tau)\big|$$
$$\leq \frac{\|\mathcal{U}'\|_p\|\mathcal{V}'\|_q\omega^{-2}(\theta)}{\Gamma^2(\kappa)}\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}$$
$$\times \mathcal{G}'(\vartheta)\omega(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta,$$

which completes the desired proof. □

By considering $\omega(\theta)=1$ in Theorem 6, we get the following new result in terms of generalized fractional integral concerning another function $\mathcal{G}$ in the kernel.

Corollary 7 *Let the two functions $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ and the function $\hbar$ be positive and integrable on $[0,\infty)$. Assume that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$. If $\mathcal{U}'\in L^p([0,\infty[)$, $\mathcal{V}'\in L^q([0,\infty[)$, $p,q,r>1$ with $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$, then the following fractional integral inequality holds for all $\theta>0$, $\kappa>0$;*

$$
\begin{aligned}
&2\left|{}^{\mathcal{G}}\mathcal{I}_0^\kappa\hbar(\theta)\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}\mathcal{I}_0^\kappa\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}_0^\kappa\hbar\mathcal{V}(\theta)\right|\\
&\quad\le\left(\frac{\|\mathcal{U}'\|_p^r}{\Gamma^r(\kappa)}\int_0^\theta\int_0^\theta\bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1}\mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar(\zeta)\right.\\
&\qquad\left.\times|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right)^{\frac{1}{r}}\\
&\qquad\times\left(\frac{\|\mathcal{V}'\|_q^{r'}}{\Gamma^{r'}(\kappa)}\int_0^\theta\int_0^\theta\bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1}\mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar(\zeta)\right.\\
&\qquad\left.\times|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right)^{\frac{1}{r'}}\\
&\quad\le\frac{\|\mathcal{U}'\|_p\|\mathcal{V}'\|_q}{\Gamma^2(\kappa)}\left(\int_0^\theta\int_0^\theta\bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1}\mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar(\zeta)\right.\\
&\qquad\left.\times|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right).
\end{aligned}
$$

Remark 4.1 Setting $\omega(\theta)=1$ and $\mathcal{G}(\theta)=\theta$ in Theorem 6, we get the inequality proved by Dahmani et al. [50]. Similarly, if we consider $\kappa=\omega(\theta)=1$ and $\mathcal{G}(\theta)=\theta$ in Theorem 6, we get the inequality (1.3) on $[0,\theta]$.

Theorem 7 *Let the two functions $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ and let the two functions $\hbar$ and $\hbar'$ be positive and integrable on $[0,\infty)$. Suppose that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ such that $\mathcal{G}(0)=0$. If $\mathcal{U}'\in L^p([0,\infty[)$, $\mathcal{V}'\in L^q([0,\infty[)$, $p,q,r>1$ with $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$, then the following weighted fractional integral inequality holds for all $\tau>0$, $\alpha,\beta>0$:*

$$
\begin{aligned}
&2\left|{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar\mathcal{U}\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar'\mathcal{V}(\theta)-{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar'\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar\mathcal{V}(\theta)\right.\\
&\qquad\left.+{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}_0^\kappa\hbar'\mathcal{U}\mathcal{V}(\theta)\right|\\
&\quad\le\left(\frac{\|\mathcal{U}'\|_p^r\omega^{-r}(\theta)}{\Gamma^r(\kappa)}\int_0^\theta\int_0^\theta\bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\right.\\
&\qquad\left.\times\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar'(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right)^{\frac{1}{r}}\\
&\qquad\times\left(\frac{\|\mathcal{V}'\|_q^{r'}\omega^{-r'}(\theta)}{\Gamma^{r'}(\kappa)}\int_0^\theta\int_0^\theta\bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1}\bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1}\mathcal{G}'(\vartheta)\omega(\vartheta)\right.\\
&\qquad\left.\times\mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar'(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right)^{\frac{1}{r'}}
\end{aligned}
$$

$$\leq \frac{\|\mathcal{U}'\|_p \|\mathcal{V}'\|_q \omega^{-2}(\theta)}{\Gamma^2(\kappa)} \biggl(\int_0^\theta \int_0^\theta \bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1} \bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1} \mathcal{G}'(\vartheta)\omega(\vartheta) \\ \times \mathcal{G}'(\zeta)\omega(\zeta)\hbar(\vartheta)\hbar'(\zeta)|\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta \biggr). \tag{4.13}$$

Proof Multiplying (4.3) by $\frac{\omega^{-1}(\theta)}{\Gamma(\kappa)}(\mathcal{G}(\theta)-\mathcal{G}(\zeta))^{\kappa-1}\mathcal{G}'(\zeta)\omega(\zeta)\hbar'(\zeta)$ and then integrating with respect to ζ over $(0,\theta)$ and using (2.7), we get

$$\frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)} \int_0^\theta \int_0^\theta \bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1} \bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1} \mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta) \\ \times \mathcal{G}'(\zeta)\omega(\zeta)\hbar'(\zeta)H(\vartheta,\zeta)\,d\vartheta\,d\zeta \\ = {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{V}(\theta) \\ - {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{V}(\theta) + {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}\mathcal{V}(\theta). \tag{4.14}$$

Using (4.8) in (4.14), we obtain

$$\bigl| {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{V}(\theta) - {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar\mathcal{V}(\theta) \\ + {}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}_{\omega}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}\mathcal{V}(\theta)\bigr| \\ = \frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)} \int_0^\theta \int_0^\theta \bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1} \bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1} \mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta) \\ \times \mathcal{G}'(\zeta)\omega(\zeta)\hbar(\zeta)\bigl|H(\vartheta,\zeta)\bigr|\,d\vartheta\,d\zeta \\ \leq \frac{\omega^{-2}(\theta)}{\Gamma^2(\kappa)} \int_0^\theta \int_0^\theta \bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1} \bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1} \mathcal{G}'(\vartheta)\omega(\vartheta)\hbar(\vartheta)\mathcal{G}'(\zeta)\omega(\zeta)\hbar'(\zeta) \\ \times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}} \left| \int_\zeta^\vartheta \bigl|\mathcal{U}'(u)\bigr|^p\,du \right|^{\frac{1}{p}} \left| \int_\zeta^\vartheta \bigl|\mathcal{V}'(v)\bigr|^q\,dv \right|^{\frac{1}{q}} d\vartheta\,d\zeta. \tag{4.15}$$

Applying similar arguments to those used in the proof of Theorem 6, we obtain the desired proof. □

Applying Theorem 7 for $\omega = 1$, then we are led to the following new result in terms of a generalized fractional integral concerning another function $\mathcal{G}$ in the kernel.

Corollary 8 *Let the two functions $\mathcal{U}$ and $\mathcal{V}$ be differentiable on $[0,\infty)$ and let the two functions $\hbar$ and $\hbar'$ be positive and integrable on $[0,\infty)$. Suppose that the function $\mathcal{G}$ is increasing, positive and monotone on $[0,\infty[$ and having continuous derivative on $[0,\infty[$ with $\mathcal{G}(0)=0$. If $\mathcal{U}' \in L^p([0,\infty[)$, $\mathcal{V}' \in L^q([0,\infty[)$, $p,q,r>1$ with $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$, then the following fractional integral inequality holds for all $\tau>0$, $\alpha,\beta>0$:*

$$2\bigl| {}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}\mathcal{V}(\theta) - {}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar'\mathcal{V}(\theta) - {}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}(\theta)\,{}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar\mathcal{V}(\theta) \\ + {}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar'(\theta)\,{}^{\mathcal{G}}\mathcal{I}^{\kappa}_0\hbar'\mathcal{U}\mathcal{V}(\theta)\bigr| \\ \leq \biggl(\frac{\|\mathcal{U}'\|_p^r}{\Gamma^r(\kappa)} \int_0^\theta \int_0^\theta \bigl(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\bigr)^{\kappa-1} \bigl(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\bigr)^{\kappa-1} \mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar'(\zeta) \\ \times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta \biggr)^{\frac{1}{r}}$$

$$\times \left(\frac{\|\mathcal{V}'\|_q^{r'}}{\Gamma^{r'}(\kappa)} \int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar'(\zeta)\right.$$

$$\left.\times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right)^{\frac{1}{r'}}$$

$$\leq \frac{\|\mathcal{U}'\|_p\|\mathcal{V}'\|_q}{\Gamma^2(\kappa)}\left(\int_0^\theta \int_0^\theta \big(\mathcal{G}(\theta)-\mathcal{G}(\vartheta)\big)^{\kappa-1}\big(\mathcal{G}(\theta)-\mathcal{G}(\zeta)\big)^{\kappa-1}\mathcal{G}'(\vartheta)\mathcal{G}'(\zeta)\hbar(\vartheta)\hbar'(\zeta)\right.$$

$$\left.\times |\vartheta-\zeta|^{\frac{1}{p'}+\frac{1}{q'}}\,d\vartheta\,d\zeta\right).$$

Remark 4.2 Setting $\omega(\theta) = 1$ and $\mathcal{G}(\theta) = \theta$ in Theorem 7, then we obtain the result of Dahmani [50]. Similarly, if we take $\kappa = \omega = 1$ and $\mathcal{G}(\theta) = \theta$ in Theorem 7, then we get the inequality (1.4) on $[0, \theta]$.

5 Concluding remarks

We presented some new weighted fractional integral inequalities for a class of differentiable functions connected with Chebyshev's, weighted Chebshev's and extended Chebyshev's functionals by utilizing weighted fractional integral operator recently introduced by Jarad et al. [45]. These inequalities are more general than the existing classical inequalities given in the literature. The special cases of our result can be found in [5, 15, 16, 48–50]. Also, one can easily obtain new fractional integral inequalities associated with Chebyshev"s, weighted Chebshev's and extended Chebyshev's functionals for another type of weighted and classical fractional integrals such as Katugampola, generalized Riemann–Liouville, classical Riemann–Liouville, generalized conformable and conformable fractional integrals with certain conditions on ω and $\mathcal{G}$ given in Remarks 2.2 and 2.3.

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