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Revisiting the Meir-Keeler Contraction via Simulation Function

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Abstract. In this paper, we aim to obtain a fixed point theorem which guarantee the existence of a fixed point for both the continuous and discontinuous mappings that fullfill certain conditions in the context of metric space. We also consider some examples to illustrate our results.

1. Introduction and preliminaries

Nonlinear integral equations play a key role in describing many real-world events [18–20]. In a nonlinear analysis, we are always looking for conditions that guarantee the existence of solutions of integral equations in various function spaces. It is worthwhile mentioning that the Fixed-point theory creates a powerful, instrumental and convenient branch of nonlinear analysis which is very applicable in proving existence theorems for several types of operator equations. Further, Fixed-point theory is one of the most thought-provoking research fields in nonlinear analysis. The many authors have been published papers and have been expanded frequently in the last decades. The main reason for this development can be observed easily for application point of view. Fixed point theory has an application in many disciplines such as chemistry, physics, biology, computer science and many branches of mathematics like Game theory and Economics (for details see [21, 22]). Banach contraction mapping principle or Banach fixed-point theorem is the most celebrated and pioneer result in this direction: In a complete metric space, each contraction mapping has a unique fixed point. Also, this principle has many generalizations see [24, 25] and others. For example, One of the important and peculiar generalizations is due to Meir and Keeler [26]. Their result can be stated as follows:

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Theorem 1.1. *(cf. [26])) Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction (MKC) on X, that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon$$

for all $x, y \in X$. Then T has a unique fixed point.

The class of Meir-Keeler contractions consists of the class of Banach contractions and many other classes of nonlinear contractions (see for example, [24]). Meir and Keeler's theorem was originator of further exploration in metric fixed point theory.

Khojasteh et al. [11] introduced the notion of simulation function.

Definition 1.2. *(cf. [11]) A* weak simulation function *is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ *for all* $t, s > 0$;

Let $\mathcal{Z}_w$ denote the family of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$. Due to the axiom ($\zeta_2$), we have

$$\zeta(t, t) < 0 \text{ for all } t > 0.$$

Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. Define a mapping $M : X \times X \to [0, \infty)$ as follows:

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Let $p : X \times X \to [0, \infty)$ a mapping. Consider the following conditions that were defined by Suzuki [17] to extend the coverage of Meir-Keeler theorem in the setting of metric spaces.

$(P_p^1 : M)$ $x \neq y$ and $d(x, Tx) \leq d(x, y) \Longrightarrow p(x, y) \leq M(x, y)$,

$(P_p^2 : c)$ $x_n \neq y$, $\lim_{n\to\infty} d(x_n, y) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ imply

$$\limsup_{n\to\infty} p(x_n, y) \leq c d(y, Ty), \text{ where } c \in [0, 1).$$

Very recently, Suzuki [17] proved the following interesting result:

Theorem 1.3. *[17] Let T be a self-mapping on a complete metric space (X, d). Let $p : X \times X \to [0, \infty)$ be mapping that satisfies the conditions $(P_p^1 : M)$ and $(P_p^2 : c)$ defined above. Suppose also that the followings are satisfied:*

(i) For any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and $p(x, y) < \varepsilon + \delta(\varepsilon)$ imply $d(Tx, Ty) \leq \varepsilon$,

(ii) $x \neq y$ and $p(x, y) > 0$ imply $d(Tx, Ty) < p(x, y)$.

Then T has a unique fixed point z. Moreover, the sequence $\{T^n x\}$ converges to z for all $x \in X$.

2. Main Results

Definition 2.1. *Let T be a self-mapping on a metric space (X, d) and $\zeta \in \mathcal{Z}_w$. Suppose that $p : X \times X \to [0, \infty)$ is a function that satisfies only $(P_p^1 : M)$. Then T is called hybrid contraction of type I if the following conditions are fulfilled:*

(a) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and

$$p(x, y) < \varepsilon + \delta(\varepsilon) \text{ imply } d(Tx, Ty) \leq \varepsilon.$$

(b) $x \neq y$ and $p(x, y) > 0$ imply

$$\zeta(\alpha(x, y)d(Tx, Ty), p(x, y)) \geq 0.$$

Remark 2.2. *If T is a hybrid contraction of type I then*

$$\alpha(x, y)d(Tx, Ty) < p(x, y), \tag{1}$$

for all distinct $x, y \in X$. Indeed, we have $d(x, y) > 0$ since $x \neq y$. If $p(x, y) = 0$, from (b) we have $d(Tx, Ty) < \varepsilon$ for any $\varepsilon > 0$. But, $\varepsilon > 0$ is arbitrary, then we obtain $Tx = Ty$. In this case $\alpha(x, y)d(Tx, Ty) = 0 \leq p(x, y)$. Otherwise, $p(x, y) > 0$ and if $Tx \neq Ty$ then $d(Tx, Ty) > 0$. If $\alpha(x, y) = 0$, the inequality (1) is satisfies. In the contrary, from (ζ_2) and (b) we get

$$0 \leq \zeta(\alpha(x, y)d(Tx, Ty), p(x, y)) < p(x, y) - \alpha(x, y)d(Tx, Ty),$$

so (1) holds.

Theorem 2.3. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a hybrid contraction of type I. Assume that the following conditions are satisfied:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(iii) T is continuous.

Then T has a fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. On account of the assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Owing to the fact that T is $\alpha-$orbital admissible, we can easily derive that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{2}$$

Again by using the assumption that T is triangular $\alpha-$orbital admissible, for any $n \in \mathbb{N}$, (2) yields that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}).,$$

Recursively, we conclude that

$$\alpha(x_n, x_{n+j}) \geq 1, \text{ for all } n, j \in \mathbb{N}. \tag{3}$$

Without loss of generality, we shall assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Indeed, if $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then $x^* = x_{n_0}$ forms a fixed point for T. It finishes the proof and hence we exclude this simple case.

On what follows we shall prove that the sequence $\{d(x_n, x_{n+1})\}$ is monotone. Hence, letting $x = x_n$ and $y = x_{n+1}$ in $(P_p^1 : M)$, we get that

$$0 < d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_n, x_{n+1}),$$

that implies

$$p(x_n, x_{n+1}) \le M(x_n, x_{n+1})$$

where,

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \tfrac{d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\}. \end{aligned} \tag{5}$$

On the other hand, by taking the triangle inequality into accounts, we observe that

$$\frac{d(x_n, x_{n+2})}{2} \le \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{+n+2})}{2} \le \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

According the observation above, we conclude that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

In the view of such information, by Definition 2.1 (b), we find that

$$0 \le \zeta(\alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}), p(x_n, x_{n+1})) < p(x_n, x_{n+1}) - \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1})$$

that is equivalent to

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \le \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \\ &< p(x_n, x_{n+1}) \le M(x_n, x_{n+1}). \end{aligned} \tag{6}$$

Notice that (6) yields a contradiction for the case $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$. Thus, we have

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}\} = d(x_n, x_{n+1}). \tag{7}$$

Moreover, by (6), we deduce that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of non-negative reals. Accordingly, there is some $\ell \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \ell$. Let $0 < \varepsilon = \ell$. We also note that

$$\varepsilon = \ell < d(x_n, x_{n+1}). \tag{8}$$

On the other hand, from (6) and (7), we have $p(x_n, x_{n+1}) \le d(x_n, x_{n+1}) < \varepsilon + \delta(\varepsilon)$ for n sufficiently large. So, it implies, from Definition 2.1 (a), that

$$d(Tx_n, Tx_{n+1}) \le \varepsilon.$$

Combining (8) together with the inequality above, we get that

$$\varepsilon < d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \le \varepsilon,$$

a contradiction. Then, we conclude that $\varepsilon = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon_1 > 0$ fixed. From (9), we can choose $k \in \mathbb{N}$ large enough such that there exists $\delta_1 > 0$, with

$$d(x_k, x_{k+1}) < \frac{\delta_1}{2}. \tag{10}$$

Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon_1) < \varepsilon_1$. By using the induction method to prove that

$$d(x_k, x_{k+m}) < \varepsilon_1 + \frac{\delta_1}{2}, \tag{11}$$

for all $k, m \in \mathbb{N}$. We already have (11) for $m = 1$. Suppose that (11) is satisfied for some $m = j$. We shall show that (11) holds for $m := j + 1$. On account of (10) and (11) we, first, observe that

$$\begin{aligned} \frac{d(x_k, x_{k+j+1}) + d(x_{k+j}, x_{k+1})}{2} &\leq \frac{1}{2}\Big(d(x_k, x_{k+j}) + d(x_{k+j}, x_{k+j+1}) + d(x_{k+j}, x_k) + d(x_k, x_{k+1})\Big) \\ &< \frac{1}{2}\left(2\varepsilon_1 + 2\frac{\delta_1}{2} + 2\frac{\delta_1}{2}\right) \\ &\leq \varepsilon_1 + \delta_1. \end{aligned} \tag{12}$$

Thus, we have

$$\begin{aligned} M(x_k, x_{k+j}) &= \max\left\{d(x_k, x_{k+j}), d(x_k, x_{k+1}), d(x_{k+j}, x_{k+j+1}), \tfrac{d(x_k, x_{k+j+1}) + d(x_{k+j}, x_{k+1})}{2}\right\} \\ &< \max\left\{\varepsilon_1 + \tfrac{\delta_1}{2}, \tfrac{\delta_1}{2}, \varepsilon_1 + \delta_1\right\} = \varepsilon_1 + \delta_1. \end{aligned} \tag{13}$$

Definition 2.1 (a), the above inequality implies that $d(x_{k+1}, x_{k+j+1}) = d(Tx_k, Tx_{k+j}) \leq \varepsilon_1$. By employing the triangle inequality, together with (3) we get

$$\begin{aligned} d(x_k, x_{k+j+1}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+j+1}) = d(x_k, x_{k+1}) + d(Tx_k, Tx_{k+j}) \\ &\leq d(x_k, x_{k+1}) + d(Tx_k, Tx_{k+j}) < \tfrac{\delta_1}{2} + \varepsilon_1. \end{aligned} \tag{14}$$

Therefore, (11) holds for $m := j + 1$. Hence, $d(x_k, x_{k+m}) < \varepsilon_1$ for all $k, m \in \mathbb{N}$. In other words, for $m > n$ we have $\limsup_{n\to\infty} d(x_n, x_m) = 0$ and the sequence $\{x_n\}$ is Cauchy. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

To finalize the proof, we shall indicate that u is the fixed point of T. Indeed, by using the definition $x_{n+1} = Tx_n$ and taking, the continuity of T, into account, we obtain $u = Tu$, that is, u is a fixed point of T. □

In the following, we will give another variant of the theorem in order to weaken the conditions for the existence of fixed points (often due the continuity of mapping T see e.g. Bisht in [16]).

Theorem 2.4. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a hybrid contraction of type I satisfying the following conditions:*

(i) T is triangular α-orbital admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(iii) T^2 is continuous,

Then $\{T^n x\}$ converges to u for all $x \in X$. Moreover if $\alpha(u, Tu) \geq 1$ then u is a fixed point for T.

Remark 2.5. *T is discontinuous at u if and only if $\lim_{n\to\infty} M(x, u) \neq 0$.*

Proof. By following the lines in the proof of Theorem 2.3, we derive that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, and that there exist $u \in X$ such that $x_n \to u$. Regarding the fact that any subsequence of $\{x_n\}$ converges to the same limit point u, we get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = u \text{ and } \lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2 x_n = u. \tag{15}$$

On the other hand, due the continuity of T^2, (hypothesis (iii)), $T^2 u = \lim_{n\to\infty} T^2 x_n = u$. We claim that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$ and $p(u, Tu) > 0$ we have

$$p(u, Tu) \leq M(u, Tu) = \max\left\{d(u, Tu), d(Tu, T^2 u), \frac{d(u, T^2 u) + d(Tu, Tu)}{2}\right\} = d(u, Tu). \tag{16}$$

Therefore, together with supplementary hypothesis $\alpha(u,Tu) \geq 1$ (since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T^n x_0 = u$) we obtain

$$0 \leq \zeta\left(\alpha(u,Tu)d(Tu,T^2u), p(u,Tu)\right),$$

and also

$$0 < d(Tu,u) = d(Tu,T^2u) \leq \alpha(u,Tu)d(Tu,T^2u) < p(u,Tu) \leq M(u,Tu) = d(u,Tu), \tag{17}$$

which is an contradiction. Hence u is a fixed point of T. □

Definition 2.6. *A metric space (X,d) is called regular if for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n,u) = 0$ and satisfying $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n,u) \geq 1$ for all $n \in \mathbb{N}$.*

Theorem 2.7. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a hybrid contraction of type I. Suppose that $(P_p^2 : c)$ holds and the followings are satisfied:*

(i) T is triangular α-orbital admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,

(iii) (X,d) is regular

Then $\{T^n x\}$ converges to u for all $x \in X$. Moreover u is a fixed point for T.

Proof. By following the lines in the proof of Theorem 2.3, we get a convergent sequence $\{x_n\}$ with a limit $u \in X$. Notice also that all adjacent terms in $\{x_n\}$ are distinct. Moreover, we note $T^n x \neq u$ for all $n \in \mathbb{N} \cup \{0\}$. Regarding the limits $\lim_{n\to\infty} d(x_n,u) = 0$ and $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we derive from $(P_p^2 : c)$ that

$$\limsup_{n\to\infty} \rho(x_n,u) \leq cd(u,Tu) \text{ for any } c \in [0,1). \tag{18}$$

Again by the proof of Theorem 2.3 $\alpha(x_n,x_{n+1}) \geq 1$. So, by the assumption (iii), we get $\alpha(x_n,u) \geq 1$.

On what follows we prove that u is a fixed point of T. Suppose that, on the contrary, $Tu \neq u$. By substituting $x = x_n$ and $y = u$ in Definition 2.1 (b), we obtain

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(x_n,u)d(Tx_n,Tu), p(x_n,u)\right) \\ &= \zeta\left(\alpha(x_n,u)d(Tx_n,Tu), p(x_n,u)\right) \\ &< p(x_n,u) - \alpha(x_n,u)d(Tx_n,Tu) \end{aligned}$$

which is equivalent to

$$d(x_{n+1},Tu) = d(Tx_n,Tu) \leq \alpha(x_n,u)d(Tx_n,Tu) < p(x_n,u). \tag{19}$$

By taking into account (18) together with letting $n \to \infty$ in (19), we find

$$d(u,Tu) = \limsup_{n\to\infty} d(x_n,u) < \limsup_{n\to\infty} p(x_n,u) \leq cd(u,Tu) \text{ for any } c \in [0,1),$$

a contradiction. Hence, u is a fixed point of T. □

Theorem 2.8. *Suppose an extra condition*

(U) $\alpha(u,v) \geq 1$ for $u,v \in Fix(T)$

in additional to the hypotheses of Theorem 2.3 (resp. Theorem 2.4 and Theorem 2.7). Then, the mapping T has a unique fixed point.

Proof. Suppose, on the contrary, that there is another fixed point v of T with $u \neq v$. Notice, first, that the case $p(u,v) = 0$ is impossible since we have $Tu = Tv$ and $0 < d(u,v) = d(Tu,Tv) = 0$ a contradiction. Accordingly, we get that $p(u,v) > 0$. Since $0 = d(u,Tu) \leq d(u,v)$ then by $(P_p^1 : M)$, we get $p(u,v) \leq M(u,v)$, where

$$M(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2}\right\} = d(u,v).$$

So, by Definition 2.1 (b), we have

$$0 \leq \zeta(\alpha(u,v)d(Tu,Tv), p(u,v)) < p(u,v) - \alpha(u,v)d(Tu,Tv), \tag{20}$$

and the condition U implies

$$0 < d(u,v) = d(Tu,Tv) \leq \alpha(u,v)d(Tu,Tv) < p(u,v) \leq d(u,v). \tag{21}$$

So, we get $d(u,v) = 0$ which completes the proof. □

We define a a mapping $N : X \times X \to [0,\infty)$ as follows

$$N(x,y) = \max\left\{d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}, d(x,y)\right\},$$

where T is a self-mapping defined on a metric space (X,d). Notice that for any $x,y \in X$ with $x = y$ we have $0 = d(Tx,Ty) \leq N(x,y)$. Moreover, if $x \neq y$, then $N(x,y) > 0$.

Definition 2.9. *Let T be a self-mapping on a metric space (X,d) and $\zeta \in \mathcal{Z}_w$. Suppose that $p : X \times X \to [0,\infty)$ is a function that satisfies $(P_p^1 : N)$ and $(P_p^2 : c)$, for all $c \in [0,1)$. Then T is called hybrid contraction of type II if the following conditions are satisfies:*

(a) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and

$$p(x,y) < \varepsilon + \delta(\varepsilon) \text{ imply } d(Tx,Ty) \leq \varepsilon.$$

(b) $x \neq y$ and $p(x,y) > 0$ imply

$$\zeta(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0. \tag{22}$$

Theorem 2.10. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a hybrid contraction of type II. Assume that the following conditions are fulfilled:*

- (i) *T is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
- (iii) *either T is continuous*
- $(iii)'$ *or T^2 is continuous and $\alpha(u,Tu) \geq 1$*
- $(iii)'$ *or (X,d) is regular.*

Then T has a fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. The proof is the mimic of the proof of Theorem 2.3. As in the proof of Theorem 2.3 we shall built a recursive sequence $\{x_n\}$, for an arbitrary initial value $x_0 \in X$ as follows:

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{23}$$

One can conclude also that $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, from (i) and (ii), as in the proof of Theorem 2.3. Throughout the proof, we assume

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}. \tag{24}$$

Indeed, as it is discussed in the proof of Theorem 2.3, the other case is trivial and it is excluded.

Now, by letting $x = x_n$ and $y = x_{n+1}$ in $(P_p^1 : N)$, we have $d(x_n, Tx_n) \leq d(x_n, x_{n+1})$ which implies $p(x_n, x_{n+1}) \leq N(x_n, x_{n+1})$, where

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{d(x_{n+1}, Tx_{n+1})\frac{1 + d(x_n, Tx_n)}{1 + d(x_n, x_{n+1})}, d(x_n, x_{n+1})\right\} \\ &= \max\left\{d(x_{n+1}, x_{n+2})\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}, d(x_n, x_{n+1})\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}. \end{aligned}$$

Regarding that T is a hybrid contraction of type II, we have

$$0 \leq \zeta(\alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}), p(x_n, x_{n+1})), \tag{25}$$

by replacing the pair x, y with the pair x_n, x_{n+1} in (22). Consequently, the inequality (25) yields that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) \\ &< p(x_n, x_{n+1}) \leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}. \end{aligned}$$

Thus, the above inequality implies that

$$N(x_n, x_{n+1}) = \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} = d(x_n, x_{n+1}), \tag{26}$$

and hence $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Consequently, there exists a real n.umber ℓ such that $d(x_n, x_{n+1}) \to \ell$ as $n \to \infty$. Suppose that $\ell = \varepsilon > 0$. First, we note that $\varepsilon = \ell < d(x_n, x_{n+1})$. On the other hand, from (26) there exists $\delta > 0$ such that $p(x_n, x_{n+1}) \leq N(x_n, x_{n+1}) = d(x_n, x_{n+1}) < \varepsilon + \delta(\varepsilon)$ for n sufficiently large. Keeping the observations above, Definition 2.9 (a) yields that

$$d(Tx_n, Tx_{n+1}) \leq \varepsilon.$$

Therefore, we have

$$\varepsilon < d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \varepsilon,$$

a contradiction. So, we derive that $\varepsilon = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{27}$$

On what follows we shall indicate that the sequence $\{x_n\}$ is Cauchy. For this aim, let $m \in \mathbb{N}$ large enough to satisfy $d(x_m, x_{m+1}) < \delta_1$. We will show, by induction, that

$$d(x_m, x_{m+k}) < \varepsilon_1 + \delta_1, \tag{28}$$

for all $k \in \mathbb{N}$. Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon) < \varepsilon$. We have already proved for $k = 1$, so, consider the following two situations.

(1) If $d(x_{m+k}, x_{m+k+1}) \leq d(x_m, x_{m+k})$ then we find

$$\frac{d(x_{m+k}, x_{m+k+1})}{1 + d(x_m, x_{m+k})} \leq d(x_{m+k}, x_{m+k+1})$$

and

$$\frac{d(x_{m+k}, x_{m+k+1})d(x_m, x_{m+1})}{1 + d(x_m, x_{m+k})} < d(x_m, x_{m+1}).$$

Hence, we have

$$\begin{aligned} p(x_m, x_{m+k}) &\leq N(x_m, x_{m+k}) = \max\left\{\tfrac{d(x_{m+k},x_{m+k+1})[1+d(x_m,x_{m+1})]}{1+d(x_m,x_{m+k})}, d(x_m, x_{m+k})\right\} \\ &\leq \max\{d(x_{m+k}, x_{m+k+1}) + d(x_m, x_{m+1}), d(x_m, x_{m+k})\} < \max\{2\delta_1, \varepsilon_1 + \delta_1\} \\ &< \varepsilon_1 + \delta_1, \end{aligned}$$

and from Definition 2.9, we get that $d(Tx_m, Tx_{m+k}) \leq \varepsilon_1$. So, we have,

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) = d(x_m, x_{m+1}) + d(Tx_m, Tx_{m+k}) < \varepsilon_1 + \delta_1. \tag{29}$$

(2) If $d(x_{m+k}, x_{m+k+1}) > d(x_m, x_{m+k})$ then

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+1}) < 2d(x_{m+k}, x_{m+k+1}) < 2\delta_1 < \varepsilon_1 + \delta_1. \tag{30}$$

Thus, by induction, (28) holds for every $k \in \mathbb{N}$. Since $\varepsilon_1 > 0$ is arbitrary, we get

$$\lim_{p\to\infty} \sup d(x_m, x_{m+p}) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in a complete metric space (X, d).

Hence, $\{x_n\}$ converges to some $u \in X$. Next, we will prove that u is a fixed point of T. For first assumption, since T is continuous, we derive that

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = 0$$

that is, the sequence $\{x_n\}$ converges to Tu as well. Since the limit is unique, we conclude that $Tu = u$ which completes the proof.

For the second assumption, since the sequence $x_n \to u$ we get that any subsequence of $\{x_n\}$ converges to the same limit point u, so

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = u \text{ and } \lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2x_n = u. \tag{31}$$

On the other hand, due the continuity of T^2, $T^2u = \lim_{n\to\infty} T^2x_n = u$. We claim that $Tu = u$. In the contrary, if $Tu \neq u$, we have $p(u, Tu) > 0$ and

$$p(u, Tu) \leq N(u, Tu) = \max\left\{d(Tu, T^2u)\frac{1 + d(u, Tu)}{1 + d(u, Tu)}, d(u, Tu)\right\} = d(u, Tu).$$

Therefore, together with supplementary hypothesis $\alpha(u, Tu) \geq 1$ (since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T^nx_0 = u$) we have

$$0 \leq \zeta(\alpha(u, Tu)d(Tu, T^2u), p(Tu, T^2u)) < p(Tu, T^2u) - \alpha(u, Tu)d(Tu, T^2u),$$

and from here,

$$0 < d(Tu, u) = d(Tu, T^2u) \leq \alpha(u, Tu)d(Tu, T^2u) < p(u, Tu) \leq N(u, Tu) = d(u, Tu), \tag{32}$$

which is an contradiction. Hence u is a fixed point of T. For the last alternative assumption, we deduce that $d(u, Tu) = 0$, using the same arguments as in Theorem 2.7. This means that u is a fixed point of T. □

Theorem 2.11. *Suppose an extra condition*

(U) $\alpha(u,v) \geq 1$ for $u,v \in Fix(T)$

in additional to the hypotheses of Theorem 2.10 we obtain that the mapping T has a unique fixed point.

We skip the proof by regarding the analogy with the proof of Theorem 2.8.

Example 2.12. *Let $X = [0,4]$, $d : X \times X \to [0,\infty)$ defined by $d(x,y) = \left|x-y\right|$ and let a continuous mapping $T : X \to X$ defined by*

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [2,4] \\ 1, & \text{if } x \in [0,2) \end{cases}$$

Let also a function $\alpha : X \times X \to [0,\infty)$,

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [2,4] \\ 2, & \text{if } x,y \in [0,2) \\ 0, & \text{otherwise} \end{cases}$$

and $q : X \times X \to [0,\infty)$, where $q(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, d(x,y)\right\}$.
First of all, we note that q satisfies condition $(P : q : N)$ and $q(x,y) > 0$ for all $x \neq y$. Since, for $x = 0$ we have $T0 = 1$ and $\alpha(0,T0) = \alpha(0,1) = 2 > 1$ so assumption (ii) by Theorem 2.10 is satisfied. Also, is easy to see that T is triangular α-orbital admissible. Let $\zeta \in \mathcal{Z}_w$, for example, $\zeta(t,s) = \frac{4s}{5} - t$. We consider the following cases:
(1) For $x,y \in [0,2)$, $x \neq y$ we have $d(Tx,Ty) = 0$, so

$$\zeta\left(\alpha(x,y)d(Tx,Ty), q(x,y)\right) = \frac{4q(x,y)}{5} > 0.$$

(2) For $x,y \in [2,4]$, $x \neq y$ we have $d(Tx,Ty) = \frac{\left|x-y\right|}{4}$, $q(x,y) = \max\left\{\frac{\frac{3x}{4}\cdot\frac{3y}{4}}{1+\left|x-y\right|}, \left|x-y\right|\right\}$, so

$$\zeta\left(\alpha(x,y)d(Tx,Ty), q(x,y)\right) = \frac{4q(x,y)}{5} - \frac{\left|x-y\right|}{4} \geq 0.$$

(3) In all other cases, we have $\alpha(x,y) = 0$, and

$$\zeta\left(\alpha(x,y)d(Tx,Ty), q(x,y)\right) = q(x,y) > 0.$$

Thus, T satisfies the conditions of Theorem 2.10 and has a unique point of $x = 1$.

3. Consequences

Theorem 3.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a hybrid contraction of type I, whith p(x,y)=d(x,y). Assume that the following conditions are satisfies:*

(i) *T is triangular α-orbital admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,*

(iii) *either T is continuous or T^2 is continuous and $\alpha(u,Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Remark 3.2. *In this case, since $x \neq y$ implies $d(x,y) > 0$ it is obviously that (b) from Definition 2.1 is equivalent to the following:*
(b') $d(x,y) > 0$ implies $\zeta(\alpha(x,y)d(Tx,Ty), d(x,y)) \geq 0$.

Proof. It is clear that d satisfies the conditions $(P : d : M)$, respectively $(P : d, 0)$ and so all assumptions by Theorem 2.3, Theorem 2.4, Theorem 2.7 are also satisfied. □

Example 3.3. *Let $X = [0,4]$ and we endow X with usual metric. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by*

$$T(x) = \begin{cases} 1, & \text{if } x \in [0,4) \\ 0, & \text{if } x = 4 \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 3, & \text{if } (x,y) \in [0,4) \times [0,4) \\ 2, & \text{if } (x,y) \in \{\{4\} \times [0,2)\} \cup \{[0,2) \cup \{4\}\} \\ 4, & \text{if } x = y = 4 \\ 0, & \text{otherwise} \end{cases}$$

Since $Tx \in \{0,1\}$ for all $x \in [0,4]$ we have $\alpha(x,Tx) \geq 1$. It is clear that T is triangular α-orbital admissible. Also, for $x_0 = 0$, $\alpha(0,T0) = \alpha(0,1) = 3 > 1$ so assumption (ii) is satisfied. The mapping T is not continuous, but $T^2x = 1$ for any $x \in [0,4]$ which shows that T^2 is continuous. Since $d(x,y) \leq M(x,y)$ for all $x,y \in [0,4]$, it remains to be verified that (Definition 2.1 (b)) holds. We will consider the following cases:
(1) If $x,y \in [0,4)$, $x \neq y$ then $d(Tx,Ty) = 0$ and

$$\alpha(x,y)d(Tx,Ty) = 0 < d(x,y).$$

Therefore,

$$\zeta(\alpha(x,y)d(Tx,Ty), d(x,y)) \geq 0,$$

for all $x,y \in [0,4)$, $x \neq y$.
(2) If $x \in [0,2)$ and $y = 4$ then $d(Tx,T4) = 1$ and $d(x,4) = 4 - x$. We have in this case:

$$\alpha(x,4)d(Tx,T4) = 2 < 4 - x = d(x,4) \Longleftrightarrow x < 2.$$

Then,

$$\zeta(\alpha(x,4)d(Tx,T4), d(x,4)) \leq 0.$$

(3) If $x \in [2,4)$ and $y = 4$ then $\alpha(x,4) = 0$ and Definition 2.1 (b) hold. Note that $d(x,y) \leq M(x,y)$ for any $x,y \in X$, $x \neq y$. For this reasons, we conclude that all assumptions of Theorem 3.1 are satisfied and T has a fixed point, $x = 1$.

Theorem 3.4. *Let (X,d) be a complete metric space, $T : X \to X$ be a hybrid contraction of type I. Let $\rho : X \times X \to [0,\infty)$ defined by*

$$\rho(x,y) = a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty),$$

where $a_1, a_2, a_3 \in [0,1)$, $a_1 + a_2 \leq \frac{1}{2}$ and $a_3 \leq \frac{1}{2}$. Assume also that:

- (i) *T is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,*
- (iii) *either T is continuous or T^2 is continuous and $\alpha(u,Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.

Proof. Let $x, y \in X$ such that $x \neq y$ and $d(x, Tx) \leq d(x, y)$. Then,

$$\begin{aligned} \rho(x,y) &= a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) \leq (a_1 + a_2) d(x,y) + a_3 d(y,Ty) \\ &\leq \frac{d(x,y) + d(y,Ty)}{2} \leq M(x,y) \end{aligned}$$

which shows that $(P : \rho : M)$ holds. On the other hand, if $x_n \neq y$, $\lim_{n\to\infty} d(x_n, y) = 0$, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ hold, then we have

$$\limsup_{n\to\infty} \rho(x_n, y) = \limsup_{n\to\infty} [a_1 d(x_n, y) + a_2 d(x_n, x_{n+1}) + a_3 d(y, Ty)] = a_3 d(y, Ty),$$

Thus, $(P : \rho, a_3)$ holds. □

Example 3.5. *Let $X = \{0, 1, 3\}$ and $d_X \times X \to [0, \infty)$, $d(x, y) = \left|x - y\right|$. Then (X, d) is a complete metric space. Let mapping $T : X \to X$ defined by $T0 = T1 = 0$ and $T3 = 1$. Define $\alpha : X \times X \to [0, \infty)$, by*

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in \{1, 3\} \\ 1, & \text{if } x, y \in \{0, 3\} \\ 3, & \text{if } x, y \in \{0, 1\} \end{cases}$$

and $\rho(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$, $a_1 = \frac{1}{4}, a_2 = \frac{1}{8}, a_3 = \frac{3}{8}$. Clearly, T is triangular α-orbital admissible, and since $\alpha(0, T0) = \alpha(0, 0) = 1$ then (ii) holds. Also, is easy to see that (X, d) is regular, since if $T^n x = 0$ and $\alpha(T^n x, 0) = \alpha(0, 0) = 1$. We have:

$$\begin{aligned} d(0,T0) &= 0, d(1,T1) = 1, d(3,T3) = 2, d(T0,T1) = 0, \\ d(T0,T3) &= 1.d(T1,T3) = 1, d(0,1) = 1, d(0,3) = 3, d(1,3) = 2 \end{aligned}$$

so is easy to see that for $x \neq y$ and $d(x, Tx) \leq d(x, y)$, hence condition $(P1 : \rho : M)$ hold. For $x = 0, y = 1$ we have $d(T0, T1) = 0$, so conditions (a) and (b) by Definition 2.1 are satisfied, which shows that T is a ρ-α-orbital admissible contraction.
For $x = 0$, $y = 3$, we have $\rho(0, 3) = \frac{1}{4} \cdot 3 + \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 2 = \frac{6}{4}$ and

$$\alpha(0,3) d(T0,T3) = 1 \cdot 1 < \frac{6}{4} = \rho(0,3),$$

so

$$\zeta(\alpha(0,3) d(T0,T3), \rho(0,3)) \geq 0$$

For $x = 1$, $y = 3$, we have $\rho(1, 3) = \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 1 + \frac{3}{8} \cdot 2 = \frac{11}{8}$ and

$$\alpha(1,3) d(T1,T3) = 1 < \frac{11}{8} = \rho(1,3),$$

so

$$\zeta(\alpha(1,3) d(T1,T3), \rho(1,3)) \geq 0$$

T satisfies the condition of Theorem 3.4 and has a unique point of $x = 0$.

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