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QUALITATIVE STUDY OF NONLINEAR COUPLED PANTOGRAPH DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract

In this paper, we investigate a nonlinear coupled system of fractional pantograph differential equations (FPDEs). The respective results address some adequate results for existence and uniqueness of solution to the problem under consideration. In light of fixed point theorems like Banach and Krasnoselskii's, we establish the required results. Considering the tools of nonlinear analysis, we develop some results regarding Ulam–Hyers (UH) stability. We give three pertinent examples to demonstrate our main work.

Keywords: FPDEs; Qualitative Study; Banach and Krasnoselskii's Fixed Point Theorems; UH Stability Theory; Coupled System.

1. INTRODUCTION

The fractional calculus is a type of generalization of the classical calculus. With the development and advancement in nano-technology, this branch of mathematics has attracted the researchers. The fractional calculus is widely used in the process of modeling problems related to engineering and scientific fields, like chemistry, biology, image processing, physics, control theory, economics and signal processing (see Refs. 1–12). Keeping in mind the above importance, researchers have considered fractional order differential equations (FODEs) from various facets, including optimization, stability theory, qualitative theory and numerical simulations. For these studies, they have used classical fixed point theory, tools of nonlinear analysis and optimization theory (we refer Refs. 13–18). The mentioned area has many applications in science, like probability theory, physics, control theory, light absorption model and cell division, we refer few papers such as Refs. 19 and 20.

Another important class, known as pantograph differential equations (PDEs), has not been properly investigated under fractional derivatives. Pantograph is basically a device used for drawing and scaling. However, these days, this device is used in electric trains.[21,22] PDEs constitute a subclass of delay-type differential equations which provide changes in the dependent value at a previous time.[23]

Stability analysis is another important aspect in fractional calculus that has received proper attention in the sense of Ulam–Hyers (UH), see for details Refs. 24 and 25. Further, UH stability was modified to more general types by other researchers. Due to stability theory, very useful results were established in the fractional calculus (we refer to Ref. 26 and the references cited there). The UH stability has been studied for both ordinary differential equations and FODEs in the last two decades (see Refs. 27–32).

Keeping in mind the aforementioned literature, we claim that qualitative analysis for coupled system of fractional pantograph differential equations (FPDEs) using fixed point theory with nonlocal conditions has been investigated rarely as far as our knowledge is concerned. Therefore, we are going to carry out a qualitative analysis to the following system of FPDEs under nonlocal conditions as

$$\begin{cases} {}^cD^a_{+0}q(t) = f_1(t, q(t), q(\lambda t), r(t), s(t)),\\ {}^cD^b_{+0}r(t) = f_2(t, q(t), r(t), r(\lambda t), s(t)),\\ {}^cD^\gamma_{+0}s(t) = f_3(t, q(t), r(t), s(t), s(\lambda t)),\\ q(0) = f(q) + q_0, r(0) = g(r) + r_0,\\ s(0) = h(s) + s_0,\ q_0, r_0, s_0 \in R, \end{cases} \tag{1}$$

where $t \in I_1 = [0, v]$ and $a, b, \gamma \in (0, 1],\ \lambda \in (0, 1) = I_2$. Also, the functions $f_1, f_2, f_3 : I_1 \times R^4 \to R$ are nonlinear continuous, $f, g, h : X \to R$ are continuous functions and D is Caputo's derivative. Through Banach and Krasnoselskii's fixed point theorems, the results that we aim to establish are investigated. In the end, some examples are given for the justification of our work. Further, we remark that the system under consideration (1) includes the following three species prey–predator delay model which is a special case under nonlocal conditions

$$\begin{cases} {}^cD^a_{+0}q(t) = \Lambda q(t)\left(1 - \dfrac{q(t)}{\kappa}\right) - \dfrac{\beta q(t)r(t)}{\alpha + q(t)}\\ \qquad\qquad - \mu q(\lambda t)s(t),\\ {}^cD^b_{+0}r(t) = \dfrac{\sigma\beta r(t)}{\alpha + q(t)} - \delta r(t)s(t) - \nu_1 s(\lambda t),\\ {}^cD^\gamma_{+0}s(t) = -\nu_2 s(t) + \delta r(\lambda t)s(t) + \delta\sigma q(t)s(t),\\ \qquad q(0) = q_0 + f(q), r(0) = r_0 + g(r),\\ \qquad s(0) = s_0 + h(s),\ q_0, r_0, s_0 \in R, \end{cases} \tag{2}$$

where q denotes susceptible, r infected and s is population density in the concerned model, while δ is the disease transmission coefficient, κ the carrying capacities of prey population, ν_2 is the death rate of infected predator. Also, Λ is the growth rate of prey population, μ_1 is the death rate of susceptible predator, β is the search rate of the prey toward susceptible predator, α is saturation constant while susceptible predators attack the prey, σ is the conversion rate of susceptible predator due to prey. Further, $t \in I_1,\ ab, \gamma \in (0, 1]$.

2. BASIC DEFINITIONS AND NOTIONS

Some useful definitions and results for the related work which may be found in Refs. 2–6 are presented.

Definition 1. Let $\gamma \in R^+$, then integral of fractional order to a function $x : (0,\infty) \to R$ is defined as

$$I^{\gamma}_{0+}x(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-y)^{\gamma-1}x(y)\,dy, \quad \gamma > 0.$$

Definition 2. Caputo-type fractional derivative to a function $x : (0,\infty) \to R$ is recalled as

$${}^{c}D^{\gamma}_{+0}x(t) = \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-y)^{m-\gamma-1}x^{(m)}(y)\,dy,$$

where $m = [\gamma] + 1$ and the integral is pointwise defined on $(0,\infty)$.

Lemma 1. *The solution of fractional differential equation (FDE)*

$${}^{c}D^{\gamma}_{+0}x(t) = 0,\ \gamma \in (m-1, m]$$

is expressed as

$$x(t) = \sum_{\eta=1}^{m} d_\eta t^{\eta-1},\ \eta = 1, 2, \ldots, m,\ \text{where } d_\eta \in R.$$

where $m = [\gamma] + 1$.

Lemma 2. *For an FDE, the following result holds*:

$$I^{\gamma}_{+0}[{}^{c}D^{\gamma}_{+0}x(t)] = x(t) + \sum_{\eta=1}^{m} d_\eta t^{m-1},\ \eta = 1, 2, 3, \ldots, m$$

with $d_\eta \in R$ *and* $m = [\gamma] + 1$.

Suppose we denote the Banach space as $(X, \|, \|)$ with norm $\|x\| = \max_{t\in I_1}|x(t)|$. As a result, $Y = X \times X \times X$ is a Banach space with a norm defined by $\|(q, r, s)\| = \|q\| + \|r\| + \|s\|$ and $\|(q, r, s)\| = \max\{\|q\|, \|r\|, \|s\|\}$. Both norms are equivalent.

Definition 3 (Ref. 33). Let for two operators $S_1, S_2, S_3 \ni S_1, S_2, S_3 : Y \to X$, defined as

$$\begin{cases} q(t) = S_1(q, r, s)(t), \\ r(t) = S_2(q, r, s)(t), \\ s(t) = S_3(q, r, s)(t) \end{cases} \tag{3}$$

is called UH if for positive numbers $\ell_i\,(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$, $\Delta_i\,(i = 1, 2, 3)$ and for every solution $(q^*, r^*, s^*) \in Y$, we have

$$\begin{cases} \|q^* - S_1(q^*, r^*, s^*)\| \le \Delta_1, \\ \|r^* - S_2(q^*, r^*, s^*)\| \le \Delta_2, \\ \|s^* - S_3(q^*, r^*, s^*)\| \le \Delta_3 \end{cases} \tag{4}$$

$\exists$ a solution $(\bar{q}, \bar{r}, \bar{s}) \in Y$ of (3), such that

$$\begin{cases} \|q^* - \bar{q}\| \le \ell_1\Delta_1 + \ell_2\Delta_2 + \ell_3\Delta_3, \\ \|r^* - \bar{r}\| \le \ell_4\Delta_1 + \ell_5\Delta_2 + \ell_6\Delta_3, \\ \|s^* - \bar{s}\| \le \ell_7\Delta_1 + \ell_8\Delta_2 + \ell_9\Delta_3. \end{cases} \tag{5}$$

Definition 4. If for $\beta_i\,(i = 1, 2, 3, \ldots, n)$ are "eigenvalues" of $\mathscr{M}$ of order $n \times n$, with spectral radius of $\Upsilon(\mathscr{M})$ may be defined by

$$\Upsilon(\mathscr{M}) = \max\{|\beta_i|, \text{for } i = 1, 2, \ldots, n\}.$$

Moreover, if $\Upsilon(\mathscr{M}) < 1$ yields that $\mathscr{M}$ converges to 0.

Theorem 1 (Ref. 33). *For the two operators* S_1, S_2, $S_3 \ni S_1, S_2, S_3 : Y \to X \ni$

$$\begin{cases} \|S_1(q, r, s) - S_1(q^*, r^*, s^*)\| \le \ell_1\|q - q^*\| \\ \qquad +\ell_2\|r - r^*\| + \ell_3\|s - s^*\|, \\ \|S_2(q, r, s) - S_2(q^*, r^*, s^*)\| \le \ell_4\|q - q^*\| \\ \qquad +\ell_5\|r - r^*\| + \ell_6\|s - s^*\|, \\ \|S_3(q, r, s) - S_3(q^*, r^*, s^*)\| \le \ell_7\|q - q^*\| \\ \qquad +\ell_8\|r - r^*\| + \ell_9\|s - s^*\|, \\ \forall (q, r, s)\,(q^*, r^*, s^*) \in Y \end{cases} \tag{6}$$

and if the following matrix:

$$\mathscr{M} = \begin{bmatrix} \ell_1\ \ell_2\ \ell_3 \\ \ell_4\ \ell_5\ \ell_6 \\ \ell_7\ \ell_8\ \ell_9 \end{bmatrix} \tag{7}$$

converges to 0, *then the fixed points for* (3) *are UH stable.*

For the remaining work, we have the following assumptions:

(M_1) For any $q, r, s \in C(I_1, R), \exists\ K_f, K_g, K_h > 0$ such that $|f(q) - f(\bar{q})| \leq K_f|q - \bar{q}|$, $|g(r) - g(\bar{r})| \leq K_g|r - \bar{r}|$, $|h(s) - h(\bar{s})| \leq K_h|s - \bar{s}|$.

(M_2) $\forall\ q, \bar{q}, r, \bar{r}, s, \bar{s} \in C(I_1, R), \forall\, t \in I_1, \exists\ L_{f_1} > 0 \ni$ $|f_1(t, q(t), q(\lambda t), r(t), s(t)) - f_1(t, \bar{q}(t), \bar{q}(\lambda t), \bar{r}(t), \bar{s}(t))| \leq L_{f_1}[2|q - \bar{q}| + |r - \bar{r}| + |s - \bar{s}|]$.

(M_3) $\forall\ q, \bar{q}, r, \bar{r}, s, \bar{s} \in C(I_1, R), \forall\, t \in I_1, \exists\ L_{f_2} > 0 \ni$ $|f_2(t, q(t), r(t), r(\lambda t), s(t)) - f_2(t, \bar{q}(t), \bar{r}(t), \bar{r}(\lambda t), \bar{s}(t))| \leq L_{f_2}[|q - \bar{q}| + 2|r - \bar{r}| + |s - \bar{s}|]$.

(M_4) $\forall\ q, \bar{q}, r, \bar{r}, s, \bar{s} \in C(I_1, R), \forall\, t \in I_1, \exists\ L_{f_3} > 0 \ni$ $|f_3(t, q(t), r(t), s(t), s(\lambda t)) - f_3(t, \bar{q}(t), \bar{r}(t), \bar{s}(t), \bar{s}(\lambda t))| \leq L_{f_3}[|q - \bar{q}| + |r - \bar{r}| + 2|s - \bar{s}|]$.

(M_5) For some positive real numbers $C_{f_1}, D_{f_1}, E_{f_1}$ and $M_{f_1}, M_{f_2}, M_{f_3}$ such that $|f_1(t, q(t), q(\lambda t), r(t), s(t))| \leq 2C_{f_1}|q| + D_{f_1}|r| + E_{f_1}|s| + M_{f_1}$, $|f_2(t, q(t), r(t), r(\lambda t), s(t))| \leq C_{f_2}|q| + 2D_{f_2}|r| + E_{f_2}|s| + M_{f_2}$, $|f_3(t, q(t), r(t), s(t), s(\lambda t))| \leq C_{f_3}|q| + D_{f_3}|r| + 2E_{f_3}|s| + M_{f_3}$.

(M_6) For some positive real numbers $\vartheta_i\,(i = 1, 2, 3)$, $\delta_f, \delta_g, \delta_h$ such that $|f(q)| \leq \vartheta_1|q| + \delta_f$, $|g(r)| \leq \vartheta_2|r| + \delta_g$, $|h(s)| \leq \vartheta_3|s| + \delta_h$.

Lemma 3 (Refs. 34–36). *Let $E \neq \emptyset$ be closed convex subset of the Banach space Y and a, there exist two operators such that $z = Gz + Hz \ni$ (a) $Gx + Hy \in E, \forall\, x, y \in E$, (b) G is continuous and compact, (c) H is contraction. Subsequently for $z \in E$, we have $Gz + Hz = z$, where $z = (q, r, s)$ belong to Y.*

3. EXISTENCE RESULTS OF THE SOLUTION

In this section, we present results about existence theory to the considered problem.

Theorem 2. *Let $q(t) \in C[0, v]$, and $z \in L[0, v]$, the solution for linear problem*

$$\begin{aligned} {}^cD^a_{+0}q(t) &= z(t),\ t \in I_1,\ \gamma_1 \in (0, 1], \\ q(0) &= f(q) + q_0, \end{aligned} \tag{8}$$

is given by

$$q(t) = q_0 + f(q) + \frac{1}{\Gamma(a)} \int_0^t (t - y)^{a-1} z(y)dy. \tag{9}$$

Proof. Thanks to Lemma 2, the solution of (8) under the given nonlocal condition is easily obtained. □

Corollary 1. *In view of Theorem 2, the solution of the considered (1) is given by*

$$\begin{cases} q(t) = q_0 + f(q) + \dfrac{1}{\Gamma(a)} \displaystyle\int_0^t (t - y)^{a-1} \\ \qquad f_1(y, q(y), q(\lambda y), r(y), s(y))dy, \\ r(t) = r_0 + g(r) + \dfrac{1}{\Gamma(b)} \displaystyle\int_0^t (t - y)^{b-1} \\ \qquad f_2(y, q(y), r(y), r(\lambda y), s(y))dy, \\ s(t) = s_0 + h(s) + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t - y)^{\gamma-1} \\ \qquad f_3(y, q(y), r(y), s(y), s(\lambda y))dy. \end{cases} \tag{10}$$

Theorem 3. *Let f_1, f_2, f_3 be continuous, subsequently $(q, r, s) \in Y$ is a solution of (1), if and only if (q, r, s) is the solution of the integral equations given (10).*

Proof. If for (1), (q, r, s) is a solution, then (q, r, s) is the solution of the system (10). On the other hand, if (q, r, s) is the solution of the system (10), then taking derivatives of both sides of (10), we can obtain (1). □

We define S_1, S_2, $S_3 : Y \to Y$ by

$$\begin{aligned} S_1(q, r, s) &= q_0 + f(q) + \frac{1}{\Gamma(a)} \int_0^t (t - y)^{a-1} \\ &\qquad \times f_1(y, q(y), q(\lambda y), r(y), s(y))dy, \\ S_2(q, r, s) &= r_0 + g(r) + \frac{1}{\Gamma(b)} \int_0^t (t - y)^{b-1} \\ &\qquad \times f_2(y, q(y), r(y), r(\lambda y), s(y))dy, \\ S_3(q, r, s) &= s_0 + h(s) + \frac{1}{\Gamma(\gamma)} \int_0^t (t - y)^{\gamma-1} \\ &\qquad \times f_3(y, q(y), r(y), s(y), s(\lambda y))dy \end{aligned}$$

and $S(q, r, s) = \begin{pmatrix} S_1(q, r, s) \\ S_2(q, r, s) \\ S_3(q, r, s) \end{pmatrix}$. So, the solutions of (10) are the fixed points of S.

Theorem 4. *If $d < 1$, with the help of assumptions (M_1)–(M_4), the system (1) has unique solution.*

Proof. Let $q, \bar{q}, r, \bar{r}, s, \bar{s} \in X$ and every $t \in I_1$, let

$$\begin{aligned}
&\|S_1(q,r,s) - S_1(\bar{q},\bar{r},\bar{s})\| \\
&\quad \le \max_{t\in I_1} |f(q) - f(\bar{q})| \\
&\qquad + \max_{t\in I_1} \frac{1}{\Gamma(a)} \int_0^t (t-y)^{a-1} \\
&\qquad \times |f_1(y, q(y), q(\lambda y), r(y), s(y)) \\
&\qquad - f_1(y, \bar{q}(y), \bar{q}(\lambda y), \bar{r}(y), \bar{s}(y))| \\
&\quad \le K_f \|q - \bar{q}\| + \frac{v^a}{\Gamma(a+1)} [L_{f_1}(2\|q-\bar{q}\| \\
&\qquad + \|r - \bar{r} + \|s - \bar{s}\|)] \\
&\quad \le \left(K_f + \frac{2L_{f_1} v^a}{\Gamma(a+1)}\right) ||q - \bar{q}|| + \frac{L_{f_1} v^a}{\Gamma(a+1)} \\
&\qquad \times (||r - \bar{r}|| + ||s - \bar{s}||) \\
&\quad \le \left(K_f + \frac{2L_{f_1} v^a}{\Gamma(a+1)}\right) (||q - \bar{q}|| + ||r - \bar{r}|| \\
&\qquad + ||s - \bar{s}||) \\
&\quad \le d_1(||q - \bar{q}|| + ||r - \bar{r}|| + ||s - \bar{s}||), \\
&\qquad \text{where } d_1 = \left(K_f + \frac{2L_{f_1} v^a}{\Gamma(a+1)}\right). \qquad (11)
\end{aligned}$$

In the same way, we can prove that

$$\begin{aligned}
&\|S_2(q,r,s) - S_2(\bar{q},\bar{r},\bar{r})\| \le d_2(||q - \bar{q}|| \\
&\qquad + ||r - \bar{r}|| + ||s - \bar{s}||), \\
&\qquad \text{where } d_2 = \left(K_g + \frac{2L_{f_2} v^b}{\Gamma(b+1)}\right), \qquad (12) \\
&\|S_3(q,r,s) - S_3(\bar{q},\bar{r},\bar{r})\| \le d_3(||q - \bar{q}|| \\
&\qquad + ||r - \bar{r}|| + ||s - \bar{s}||), \\
&\qquad \text{where } d_3 = \left(K_h + \frac{2L_{f_3} v^\gamma}{\Gamma(\gamma+1)}\right). \qquad (13)
\end{aligned}$$

Hence, from (11)–(13), one has

$$\begin{aligned}
&\|S(q,r,s) - S(\bar{q},\bar{r},\bar{s})| \le \max(d_1, d_2, d_3)(\|q - \bar{q}\| \\
&\qquad + \|r - \bar{r}\| + \|s - \bar{s}\|) \\
&= d(\|u - \bar{q}\| + \|r - \bar{r}\| + \|s - \bar{s}\|), \qquad (14)
\end{aligned}$$

where $d = \max_{t\in I_1}\{d_1, d_2, d_3\}$. Therefore, S is contraction, so it has unique fixed point. Consequently the corresponding system (1) has unique solution. □

Theorem 5. *Under the assumptions* (M_1), (M_5), (M_6) *and if* $\max\{K_f, K_g, K_h\} < 1$ *holds, then the system* (1) *possesses at least one solution.*

Proof. Let B be closed subset of Y that is

$$B = \{(q,r,s) \in Y : \|(q,r,s)\| \le \mathbb{R}\}.$$

Here, $\widetilde{S} = G + H$, where $G = (G_1, G_2, G_3)$, $H = (H_1, H_2, H_3)$ and

$$\begin{aligned}
\max\Bigg\{ &\frac{v^a}{\Gamma(a+1)} (2\mathbb{R}C_{f_1} + \mathbb{R}D_{f_1} + \mathbb{R}E_{f_1} + M_{f_1}) \\
&+ |q_0| + \mathbb{R}\vartheta_1 + \delta_f, \\
&\frac{v^b}{\Gamma(b+1)} (\mathbb{R}C_{f_2} + 2\mathbb{R}D_{f_2} + \mathbb{R}E_{f_2} + M_{f_2}) \\
&+ |r_0| + \mathbb{R}\vartheta_2 + \delta_g, \\
&\frac{v^\gamma}{\Gamma(\gamma+1)} (\mathbb{R}C_{f_3} + \mathbb{R}D_{f_3} + 2\mathbb{R}E_{f_2} + M_{f_3}) \\
&+ |s_0| + \mathbb{R}\vartheta_3 + \delta_h \Bigg\} \le \mathbb{R}.
\end{aligned}$$

To establish the required results, we define the operators as

$$\begin{aligned}
G_1(q,r,s) &= \frac{1}{\Gamma(a)} \int_0^t (t-y)^{a-1} \\
&\qquad f_1(y, q(y), q(\lambda y), r(y), s(y))dy, \\
G_2(q,r,s) &= \frac{1}{\Gamma(b)} \int_0^t (t-y)^{b-1} \\
&\qquad f_2(y, q(y), r(y), r(\lambda y), s(y))dy, \\
G_3(q,r,s) &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-y)^{\gamma-1} \\
&\qquad f_3(y, q(y), r(y), s(y), s(\lambda y))dy
\end{aligned}$$

and

$$H_1 q(t) = q_0 + f(q),\ H_2 r(t) = r_0 + g(r),$$
$$H_3 s(t) = s_0 + h(s).$$

So, it is quite visible that $\widetilde{S}_1 = G_1 + H_1$, $\widetilde{S}_2 = G_2 + H_2$, $\widetilde{S}_3 = G_3 + H_3$. In the next step, we prove that

$$\widetilde{S}(q,r,s) = G(q,r,s) + H(q,r,s) \in B,$$
$$\text{for all } (q,r,s) \in B.$$

For any $(q, r, s) \in B$, we have

$$\begin{aligned}
&|\widetilde{S}_1(q, r, s)| \\
&\quad = \left| q_0 + f(q) + \frac{1}{\Gamma(a)} \times \int_0^t (t-y)^{a-1} f_1(y, q(y), q(\lambda y), r(y)) \right| dy \\
&\quad \leq \frac{v^a}{\Gamma(a+1)} |f_1(y, q(y), q(\lambda y), r(y))| + |q_0| + |f(q)| \\
&\quad \leq \frac{v^a}{\Gamma(a+1)} (2C_{f_1}|q| + D_{f_1}|r| + E_{f_1}|s| + M_{f_1}) + |q_0| + \vartheta_1 |q| + \delta_f \\
&\quad \leq \frac{v^a}{\Gamma(a+1)} (2\mathbb{R}C_{f_1} + \mathbb{R}D_{f_1} + \mathbb{R}E_{f_1} + M_{f_1}) + |q_0| + \mathbb{R}\vartheta_1 + \delta_f \\
&\quad \leq \mathbb{R}.
\end{aligned}$$

In the same way, one has

$$\begin{aligned}
\|\widetilde{S}_2(q, r, s)\| &\leq \frac{v^b}{\Gamma(b+1)} (\mathbb{R}C_{f_2} + 2\mathbb{R}D_{f_2} + \mathbb{R}E_{f_2} + M_{f_2}) + |r_0| + \mathbb{R}\vartheta_2 + \delta_g \\
&\leq \mathbb{R}, \\
\|\widetilde{S}_3(q, r, s)\| &\leq \frac{v^\gamma}{\Gamma(\gamma+1)} (\mathbb{R}C_{f_3} + \mathbb{R}D_{f_3} + 2\mathbb{R}E_{f_2} + M_{f_3}) + |s_0| + \mathbb{R}\vartheta_3 + \delta_h \\
&\leq \mathbb{R}.
\end{aligned}$$

Thus, $\|\widetilde{S}(q, r, s)\| \leq \mathbb{R}$ which indicates that $\widetilde{S}(B) \subseteq B$. In the upcoming portion, our target is to prove that H is contraction. Through (M_1) for any two solutions (q, r, s) and $(\bar{q}, \bar{r}, \bar{s}) \in B$, we have

$$\begin{aligned}
\|H_1(q) - H_1(\bar{q})\| &\leq \max_{t \in I_1} [|f(q) - f(\bar{q})|] \\
&\leq K_f \|q - \bar{q}\|. \qquad (15)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|H_2(v) - H_2(\bar{r})\| &\leq \max_{t \in I_1} [|g(r) - g(\bar{r})|] \\
&\leq K_g \|r - \bar{r}\|, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\|H_3(s) - H_3(\bar{s})\| &\leq \max_{t \in I_1} [|h(s) - h(\bar{s})|] \\
&\leq K_h \|s - \bar{s}\|. \qquad (17)
\end{aligned}$$

From (15)–(17), we have that H is contraction to show that G is relatively compact. The continuity of f_1, f_2, f_3 yields the continuity of G. For $(q, r, s) \in B$, we have

$$\begin{aligned}
|G_1(q, r, s)| &\leq \frac{1}{\Gamma(a)} \int_0^t (t-y)^{a-1} \times |f_1(y, q(y), q(\lambda y), r(y), s(y))| dy \\
&\leq \frac{v^a}{\Gamma(a+1)} (2C_{f_1}|q| + D_{f_1}|r| + E_{f_1}|s| + M_{f_1}) \\
&\leq \frac{v^a}{\Gamma(a+1)} (2\mathbb{R}C_{f_1} + \mathbb{R}D_{f_1} + \mathbb{R}E_{f_1} + M_{f_1}). \qquad (18)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|G_2(q, r, s)| &\leq \frac{v^b}{\Gamma(b+1)} (\mathbb{R}C_{f_2} + 2\mathbb{R}D_{f_2} + \mathbb{R}E_{f_2} + M_{f_2}), \\
|G_3(q, r, s)| &\leq \frac{v^\gamma}{\Gamma(\gamma+1)} (\mathbb{R}C_{f_3} + \mathbb{R}D_{f_3} + 2\mathbb{R}E_{f_3} + M_{f_3}).
\end{aligned} \qquad (19)$$

Therefore, from (18) and (19), one has

$$\mathbb{R} \geq \|G(q, r, s)\|. \qquad (20)$$

Thus, (20) deduces uniform boundedness of G on B. Let $(q, r, s) \in C$, where C is a bounded set of B. Then for ξ, t belongs to I_1 with $1 \geq \xi \geq t \geq 0$, through (M_5), we get

$$\begin{aligned}
&|G_1(q(t), r(t), s(t)) - G_1(q(\xi), r(\xi), s(\xi))| \\
&\quad = \left| \frac{1}{\Gamma(a)} \int_0^t (t-y)^{a-1} \times f_1(y, q(y), q(\lambda y), r(y), s(y)) dy - \frac{1}{\Gamma(a)} \int_o^\xi (\xi - y)^{a-1} \times f_1(y, q(y), q(\lambda y), r(y), s(y)) dy \right| \\
&\quad \leq \frac{(2\mathbb{R}C_{f_1} + \mathbb{R}D_{f_1} + \mathbb{R}E_{f_1} + M_{f_1})}{\Gamma(a)} \times \left[(t-y)^{a-1} dy - \int_0^\xi (\xi - y)^{a-1} dy \right] \\
&\quad \leq \frac{(2\mathbb{R}C_{f_1} + \mathbb{R}D_{f_1} + \mathbb{R}E_{f_1} + M_{f_1})}{\Gamma(a+1)} (t^a - \xi^a). \qquad (21)
\end{aligned}$$

Obviously, right side in above inequality (21) tends to zero on $t \to \xi$. Also, G_1 is bounded and continuous. Thus it is uniformly bounded. Hence $\|G_1(q(t), r(t), s(t)) - G_1(q(\xi), r(\xi), s(\xi))\| \to 0$ as t tends to ξ. In a similar way, one can also show that $\|G_2(q(t), r(t), s(t)) - G_2(u(\xi), r(\xi), s(\xi))\| \to 0$ as t tends to ξ and $\|G_3(q(t),\ r(t),\ s(t)) - G_3(q(\xi),$

$r(\xi), s(\xi))\| \to 0$ as t tends to ξ. Consequently, G is equicontinuous and so by Arzelá–Ascoli theorem, G is relatively compact. By Krasnoselskii's fixed point theorem (Theorem 3), the system (1), under consideration has at least one solution. □

4. RESULTS REGARDING STABILITY

Here, in this section, we discuss UH stability results for the concerned problem.

Theorem 6. *Let under the assumptions* (M_1)–(M_4), $d < 1$ *hold and also the matrix* $\mathscr{M}$ *converge to* 0. *Then the results of* (1) *are UH stable.*

Proof. To prove the above theorem, let for any two solutions (q, r, s), $(\bar{q}, \bar{r}, \bar{s})$, we have

$$\begin{aligned}
&\|S_1(q,r,s) - S_1(\bar{q},\bar{r},\bar{s})\| \\
&\quad \le |f(q) - f(\bar{q})| + \frac{1}{\Gamma(a)}\int_0^t (t-y)^{a-1}| \\
&\qquad \times f_1(y, q(y), q(\lambda y), r(y), s(y)) \\
&\qquad - f_1(y, \bar{q}(y), \bar{q}(\lambda y), \bar{r}(y), \bar{s}(y))| \\
&\quad \le K_f|q-\bar{q}| + \frac{v^a}{\Gamma(a+1)}[L_{f_1}(2|q-\bar{q}| \\
&\qquad + |r - \bar{r} + |s - \bar{s}||)] \\
&\quad \le \left(K_f + \frac{2L_{f_1}v^a}{\Gamma(a+1)}\right)||q-\bar{q}|| \\
&\qquad + \frac{L_{f_1}v^a}{\Gamma(a+1)}(||r-\bar{r}|| + ||s-\bar{s}||) \\
&\quad \le \ell_1\|x_1 - \bar{q}\| + \ell_2\|v - \bar{v}\| + \ell_3||s-\bar{s}||, \quad (22)
\end{aligned}$$

where

$$\ell_1 = \left(K_f + \frac{2L_{f_1}v^a}{\Gamma(a+1)}\right), \quad \ell_2 = \ell_3 = \frac{L_{f_1}v^a}{\Gamma(a+1)}.$$

In a similar way, we may have

$$\begin{aligned}
&\|S_2(q,r,s) - S_2(\bar{q},\bar{r},\bar{s})\| \le \ell_4\|q-\bar{q}\| \\
&\quad + \ell_5\|r-\bar{r}\| + \ell_6||s-\bar{s}||, \\
&\|S_3(q,r,s) - S_3(\bar{q},\bar{r},\bar{s})\| \le \ell_7\|q-\bar{q}\| \\
&\quad + \ell_8\|r-\bar{r}\| + \ell_9||s-\bar{s}||, \qquad (23)
\end{aligned}$$

where

$$\ell_4 = \ell_6 = \frac{L_{f_2}v^b}{\Gamma(b+1)}, \quad \ell_5 = \left(K_g + \frac{2L_{f_2}v^b}{\Gamma(b+1)}\right),$$

and

$$\ell_7 = \ell_8 = \frac{L_{f_3}v^\gamma}{\Gamma(\gamma+1)}, \quad \ell_9 = \left(K_h + \frac{2L_{f_3}v^\gamma}{\Gamma(\gamma+1)}\right).$$

So, from (22) and (23), we get

$$\begin{aligned}
&\|S_1(q,r,s) - S_1(\bar{q},\bar{r},\bar{s})\| \le \ell_1\|q-\bar{q}\| \\
&\quad + \ell_2\|r-\bar{r}\| + \ell_3||s-\bar{s}||, \\
&\|S_2(q,r,s) - S_2(\bar{q},\bar{r},\bar{s})\| \le \ell_4\|q-\bar{q}\| \\
&\quad + \ell_5\|r-\bar{r}\| + \ell_6||s-\bar{s}||, \\
&\|S_3(q,r,s) - S_3(\bar{q},\bar{r},\bar{s})\| \le \ell_7\|q-\bar{q}\| \\
&\quad + \ell_8\|r-\bar{r}\| + \ell_9||s-\bar{s}||.
\end{aligned} \qquad (24)$$

Now, it is given that the matrix $\mathscr{M}$ from (24) converges to zero given by

$$\mathscr{M} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \\ \ell_7 & \ell_8 & \ell_9 \end{bmatrix}.$$

Hence, the solution of (1) is UH stable. □

5. EXAMPLES

Example 1. Consider the following system of FPDEs:

$$\begin{cases}
{}^cD_{+0}^{\frac{1}{2}}q(t) = \dfrac{|q(t)|+|q(\frac{t}{2})|}{(t+9)^2(1+|q(t)|)} + \dfrac{1}{65(1+r^2(t))} \\
\qquad + \dfrac{\sin|s(t)|}{e^{t^2+4}}, \; t \in I_1 = [1, 10], \\
{}^cD_{+0}^{\frac{1}{3}}r(t) = \dfrac{3 + |q(t)| + |r(t)| + |r(\frac{t}{2})| + |s(t)|}{2e^{t+4}(1 + |q(t)| + |r(t)| + |s(t)|)} \\
\qquad + \dfrac{1}{2(t^3+4)}, \; t \in I_1, \\
{}^cD_{+0}^{\dfrac{1}{4}}s(t) = \dfrac{\sin|q(t)|}{e^{t+4}} + \dfrac{|r(t)|}{(t+9)^2(1+|r(t)|)} \\
\qquad + \dfrac{1}{65(1+s^2(t))} + \dfrac{1}{65(1+s^2(\frac{t}{2}))} \\
\qquad + \dfrac{1}{20e^t}, \; t \in I_1, \\
q(0) = \dfrac{\sin|q|}{30}, \; r(0) = \dfrac{\cos|r|}{20}, \; s(0) = \dfrac{\sin|s|}{25}.
\end{cases} \qquad (25)$$

For $t \in I_1, q, r, s, \bar{q}, \bar{r}, \bar{s} \in R$, we get

$$\begin{aligned}
&|f_1(t, q(t), q(\lambda t), r(t), s(t)) \\
&-f_1(t, \bar{q}(t), \bar{r}(\lambda t), \bar{r}(t), \bar{s}(t))| \\
&\quad \leq 0.0183[2|q-\bar{q}| + |r-\bar{r}| + |s-\bar{s}|], \\
&|f_2(t, q(t), r(t), r(\lambda t), s(t)) \\
&-f_2(t, \bar{q}(t), \bar{r}(t), \bar{r}(\lambda t), \bar{s}(t))| \\
&\quad \leq 0.0183[|q-\bar{q}| + 2|r-\bar{r}| + |s-\bar{s}|], \\
&|f_3(t, q(t), r(t), s(t), s(\lambda t)) \\
&-f_3(t, \bar{q}(t), \bar{r}(t), \bar{s}(t), \bar{s}(\lambda t))| \\
&\quad \leq 0.0183[2|q-\bar{q}| + |r-\bar{r}| + |s-\bar{s}|].
\end{aligned}$$

Also, one has

$$\begin{aligned}
|f(q) - f(\bar{q})| &\leq \frac{1}{30}\|q-\bar{q}\|, \ |g(r) - g(\bar{r})| \\
&\leq \frac{1}{20}\|r-\bar{r}\|, \ |h(s) - h(\bar{s})| \\
&\leq \frac{1}{25}\|s-\bar{s}\|,
\end{aligned}$$

after calculation we have $K_f = \frac{1}{30}, K_g = \frac{1}{20}, K_h = \frac{1}{25}, L_{f_1} = L_{f_2} = L_{f_3} = 0.0183$,

$$\begin{aligned}
d_1 &= \left(K_f + \frac{2L_{f_1}v^a}{\Gamma(a+1)}\right) \\
&= 0.1683 < 1, \ d_2 = 0.1382 < 1, \text{ and } d_3 = 0.1117.
\end{aligned}$$

Therefore, by Theorem 4, system (25) has unique solution and

$$\mathscr{M} = \begin{bmatrix} 0.1683 & 0.0652 & 0.0652 \\ 0.0441 & 0.1382 & 0.0441 \\ 0.0358 & 0.0358 & 0.1117 \end{bmatrix}. \tag{26}$$

The eigenvalues are $\beta_1 = 0.2395$, $\beta_2 = 0.0978$ and $\beta_3 = 0.0808$, and $\Upsilon(\mathscr{M}) = \max\{|\beta_i|, \ i = 1,2,3\} < 1$, which implies that $\mathscr{M}$ converges to zero. Hence, according to Theorem 6, the solution of system (25) is UH stable.

Example 2. Consider the following system of FPDEs:

$$\begin{cases}
{}^cD_{+0}^{\frac{1}{2}}q(t) = \dfrac{t^3 + \sin|q(t)| + \cos|r(t)| + |s(t)|}{90} \\
\qquad + \dfrac{\sin|q(\frac{t}{3})|}{100}, \ t \in I_1, \\
{}^cD_{+0}^{\frac{1}{3}}r(t) = \dfrac{\cos|q(t) + r(t) + s(t)|}{50} \\
\qquad + \dfrac{|r(\frac{t}{3})|}{(t^3+5)^4}, \ t \in I_1, \\
{}^cD_{+0}^{\frac{1}{4}}s(t) = \dfrac{\sin|q(t)| + |r(t)| + |s(t)|}{e^t(t^3+40)} \\
\qquad + \dfrac{|s(\frac{t}{3})|}{30e^t}, \ t \in I_1, \\
q(0) = \dfrac{\sin|q|}{20e^{\pi}}, \ r(0) = \dfrac{|r|}{e^{7\pi}}, \ s(0) = \dfrac{\cos|s|}{30}.
\end{cases} \tag{27}$$

For $t \in I_1, q, r, s, \bar{q}, \bar{r}, \bar{s} \in R$, we get

$$\begin{aligned}
&|f_1(t, q(t), q(\lambda t), r(t), s(t)) \\
&\quad -f_1(t, \bar{q}(t), \bar{r}(\lambda t), \bar{r}(t), \bar{s}(t))| \\
&\leq 0.0111[2|q-\bar{q}| + |r-\bar{r}| + |s-\bar{s}|], \\
&|f_2(t, q(t), r(t), r(\lambda t), s(t)) \\
&\quad -f_2(t, \bar{q}(t), \bar{r}(t), \bar{r}(\lambda t), \bar{s}(t))| \\
&\leq 0.020[|q-\bar{q}| + 2|r-\bar{r}| + |s-\bar{s}|], \\
&|f_3(t, q(t), r(t), s(t), s(\lambda t)) \\
&\quad -f_3(t, \bar{q}(t), \bar{r}(t), \bar{s}(t), \bar{s}(\lambda t))| \\
&\leq 0.0333[2|q-\bar{q}| + |r-\bar{r}| + |s-\bar{s}|].
\end{aligned}$$

Also one has

$$\begin{aligned}
|f(q) - f(\bar{q})| &\leq \frac{1}{20}\|q-\bar{q}\|, \ |g(r) - g(\bar{r})| \\
&\leq \frac{1}{e^7}\|r-\bar{r}\|, \ |h(s) - h(\bar{s})| \\
&\leq \frac{1}{49}\|s-\bar{s}\|,
\end{aligned}$$

after calculation we have $K_f = \frac{1}{20}, K_g = \frac{1}{e^7}, K_h = \frac{1}{49}, L_{f_1} = 0.0111, L_{f_2} = 0.020, L_{f_3} = 0.0333$,

$$\begin{aligned}
d_1 &= \left(K_f + \frac{2L_{f_1}v^a}{\Gamma(a+1)}\right) \\
&= 0.1201 < 1, \\
d_2 &= 0.0973 < 1, \text{ and} \\
d_3 &= 0.1509.
\end{aligned}$$

Therefore, by Theorem 4, system (27) has unique solution and

$$\mathscr{M} = \begin{bmatrix} 0.1201 & 0.0396 & 0.0396 \\ 0.0482 & 0.0973 & 0.0482 \\ 0.0653 & 0.0653 & 0.1509 \end{bmatrix}. \tag{28}$$

The eigenvalues are $\beta_1 = 0.2275$, $\beta_2 = 0.0825$ and $\beta_3 = 0.0584$, and $\Upsilon(\mathscr{M}) = \max\{|\beta_i|,\ i = 1, 2, 3\} < 1$ which implies that $\mathscr{M}$ converges to zero. Hence, according to Theorem 6, the solution of system (27) is UH stable.

We present the application of the aforesaid analysis.

Example 3. Consider the famous prey–predator model as

$$\begin{cases} {}^cD^a_{+0}q(t) = 0.005q(t)(1 - 0.002q(t)) \\ \qquad -\dfrac{0.003q(t)r(t)}{0.05 + q(t)} \\ \qquad -0.003q(0.5t)s(t), \\ {}^cD^b_{+0}r(t) = \dfrac{0.02 \times 0.003r(t)}{0.05 + q(t)} \\ \qquad -0.004r(t)s(t) \\ \qquad -0.01s(0.5t), \\ {}^cD^\gamma_{+0}s(t) = -0.007s(t) + 0.003r(0.5t)s(t) \\ \qquad +0.003 \times 0.02q(t)s(t), \\ q(0) = 2 + \dfrac{\sin|q|}{10}, r(0) = 1 \\ \qquad + \dfrac{\sin|r|}{20},\ s(0) = \dfrac{\sin|s|}{30}. \end{cases} \tag{29}$$

On calculation and taking $v = 10$ and using $a = b = \gamma = \frac{1}{2}$, after calculation, we have $K_f = \frac{1}{10}, K_g = \frac{1}{20}, K_h = \frac{1}{30}, L_{f_1} = 0.005, L_{f_2} = 0.020, L_{f_3} = 0.02,$

$$d_1 = 0.11784,\ d_2 = 0.1927,\ d_3 = 0.2260.$$

Now, $\max\{d_1, d_2, d_3\} = 0.2260 < 1$. Hence, the system (29) has unique solution by Theorem 4. Further, calculating the values for ℓ_i $(i = 1, 2, \ldots, 9)$, we have the given matrix as

$$\mathscr{M} = \begin{bmatrix} 0.11784 & 0.01784 & 0.01784 \\ 0.07136 & 0.1927 & 0.07136 \\ 0.07136 & 0.07136 & 0.2260 \end{bmatrix}. \tag{30}$$

On calculation, the eigenvalues are $\beta_1 = 0.2967$, $\beta_2 = 0.1029$, $\beta_3 = 0.1370$. Therefore, $\Upsilon(\mathscr{M}) = 0.2967 < 1$. Thus, the given prey–predator system under delay term with respect to the given fractional order is HU stable by using Theorem 6.

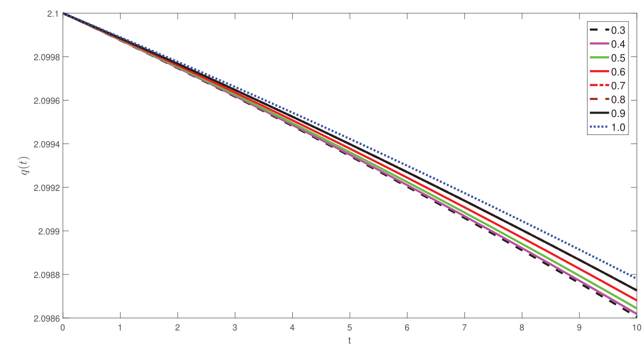

Fig. 1 Graphical presentation of $q(t)$ for various fractional values of a at the given value of $v = 10$.

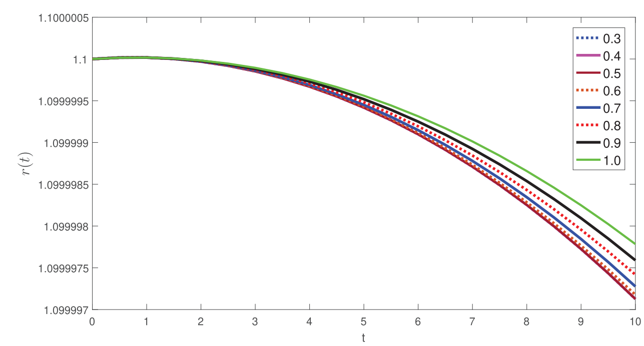

Fig. 2 Graphical presentation of $r(t)$ for various fractional values of b at the given value of $v = 10$.

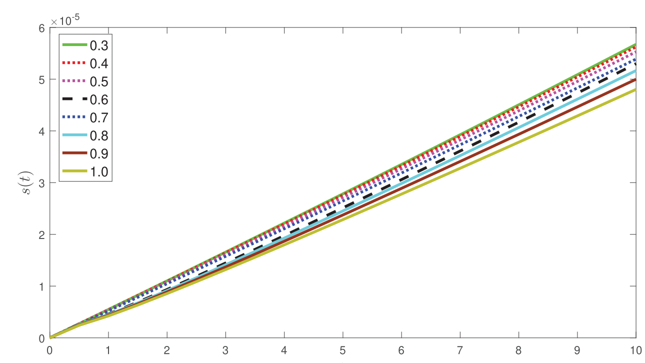

Fig. 3 Graphical presentation of $s(t)$ for various fractional values of γ at the given value of $v = 10$.

We have plotted the results for different fractional order by using Matlab-16, the results of this example in Figs. 1–3. In Figs. 1 and 2, the density of population of species q, r is decreasing with various rates due to fractional order while the species depends on both so its density of population is increasing. The concerned functions q, r, s approach to their stable position at various rates due to fractional order derivative.

6. CONCLUSION

Due to the important applications of delay problems under fractional derivatives, we have established a qualitative analysis regarding the existence of solutions to FPDEs. The mentioned problem has been investigated with nonlocal conditions involving proportional delay. With the help of classical fixed point theory, the mentioned analysis has been established. Some results regarding stability of Ulam's type have been developed via using nonlinear analysis. In last by a pertinent example, the results were justified. Hence, we concluded that classical fixed point theory has the ability to deal such like complicated problems for mathematical analysis. The aforementioned analysis can be carried out for problems involving delay under various nonlocal conditions. An interesting example has been verified for our analysis.

REFERENCES

1. A. A. Kilbas, O. I. Maricheva and S. G. Samko, *Fractional Integrals and Derivatives* (*Theory and Applications*) (Gordon and Breach, Switzerland, 1993).
2. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993).
3. I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering* (Academic Press, New York, 1999).
4. V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems* (Cambridge Academic Publishers, Cambridge, UK, 2009).
5. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
6. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
7. K. Shah, H. Khalil and R. A. Khan, Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations, *Chaos Solitons Fractals* **77** (2015) 240–246.
8. Y. Li, Y. Sang and H. Zhang, Solvability of a coupled system of nonlinear fractional differential equations with fractional integral conditions, *J. Appl. Math. Comput.* **50** (2015) 73–91.
9. H. Y. Alfifi, I. Ben Saad, S. Turki and Z. Z. El Abidine, Existence and asymptotic behavior of positive solutions for a coupled system of semilinear fractional differential equations, *Results Math.* **71** (2017) 705–730.
10. A. Atangana and E. Alabaraoye, Solving a system of fractional partial differential equations arising in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller–Segel equations, *Adv. Differ. Equ.* **2013** (2013) 94.
11. A. Atangana, A novel model for the lassa hemorrhagic fever: Deathly disease for pregnant women, *Neural Comput. Appl.* **26**(8) (2015) 1895–1903.
12. A. Atangana and R. T. Alqahtani, Modelling the spread of river blindness disease via the caputo fractional derivative and the beta-derivative, *Entropy* **18**(2) (2016) 40.
13. M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (2010) 851–863.
14. A. Ali, K. Shah and T. Abdeljawad, Study of implicit delay fractional differential equations under anti-periodic boundary conditions, *Adv. Differ. Equ.* **2020**(1) (2020) 1–16.
15. T. Abdeljawad and D. Baleanu, Monotonicity results for a nabla fractional difference operator with discrete Mittag-Leffler kernels, *Chaos Solitons Fractals* **102** (2017) 106–110.
16. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.* **2017** (2017) 313.
17. J. Wang, H. Xiang and Z. Liu, Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations, *Int. J. Difference Equ.* **2010** (2010) 12pp.
18. M. Benchohra, S. Bouriah and J. J. Nieto, Existence and Ulam stability for nonlinear implicit differential equations with Riemann–Liouville fractional derivative, *Demonstr. Math.* **52**(1) (2019) 437–450.
19. V. A. Ambartsumyan, On the fluctuation of the brightness of the milky way, *Dokl. Akad. Nauk.* **44** (1944) 223–226.
20. B. Brunt, A. A. Zaidi and T. Lynch, Cell division and the pantograph equation, *ESAIM: Proc. Surv.* **62** (2018) 158–167.
21. S. Sedaghat, Y. Ordokhani and M. Dehghan, Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials, *Commun. Nonlinear Sci. Numer. Simul.* **17**(12) (2012) 4815–4830.
22. M. Bahsi, M. Cevik and M. Sezer, Orthoexponential polynomial solutions of delay pantograph differential equations with residual error estimation, *Appl. Math. Comput.* **271** (2015) 11–21.
23. J. K. Hale and S. M. Lunel, *Introduction to Functional Differential Equations* (Springer Science and Business Media, New York, 2013).

24. D. H. Hyers, On the stability of the linear functional equations, *Proc. Natl. Acad. Sci. USA* **27**(4) (1941) 222–224.
25. S. M. Ulam, *A Collection of the Mathematical Problems* (Interscience Publishers, New York, USA, 1960).
26. S. M. Jung, Hyers–Ulam stability of linear differential equations of first order, *Appl. Math. Lett.* **17**(10) (2004) 1135–1140.
27. A. Zada, W. Ali and Choonkil, Ulam type stability of higher order nonlinear delay differential equations via integral inequality of Grönwall–Bellman–Biharis type, *Appl. Math. Comput.* **350** (2019) 60–65.
28. J. Wang, L. Lv and W. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equ.* **63** (2011) 1–10.
29. Z. Xia and H. Zhou, Pseudo almost periodicity of fractional integro- differential equations with impulsive effects in banach spaces, *Czechoslovak Math. J.* **67**(142) (2017) 123–141.
30. Y. Zhao, S. Sun, S. Han and Z. Li, Qiuping Theory of fractional hybrid differential equations, *Comput. Math. Appl.* **62**(3) (2011) 1312–1324.
31. J. Wang, K. Shah and A. Ali, Existence and Hyers–Ulam stability of fractional nonlinear impulsive switched coupled evolution equations, *Math. Meth. Appl. Sci.* **41**(6) (2018) 2392–2402.
32. Z. Ali, A. Zada and K. Shah, On Ulam's stability for a coupled systems of nonlinear implicit fractional differential equations, *Bull. Malays. Math. Sci. Soc.* **42**(5) (2018) 2681–2699.
33. C. Urs, Coupled fixed point theorem and applications to periodic boundary value problem, *Miskolc Math. Notes* **14**(1) (2013) 323–333.
34. V. A. Yurko, Boundary value problems with discontinuity conditions in an interior point of the interval, *J. Diff. Equ.* **36**(8) (2000) 1266–1269.
35. M. Altman, A fixed point theorem for completely continuous operators in Banach spaces, *Bull. Acad. Polon. Sci.* **3** (1955) 409–413.
36. T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii's type, *J. Math. Nachr.* **189** (1998) 23–31.