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Research Article

Qualitative Analysis of Implicit Dirichlet Boundary Value Problem for Caputo-Fabrizio Fractional Differential Equations

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This article studies a class of implicit fractional differential equations involving a Caputo-Fabrizio fractional derivative under Dirichlet boundary conditions (DBCs). Using classical fixed-point theory techniques due to Banch's and Krasnoselskii, a qualitative analysis of the concerned problem for the existence of solutions is established. Furthermore, some results about the stability of the Ulam type are also studied for the proposed problem. Some pertinent examples are given to justify the results.

1. Introduction and Preliminaries

The concerned area of fractional order differential equations (FODEs) have many concentrations in real-world problems and have paid close attention to numerous researchers in the past few decades [1–5]. The mentioned area has been studied from several aspects, such as the existence and uniqueness of solutions via using the classical fixed-point theory, the numerical analysis, the optimization theory, and also the theory of stability corresponding to various fractional differential operators like Caputo, Hamdard, and Riemann-Liouville (we refer few as [6–9]). In the aforementioned operators, there exists a singular kernel. Therefore, recently some authors introduced some new types of fractional derivative operators in which they have replaced a singular kernel by a nonsingular kernel. The nonsingular kernel derivative has been proved as a good tool to model real-world problems in different fields of science and engineering [10, 11]. In fractional, it is called nonsingular exponential type or Caputo-Fabrizio fractional differential (CFFD) operator. The CFFD operator introduced two researchers, Caputo and Fabrizio for the first time in 2015 [12]. They replaced the singular kernel in the usual Caputo and Riemann-Liouville derivative by an exponential nonsingular kernel. The new operator of this type was found to be more practical than the usual Caputo and Riemann-Liouville fractional differential operators in some problems (see some detailed references such as [13–15]). Recently, many researchers have studied the existence and uniqueness of the solutions at the initial value problems for FODEs under the said operator. But the investigation has been limited to initial value problems only. On the other hand, boundary value problems have significant applications in engineering and other physical sciences during modeling numerous phenomena (we refer to see [16–19]). Furthermore, during optimization and numerical analysis of the mentioned problems, researchers need stable results from theoretical as well as practical sides. A stable result may lead us to a stable process. Therefore, the stability theory has also got proper attention during the last many decades. It is well known fact that stability analysis plays an important role. Various stability concepts such as exponential stability, Mittag-Lefler stability and Hayers-Ulam's stability have been adopted in literature to study the stability of different systems of FODEs. The analysis of

Hyers-Ulam's stability has been recognized as a simple form of investigation. For historical background on the stability of Hyers-Ulam, we refer to see previous articles [20–23]. But recently, that type of problem has not been adequately studied for a new type of CFFD operator. Therefore, in this work, we will investigate an implicit class of FODEs involving the CFFD operator under DBCs

$$\begin{cases} {}_c^{\mathrm{CF}}\mathbf{D}_w^{\mu} z(w) = f\left(w, z(w), {}_c^{\mathrm{CF}}\mathbf{D}_w^{\mu} z(w)\right), 1 < \mu \leq 2, w \in [c, d], \\ z(c) = 0, z(d) = 0 \text{ and } c, d \in R, \end{cases} \tag{1}$$

where ${}^{\mathrm{CF}}\mathbf{D}$ is used for CFFD and $I = [c, d]$, $f : I \times R \times R \rightarrow R$ is a continuous function. In this article, we investigate uniqueness and existence of solutions to the proposed problem (1) by classical fixed-point theorems due to Banach's and Krasnoselskii. Further, we investigate some pertinent analysis about the stability theory due to Ulam, and Hyers is investigated for the mentioned problem (1). For the authenticity of the presented work, two concrete examples are also studied.

Throughout the paper, $C[I, R]$ is a Banach space with norm $\|z\| = \max_{w \in I} |z(w)|$.

Definition 1 (see [24]). For any $z(w) \in C[I, R]$, we defined the derivative of Caputo-Fabrizio for nonsingular kernel as

$$ {}_c^{\mathrm{CF}}\mathbf{D}_w^{\mu} z(w) = \frac{D(\mu)}{1-\mu} \int_c^w z'(w) \exp^{\frac{-\mu(w-\zeta)}{1-\mu}} d\zeta, \tag{2}$$

where $D(\mu) > 0$ is the normalization function with $D(0) = D(1) = 1$ satisfying.

Definition 2 (see [24]). The integral of Caputo-Fabrizio for nonsingular kernel type is given by

$$ {}_c^{\mathrm{CF}}\mathbf{I}_w^{\mu} z(w) = \frac{1-\mu}{D(\mu)} z(w) + \frac{\mu}{D(\mu)} \int_c^w z(\zeta) d\zeta, \tag{3}$$

where ${}^{CF}\mathbf{I}$ is used for Caputo-Fabrizio integral operator.

Definition 3 (see [25]). Let $n < \mu \leq n+1$ and f be such that $f^{(n)} \in H^1(c, d)$. Set $\alpha = \mu - n$. Then, $\alpha \in [0, 1]$ and we define

$$\begin{aligned} {}_c^{\mathrm{CFC}}\mathbf{D}_w^{\mu} f(w) &= {}_c^{\mathrm{CFC}}\mathbf{D}_w^{\alpha} f^{(n)}(w), \\ {}_c^{\mathrm{CFR}}\mathbf{D}_w^{\mu} f(w) &= {}_c^{\mathrm{CFR}}\mathbf{D}_w^{\alpha} f^{(n)}(w), \\ {}_c^{\mathrm{CF}}\mathbf{I}_w^{\mu} f(w) &= {}_c\mathbf{I}_{w c}^{n}\, {}^{\mathrm{CF}}\mathbf{I}_w^{\alpha} f^{(n)}(w). \end{aligned} \tag{4}$$

Lemma 4. *For $z(w)$ defined on $[c, d]$ and $\mu \in [n, n+1]$, for some $n \in N_0$, we have*

$$ {}_c^{CF}\mathbf{I}_{w c}^{\mu}\, {}^{CF}\mathbf{D}_w^{\mu} z(w) = z(w) - \sum_{k=0}^{n} \frac{z^k(c)}{k!} (w-c)^k. \tag{5}$$

2. Results and Discussion

In this part, we investigate the solution of the proposed problem (1) and also study the uniqueness and existence of the solutions.

Lemma 5. *The solution of*

$$\begin{cases} {}_c^{CF}\mathbf{D}_w^{\mu} z(w) = \psi(w), 1 < \mu \leq 2, w \in [c, d], \\ z(c) = 0, z(d) = 0 \text{ and } c, d \in R, \end{cases} \tag{6}$$

is given by

$$\begin{aligned} z(w) = &- \frac{2-\mu(w-c)}{(d-c)D(\mu-1)} \int_c^d \Psi(\zeta) d\zeta \\ &- \frac{\mu-1(w-c)}{(d-c)D(\mu-1)} \int_c^d (d-\zeta)\Psi(\zeta) d\zeta \\ &+ \frac{2-\mu}{D(\mu-1)} \int_c^w \Psi(\zeta) d\zeta \\ &+ \frac{\mu-1}{D(\mu-1)} \int_c^w (w-\zeta)\Psi(\zeta) d\zeta. \end{aligned} \tag{7}$$

Proof. Let $z(w)$ be a solution to problem (6). Applying Caputo-Fabrizio integral on both sides and then using Lemma 4 and Definition 3, we have

$$ {}_c^{\mathrm{CF}}\mathbf{I}_{w c}^{\mu}\, {}^{\mathrm{CF}}\mathbf{D}_w^{\mu} z(w) = {}_c^{\mathrm{CF}}\mathbf{I}_w^{\mu} \Psi(w), \tag{8}$$

which implies that

$$\begin{aligned} z(w) = &\, c_0 + c_1(w-c) + \frac{2-\mu}{D(\mu-1)} \int_c^w \Psi(\zeta) d\zeta \\ &+ \frac{\mu-1}{D(\mu-1)} \int_c^w (w-\zeta)\Psi(\zeta) d\zeta. \end{aligned} \tag{9}$$

Using boundary conditions $z(c) = z(d) = 0$, we have

$$\begin{aligned} c_0 &= 0, \\ c_1 &= - \frac{2-\mu}{(d-c)D(\mu-1)} \int_c^d \Psi(\zeta) d\zeta \\ &\quad - \frac{\mu-1}{(d-c)D(\mu-1)} \int_c^d (d-\zeta)\Psi(\zeta) d\zeta. \end{aligned} \tag{10}$$

Putting c_0, c_1 in (9), we get

$$\begin{aligned} z(w) = &- \frac{2-\mu(w-c)}{(d-c)D(\mu-1)} \int_c^d \Psi(\zeta) d\zeta \\ &- \frac{\mu-1(w-c)}{(d-c)D(\mu-1)} \int_c^d (d-\zeta)\Psi(\zeta) d\zeta \\ &+ \frac{2-\mu}{D(\mu-1)} \int_c^w \Psi(\zeta) d\zeta \\ &+ \frac{\mu-1}{D(\mu-1)} \int_c^w (w-\zeta)\Psi(\zeta) d\zeta. \end{aligned} \tag{11}$$

For simplification, use some notations; we use $G_\mu = (2-\mu)/D(\mu-1)$, $G^*_\mu = (\mu-1)/D(\mu-1)$ and give the solution of (1) as bellow.

Corollary 6. *In view of 6, the solution of the considered problem (1) is given by*

$$\begin{aligned} z(w) = &-\frac{G_\mu(w-c)}{(d-c)}\int_c^d f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &- \frac{G^*_\mu(w-c)}{(d-c)}\int_c^d (d-\zeta) f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &+ G_\mu \int_c^w f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &+ G^*_\mu \int_c^w (w-\zeta) f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta. \end{aligned} \tag{12}$$

Further, for the existence and uniqueness of the solution of problem (1), we use some fixed point theorems. For this, we need to define an operator as $N : C[I,R] \to C[I,R]$ *by*

$$\begin{aligned} N[z(w)] = &-\frac{G_\mu(w-c)}{(d-c)}\int_c^d f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &- \frac{G^*_\mu(w-c)}{(d-c)}\int_c^d (d-\zeta) f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &+ G_\mu \int_c^w f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta \\ &+ G^*_\mu \int_c^w (w-\zeta) f\left(\zeta, z(\zeta), {}_c^{CF}\mathbf{D}_\zeta^\mu z(\zeta)\right)d\zeta. \end{aligned} \tag{13}$$

To proceed further, using Corollary (6) to convert the proposed problem (1) is to a fixed point problem as $Nz(w) = z(w)$, *where the operator N is given by (13). Therefore, Problem (1) has a solution if and only if the operator N has a fixed point, where* $\lambda(w) = f(w, z(w), \lambda(w))$ *and* $\lambda(w) = {}_c^{CF}\mathbf{D}_w^\mu z(w)$. *We assume that*

(H_1) There exist certain constant $D_f > 0$ *and* $0 < E_f < 1$, *such that*

$$\begin{aligned} &\left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &\quad \le D_f|z(w) - \bar{z}(w)| + E_f\left|\lambda(w) - \bar{\lambda}(w)\right|, \end{aligned} \tag{14}$$

for all $z, \bar{z}, \lambda, \bar{\lambda} \in R$.

Theorem 7. *Under the hypothesis (H_1), the mentioned problem (1) has a unique solution if*

$$\left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\frac{D_f}{1-E_f} < 1. \tag{15}$$

Proof. Suppose $z(w), \bar{z}(w) \in C[I,R]$, we have

$$\begin{aligned} |Nz(w) - N\bar{z}(w)| \le &\frac{G_\mu(w-c)}{(d-c)}\int_c^d \left|\lambda(\zeta) - \bar{\lambda}(\zeta)\right|d\zeta \\ &+ \frac{G^*_\mu(w-c)}{(d-c)}\int_c^d (d-\zeta)\left|\lambda(\zeta) - \bar{\lambda}(\zeta)\right|d\zeta \\ &+ G_\mu\int_c^w |\lambda(\zeta) - \bar{\lambda}(\zeta)|d\zeta \\ &+ G^*_\mu\int_c^w (w-\zeta)|\lambda(\zeta) - \bar{\lambda}(\zeta)|d\zeta. \end{aligned} \tag{16}$$

where $\lambda(w), \bar{\lambda}(w) \in C[I,R]$ are given by $\lambda(w) = f(w, z(w), \lambda(w))$ and $\bar{\lambda}(w) = f(w, \bar{z}(w), \bar{\lambda}(w))$ by using hypothesis (H_1), we have

$$\begin{aligned} \left|\lambda(w) - \bar{\lambda}(w)\right| &= \left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &\le D_f|z(w) - \bar{z}(w)| + E_f\left|\lambda(w) - \bar{\lambda}(w)\right|. \end{aligned} \tag{17}$$

Repeating the above process, we get

$$|\lambda(w) - \bar{\lambda}(w)| \le \frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|. \tag{18}$$

Using (18) in (16), we have

$$\begin{aligned} |Nz(w) - N\bar{z}(w)| \le &G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)| \\ &- \frac{G^*_\mu(w-c)(d-c)}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)| \\ &+ G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)| \\ &- \frac{G^*_\mu(w-c)^2}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|. \end{aligned} \tag{19}$$

Applying maximum on both sides, we have

$$\begin{aligned} &\max_{w\in I}|Nz(w) - N\bar{z}(w)| \\ &\quad \le \max_{w\in I}\left(G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right) \\ &\quad\quad - \max_{w\in I}\left(\frac{G^*_\mu(w-c)(d-c)}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right) \\ &\quad\quad + \max_{w\in I}\left(G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right) \\ &\quad\quad - \max_{w\in I}\left(\frac{G^*_\mu(w-c)^2}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right), \\ &\|Nz - N\bar{z}\| \le \left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\frac{D_f}{1-E_f}\|z - \bar{z}\|. \end{aligned} \tag{20}$$

Thus, operator N is a contraction; therefore, the operator N has a unique fixed point. Hence, the corresponding problem (1) has a unique solution.

Our next result is to show the existence of the solution to the proposed problem (1) which is based on Krasnoselskii's fixed-point theorem. Therefore, the given hypothesis hold.

(H_2) There exist constant $p_f, q_f, r_f > 0$ with $0 < r_f < 1$ such that

$$|f(w, z(w), \lambda(w))| \leq p_f + q_f|z(w)| + r_f|\lambda(w)|. \quad (21)$$

Theorem 8 (see [26]). *Let $H \subset C[I, R]$ be a closed, convex nonempty subset of $C[I, R]$; then, there exist N_1, N_2 operators such that*

(1) $N_1 z_1 + N_2 z_2 \in H$ for all $z_1, z_2 \in H$

(2) N_1 is a contraction, and N_2 is compact and continuous

Then, there exist at least one solution $z \in H$ such that $N_1 z + N_2 z = z$.

Theorem 9. *If the hypothesis (H_2) is satisfied, then (1) has at least one solution if*

$$0 < \left(\frac{4G_\mu(d-c) - G_\mu^*(d-c)^2}{2}\right)\frac{D_f}{1-E_f} < 1. \quad (22)$$

Proof. Suppose we define two operators from (13) as

$$\begin{aligned} N_1 z(w) &= -\frac{G_\mu(w-c)}{(d-c)}\int_c^d \lambda(\zeta)d\zeta \\ &\quad - \frac{G_\mu^*(w-c)}{(d-c)}\int_c^d (d-\zeta)\lambda(\zeta)d\zeta + G_\mu\int_c^w \lambda(\zeta)d\zeta. \\ N_2 z(w) &= G_\mu^*\int_c^w (w-\zeta)\lambda(\zeta)d\zeta. \end{aligned} \quad (23)$$

Let us define a set $F = \{z \in C[I, R]: \|z\| \leq r\}$, since f is continuous, so we show that the operator N_1 is contraction. For this $z, \bar{z} \in C[I, R]$, we have

$$\begin{aligned} |N_1 z(w) - N_1\bar{z}(w)| &\leq \frac{G_\mu(w-c)}{(d-c)}\int_c^d |\lambda(\zeta) - \bar{\lambda}(\zeta)|d\zeta \\ &\quad + \frac{G_\mu^*(w-c)}{(d-c)}\int_c^d (d-\zeta)|\lambda(\zeta) - \bar{\lambda}(\zeta)|d\zeta \\ &\quad + G_\mu\int_c^w |\lambda(\zeta) - \bar{\lambda}(\zeta)|d\zeta, \end{aligned} \quad (24)$$

using (18), and then taking the maximum on both sides, we have

$$\begin{aligned} &|N_1 z(w) - N_1\bar{z}(w)| \\ &\quad \leq 2G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)| \\ &\quad\quad - \frac{G_\mu^*(w-c)(d-c)}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)| \\ &\max_{w\in I}|N_1 z(w) - N_1\bar{z}(w)| \\ &\quad \leq \max_{w\in I}\left(2G_\mu(w-c)\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right) \\ &\quad\quad - \max_{w\in I}\left(\frac{G_\mu^*(w-c)(d-c)}{2}\frac{D_f}{1-E_f}|z(w) - \bar{z}(w)|\right) \\ &\|N_1 z - N_1\bar{z}\| \leq \left(\frac{4G_\mu(d-c) - G_\mu^*(d-c)^2}{2}\right)\frac{D_f}{1-E_f}\|z - \bar{z}\|. \end{aligned} \quad (25)$$

Hence, N_1 is contraction. Next, to prove that the operator N_2 is compact and continuous, for this $z(w) \in C[I, R]$, we have

$$|N_2 z(w)| = |G_\mu^*\int_c^w (w-\zeta)\lambda(\zeta)d\zeta| \leq G_\mu^*\int_c^w (w-\zeta)|\lambda(\zeta)|d\zeta, \quad (26)$$

where $\lambda(w) \in R$, $\lambda(w) = f(w, z(w), \lambda(w))$; now, using hypothesis (H_2), we have

$$\begin{aligned} |\lambda(w)| &= |f(w, z(w), \lambda(w)|, \\ &\leq p_f + q_f|z(w)| + r_f|\lambda(w)|, \end{aligned} \quad (27)$$

repeating the above process, so we get

$$|\lambda(w)| \leq \frac{p_f + q_f}{1-r_f}|z(w)|. \quad (28)$$

Now, using (28) in (26) and then taking the maximum on both sides, we have

$$\begin{aligned} |N_2 z(w)| &\leq \frac{G_\mu^*(w-c)^2}{2}\left(\frac{p_f+q_f}{1-r_f}\right)|z(w)| \\ \max_{w\in I}|N_2 z(w)| &\leq \max_{w\in I}\left(\frac{G_\mu^*(w-c)^2}{2}\left(\frac{p_f+q_f}{1-r_f}\right)|z(w)|\right) \\ \|N_2 z\| &\leq \frac{G_\mu^*(d-c)^2}{2}\left(\frac{p_f+q_f}{1-r_f}\right)\|z\|. \end{aligned} \quad (29)$$

Which implies that

$$\|N_2 z\| \le \frac{G_\mu^*(d-c)^2}{2}\left(\frac{p_f+q_f}{1-r_f}\right) r \le A^*. \quad (30)$$

Therefore, N_2 is bounded. Next, let $w_1 < w_2$ in I, we have

$$\begin{aligned}
&|N_2 z(w_2) - N_2 z(w_1)| \\
&= \left| G_\mu^* \int_c^{w_2} (w_2-\zeta)\lambda(\zeta)d\zeta - G_\mu^* \int_c^{w_1} (w_1-\zeta)\lambda(\zeta)d\zeta \right| \\
&= | G_\mu^* \int_c^{w_2} (w_2-\zeta)\lambda(\zeta)d\zeta + G_\mu^* \int_{w_1}^{c} (w_1-\zeta)\lambda(\zeta)d\zeta | \\
&\le G_\mu^* \left(\int_c^{w_2} (w_2-\zeta) \mid \lambda(\zeta) \mid d\zeta + \int_{w_1}^{c} (w_1-\zeta) \mid \lambda(\zeta) \mid d\zeta \right).
\end{aligned} \quad (31)$$

Now, using (28) in (31), we have

$$\begin{aligned}
&|N_2 z(w_2) - N_2 z(w_1)| \\
&\le \frac{G_\mu^*}{2}\left(\frac{p_f+q_f}{1-r_f}\right)\left((w_1-c)^2-(w_2-c)^2\right)|z(w)|.
\end{aligned} \quad (32)$$

Applying maximum on right-hand side of the above inequality, we take

$$\begin{aligned}
&|N_2 z(w_2) - N_2 z(w_1)| \\
&\le \frac{G_\mu^*}{2}\left(\frac{p_f+q_f}{1-r_f}\right) \max_{w\in I}|z(w)|\left((w_1-c)^2-(w_2-c)^2\right) \\
&\le \frac{G_\mu^*}{2}\left(\frac{p_f+q_f}{1-r_f}\right) \|z\|\left((w_1-c)^2-(w_2-c)^2\right) \\
&\le \frac{G_\mu^*}{2}\left(\frac{p_f+q_f r}{1-r_f}\right)\left((w_1-c)^2-(w_2-c)^2\right).
\end{aligned} \quad (33)$$

Obviously, from (33), we see that $w_1 \to w_2$; then, the right-hand side of (33) goes to zero, so $|N_2 z(w_2) - N_2 z(w_1)| \to 0$ as $w_1 \to w_2$. Hence, the operator N_2 is continuous. Also, $N(H) \subset H$; therefore, the operator N_2 is compact, and by the Arzela-Ascoli theorem, the operator N has at least one fixed point. Therefore, the mentioned problem (1) has at least one solution.

3. Stability Theory

In this portion, we develop several consequences concerning the stability of Hyers-Ulam and generalize Hyers-Ulam type. Before progressing further, we provide various notions and definitions:

Definition 10. The proposed problem (1) is Hyers-Ulam stable if at any $\varepsilon > 0$ for the given inequality

$$|{}_c^{CF}\mathbf{D}_w^\mu z(w) - f\left(w, z(w), {}_c^{CF}\mathbf{D}_w^\mu z(w)\right)| < \varepsilon, \text{for all } w \in I, \quad (34)$$

there exist a unique solution $\bar{z}(w)$ with a constant K_f such that

$$|z(w) - \bar{z}(w)| \le K_f \varepsilon, \text{for all } w \in I. \quad (35)$$

Further, the considered problem (1) will generalize Hyers-Ulam stable if there exists nondecreasing function $\phi : (c, d) \to (0,\infty)$ such that

$$|z(w) - \bar{z}(w)| \le K_f \phi(\varepsilon), \text{for all } w \in I, \quad (36)$$

with $\phi(c) = 0$ and $\phi(d) = 0$.

Also, we state an important remark as:

Remark 11. Let there exist a function $\psi(w)$ which depends on $z \in C[I, R]$ with $\psi(c) = 0$ and $\psi(d) = 0$ such that

$$\begin{aligned}
&|\psi(w)| \le \varepsilon, \text{for all } w \in I, \\
&{}_c^{CF}\mathbf{D}_w^\mu z(w) = f\left(w, z(w), {}_c^{CF}\mathbf{D}_w^\mu z(w)\right) + \psi(w), \text{for all } w \in I.
\end{aligned} \quad (37)$$

Lemma 12. *The solution of the given proposed problem*

$$\begin{aligned}
&{}_c^{CF}\mathbf{D}_w^\mu z(w) = f\left(w, z(w), {}_c^{CF}\mathbf{D}_w^\mu z(w)\right) + \psi(w), \textit{for all } w \in I, \\
&z(c) = 0, z(d) = 0.
\end{aligned} \quad (38)$$

is

$$\begin{aligned}
z(w) = &-\frac{G_\mu(w-c)}{(d-c)}\int_c^d \lambda(\zeta)d\zeta - \frac{G_\mu^*(w-c)}{(d-c)}\int_c^d (d-\zeta)\lambda(\zeta)d\zeta \\
&- \frac{G_\mu(w-c)}{(d-c)}\int_c^d \psi(\zeta)d\zeta - \frac{G_\mu^*(w-c)}{(d-c)}\int_c^d (d-\zeta)\psi(\zeta)d\zeta \\
&+ G_\mu \int_c^w \lambda(\zeta)d\zeta + G_\mu^* \int_c^w (w-\zeta)\lambda(\zeta)d\zeta + G_\mu \int_c^w \psi(\zeta)d\zeta \\
&+ G_\mu^* \int_c^w (w-\zeta)\psi(\zeta)d\zeta, \textit{for all } w \in I,
\end{aligned} \quad (39)$$

where $G_\mu = (2-\mu)/D(\mu-1)$, $G_\mu^* = (\mu-1)/D(\mu-1)$, *and* $\lambda(w) = f(w, z(w), \lambda(w))$. *Moreover, the solution of the given inequality, we have*

$$\left|z(w)-\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\lambda(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)\lambda(\zeta)d\zeta\right.\right.$$
$$\left.\left.+G_\mu\int_c^w\lambda(\zeta)d\zeta+G_\mu^*\int_c^w(w-\zeta)\lambda(\zeta)d\zeta\right]\right|$$
$$\leq 2\left(G_\mu(d-c)-G_\mu^*(d-c)^2\right)\varepsilon. \quad (40)$$

Proof. The solution of (39) can be acquired straightforward by using Lemma 5. Although from the solution, it is clear to become result (40) by using Remark 11.

Theorem 13. *Under the Lemma 12, the solution of the proposed problem (1) is Hyers-Ulam stable and also generalized Hyers-Ulam stable if* $(2G_\mu(d-c)-G_\mu^*(d-c)^2)(D_f/(1-E_f))<1$.

Proof. Let $z(w)\in C[I,R]$ be any solution of the considered problem (1) and $\bar{z}(w)\in C[I,R]$ be a unique solution of the said problem; then, we take,

$$\begin{aligned}|z(w)-\bar{z}(w)| &= \left|z(w)-\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\bar{\lambda}(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)\bar{\lambda}(\zeta)d\zeta+G_\mu\int_c^w\bar{\lambda}(\zeta)d\zeta+G_\mu^*\int_c^w(w-\zeta)\bar{\lambda}(\zeta)d\zeta\right]\right|,\\
&=\left|z(w)-\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\lambda(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)\lambda(\zeta)d\zeta+G_\mu\int_c^w\lambda(\zeta)d\zeta+G_\mu^*\int_c^w(w-\zeta)\lambda(\zeta)d\zeta\right]\right.\\
&\quad+\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\lambda(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)\lambda(\zeta)d\zeta+G_\mu\int_c^w\lambda(\zeta)d\zeta+G_\mu^*\int_c^w(w-\zeta)\lambda(\zeta)d\zeta\right]\\
&\quad-\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\bar{\lambda}(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\cdot\int_c^d(d-\zeta)\bar{\lambda}(\zeta)d\zeta+G_\mu\int_c^w\bar{\lambda}(\zeta)d\zeta\right.\\
&\quad\left.\left.+G_\mu^*\int_c^w(w-\zeta)\bar{\lambda}(\zeta)d\zeta\right]\right|,\\
&\leq\left|z(w)-\left[-\frac{G_\mu(w-c)}{(d-c)}\int_c^d\lambda(\zeta)d\zeta-\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)\lambda(\zeta)d\zeta+G_\mu\int_c^w\lambda(\zeta)d\zeta+G_\mu^*\int_c^w(w-\zeta)\lambda(\zeta)d\zeta\right]\right|\\
&\quad+\frac{G_\mu(w-c)}{(d-c)}\int_c^d|\lambda(\zeta)-\bar{\lambda}(\zeta)|d\zeta+\frac{G_\mu^*(w-c)}{(d-c)}\int_c^d(d-\zeta)|\lambda(\zeta)-\bar{\lambda}(\zeta)|d\zeta\\
&\quad+G_\mu\int_c^w|\lambda(\zeta)-\bar{\lambda}(\zeta)|d\zeta+G_\mu^*\int_c^w(w-\zeta)|\lambda(\zeta)-\bar{\lambda}(\zeta)|d\zeta.\end{aligned} \quad (41)$$

Using (40) and (18) in the above inequality, then taking maximum on both sides, we have

$$\begin{aligned}|z(w)-\bar{z}(w)| &\leq 2\left(G_\mu(d-c)-G_\mu^*(d-c)^2\right)\varepsilon+G_\mu(w-c)\left(\frac{D_f}{1-E_f}\right)|z(w)-\bar{z}(w)|\\
&\quad-\frac{G_\mu^*(w-c)(d-c)}{2}\left(\frac{D_f}{1-E_f}\right)|z(w)-\bar{z}(w)|+G_\mu(w-c)\left(\frac{D_f}{1-E_f}\right)|z(w)-\bar{z}(w)|\\
&\quad-\frac{G_\mu^*(w-c)^2}{2}\left(\frac{D_f}{1-E_f}\right)|z(w)-\bar{z}(w)|,\\
\|z-\bar{z}\| &\leq 2\left(G_\mu(d-c)-G_\mu^*(d-c)^2\right)\varepsilon+2G_\mu(d-c)\left(\frac{D_f}{1-E_f}\right)\|z-\bar{z}\|-G_\mu^*(d-c)^2\left(\frac{D_f}{1-E_f}\right)\|z-\bar{z}\|,\\
\|z-\bar{z}\| &\leq 2\left(G_\mu(d-c)-G_\mu^*(d-c)^2\right)\varepsilon+\left(2G_\mu(d-c)-G_\mu^*(d-c)^2\right)\left(\frac{D_f}{1-E_f}\right)\|z-\bar{z}\|.\end{aligned} \quad (42)$$

Hence, from the above inequality, we have

$$\|z-\bar{z}\|\leq\frac{2\left(G_\mu(d-c)-G_\mu^*(d-c)^2\right)\varepsilon}{1-\left(2G_\mu(d-c)-G_\mu^*(d-c)^2\right)\left(D_f/(1-E_f)\right)}. \quad (43)$$

Therefore, the solution is Hyers-Ulam stable. Further, let

$$K_f=\frac{2\left(G_\mu(d-c)-G_i^*(d-c)^2\right)}{1-\left(2G_\mu(d-c)-G_\mu^*(d-c)^2\right)\left(D_f/(1-E_f)\right)}, \quad (44)$$

and there exist nondecreasing function $\phi\in C((c,d),(0,\infty))$. Then, from (43) we can write as

$$\|z-\bar{z}\|\leq k_f\phi(\varepsilon),\ \text{with}\ \phi(c)=0,\ \phi(d)=0. \quad (45)$$

4. Examples of Our Analysis

In this part of our analysis, we justify certain obtained results through some counter examples which are given below.

Example 14. Suppose, we take the boundary value problem of implicit type as

$$\begin{cases} {}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w} z(w) = \dfrac{w^3}{35} + \dfrac{\cos|z(w)| + \cos\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w} z(w)\right|}{55+w^3}, w \in [0,1], \\ z(0) = 0,\ z(1) = 0. \end{cases} \tag{46}$$

Clearly, $c=0, d=1$ and $f(w, z(w), \lambda(w)) = (w^3/35) + (\cos|z(w)| + \cos|{}^{CF}_{c}\mathbf{D}^{1/3}_{w}\lambda(w)|/55+w^3)$ is a continuous function for all $x \in [0,1]$. Further, suppose that $z, \bar{z}, \lambda, \bar{\lambda} \in C[I, R]$; then, we consider as

$$\begin{aligned} &\left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &= \left|\frac{w^3}{35} + \frac{\cos|z(w)| + \cos\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w}\lambda(w)\right|}{55+w^3} - \frac{w^3}{35} \right. \\ &\quad \left. - \frac{\cos|\bar{z}(w)| - \cos\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w}\bar{\lambda}(w)\right|}{55+w^3}\right|, \\ &\le \left|\frac{\cos|z(w)| - \cos|\bar{z}(w)|}{55+w^3}\right| + \left|\frac{\cos|\lambda(w)| - \cos\left|\bar{\lambda}(w)\right|}{55+w^3}\right|, \end{aligned} \tag{47}$$

which implies that

$$\begin{aligned} &\left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &\le \frac{1}{55}\left(|z(w) - \bar{z}(w)| + \left|\lambda(w) - \bar{\lambda}(w)\right|\right). \end{aligned} \tag{48}$$

Since from (48), one has $D_f = 1/55$, $E_f = 1/55$, and $\mu = 1/3$. Further, also consider

$$\begin{aligned} |f(w, z(w), \lambda(w))| &= \left|\frac{w^3}{35} + \frac{\cos|z(w)| + \cos\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w}\lambda(w)\right|}{55+w^3}\right| \\ &\le \left|\frac{w^3}{35}\right| + \left|\frac{\cos|z(w)|}{55+w^3}\right| + \left|\frac{\cos\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{1/3}_{w}\lambda(w)\right|}{55+w^3}\right| \\ &\le \frac{1}{35} + \frac{1}{55}|z(w)| + \frac{1}{55}|\lambda(w)|. \end{aligned} \tag{49}$$

Therefore, $p_f = 1/35$, $q_f = 1/55$, $r_f = 1/55$. and $G_\mu = 1/3$, $G^*_\mu = 1/3$, $c=0$, and $d=1$. Then

$$\left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\left(\frac{D_f}{1-E_f}\right) = \frac{1}{27} < 1. \tag{50}$$

Therefore, the conditions of Theorem 7 are satisfied. Thus, the problem (46) has a unique solution. Further, we need to satisfy some conditions of theorem (9).

$$0 < \left(\frac{4G_\mu(d-c) - G^*_\mu(d-c)^2}{2}\right)\frac{D_f}{1-E_f} = \frac{1}{18} < 1. \tag{51}$$

Hence, the conditions of Theorem 9 also hold. Therefore, (46) has at least one solution. Furthermore, proceed to verify the stability results; we see that

$$\left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\left(\frac{D_f}{1-E_f}\right) = 0.370 < 1. \tag{52}$$

Hence, the solution of the mentioned problem (46) is Hyers-Ulam stable and consequently generalized Hyers-Ulam stable.

Example 15. Take another boundary value problem of implicit type as

$$\begin{cases} {}^{\mathrm{CF}}_{c}\mathbf{D}^{3/7}_{w} z(w) = \dfrac{w+e^{2w}}{15} + \dfrac{e^{3w}\sin|z(w)|}{45+w^2} + \dfrac{3w^2\sin\left|{}^{\mathrm{CF}}_{c}\mathbf{D}^{3/7}_{w} z(w)\right|}{65}, w \in [0,1], \\ z(0) = 0,\ z(1) = 0. \end{cases} \tag{53}$$

Clearly $c=0, d=1$ and $f(w, z(w), \lambda(w)) = ((w+e^{2w})/15) + ((e^{3w}\sin|z(w)|)/(45+w^2)) + ((3w^2\sin|{}^{\mathrm{CF}}_{c}\mathbf{D}^{3/7}_{w}\lambda(w)|)/65)$ is a continuous function for all $w \in [0,1]$. Further let $z, \bar{z}, \lambda, \bar{\lambda} \in C[I, R]$, then consider, we have

$$\begin{aligned} &\left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &= \left|\frac{w+e^{2w}}{15} + \frac{e^{3w}\sin|z(w)|}{45+w^2} + \frac{3w^2\sin\left|{}^{CF}_{c}\mathbf{D}^{3/7}_{w}\lambda(w)\right|}{65} \right. \\ &\quad \left. - \frac{w+e^{2w}}{15} - \frac{e^{3w}\sin|\bar{z}(w)|}{45+w^2} - \frac{3w^2\sin\left|{}^{CF}_{c}\mathbf{D}^{3/7}_{w}\bar{\lambda}(w)\right|}{65}\right| \\ &\le \frac{e^{3x}}{45+w^2}|z(w) - \bar{z}(w)| + \frac{3w^2}{65}\left|\lambda(w) - \bar{\lambda}(w)\right|, \end{aligned} \tag{54}$$

which implies that the maximum on right side to the above inequality, we have

$$\begin{aligned} &\left|f(w, z(w), \lambda(w)) - f\left(w, \bar{z}(w), \bar{\lambda}(w)\right)\right| \\ &\le \frac{1}{45}|z(w) - \bar{z}(w)| + \frac{3}{65}\left|\lambda(w) - \bar{\lambda}(w)\right|. \end{aligned} \tag{55}$$

Thus from (55), one has $D_f = 1/45$, $E_f = 3/65$, and $\mu = 3/7$. And also consider we have

$$\begin{aligned}
&|f(w, z(w), \lambda(w))| \\
&= \left|\frac{w+e^{2w}}{15} + \frac{e^{3w}\sin|z(w)|}{45+w^2} + \frac{3w^2 \sin\left|{}_{c}^{CF}\mathbf{D}_w^{3/7}\lambda(w)\right|}{65}\right| \\
&\le \left|\frac{w+e^{2w}}{15}\right| + \left|\frac{e^{3w}\sin|z(w)|}{45+w^2}\right| + \left|\frac{3w^2 \sin\left|{}_{c}^{CF}\mathbf{D}_w^{3/7}\lambda(w)\right|}{65}\right| \\
&\le \frac{1}{15} + \frac{1}{45}|z(w)| + \frac{3}{65}|\lambda(w)|,
\end{aligned} \tag{56}$$

where $p_f = 1/15$, $q_f = 1/45$, $r_f = 3/65$, and then $G_\mu = 1/200$, $G^*_\mu = 1/150$. Then

$$\left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\left(\frac{D_f}{1-E_f}\right) = \frac{13}{167400} < 1. \tag{57}$$

Therefore, the conditions of Theorem 7 are satisfied. Thus, the problem (53) has a unique solution. Further, we need to satisfy some conditions of Theorem (9), we have

$$0 < \left(\frac{4G_\mu(d-c) - G^*_\mu(d-c)^2}{2}\right)\frac{D_f}{1-E_f} = \frac{13}{83{,}700} < 1. \tag{58}$$

Hence, the conditions of Theorem (9) also hold. Therefore, (53) has at least one solution. Furthermore, proceed to verify stability results; we see that

$$\left(2G_\mu(d-c) - G^*_\mu(d-c)^2\right)\left(\frac{D_f}{1-E_f}\right) = 0.00007765 < 1. \tag{59}$$

Hence, the solution of the mentioned problem (53) is Hyers-Ualm stable and consequently generalized Hyers-Ulam stable.

5. Conclusion

We have successfully attained several essential conditions consistent to existence theory and stability theory for implicit type problem of DBCs with involving Caputo-Fabrizio fractional operator. By classical fixed point theory, we used some fixed point theorem like Krasnoselskii's fixed-point and Banach's contraction. Further, we studied certain stability results of Hyers-Ulam and generalized Hyers-Ulam stability. By appropriate illustrations, we have established the obtained investigation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest regarding this manuscript.

Authors' Contributions

All authors contribute equally to the writing of this manuscript. All authors read and approve the final version.

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