On time fractional pseudo-parabolic equations with nonlocal integral conditions

Article *in* Evolution Equations and Control Theory · January 2020

DOI: 10.3934/eect.2020109

EVOLUTION EQUATIONS AND
CONTROL THEORY
doi:10.3934/eect.2020109

ON TIME FRACTIONAL PSEUDO-PARABOLIC EQUATIONS WITH NONLOCAL INTEGRAL CONDITIONS

Nguyen Anh Tuan

Division of Applied Mathematics
Thu Dau Mot University, Binh Duong Province, Vietnam

Donal O'Regan

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland

Dumitru Baleanu

Department of Mathematics, Faculty of Arts and Sciences
Cankaya University, 06530 Ankara, Turkey
Institute of Space Sciences
P.O.Box, MG-23, R 76900, Magurele-Bucharest, Romania

Nguyen H. Tuan*

Division of Applied Mathematics
Thu Dau Mot University, Binh Duong Province, Vietnam

(Communicated by Juan J. Nieto)

Abstract. In this paper, we study the nonlocal problem for pseudo-parabolic equation with time and space fractional derivatives. The time derivative is of Caputo type and of order $\sigma,\ 0<\sigma<1$ and the space fractional derivative is of order $\alpha,\beta>0$. In the first part, we obtain some results of the existence and uniqueness of our problem with suitably chosen α,β. The technique uses a Sobolev embedding and is based on constructing a Mittag-Leffler operator. In the second part, we give the ill-posedness of our problem and give a regularized solution. An error estimate in L^p between the regularized solution and the sought solution is obtained.

1. **Introduction.** Fractional differential equations (FDEs) have been extensively studied during the past two decades by many researchers because of their diverse applications in physics, electrochemistry, viscoelasticity, etc. Time-fractional PDEs have been used as a major tool for modeling various practical fields and there are a number of publications devoted to the study of time-fractional PDEs and their applications (we refer the reader to [2, 28, 18, 20]).

2020 *Mathematics Subject Classification.* 26A33, 35B65, 35R11.
Key words and phrases. Fractional partial differential equation, Caputo fractional, well-posedness, pseudo-parabolic equation, nonlocal conditions, nonlocal in time.
* Corresponding author: Nguyen H. Tuan.

In this paper, for $\alpha, \beta \in (0,1)$, we consider the integral boundary problem for the fractional differential equation as follows

$$\begin{cases} \partial_t^\sigma (u + \kappa(-\Delta)^\alpha u)(x,t) + (-\Delta)^\beta u(x,t) = F(x,t), & \text{in } \Omega \times (0,T], \\ u = 0 & \text{in } \partial\Omega \times (0,T], \\ \rho_1 u(x,T) + \rho_2 \int_0^T u(x,t)dt = f(x), & \text{in } \Omega. \end{cases} \tag{P}$$

Here we consider a domain $\Omega \subset \mathbb{R}^N$ with the smooth boundary $\partial\Omega$ and the constants $\rho_1, \rho_2 \geq 0$ satisfying $\rho_1^2 + \rho_2^2 > 0$. Our problem is studied with the time fractional derivative of order $\sigma \in (0,1)$ in the sense of Caputo which is denoted by ∂_t^σ.

In the main equation of problem (P), if we take $\kappa = 0$, and $\sigma = \alpha = \beta = 1$, we have the usual parabolic equation, which has been investigated by many researchers; see [15],[23],[19] and the references therein. If $\sigma = \alpha = \beta = 1$, and $\kappa > 0$, our main equation becomes the pseudo-parabolic equation. This type of FDEs can be used to model many phenomena in many fields of science. In [8], Peter J. Chen and Morton E. Gurtin presented a theory about a non-simple material for which the conductive temperature and the thermodynamic temperature do not coincide. The nonstationary processes in semiconductors in the presence of sources can be analyzed by the equation (see [31])

$$\frac{d}{dt}u - \frac{d}{dt}\Delta u - \Delta u = u^q.$$

The unidirectional propagation of nonlinear, dispersive, long waves is also described by the classical pseudo-parabolic (see [4]). For more applications, we refer the reader to [9],[24].

Problems with the usual Cauchy conditions such as the initial condition $u(x,0) = \phi(x)$ or the condition at the terminal time $t = T$ are familiar. Usually, we can obtain well-posedness results for problems with initial conditions. In contrast, problems with Cauchy conditions at the terminal time are often ill-posed; we refer the reader to some recent results [17, 25] on the terminal value problem.

Our paper considers a non-local in time condition replacing the usual Cauchy conditions, that is

$$\rho_1 u(x,T) + \rho_2 \int_0^T u(x,t)dt = f(x). \tag{1}$$

In [12],[30], we can find two types of condition similar to the above such as

$$u(x,0) = \sum_{j=1}^m \alpha_j u(x,T_j) + \int_0^T v(\tau)u(x,\tau)d\tau + \varphi(x) \quad \text{or} \quad \int_0^T u(x,t)dt = \phi(x). \tag{2}$$

M. Beshtokov [7, 5, 6] considered boundary value problem for FPPDE. In practice, some phenomena will be simulated more effectively if we investigate the problems with the non-local condition. Indeed, in some models for meteorology, the time-averaged data help us to get a more reliable long-term weather forecast (see [3]). The problem with this type of condition can also be used when investigating radionuclides propagation in Stokes fluid, diffusion and flow in porous media ([13],[22],[26]). Compared with usual local initial/final value conditions, non-local conditions are more difficult to handle and motivated by this reason and their high application value, we work on time-fractional pseudo-parabolic equations with non-local final conditions, and our paper provides new results for the linear source term case (to the best of the authors' knowledge, it seems that a problem like (P) has not really been studied). Our paper will investigate problem (P) and the main results of this work are as follows:

- The regularity results for the mild solution.
- The proof for the instability of the solution to the initial data recovery problem.
- The regularization of the initial data recovery problem.

This paper is organized as follows. Section 2 gives some preliminaries that are needed throughout the paper. In section 3, we give the regularity result for the mild solution, and an example which shows that our solution is unstable in the case $t = 0$, and moreover, we give a regularized solution for the initial data recovery problem.

2. **Preliminary.** Before we introduce the main results of our works, some preliminary materials are given.

Definition 2.1. Let $\|\cdot\|_B$ be a norm on a Banach space B. Then, we define the following spaces

- For $1 \le q < \infty$

$$L^q(0,T;B) = \left\{ f:(0,T) \to B \middle| \|f\|_{L^q(0,T;B)} = \left(\int_0^T \|f(t)\|_B^q dt \right)^{\frac{1}{q}} < \infty \right\}.$$

- For $q = \infty$

$$L^\infty(0,T;B) = \left\{ f:(0,T) \to B \middle| \|f\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_B < \infty \right\}.$$

Remark 1. The spaces $L^q(0,T;B), L^\infty(0,T;B)$ with the corresponding norms $\|\cdot\|_{L^q(0,T;B)}$ and $\|\cdot\|_{L^\infty(0,T;B)}$, respectively, are Banach spaces.

Definition 2.2. Let $(B, \|\cdot\|_B)$ be a Banach space. Then we use the notation $C^k([0,T];B)$ to denote the Banach space of all $k \in \mathbb{N}$ time's derivative continuous functions eqquipped with the norm

$$\|f\|_{C^k([0,T];B)} = \sum_{j=1}^{n} \sup_{t\in[0,T]} \|f^j(t)\|_B.$$

Definition 2.3. (see [16]) The fractional Laplace operator of order $\alpha \in (0,1)$ is defined as a Fourier multiplier with symbol $-|\xi|^{2\alpha}$ given by

$$\mathscr{F}\left[(-\Delta)^\alpha u\right](\xi) = -|\xi|^{2\alpha}\mathscr{F}u(\xi), \tag{3}$$

and it is equivalent to

$$(-\Delta)^\alpha u = \mathscr{F}^{-1}\left(-|\xi|^{2\alpha}\mathscr{F}u\right), \tag{4}$$

where the notations $\mathscr{F}$ and $\mathscr{F}^{-1}$ stand for the Fourier transform and the inverse Fourier transform, respectively.

Next, let us recall the spectral problem for the fractional Laplace operator on the bounded domain Ω as follows

$$\begin{cases} (-\Delta)^\alpha \psi_j(x) = \lambda_j^\alpha \psi_j(x), & x \in \Omega, \forall j \in \mathbb{N}, \\ \psi_j(x) = 0, & x \in \partial\Omega, \forall j \in \mathbb{N}, \end{cases}$$

where the sequence of positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ satisfy

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \nearrow \infty,$$

whose corresponding set of real eigenfunctions $\{\psi_j(x)\}_{j\in\mathbb{N}}$ is orthogonal and complete.

Remark 2. If the inner product on $L^2(\Omega)$ is denoted by $(\cdot,\cdot)_{L^2(\Omega)}$, then the Fourier series of a function u in $L^2(\Omega)$ can be formulated as

$$u(x,t) = \sum_{j=1}^{\infty} (u(t),\psi_j)_{L^2(\Omega)}\, \psi_j(x).$$

Definition 2.4. For any $\eta > 0$, we define the fractional Hilbert scale space by

$$\mathcal{H}^\eta(\Omega) = \left\{ w \in L^2(\Omega) : \|w\|^2_{\mathcal{H}^\eta(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^{2\eta} (w,\psi_j)^2_{L^2(\Omega)} < \infty \right\}.$$

We denote the dual space $\mathcal{H}^\eta(\Omega)$ by $\mathcal{H}^{-\eta}(\Omega)$ provided that the dual space of $L^2(\Omega)$ is identified with itself. The space $\mathcal{H}^{-\eta}(\Omega)$ is a Hilbert space, equipped with the norm

$$\|f\|_{\mathcal{H}^{-\eta}} = \left(\sum_{j=1}^{\infty} \lambda_j^{-2\eta} (w,\psi_j)^2_* \right)^{\frac{1}{2}}, \tag{5}$$

for $w \in \mathcal{H}^{-\eta}$ where $(\cdot,\cdot)_*$ represents the dual product between $\mathcal{H}^\eta(\Omega)$ and $\mathcal{H}^{-\eta}(\Omega)$. For $1 \le p < \infty, m > 0$, if the Sobolev-Slobodecki space and the Lions–Magenes space are denoted by $W^{m,p}(\Omega), W_{00}^{1/2,2}(\Omega)$, respectively. Then we have

$$\mathcal{H}^m(\Omega) = \begin{cases} W^{m,p}(\Omega), & \text{if } m \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right), \\ W_{00}^{1/2,2}(\Omega), & \text{if } m = \frac{1}{2} \\ H_0^1(\Omega) \cap W^{m,2}(\Omega) & \text{if } m \in (1,2]. \end{cases}$$

Lemma 2.5. *(see[1],[10]) Let $\Omega \subset \mathbb{R}^N, 1 \le p < \infty$ such that $k \ge m \ge 0$ and $(k-m)p < N$. Then we have*

- $W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$, *for* $1 \le q < \dfrac{pN}{N-(k-m)p}$,
- $\mathcal{H}^\eta(\Omega) \hookrightarrow W^{\eta,2}$, *for* $\eta > 0$,
- $L^p(\Omega) \hookrightarrow \mathcal{H}^\eta(\Omega)$, *for* $-\dfrac{N}{2} < \eta \le 0, \quad p \ge \dfrac{2N}{N-2\eta}$,
- $\mathcal{H}^\eta(\Omega) \hookrightarrow L^p(\Omega)$, *for* $0 \le \eta < \dfrac{N}{2}, \quad p \le \dfrac{2N}{N-2\eta}$.

For more details about the definition of the fractional Sobolev spaces, we refer the reader to [10].

Definition 2.6. For $\alpha > 0$, and a arbitrary constant $\beta \in \mathbb{R}$, the Mittag-Leffler function can be defined by (see [14])

$$E_{\alpha,\beta}(z) = \sum_{j=1}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathbb{C}, \tag{6}$$

where Γ is the usual Gamma function.

Lemma 2.7. *(see [14])For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, there exist positive constants m_α and M_α, depending only on α such that*

(i) $E_{\alpha,1}(-z) > 0$, *for* $z > 0$.

(ii) $\dfrac{m_\alpha}{1+z} \le E_{\alpha,\beta}(-z) \le \dfrac{M_\alpha}{1+z}$, *for* $\beta \in \mathbb{R}, z > 0$.

3. **Main results.** Using the Laplace transform method, we can find the formula of the solution to the first equation of (P) as follows

$$
\begin{aligned}
u(x,t) &= \sum_{j=1}^{\infty} E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(u(0),\psi_j\right)_{L^2(\Omega)} \psi_j(x) \\
&\quad + \sum_{j=1}^{\infty} \left(\int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(t-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} ds \right) \psi_j(x). \qquad (7)
\end{aligned}
$$

To find the formula of the mild solution to problem (P), we need to find the representation of the initial data $u(x,0)$. Using our non-local final condition, we have

$$
\begin{aligned}
\left(f,\psi_j\right)_{L^2(\Omega)} &= \left(u(0),\psi_j\right)_{L^2(\Omega)} \left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} T^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) dt \right] \\
&\quad + \rho_1 \int_0^T \frac{(T-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(T-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} ds \qquad (8) \\
&\quad + \rho_2 \int_0^T \int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(t-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} dsdt.
\end{aligned}
$$

Therefore, the formula of the mild solution to the problem (P) can be given by

$$
\begin{aligned}
u(x,t) &= \sum_{j=1}^{\infty} \frac{E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(f,\psi_j\right)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} T^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) dt} \psi_j(x) \\
&- \sum_{j=1}^{\infty} \frac{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \int_0^T \frac{(T-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(T-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} ds}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} T^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) dt} \psi_j(x) \\
&- \sum_{j=1}^{\infty} \frac{\rho_2 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \int_0^T \int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(t-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} dsdt}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} T^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^{\beta} t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) dt} \psi_j(x) \\
&+ \sum_{j=1}^{\infty} \left(\int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}} E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(t-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right) \left(F(s),\psi_j\right)_{L^2(\Omega)} ds \right) \psi_j(x) \\
&:= \mathcal{Q}_1(x,t) + \mathcal{Q}_2(x,t) + \mathcal{Q}_3(x,t) + \mathcal{Q}_4(x,t). \qquad (9)
\end{aligned}
$$

Next we introduce the structure for this section.

Part 1: Regularity of the mild solution.

Part 2: The ill-posedness of the initial data recovery problem.

Part 3: Regularization and L^p error estimate for the initial data recovery problem.

3.1. **Regularity result.**

Lemma 3.1. *Let $0 < \alpha, \beta < 1$. Then we can find a constant $C_0 > 0$ such that*

(i) If $0 < \beta \le \dfrac{\alpha}{2}$, we have

$$\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt\right]^{-1} \le \frac{1}{C_0}. \tag{10}$$

(ii) If $\dfrac{\alpha}{2} < \beta < 1$, we have

$$\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt\right]^{-1} \le \frac{\lambda_j^{\beta-\frac{\alpha}{2}}}{C_0}. \tag{11}$$

Proof. First, we use the Cauchy inequality to get

$$\frac{\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha} \le \frac{t^\sigma}{2\sqrt{\kappa}}\lambda_j^{\beta-\frac{\alpha}{2}}. \tag{12}$$

Using Lemma 2.7, it follows that

$$E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) \ge \frac{m_\sigma}{1+\frac{\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}} \ge \frac{m_\sigma}{1+\frac{T^\sigma}{2\sqrt{\kappa}}\lambda_j^{\beta-\frac{\alpha}{2}}}. \tag{13}$$

From the above estimate, we find that

- If $0 < \beta \le \dfrac{\alpha}{2}$, we have

$$\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt \ge \frac{m_\sigma(\rho_1+\rho_2 T)}{1+\frac{T^\sigma}{2\sqrt{\kappa}}\lambda_1^{\beta-\frac{\alpha}{2}}}. \tag{14}$$

- If $\dfrac{\alpha}{2} < \beta < 1$, we have

$$\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt \ge \frac{m_\sigma(\rho_1+\rho_2 T)}{\lambda_j^{\beta-\frac{\alpha}{2}}\left(\frac{1}{\lambda_1^{\beta-\frac{\alpha}{2}}}+\frac{T^\sigma}{2\sqrt{\kappa}}\right)}. \tag{15}$$

From the above estimates, our lemma is proved. □

Theorem 3.2. *We assume that the constants $\sigma, \alpha, \beta, \theta, \eta, p, k, m$ satisfy*

$$1/2 < \sigma < 1, \qquad \alpha/2 < \beta < 1, \qquad p < 1/\alpha, \qquad 0 < \theta < \frac{2\sigma-1}{2\sigma}$$
$$0 < \eta < 2, \qquad k \le \eta, \qquad 1 \le m \le \frac{2N}{N+2k-2\eta}.$$

Then the mild solution u of the Problem (P) will belong to $L^p(0,T;W^{k,m}(\Omega))$ and the following holds

$$\|u\|_{L^p(0,T;W^{k,m}(\Omega))} \lesssim \|f\|_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)} + \|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} + \|F\|_{L^2(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))}, \tag{16}$$

where we use the notation $a \lesssim b$ if we can find a positive constant K such that $a \le Kb$.

Proof. By the triangle inequality, the following holds

$$\|u(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} \le \|\mathcal{Q}_1(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} + \|\mathcal{Q}_2(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} + \|\mathcal{Q}_3(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} + \|\mathcal{Q}_4(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)}. \tag{17}$$

Hence, we need to estimate the four terms on the right-hand side of the above to obtain the regularity results for our mild solution.

- **Estimate of the first term**. Parseval's identity gives us

$$\|\mathcal{Q}_1(\cdot,t)\|^2_{\mathcal{H}^\eta(\Omega)} = \sum_{j=1}^\infty \lambda_j^{2\eta} \left[\frac{E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)(f,\psi_j)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt} \right]^2. \tag{18}$$

Thanks to Lemma 3.1 and Lemma 2.7, we obtain

$$\begin{aligned}
&\lambda_j^{2\eta} \left[\frac{E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)(f,\psi_j)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt} \right]^2 \\
&\le t^{-2\sigma} \sum_{j=1}^\infty \frac{\lambda_j^{2\eta+2\beta-\alpha}}{C_0^2} \frac{\left(1+\kappa\lambda_j^\alpha\right)^2 M_\sigma^2}{\lambda_j^{2\beta}} (f,\psi_j)^2_{L^2(\Omega)} \\
&\lesssim t^{-2\sigma} \sum_{j=1}^\infty \left(\lambda_j^{2\eta-\alpha} + \kappa\lambda_j^{2\eta+\alpha}\right)(f,\psi_j)^2_{L^2(\Omega)} \\
&\lesssim t^{-2\sigma}\left(\|f\|^2_{\mathcal{H}^{\eta-\frac{\alpha}{2}}(\Omega)} + \|f\|^2_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)}\right).
\end{aligned} \tag{19}$$

The Sobolev embedding $\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega) \hookrightarrow \mathcal{H}^{\eta-\frac{\alpha}{2}}(\Omega)$ enables us to get

$$\|\mathcal{Q}_1(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} \le t^{-\sigma} \|f\|_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)}. \tag{20}$$

- **Estimate of the second term**.

$$\begin{aligned}
&\|\mathcal{Q}_2(\cdot,t)\|^2_{\mathcal{H}^\eta(\Omega)} \\
&\le \sum_{j=1}^\infty \lambda_j^{2\eta} \left[\frac{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) \int_0^T \frac{(T-s)^{\sigma-1}}{1+\kappa\lambda_j^\alpha} E_{\sigma,\sigma}\left(\frac{-\lambda_j^\beta (T-s)^\sigma}{1+\kappa\lambda_j^\alpha}\right)(F(s),\psi_j)_{L^2(\Omega)} ds}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt} \right]^2.
\end{aligned} \tag{21}$$

Using Lemma 2.7 we have

$$\begin{aligned}
&\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) \int_0^T \frac{(T-s)^{\sigma-1}}{1+\kappa\lambda_j^\alpha} E_{\sigma,\sigma}\left(\frac{-\lambda_j^\beta (T-s)^\sigma}{1+\kappa\lambda_j^\alpha}\right)(F(s),\psi_j)_{L^2(\Omega)} ds\right]^2 \\
&\le \rho_1^2 \frac{M_\sigma^4}{\left[1+\frac{\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right]^2} \left[\int_0^T \frac{(T-s)^{\sigma-1}}{1+\kappa\lambda_j^\alpha + \lambda_j^\beta (T-s)^\sigma}(F(s),\psi_j)_{L^2(\Omega)} ds\right]^2
\end{aligned} \tag{22}$$

$$\lesssim \frac{\left(1+\kappa\lambda_j^{\alpha}\right)^2}{t^{2\sigma}\lambda_j^{2\beta+2\beta\theta}}\left(\int_0^T (T-s)^{\sigma-\sigma\theta-1}\left(F(s),\psi_j\right)_{L^2(\Omega)} ds\right)^2.$$

Now, the Hölder inequality will be applied to get the following estimate

$$\begin{aligned}
&\left(\int_0^T (T-s)^{\sigma-\sigma\theta-1}\left(G(s),\psi_j\right)_{L^2(\Omega)} ds\right)^2 \\
&\leq \left(\int_0^T (T-s)^{\sigma-\sigma\theta-1} ds\right)\left(\int_0^T (T-s)^{\sigma-\sigma\theta-1}\left(G(s),\psi_j\right)^2_{L^2(\Omega)} ds\right) \qquad (23)\\
&\lesssim \frac{T^{\sigma-\sigma\theta}}{\sigma-\sigma\theta}\left(\int_0^T (T-s)^{\sigma-\sigma\theta-1}\left(G(s),\psi_j\right)^2_{L^2(\Omega)} ds\right).
\end{aligned}$$

Combining these and noting that $\lambda_j^{2\eta-2\beta\theta-\alpha} \lesssim \lambda_j^{2\eta-2\beta\theta+\alpha}, \forall j\in\mathbb{N}$, we deduce that

$$\begin{aligned}
\|\mathcal{Q}_2(\cdot,t)\|^2_{\mathcal{H}^\eta(\Omega)} &\lesssim t^{-2\sigma}\sum_{j=1}^{\infty}\lambda_j^{2\eta-2\beta\theta+\alpha}\left(\int_0^T (T-s)^{\sigma-\sigma\theta-1}\left(F(s),\psi_j\right)^2_{L^2(\Omega)} ds\right)\\
&= t^{-2\sigma}\int_0^T (T-s)^{\sigma-\sigma\theta-1}\|F(s)\|^2_{\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega)} ds \qquad (24)\\
&\lesssim t^{-2\sigma}\|F\|^2_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))}.
\end{aligned}$$

• **Estimate of the third term**.

In the same way as in the previous step, we obtain

$$\begin{aligned}
&\left(\rho_2 E_{\sigma,1}\left(\frac{-\lambda_j^{\beta}t^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right)\int_0^T\int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^{\alpha}}E_{\sigma,\sigma}\left(\frac{-\lambda_j^{\beta}(t-s)^{\sigma}}{1+\kappa\lambda_j^{\alpha}}\right)\left(F(s),\psi_j\right)_{L^2(\Omega)} dsdt\right)^2\\
&\lesssim \frac{\left(1+\kappa\lambda_j^{\alpha}\right)^2}{t^{2\sigma}\lambda_j^{2\beta+2\beta\theta}}\left(\int_0^T\int_0^t (t-s)^{\sigma-\sigma\theta-1}\left(F(s),\psi_j\right)_{L^2(\Omega)} dsdt\right)^2 \qquad (25)\\
&\lesssim \frac{\left(1+\kappa\lambda_j^{\alpha}\right)^2}{t^{2\sigma}\lambda_j^{2\beta+2\beta\theta}}\left(\int_0^T \left(F(s),\psi_j\right)^2_{L^2(\Omega)} ds\right).
\end{aligned}$$

Note that, we get the estimate above by using the Hölder inequality as follows

$$\begin{aligned}
&\left(\int_0^T\int_0^t (t-s)^{\sigma-\sigma\theta-1}\left(F(s),\psi_j\right)_{L^2(\Omega)} dsdt\right)^2\\
&\leq \left[\int_0^T\left(\int_0^t (t-s)^{2\sigma-2\sigma\theta-2} ds\right)^{\frac{1}{2}}\left(\int_0^t \left(F(s),\psi_j\right)^2_{L^2(\Omega)} ds\right)^{\frac{1}{2}} dt\right]^2 \qquad (26)\\
&\leq \left(\int_0^T\left[\frac{T^{2\sigma-2\sigma\theta-1}}{2\sigma-2\sigma\theta-1}\right]^{\frac{1}{2}} dt\right)^2\left(\int_0^T \left(F(s),\psi_j\right)^2_{L^2(\Omega)} ds\right).
\end{aligned}$$

We thus have

$$\begin{aligned}
\|\mathcal{Q}_3(\cdot,t)\|^2_{\mathcal{H}^\eta(\Omega)} &\lesssim t^{-2\sigma}\sum_{j=1}^{\infty}\lambda_j^{2\eta-2\beta\theta+\alpha}\left(\int_0^T \left(F(s),\psi_j\right)^2_{L^2(\Omega)} ds\right)\\
&= t^{-2\sigma}\|F\|^2_{L^2(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))}. \qquad (27)
\end{aligned}$$

• **Estimate of the fourth term**. This term can be treated more simply than the above two terms. Indeed, thanks to Lemma 2.7 and the Hölder inequality, we have

$$\begin{aligned}\|\mathcal{Q}_4(\cdot,t)\|^2_{\mathcal{H}^\eta(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2\eta} \left(\int_0^t \frac{(t-s)^{\sigma-1}}{1+\kappa\lambda_j^\alpha} E_{\sigma,\sigma}\left(\frac{-\lambda_j^\beta (t-s)^\sigma}{1+\kappa\lambda_j^\alpha} \right) (F(s),\psi_j)_{L^2(\Omega)}\, ds \right)^2 \\ &\lesssim \sum_{j=1}^{\infty} \lambda_j^{2\eta-\beta\theta} \left(\int_0^t (t-s)^{\sigma-\sigma\theta-1} (F(s),\psi_j)_{L^2(\Omega)}\, ds \right)^2 \\ &\lesssim \|F\|^2_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta}(\Omega))}. \end{aligned} \tag{28}$$

Combining these estimates with (17), we have

$$\begin{aligned}\|u(\cdot,t)\|_{\mathcal{H}^\eta(\Omega)} \lesssim\ & t^{-\sigma}\|f\|_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)} + t^{-\sigma}\|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} \\ &+ t^{-\sigma}\|F\|_{L^2(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} + \|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta}(\Omega))}. \end{aligned} \tag{29}$$

We apply the following Sobolev embeddings

$$\begin{cases} \mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega) \hookrightarrow \mathcal{H}^{\eta-\beta\theta}(\Omega), \\ \mathcal{H}^\eta(\Omega) \hookrightarrow W^{\eta,2}(\Omega), \\ W^{\eta,2}(\Omega) \hookrightarrow W^{k,m}(\Omega), \end{cases}$$

to obtain the important estimate

$$\begin{aligned}\|u\|_{L^p(0,T;W^{k,m}(\Omega))} &= \left(\int_0^T \|u(s)\|^p_{W^{k,m}(\Omega)}\, ds \right)^{\frac{1}{p}} \\ &\lesssim \left[\|f\|_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)} + \|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} + \|F\|_{L^2(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} \right] \left(\int_0^T t^{-\sigma p} dt \right)^{\frac{1}{p}} \\ &+ \|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} \left(\int_0^T dt \right)^{\frac{1}{p}}. \end{aligned} \tag{30}$$

Let us note that the integral $\left(\int_0^T t^{-\sigma p} dt \right)^{\frac{1}{p}}$ is convergent for $p < \frac{1}{\sigma}$. Hence, we can assert that

$$\begin{aligned}\|u\|_{L^p(0,T;W^{k,m}(\Omega))} \lesssim\ & \|f\|_{\mathcal{H}^{\eta+\frac{\alpha}{2}}(\Omega)} + \|F\|_{L^\infty(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))} \\ &+ \|F\|_{L^2(0,T,\mathcal{H}^{\eta-\beta\theta+\frac{\alpha}{2}}(\Omega))}. \end{aligned} \tag{31}$$

This show that u belongs to $L^p(0,T;W^{k,m}(\Omega))$ and the proof is complete. □

3.2. **The ill-posedness of the initial data recovery problem.** From now on, we will only consider our problem in the homogeneous case i.e. when $F = 0$. Furthermore, we also assume that $\beta > \alpha$ throughout the rest of the paper.

Lemma 3.3. *Let $\sigma \in (0,1)$ and $0 < \alpha < \beta < 1$. Then, we get the following estimate*

$$\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha} \right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha} \right) dt \lesssim \frac{1}{\lambda_j^\beta} + \frac{1}{\lambda_j^{\beta-\alpha}}. \tag{32}$$

Proof. Lemma 2.7 gives us

$$\begin{aligned} &\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt \\ &\le M_\sigma\left[\frac{\rho_1}{1+\frac{\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}}+\int_0^T\frac{\rho_2}{1+\frac{\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}}dt\right] \\ &\lesssim \frac{1+\kappa\lambda_j^\alpha}{\lambda_j^\beta T^\sigma}+\int_0^T\frac{1+\kappa\lambda_j^\alpha}{\lambda_j^\beta t^\sigma}dt\lesssim\frac{1}{\lambda_j^\beta}+\frac{1}{\lambda_j^{\beta-\alpha}}. \end{aligned} \tag{33}$$

In the latter inequality, we note that the integral $\int_0^T t^{-\sigma}dt$ is convergent for $\sigma\in(0,1)$. □

Theorem 3.4. *If $t=0$, the solution of the problem (P) is unstable in the sense of the $L^2(\Omega)$ norm.*

Proof. We begin the proof by setting $u(x,0)=u_0(x)$ and defining a mapping $\mathscr{T}$ from $L^2(\Omega)$ to $L^2(\Omega)$ as follows

$$\mathscr{T}u_0(x)=\int_\Omega\varphi(x,z)u_0(z)dz, \tag{34}$$

where

$$\varphi(x,z)=\sum_{j=1}^\infty\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt\right]\psi_j(x)\psi_j(z). \tag{35}$$

It's a simple matter to check that $\varphi(x,z)=\varphi(z,x)$, and $\mathscr{T}$ is a self-adjoint operator. Let us consider the following finite rank operator

$$\begin{aligned} &\mathscr{T}_M u_0(x) \\ &=\sum_{j=1}^M\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt\right](u_0,\psi_j)_{L^2(\Omega)}\psi_j(x). \end{aligned} \tag{36}$$

From Lemma 3.3, it is clear that

$$\begin{aligned} &\|\mathscr{T}u_0(x)-\mathscr{T}_M u_0(x)\|^2_{L^2(\Omega)} \\ &=\sum_{j=M+1}^\infty\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt\right]^2(u_0,\psi_j)^2_{L^2(\Omega)} \\ &\lesssim\left(\frac{1}{\lambda_M^\beta}+\frac{1}{\lambda_M^{\beta-\alpha}}\right)^2\sum_{j=M+1}^\infty(u_0,\psi_j)^2_{L^2(\Omega)}\le\left(\frac{1}{\lambda_M^\beta}+\frac{1}{\lambda_M^{\beta-\alpha}}\right)^2\|u_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{37}$$

It follows immediately that $\|\mathscr{T}-\mathscr{T}_M\|_{\mathscr{L}(L^2(\Omega);L^2(\Omega))}\longrightarrow 0$ as $M\to\infty$. We also can prove that $\mathscr{T}$ is compact. Moreover, we have

$$\mathscr{T}u_0(x)=f(x), \tag{38}$$

and then, we can conclude that our problem is ill-posed. Let us give an example to illustrate the ill-posedness of the problem. Taking the input data $f_k(x)=$

$\sqrt{\frac{1}{\lambda_k^\beta}+\frac{1}{\lambda_k^{\beta-\alpha}}}\psi_j(x)$, we can see at once that

$$\|f_k\|_{L^2(\Omega)} = \sqrt{\frac{1}{\lambda_k^\beta}+\frac{1}{\lambda_k^{\beta-\alpha}}} \xrightarrow{k\to\infty} 0. \tag{39}$$

However, the initial data $u_{0,k}$ corresponding to the final data f_k, is given by

$$\begin{aligned} u_{0,k}(x) &= \sum_{j=1}^{\infty} \frac{(f_k,\psi_j)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt}\psi_j(x) \\ &= \sqrt{\frac{1}{\lambda_k^\beta}+\frac{1}{\lambda_k^{\beta-\alpha}}}\frac{1}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_k^\beta T^\sigma}{1+\kappa\lambda_k^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_k^\beta t^\sigma}{1+\kappa\lambda_k^\alpha}\right)dt}\psi_k(x), \end{aligned} \tag{40}$$

and it follows that

$$\begin{aligned} \|u_{0,k}\|_{L^2(\Omega)} &= \left\|\sqrt{\frac{1}{\lambda_k^\beta}+\frac{1}{\lambda_k^{\beta-\alpha}}}\frac{\psi_k(\cdot)}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_k^\beta T^\sigma}{1+\kappa\lambda_k^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_k^\beta t^\sigma}{1+\kappa\lambda_k^\alpha}\right)dt}\right\|_{L^2(\Omega)} \\ &\gtrsim \sqrt{\frac{\lambda_k^{2\beta-\alpha}}{\lambda_k^\beta+\lambda_k^{\beta-\alpha}}}, \end{aligned} \tag{41}$$

The above estimate helps us to get the limit below

$$\|u_{0,k}\|_{L^2(\Omega)} \xrightarrow{k\to\infty} \infty. \tag{42}$$

From (39) and (42), we deduce that the solution to problem (P) is unstable in $L^2(\Omega)$. □

3.3. Regularization and $\mathbf{L^p}$ error estimate.

Theorem 3.5. *Let f_ε be noisy data satisfying $\|f_\varepsilon - f\|_{L^p(\Omega)} \le \varepsilon$, for $p \ge 1$, and $u_0 \in \mathcal{H}^\nu(\Omega)$ for $\nu > 0$. Then, we can find a regularized solution $u_{0,\varepsilon}$ such that*

$$\|u_{0,\varepsilon}-u_0\|_{L^{\frac{2N}{d-4\gamma}}(\Omega)} \lesssim C(p,\eta)\varepsilon^\theta + \varepsilon^{\frac{(\theta-1)(\gamma-\nu)}{\gamma+\beta}}\|u_0\|_{\mathcal{H}^\nu(\Omega)}, \quad (0<\theta<1), \tag{43}$$

where

$$\frac{-N}{2} < \eta \le \min\left\{0, \frac{(p-2)N}{2p}\right\}, \text{ and } 0 \le \gamma < \min\left\{\frac{N}{2},\nu\right\} \tag{44}$$

Proof. First, for $\theta \in (0,1)$, we set $M_\varepsilon = \varepsilon^{\frac{\theta-1}{\gamma+\beta}}$. Then, we choose the following regularized solution

$$u_{0,\varepsilon}(x) = \sum_{j=1}^{\lambda_j\le M_\varepsilon} \frac{(f_\varepsilon,\psi_j)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt}\psi_j(x), \tag{45}$$

and define the supporting series as follows

$$u_{0,\varepsilon}^*(x) = \sum_{j=1}^{\lambda_j\le M_\varepsilon} \frac{(f,\psi_j)_{L^2(\Omega)}}{\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right)+\rho_2\int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right)dt}\psi_j(x). \tag{46}$$

For the purpose of obtaining an error estimate between $u_{0,\varepsilon}$ and u_0, we need two important estimates below.

- For any $\gamma < \min\left\{\frac{N}{2}, \nu\right\}$, thanks to Lemma 3.1, we have

$$\begin{aligned}\left\|u_{0,\varepsilon} - u^*_{0,\varepsilon}\right\|^2_{\mathcal{H}^\gamma(\Omega)} &= \sum_{j=1}^{\lambda_j \le M_\varepsilon} \frac{\lambda_j^{2\gamma}\left(f_\varepsilon - f, \psi_j\right)^2_{L^2(\Omega)}}{\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt\right]^2} \\ &\lesssim \sum_{j=1}^{\lambda_j \le M_\varepsilon} \lambda_j^{2\gamma+2\beta-2\eta-\alpha}\lambda_j^{2\eta}\left(f_\varepsilon - f, \psi_j\right)^2_{L^2(\Omega)} \\ &\lesssim M_\varepsilon^{2\gamma+2\beta}\left\|f_\varepsilon - f\right\|^2_{\mathcal{H}^\eta(\Omega)}.\end{aligned} \tag{47}$$

By applying the Sobolev embedding $L^p(\Omega) \hookrightarrow \mathcal{H}^\eta(\Omega)$, we get

$$\left\|u_{0,\varepsilon} - u^*_{0,\varepsilon}\right\|_{\mathcal{H}^\gamma(\Omega)} \lesssim C(p,\eta)M_\varepsilon^{\gamma+\beta}\left\|f_\varepsilon - f\right\|_{L^p(\Omega)} \lesssim C(p,\eta)M_\varepsilon^{\gamma+\beta}\varepsilon. \tag{48}$$

- From the formula of u_0, we deduce that

$$\begin{aligned}\left\|u^*_{0,\varepsilon} - u_0\right\|^2_{\mathcal{H}^\gamma(\Omega)} &= \sum_{\lambda_j > M_\varepsilon} \frac{\lambda_j^{2\gamma}\left(f, \psi_j\right)^2_{L^2(\Omega)}}{\left[\rho_1 E_{\sigma,1}\left(\frac{-\lambda_j^\beta T^\sigma}{1+\kappa\lambda_j^\alpha}\right) + \rho_2 \int_0^T E_{\sigma,1}\left(\frac{-\lambda_j^\beta t^\sigma}{1+\kappa\lambda_j^\alpha}\right) dt\right]^2} \\ &= \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma}\left(u_0, \psi_j\right)^2_{L^2(\Omega)} = \sum_{\lambda_j > M_\varepsilon} \lambda_j^{2\gamma-2\nu}\lambda_j^{2\nu}\left(u_0, \psi_j\right)^2_{L^2(\Omega)} \\ &\le M_\varepsilon^{2\gamma-2\nu}\left\|u_0\right\|^2_{\mathcal{H}^\nu(\Omega)}.\end{aligned} \tag{49}$$

Now, by combining these estimates with the triangle ineqality and using the Sobolev embedding $\mathcal{H}^\gamma(\Omega) \hookrightarrow L^{\frac{2N}{N-2\gamma}}(\Omega)$, we can assert that

$$\begin{aligned}\left\|u_{0,\varepsilon} - u_0\right\|_{L^{\frac{2N}{N-2\gamma}}(\Omega)} &\lesssim C(p,\eta)M_\varepsilon^{\gamma+\beta}\varepsilon + M_\varepsilon^{\gamma-\nu}\left\|u_0\right\|_{\mathcal{H}^\nu(\Omega)} \\ &= C(p,\eta)\varepsilon^\theta + \varepsilon^{\frac{(\theta-1)(\gamma-\nu)}{\gamma+\beta}}\left\|u_0\right\|_{\mathcal{H}^\nu(\Omega)}.\end{aligned} \tag{50}$$

It is easily seen that when $\varepsilon \to 0$, we have

$$\begin{cases}\left\|u_{0,\varepsilon} - u_0\right\|_{L^{\frac{2N}{N-2\gamma}}(\Omega)} \longrightarrow 0, \\ M_\varepsilon \longrightarrow \infty,\end{cases} \tag{51}$$

and this finishes our proof. □

4. **Conclusion.**

This paper investigates time fractional pseudo-parabolic equations with nonlocal integral conditions. The results of this work are divided into two main parts:

Part I: The Regularity result of the mild solution for Problem (P) is given.

Part II: We show that the initial data recovery problem is ill-posed. We also give the regularized solution and estimate error in L^p between the regularized solution and the sought solution.

This paper only considers the linear case, and in the future we hope to expand Problem (P) to the case of nonlinear source terms.

Acknowledgments. The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improve the quality of this paper.

REFERENCES

[1] J. M. Arrieta and A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Amer. Math. Soc*, **352** (2000), 285–310.

[2] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm Sci*, **20** (2016), 763–769.

[3] P. N. Belov, *The Numerical Methods of Weather Forecasting*, Gidrometeoizdat, Leningrad, 1975.

[4] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc. London Ser.A*, **272** (1972), 47–78.

[5] M. K. Beshtokov, Boundary-value problems for loaded pseudoparabolic equations of fractional order and difference methods of their solving, *Russian Mathematics*, **63** (2019), 1–10.

[6] M. K. Beshtokov, Boundary value problems for a pseudoparabolic equation with the Caputo fractional derivative, *Differ. Equ*, **55** (2019), 884–893.

[7] M. K. Beshtokov, Toward boundary-value problems for degenerating pseudoparabolic equations with Gerasimov-Caputo fractional derivative, *Izv. Vyssh. Uchebn. Zaved. Mat*, **62** (2018), 3–16.

[8] P. J. Chen and M. E. Gurtin, On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Phys*, **19** (1968), 614–627.

[9] H. Chen and S. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity, *J. Differential Equations*, **258** (2015), 4424–4442.

[10] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math*, **1360** (2012), 521–573.

[11] J.-D. Djida, A. Fernandez and I. Area, Well-posedness results for fractional semi-linear wave equations, *Discrete Contin. Dyn. Syst. Ser B*, **25** (2020), 569–597.

[12] N. Dokuchaev, On recovering parabolic diffusions from their time-averages, *Calc. Var. Partial Differential Equations*, **58** (2019), 14 pp.

[13] R. E. Ewing, R. D. Lazarov and Y. Lin, Finite volume element approximations of nonlocal in time one-dimensional flows in porous media, *Computing*, **64** (2000), 157–182.

[14] R. Gorenflo, A. A. Kilbas and F. Mainardi, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, Berlin, (2014).

[15] R. Ikehata and T. Suzuki, Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J*, **26** (1996), 475–491.

[16] M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, *Fract. Calc. Appl. Anal*, **20** (2017), 7–51.

[17] B. Kaltenbacher and W. Rundell, Regularization of a backward parabolic equation by fractional operators, *Inverse Probl. Imaging*, **13** (2019), 401–430.

[18] N. H. Luc, L. N. Huynh, D. Baleanu and N. H. Can, Identifying the space source term problem for a generalization of the fractional diffusion equation with hyper-Bessel operator, *Adv. Difference Equ*, **2020** (2020), 23 pp.

[19] Y. Liu, R. Xu and T. Yu, Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations, *Nonlinear Anal*, **68** (2008), 3332–3348.

[20] T. B. Ngoc, D. Baleanu, L. T. M. Duc and N. H. Tuan, Regularity results for fractional diffusion equations involving fractional derivative with Mittag-Leffler kernel, *Math. Methods Appl. Sci*, **43** (2020), 7208–7226.

[21] E. Otárola and A. J. Salgado, Regularity of solutions to space-time fractional wave equations: A PDE approach, *Fract. Calc. Appl. Anal*, **21** (2018), 1262–1293.

[22] C. V. Pao, Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions, *J. Math. Anal. Appl.*, **195** (1995), 702–718.

[23] Q. Pavol and P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.

[24] V. Padron, Effect of aggregation on population recovery modeled by a forward-backward pseudo parabolic equation, *Trans. Amer. Math. Soc*, **356** (2004), 2739–2756.

[25] K. Sakamoto and M. Yamamoto, Initial value/boudary value problems for fractional diffusion - wave equations and applications to some inverse problems, *J. Math. Anal. Appl*, **382** (2011), 426–447.

[26] V. V. Shelukhin, A non-local (in time) model for radionuclides propagation in a Stokes fluid, *Dinamika Sploshn. Sredy*, **107** (1993), 180–193.

[27] R. E. Showalter and T. W. Ting, Pseudoparabolic partial differential equations, *SIAM J. Math. Anal*, **1** (1970), 1–26.

[28] N. H. Tuan, D. Baleanu, T. N. Thach, D. O'Regan and N. H. Can, Final value problem for nonlinear time fractional reaction-diffusion equation with discrete data, *J. Comput. Appl. Math*, **376** (2020), 25 pp.

[29] N. H. Tuan, T. B. Ngoc, Y. Zhou and D. O'Regan, On existence and regularity of a terminal value problem for the time fractional diffusion equation, *Inverse Problems*, **36**, (2020), 41 pp.

[30] J. M. Vaquero and S. Sajavicius, The two-level finite difference schemes for the heat equation with nonlocal initial condition, *Appl. Math. Comput.*, **342** (2019), 166–177.

[31] R. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Functional Analysis*, **264** (2013), 2732–2763.

Received July 2020; revised October 2020.

E-mail address: nguyenanhtuan@tdmu.edu.vn
E-mail address: donal.oregan@nuigalway.ie
E-mail address: dumitru@cankaya.edu.tr
E-mail address: nguyenhuytuan@tdmu.edu.vn