Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Simplified and improved criteria for oscillation of delay differential equations of fourth order

O. Moaaz[1*], A. Muhib[1,2], D. Baleanu[3], W. Alharbi[4] and E.E. Mahmoud[5]

*Correspondence:
o_moaaz@mans.edu.eg
[1]Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
Full list of author information is available at the end of the article

Abstract

An interesting point in studying the oscillatory behavior of solutions of delay differential equations is the abbreviation of the conditions that ensure the oscillation of all solutions, especially when studying the noncanonical case. Therefore, this study aims to reduce the oscillation conditions of the fourth-order delay differential equations with a noncanonical operator. Moreover, the approach used gives more accurate results when applied to some special cases, as we explained in the examples.

MSC: 34C10; 34K11

Keywords: Delay argument; Noncanonical operator; Fourth-order; Oscillation; Differential equations

1 Introduction and preliminaries

Delay differential equations (DDEs) are of great importance in modeling many phenomena and problems in various applied sciences, see [13]. The mounting interest in studying the qualitative properties of solutions of DDEs is easy to notice, see for example [1–12] and [14–25]. However, the equations with noncanonical operator did not receive the same attention as the equations in the canonical case. One can trace the evolution in the study of the oscillatory properties of higher-order DDEs with noncanonical operator through works of Baculikova et al. [7], Zhang et al. [23–25], and, recently, Moaaz et al. [16, 18].

This study is concerned with finding sufficient oscillation conditions for the solutions of the DDE

$$\left(a(l)\left(v'''(l)\right)^{\kappa}\right)' + f\left(l, v\left(g(l)\right)\right) = 0, \quad l \geq l_0, \tag{1.1}$$

in the noncanonical case, that is,

$$\psi_0(l_0) := \int_{l_0}^{\infty} \frac{1}{(a(\nu))^{1/\kappa}} \, \mathrm{d}\nu < \infty. \tag{1.2}$$

In this study, we suppose that $\kappa > 0$ is a ratio of odd integers, $a \in C^1(I_0, \mathbb{R}^+)$, $a'(l) \geq 0$, $g \in C(I_0, \mathbb{R}^+)$, $g(l) \leq l$, $g'(l) > 0$, $\lim_{l\to\infty} g(l) = \infty$, $I_\vartheta := [l_\vartheta, \infty)$, $f \in C(I_0 \times \mathbb{R}, \mathbb{R})$, and there exists a function $h \in C(I_0, [0,\infty))$ such that $f(l, v) \geq h(l)v^\kappa$.

By a solution of (1.1), we mean a nontrivial real-valued function $v \in C([l_\varkappa, \infty), \mathbb{R})$ for some $l_\varkappa \geq l_0$, which has the property $a(v''')^\kappa \in C^1([l_0,\infty), \mathbb{R})$ and satisfies (1.1) on $[l_0, \infty)$. We will consider only those solutions of (1.1) which exist on some half-line $[l_\varkappa, \infty)$ and satisfy the condition

$$\sup\{|v(l)| : l_c \leq l < \infty\} > 0 \quad \text{for any } l_c \geq l_\varkappa.$$

If v is either positive or negative, eventually, then v is called nonoscillatory; otherwise it is called oscillatory. Equation (1.1) itself is termed oscillatory if all its solutions are oscillatory.

Zhang et al. [25] considered the higher-order DDE

$$\left(a\left(v^{(n-1)}\right)^\kappa\right)'(l) + h(l)v^\gamma\left(g(l)\right) = 0, \tag{1.3}$$

where κ, γ are a ration of odd integers and $0 < \gamma \leq \kappa$. Moreover, Zhang et al. [23] studied the oscillation of solutions for (1.3) and improved the results [25]. For the convenience of the reader, we present some of their results below at $\kappa = \gamma$ and $n = 4$.

Theorem 1.1 ([25, Corollary 2.1]) *If*

$$\liminf_{l\to\infty} \int_{g(l)}^{l} h(s)\frac{(g^3(s))^\kappa}{a(g(s))}\,\mathrm{d}s > \frac{(3!)^\kappa}{\mathrm{e}} \tag{1.4}$$

and

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(h(s)\left(\frac{\varepsilon_1 \psi_0(s) g^2(s)}{2!}\right)^\kappa - \frac{\kappa^{\kappa+1}}{(\kappa+1)^{\kappa+1}} \frac{1}{\psi_0(s)a^{1/\kappa}(s)}\right)\mathrm{d}s = \infty \tag{1.5}$$

for some $\varepsilon_1 \in (0,1)$, *then every nonoscillatory solution of* (1.1) *tends to zero.*

Theorem 1.2 ([23, Corollary 2.1]) *If* (1.4), (1.5), *and*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(h(s)a^\kappa(s) - \frac{\kappa^{\kappa+1}}{(\kappa+1)^{\kappa+1}} \frac{(a'(s))^{\kappa+1}}{a(s)a_*^\kappa(s)}\right)\mathrm{d}s = \infty \tag{1.6}$$

for some $\varepsilon_1 \in (0,1)$, *where*

$$a(s) = \int_l^\infty (\eta - l)\psi_0(\eta)\,\mathrm{d}\eta$$

and

$$a_*(s) = \int_l^\infty \psi_0(\eta)\,\mathrm{d}\eta,$$

then (1.1) *is oscillatory.*

Dzurina and Jadlovska [9] considered the second-order DDE

$$\big(a(l)\big(v'(l)\big)^{\kappa}\big)' + h(l)v^{\kappa}\big(g(l)\big) = 0. \tag{1.7}$$

Moreover, Dzurina et al. [10] investigated the oscillation of solutions for (1.7) and improved the results [9].

Theorem 1.3 ([9, Theorem 3]) *Assume that*

$$\limsup_{l\to\infty} \psi_0^{\kappa}(l) \int_{l_0}^{l} h(s)\,\mathrm{d}s > 1.$$

Then (1.7) *is oscillatory.*

Theorem 1.4 ([10, Theorem 2.3]) *Let*

$$\int_{l_0}^{\infty} \frac{1}{a^{1/\kappa}(l)} \left(\int_{l_0}^{l} h(s)\,\mathrm{d}s \right)^{1/\kappa} \mathrm{d}l = \infty$$

hold. If

$$k := \liminf_{l\to\infty} \frac{1}{\psi(l)} \int_{l}^{\infty} \psi^{\kappa+1}(s)h(s)\,\mathrm{d}s > \kappa$$

or

$$k \le \kappa \quad \textit{and} \quad K > 1 - \frac{k}{\kappa},$$

where

$$K := \limsup_{l\to\infty} \psi(l) \left(\int_{l_0}^{l} h(s)\,\mathrm{d}s \right)^{1/\kappa} > 1,$$

then (1.7) *is oscillatory.*

The objective of this paper is to improve and simplify the oscillation criteria of the fourth-order DDE (1.1) in the noncanonical case. In the noncanonical case, it is usual to have oscillation criteria in the form of at least three independent conditions; however, in Sect. 2, we obtain only two independent conditions that guarantee the oscillation of all solutions. In Sect. 3, we take an approach that creates improved criteria for oscillation. Further, the examples provided illustrate the significance of the results.

Lemma 1.1 ([5]) *Assume that* $F \in C^m(I_0, \mathbb{R})$ *and* $F^{(m)}(l)$ *is eventually of constant sign. Then there are* $l_u \ge l_0$ *and* $\ell \in \mathbb{Z}$, $0 \le \ell \le m$, *with* $m + \ell$ *even for* $F^{(m)}(l) \ge 0$ *or* $m + \ell$ *odd for* $F^{(m)}(l) \le 0$, *such that*

$$\ell > 0 \textit{ yields} \quad F^{(k)}(l) > 0 \quad \textit{for } k = 0, 1, \ldots, \ell - 1$$

and

$$\ell \le m-1 \text{ yields } \quad (-1)^{\ell+k}F^{(k)}(l) > 0 \quad \text{for } k = \ell, \ell+1, \dots, m-1$$

for all $l \in I_u$.

2 Simplified criteria for oscillation

Lemma 2.1 *Assume that* $v \in C([l_0,\infty),(0,\infty))$ *is a solution of* (1.1). *Then* $(a(l)(v'''(l))^\kappa)' \le 0$, *and one of the following cases holds, eventually*:

(a) $v'(l)$ *and* $v'''(l)$ *are positive, and* $v^{(4)}(l)$ *is nonpositive*;
(b) $v'(l)$ *and* $v''(l)$ *are positive, and* $v'''(l)$ *is negative*;
(c) $v''(l)$ *is positive, and* $v'(l)$ *and* $v'''(l)$ *are negative*.

Proof Assume that $v \in C([l_0,\infty),(0,\infty))$ is a solution of (1.1). From (1.1), we have

$$\left(a(l)\left(v'''(l)\right)^\kappa\right)' \le -h(l)v^\kappa(l) \le 0.$$

From (1.1) and Lemma 1.1, there exist three possible cases (a), (b), and (c) for $l \ge l_1$, l_1 large enough. The proof is complete. □

Let us define

$$\psi_m(l) := \int_l^\infty \psi_{m-1}(\nu)\,\mathrm{d}\nu \quad \text{for } m = 1, 2.$$

Theorem 2.1 *Assume that* $v \in C(I_0,(0,\infty))$ *is a solution of* (1.1). *If*

$$\limsup_{l\to\infty} \int_{l_1}^{l} \left(\frac{1}{a^{1/\kappa}(u)}\left(\int_{l_1}^{u} h(s)\psi_2^\kappa\left(g(s)\right)\mathrm{d}s\right)^{1/\kappa}\right)\mathrm{d}u = \infty, \tag{2.1}$$

then v *satisfies case* (b) *in Lemma* 2.1.

Proof Assume on the contrary that $v \in C(I_0,(0,\infty))$ is a solution (1.1) and satisfies either case (a) or case (c).

First, we suppose that (c) holds on I_1. Since $(a(l)(v'''(l))^\kappa)' \le 0$, we have

$$a(l)\left(v'''(l)\right)^\kappa \le a(l_1)\left(v'''(l_1)\right)^\kappa := -L < 0, \tag{2.2}$$

which is

$$a^{1/\kappa}(l)v'''(l) \le -L^{1/\kappa}. \tag{2.3}$$

If we divide (2.3) by $a^{1/\kappa}$ and then integrate from $l$to ϱ, we find

$$v''(\varrho) \le v''(l) - L^{1/\kappa}\int_l^\varrho \frac{1}{a^{1/\kappa}(s)}\,\mathrm{d}s.$$

Letting $\varrho \to \infty$, we get

$$0 \le v''(l) - L^{1/\kappa}\psi_0(l). \tag{2.4}$$

Integrating (2.4) from l to ∞, we obtain

$$-v'(l) \geq L^{1/\kappa}\psi_1(l). \tag{2.5}$$

Integrating (2.5) from l to ∞ implies that

$$v(l) \geq L^{1/\kappa}\psi_2(l). \tag{2.6}$$

From (1.1) and (2.6), we have

$$\big(a(l)\big(v'''(l)\big)^\kappa\big)' \leq -h(l)L\psi_2^\kappa\big(g(l)\big). \tag{2.7}$$

Integrating (2.7) from l_1 to l, we obtain

$$\begin{aligned} a(l)\big(v'''(l)\big)^\kappa &\leq a(l_1)\big(v'''(l_1)\big)^\kappa - L\int_{l_1}^{l} h(s)\psi_2^\kappa\big(g(s)\big)\,\mathrm{d}s \\ &\leq -L\int_{l_1}^{l} h(s)\psi_2^\kappa\big(g(s)\big)\,\mathrm{d}s. \end{aligned} \tag{2.8}$$

Integrating (2.8) from l_1 to l, we get

$$v''(l) \leq v''(l_1) - L^{1/\kappa}\int_{l_1}^{l}\left(\frac{1}{a^{1/\kappa}(u)}\left(\int_{l_1}^{u} h(s)\psi_2^\kappa\big(g(s)\big)\,\mathrm{d}s\right)^{1/\kappa}\right)\mathrm{d}u.$$

At $l \to \infty$, we arrive at a contradiction with (2.1).

Finally, let case (a) hold on I_1. On the other hand, it follows from (2.1) and (1.2) that $\int_{l_1}^{l} h(s)\psi_2^\kappa(s)\,\mathrm{d}s$ must be unbounded. Further, since $\psi_2'(s) < 0$, it is easy to see that

$$\int_{l_1}^{l} h(s)\,\mathrm{d}s \to \infty \quad \text{as } l \to \infty. \tag{2.9}$$

Integrating (1.1) from l_2 to l, we get

$$\begin{aligned} a(l)\big(v'''(l)\big)^\kappa &\leq a(l_2)\big(v'''(l_2)\big)^\kappa - \int_{l_2}^{l} h(s)v^\kappa\big(g(s)\big)\,\mathrm{d}s \\ &\leq a(l_2)\big(v'''(l_2)\big)^\kappa - v^\kappa\big(g(l_2)\big)\int_{l_2}^{l} h(s)\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we get a contradiction with the positivity of $a(l)(v'''(l))^\kappa$. This completes the proof. □

Theorem 2.2 *Assume that $v \in C(I_0,(0,\infty))$ is a solution of* (1.1). *If*

$$\limsup_{l\to\infty}\psi_2^\kappa(l)\int_{l_1}^{l} h(s)\,\mathrm{d}s > 1, \tag{2.11}$$

then v satisfies case (b) *in Lemma* 2.1.

Proof Assume on the contrary that $v \in C(I_0,(0,\infty))$ is a solution (1.1) and satisfies case (a) or case (c).

First, we suppose that (c) holds on I_1. Then

$$v''(l) \geq -\int_l^\infty a^{-1/\kappa}(s)a^{1/\kappa}(s)v'''(s)\,ds \geq -a^{1/\kappa}(l)v'''(l)\psi_0(l). \tag{2.12}$$

Integrating (2.12) twice from l to ∞, we arrive at

$$v'(l) \leq \int_l^\infty a^{1/\kappa}(s)v'''(s)\psi(s)\,ds \leq a^{1/\kappa}(l)v'''(l)\psi_1(l) \tag{2.13}$$

and

$$v(l) \geq -\int_l^\infty a^{1/\kappa}(s)v'''(s)\psi_1(s)\,ds \geq -a^{1/\kappa}(l)v'''(l)\psi_2(l). \tag{2.14}$$

Integrating (1.1) from l_1 to l, we get

$$a(l)\big(v'''(l)\big)^\kappa \leq a(l_1)\big(v'''(l_1)\big)^\kappa - \int_{l_1}^l h(s)v^\kappa\big(g(s)\big)\,ds,$$

since $g'(l) > 0$ and $s \leq l$, we obtain

$$a(l)\big(v'''(l)\big)^\kappa \leq -v^\kappa\big(g(l)\big)\int_{l_1}^l h(s)\,ds. \tag{2.15}$$

Since $g(l) \leq l$, we have

$$a(l)\big(v'''(l)\big)^\kappa \leq -v^\kappa(l)\int_{l_1}^l h(s)\,ds. \tag{2.16}$$

From (2.14) and (2.16), we find

$$a(l)\big(v'''(l)\big)^\kappa \leq a(l)\big(v'''(l)\big)^\kappa\psi_2^\kappa(l)\int_{l_1}^l h(s)\,ds. \tag{2.17}$$

Dividing both sides of inequality (2.17) by $a(l)(v'''(l))^\kappa$ and taking the limsup, we arrive at

$$\limsup_{l\to\infty}\psi_2^\kappa(l)\int_{l_1}^l h(s)\,ds \leq 1,$$

we arrive at a contradiction with (2.11).

Next, we suppose that case (a) holds on I_1. From (2.11) and the fact that $\psi_2(l) < \infty$, we get that (2.9) holds. Then, this part of the proof is similar to that of Theorem 2.1. This completes the proof. □

Theorem 2.3 *Assume that* (2.1) *or* (2.11) *holds. If there is* $\rho \in C^1(I_0,\mathbb{R}^+)$ *such that*

$$\limsup_{l\to\infty}\frac{\psi_0^\kappa(l)}{\rho(l)}\int_{l_0}^l\left(\rho(s)h(s)\left(\frac{\lambda}{2!}g^2(s)\right)^\kappa - \frac{a(s)(\rho'(s))^{\kappa+1}}{(\kappa+1)^{\kappa+1}\rho^\kappa(s)}\right)ds > 1 \tag{2.18}$$

holds for some $\lambda_1 \in (0,1)$, *then all solutions of* (1.1) *are oscillatory.*

Proof Suppose that (1.1) has a nonoscillatory solution v in I_0. Then we assume that v is eventually positive. From Lemma 2.1, we have three cases for v and its derivatives. Using Theorems 2.1 and 2.2, we have that condition (2.1) or (2.11) ensures that solution v satisfies case (b). On the other hand, using Theorem 2.2 in [18], we find that condition (2.18) contrasts with case (b). This completes the proof. □

Example 2.1 Consider the DDE

$$\left(l^{3\kappa+1}\left(v'''(l)\right)^{\kappa}\right)' + h_0 v^{\kappa}(\epsilon l) = 0, \tag{2.19}$$

where $h_0 > 0$ and $\epsilon \in (0,1]$. Note that $a(l) := l^{3\kappa+1}$, $g(l) := \epsilon l$, $f(v) := v^{\kappa}$, and $h(l) := h_0$. Thus, we have that

$$\psi_0(l) = \frac{\kappa}{(2\kappa+1)l^{(2\kappa+1)/\kappa}}, \qquad \psi_1(l) = \frac{\kappa^2}{(2\kappa+1)(\kappa+1)l^{(\kappa+1)/\kappa}}$$

and

$$\psi_2(l) = \frac{\kappa^3}{(2\kappa+1)(\kappa+1)l^{1/\kappa}}.$$

Now, condition (2.11) reduces to

$$\frac{\kappa^{3\kappa} h_0}{((2\kappa+1)(\kappa+1))^{\kappa}} > 1. \tag{2.20}$$

Furthermore, if $\rho(l) := 1/l^{2\kappa+1}$, then condition (2.18) becomes

$$h_0\left(\frac{\lambda}{2!}\epsilon^2\right)^{\kappa} > \frac{(2\kappa+1)^{\kappa+1}}{(\kappa+1)^{\kappa+1}}. \tag{2.21}$$

Using Theorem 2.3, we have that (2.19) is oscillatory if (2.20) and (2.21) hold.

Remark 2.4 Note that, we used two conditions only for testing the oscillation of the fourth-order DDEs. Moreover, our results can also be applied to ordinary DEs when $g(l) = l$.

3 Improved criteria for oscillation

Theorem 3.1 *Assume that $v \in C(I_0,(0,\infty))$ is a solution of* (1.1). *If the DE*

$$v'(l) + \frac{1}{\psi_2(g(l))}\left(\int_l^{\infty}\int_{\varsigma}^{\infty}\frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^{u} h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u\,\mathrm{d}\varsigma\right)v\big(g(l)\big) = 0 \tag{3.1}$$

is oscillatory, then the solution v does not satisfy case (c).

Proof Suppose the contrary that v satisfies case (c). As in the proof of Theorem 2.2, we get that (2.12) and (2.15) hold. From (2.12), we have

$$\left(\frac{v''(l)}{\psi(l)}\right)' = \frac{\psi(l)v'''(l) + v''(l)a^{-1/\kappa}(l)}{\psi^2(l)} \geq 0.$$

Thus, we get that

$$-v'(l) \geq \int_l^\infty \frac{v''(s)}{\psi(s)}\psi(s)\,\mathrm{d}s \geq \frac{v''(l)}{\psi(l)}\int_l^\infty \psi(s)\,\mathrm{d}s,$$

that is, $-v'(l)\psi(l) \geq v''(l)\psi_1(l)$. Therefore,

$$\left(\frac{v'(l)}{\psi_1(l)}\right)' = \frac{\psi_1(l)v''(l) + v'(l)\psi(l)}{\psi_1^2(l)} \leq 0. \tag{3.2}$$

Using (3.2), we obtain that

$$-v(l) \leq \int_l^\infty \frac{v'(s)}{\psi_1(s)}\psi_1(s)\,\mathrm{d}s \leq \frac{v'(l)}{\psi_1(l)}\int_l^\infty \psi_1(s)\,\mathrm{d}s,$$

that is, $-\psi_1(l)v(l) \leq v'(l)\psi_2(l)$. Hence,

$$\left(\frac{v(l)}{\psi_2(l)}\right)' = \frac{\psi_2(l)v'(l) + v(l)\psi_1(l)}{\psi_2^2(l)} \geq 0. \tag{3.3}$$

Now, integrating (2.15) from l to ∞ and using (3.3), we get

$$\begin{aligned}
-v''(l) &\leq -\int_l^\infty \frac{v(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u \\
&\leq -\int_l^\infty \frac{v(g(u))}{\psi_2(g(u))}\frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u \\
&\leq -\frac{v(g(l))}{\psi_2(g(l))}\int_l^\infty \frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u.
\end{aligned} \tag{3.4}$$

Integrating (3.4) from l to ∞, we find

$$\begin{aligned}
v'(l) &\leq -\int_l^\infty \frac{v(g(\varsigma))}{\psi_2(g(\varsigma))}\int_\varsigma^\infty \frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u\,\mathrm{d}\varsigma \\
&\leq -\frac{v(g(l))}{\psi_2(g(l))}\int_l^\infty\int_\varsigma^\infty \frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u\,\mathrm{d}\varsigma.
\end{aligned}$$

Thus, it is easy to see that v is a positive solution of the first-order delay differential inequality

$$v'(l) + \frac{1}{\psi_2(g(l))}\left(\int_l^\infty\int_\varsigma^\infty \frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u\,\mathrm{d}\varsigma\right)v(g(l)) \leq 0.$$

Using [22], we have that (3.1) has also a positive solution, a contradiction. This completes the proof. □

Corollary 3.1 *Assume that* $v \in C(I_0,(0,\infty))$ *is a solution of* (1.1). *If*

$$\liminf_{l\to\infty}\int_{g(l)}^l \frac{1}{\psi_2(g(\vartheta))}\left(\int_\vartheta^\infty\int_\varsigma^\infty \frac{\psi_2(g(u))}{a^{1/\kappa}(u)}\left(\int_{l_1}^u h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}u\,\mathrm{d}\varsigma\right)\mathrm{d}\vartheta > \frac{1}{\mathrm{e}}, \tag{3.5}$$

then the solution v *does not satisfy case* (c).

Proof Using [22], we note that condition (3.5) ensures the oscillation of (3.1). This completes the proof. □

Lemma 3.1 *Assume that* $v \in C(I_0,(0,\infty))$ *is a solution of* (1.1) *and case* (c) *holds. If*

$$\int_{l_0}^{\infty}\left(\frac{1}{a(\varsigma)}\int_{l_1}^{\varsigma}h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}\varsigma=\infty, \tag{3.6}$$

then $\lim_{l\to\infty}v(l)=0$.

Proof Suppose that v satisfies case (c). Then we obtain that $\lim_{l\to\infty}v(l)=c\geq 0$. We claim that $\lim_{l\to\infty}v(l)=0$. Suppose the contrary that $c>0$. Thus, there exists $l_1\geq l_0$ such that $v(g(l))\geq c$ for $l\geq l_1$, and hence

$$-\left(a(l)\left(v'''(l)\right)^{\kappa}\right)'\geq h(l)v^{\kappa}\left(g(l)\right)\geq c^{\kappa}h(l) \tag{3.7}$$

for $l\geq l_1$. Integrating (3.7) twice from l_1 to l, we obtain

$$v'''(l)\leq -c\left(\frac{1}{a(l)}\int_{l_1}^{l}h(s)\,\mathrm{d}s\right)^{1/\kappa}$$

and

$$v''(l)\leq v''(l_1)-c\int_{l_1}^{l}\left(\frac{1}{a(\varsigma)}\int_{l_1}^{\varsigma}h(s)\,\mathrm{d}s\right)^{1/\kappa}\mathrm{d}\varsigma.$$

Letting $l\to\infty$ and using (3.6), we obtain that $\lim_{l\to\infty}v''(l)=-\infty$, which contradicts $v''(l)>0$. Thus, the proof is complete. □

Lemma 3.2 *Assume that* (3.6) *holds,* $v\in C(I_0,(0,\infty))$ *is a solution of* (1.1), *and case* (c) *holds. If there exists a constant* $\mu\geq 0$ *such that*

$$\psi_2(l)\left(\int_{l_0}^{l}h(s)\,\mathrm{d}s\right)^{1/\kappa}\geq\mu, \tag{3.8}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{v(l)}{\psi_2^{\mu}(l)}\right)\leq 0. \tag{3.9}$$

Proof Suppose that v satisfies case (c). As in the proof of Theorem 2.2, we get that (2.13) holds. Integrating (1.1) from l_1 to l and using $v'(l)<0$, we find

$$\begin{aligned}a(l)\left(v'''(l)\right)^{\kappa}&\leq a(l_1)\left(v'''(l_1)\right)^{\kappa}-\int_{l_1}^{l}h(s)v^{\kappa}\left(g(s)\right)\mathrm{d}s\\&\leq a(l_1)\left(v'''(l_1)\right)^{\kappa}-v^{\kappa}\left(g(l)\right)\int_{l_0}^{l}h(s)\,\mathrm{d}s+v^{\kappa}\left(g(l)\right)\int_{l_0}^{l_1}h(s)\,\mathrm{d}s.\end{aligned} \tag{3.10}$$

Using Lemma 3.1, we get that $\lim_{l\to\infty} v(l) = 0$. Thus, there is $l_2 \geq l_1$ such that

$$a(l_1)\big(v'''(l_1)\big)^{\kappa} + v^{\kappa}\big(g(l)\big)\int_{l_0}^{l_1} h(s)\,\mathrm{d}s < 0 \quad \text{for every } l \geq l_2,$$

which, with (3.10), gives

$$a(l)\big(v'''(l)\big)^{\kappa} \leq -v^{\kappa}\big(g(l)\big)\int_{l_0}^{l} h(s)\,\mathrm{d}s \leq -v^{\kappa}(l)\int_{l_0}^{l} h(s)\,\mathrm{d}s. \tag{3.11}$$

Next, we have that

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{v(l)}{\psi_2^{\mu}(l)}\right) = \frac{\psi_2^{\mu}(l)v'(l) + \mu\psi_2^{\mu-1}(l)\psi_1(l)v(l)}{\psi_2^{2\mu}(l)}. \tag{3.12}$$

Combining (2.13) and (3.11), we get

$$v'(l) \leq -v(l)\psi_1(l)\left(\int_{l_0}^{l} h(s)\,\mathrm{d}s\right)^{1/\kappa}.$$

This implies

$$\begin{aligned}\psi_2^{\mu}(l)v'(l) + \mu\psi_2^{\mu-1}(l)\psi_1(l)v(l) &\leq -\psi_2^{\mu}(l)\psi_1(l)v(l)\left(\int_{l_0}^{l} h(s)\,\mathrm{d}s\right)^{1/\kappa} + \mu\psi_2^{\mu-1}(l)\psi_1(l)v(l)\\ &= \left(-\psi_2(l)\left(\int_{l_0}^{l} h(s)\,\mathrm{d}s\right)^{1/\kappa} + \mu\right)\psi_2^{\mu-1}(l)\psi_1(l)v(l).\end{aligned}$$

It follows from (3.8) that $\psi_2^{\mu}(l)v'(l) + \mu\psi_2^{\mu-1}(l)\psi_1(l)v(l) \leq 0$, which, with (3.12), implies that the function $v(l)/\psi_2^{\mu}(l)$ is nonincreasing. This completes the proof. □

Theorem 3.2 *Assume that* (3.6) *holds. If there exists a constant* $\mu \geq 0$ *such that* (3.8) *holds, and the equation*

$$\left(\frac{1}{\psi_1^{\kappa}(l)}\big(v'(l)\big)^{\kappa}\right)' + h(l)\left(\frac{\psi_2(g(l))}{\psi_2(l)}\right)^{\mu\kappa} v^{\kappa}(l) = 0 \tag{3.13}$$

is oscillatory, then the solution v *does not satisfy case* (c).

Proof Assume on the contrary that (1.1) has a positive solution v which satisfies case (c). Using Theorem 2.2 and Lemma 3.2, we get that (2.13) and (3.9) hold, respectively. Integrating (3.9) from $g(l)$ to l, we obtain

$$v\big(g(l)\big) \geq \left(\frac{\psi_2(g(l))}{\psi_2(l)}\right)^{\mu} v(l),$$

which with (1.1) gives

$$\big(a(l)\big(v'''(l)\big)^{\kappa}\big)' \leq -h(l)\left(\frac{\psi_2(g(l))}{\psi_2(l)}\right)^{\mu\kappa} v^{\kappa}(l). \tag{3.14}$$

Integrating (2.13) from l to ∞ provides

$$v(l) \geq -a^{1/\kappa}(l)v'''(l)\psi_2(l). \tag{3.15}$$

Next, we define

$$w(l) := a(l)\left(\frac{v'''(l)}{v(l)}\right)^{\kappa} < 0. \tag{3.16}$$

From (3.14) and (3.16), we conclude that

$$w'(l) \leq -h(l)\left(\frac{\psi_2(g(l))}{\psi_2(l)}\right)^{\mu\kappa} - \kappa\frac{a(l)(v'''(l))^{\kappa}}{v^{\kappa+1}(l)}v'(l),$$

which, in view of (2.13), gives

$$w'(l) + h(l)\left(\frac{\psi_2(g(l))}{\psi_2(l)}\right)^{\mu\kappa} + \kappa\psi_1(l)w^{(\kappa+1)/\kappa}(l) \leq 0. \tag{3.17}$$

In view of [6], differential equation (3.13) is nonoscillatory if and only if there exists a function $w \in C([l_1,\infty),\mathbb{R})$ satisfying inequality (3.17) for $l \geq l_1$, l_1 large enough, which is a contradiction. This completes the proof. □

Using Theorems 3.2, 1.3, and 1.4, we establish the following oscillation criteria for (1.1) under the assumption $\psi_2(l_0) < \infty$.

Corollary 3.2 *Assume that* (3.6) *holds and there exists a constant* $\mu \geq 0$ *such that* (3.8) *holds. If* $\psi_2(l_0) < \infty$ *and*

$$\limsup_{l\to\infty} \psi_2^{\kappa}(l)\int_{l_0}^{l} h(s)\left(\frac{\psi_2(g(s))}{\psi_2(s)}\right)^{\mu\kappa} \mathrm{d}s > 1 \tag{3.18}$$

or

$$\liminf_{l\to\infty} \frac{1}{\psi_2(l)}\int_{l}^{\infty} \psi_2^{\kappa+1}(s)h(s)\left(\frac{\psi_2(g(s))}{\psi_2(s)}\right)^{\mu\kappa} \mathrm{d}s > \left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1} \tag{3.19}$$

hold, then the solution v *does not satisfy case* (c).

Theorem 3.3 *Assume that* (1.4), (1.5), *and* (3.5) *hold, then all solutions of equation* (1.1) *are oscillatory.*

Proof Suppose to the contrary that there exists a nonoscillatory solution v of (1.1). Without loss of generality, we suppose that there exists $l_1 \in [l_0,\infty)$ such that $v(l) > 0$ and $v(g(l)) > 0$ for $l \geq l_1$. Using Lemma 2.1, there exist three possible cases (a)–(c). Obviously, one can show that Theorem 1.1 together with (a) and (b) leads to a contradiction with (1.4) and (1.5). Therefore, v satisfies (c). From Corollary 3.1, we get a contradiction with condition (3.5). This completes the proof. □

Theorem 3.4 *Assume that* (3.6), (1.4), *and* (1.5 *hold and there exists a constant* $\mu \geq 0$ *such that* (3.8) *holds. If* $\psi_2(l_0) < \infty$ *and* (3.19) *hold, then all solutions of equation* (1.1) *are oscillatory.*

Proof Suppose to the contrary that there exists a nonoscillatory solution v of (1.1). Without loss of generality, we suppose that there exists $l_1 \in [l_0, \infty)$ such that $v(l) > 0$ and $v(g(l)) > 0$ for $l \geq l_1$. Using Lemma 2.1, there exist three possible cases (a)–(c). Obviously, one can show that Theorem 1.1 together with (a) and (b) leads to a contradiction with (1.4) and (1.5). Therefore, v satisfies (c). From Corollary 3.2, we get a contradiction with condition (3.19). This completes the proof. □

Example 3.1 Consider the delay differential equation

$$\left(e^{3l}\left(v'''(l)\right)^3\right)' + h_0 e^{3l} v^3(l-1) = 0, \tag{3.20}$$

where $h_0 > 0$. We note that $a(l) := e^{3l}$, $h(l) := h_0 e^{3l}$, $f(v) := v^3$, and $g(l) := l - 1$. Thus, we have that

$$\psi_i(l) = e^{-l} \quad \text{for } i = 0, 1, 2.$$

It is easy to verify that $\psi_2(l_0) < \infty$, (3.6), (1.4), and (1.5) are satisfied. Now, (3.5) holds if $h_0 > 0.14936$. Moreover, if we choose $\mu := (h_0/3)^{1/3}$, then we see that (3.8) is satisfied and (3.19) holds if $h_0 > 0.11505$.

Hence, by Theorem 3.3, every solution of (3.20) is oscillatory if $h_0 > 0.14936$. Further, by Theorem 3.4, every solution of (3.20) is oscillatory if $h_0 > 0.11505$.

Remark 3.5 By using [23, Corollary 2.1], equation (3.20) is oscillatory when $h_0 > 0.31641$. Thus, we note that Theorem 3.4 provides a better criterion for the oscillation of (3.20). Moreover, our oscillation criteria take into account the influence of $g(l)$, which has not been taken care of in the related results [18, 25].

4 Conclusion

In this work, we simplified and improved the oscillation criteria for a class of even-order delay differential equations. In the noncanonical case, it always sets three conditions to check the oscillation of even-order DDEs. First, we obtained a criterion with only two conditions to check the oscillation. Furthermore, we improved the three-condition oscillation criteria by creating a better estimate of the ratio $v(g(l))/v(l)$. Through the example, we compared our results with the previous results and explained the importance of our new oscillation criteria. It will be interesting to extend our results of this study to the neutral and mixed case.

Acknowledgements
The authors present their sincere thanks to the two anonymous referees. (E.E. Mahmoud) Taif University Research Supporting Project number (TURSP-2020/20), Taif University, Taif, Saudi Arabia.

Funding
Not applicable.

Availability of data and materials
There are no data and materials for this article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt. [2]Department of Mathematics, Faculty of Education – Al-Nadirah, Ibb University, Ibb, Yemen. [3]Department of Mathematics and Computer Science, Faculty of Arts and Sciences, Çankaya University Ankara, 06790 Etimesgut, Turkey. [4]Physics Department, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia. [5]Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif, 21944, Saudi Arabia.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 February 2021 Accepted: 7 June 2021 Published online: 15 June 2021

References
1. Agarwal, P., Agarwal, R.P., Ruzhansky, M.: Special Functions and Analysis of Differential Equations. CRC Press, Boca Raton (2020)
2. Agarwal, P., Akbar, M., Nawaz, R., Jleli, M.: Solutions of system of Volterra integro-differential equations using optimal homotopy asymptotic method. Math Meth Appl Sci. **44**, 2671–2681 (2021)
3. Agarwal, P., Merker, J., Schuldt, G.: Singular integral Neumann boundary conditions for semilinear elliptic PDEs. Axioms **10**, 74 (2021)
4. Agarwal, P., Sidi Ammi, M.R., Asad, J.: Existence and uniqueness results on time scales for fractional nonlocal thermistor problem in the conformable sense. Adv Differ Equ. **2021**, 162 (2021)
5. Agarwal, R.P., Grace, S.R., O'Regan, D.: Oscillation Theory for Difference and Differential Equations. Kluwer Academic, Dordrecht (2000)
6. Agarwal, R.P., Shieh, S.L., Yeh, C.C.: Oscillation criteria for second order retarded differential equations. Math. Comput. Model. **26**, 1–11 (1997)
7. Baculikova, B., Dzurina, J., Graef, J.R.: On the oscillation of higher-order delay differential equations. J. Math. Sci. **187**(4), 387–400 (2012)
8. Chatzarakis, G.E., Li, T.: Oscillation criteria for delay and advanced differential equations with non-monotone arguments. Complexity **2018**, Article ID 8237634 (2018)
9. Dzurina, J., Jadlovska, I.: A note on oscillation of second-order delay differential equations. Appl. Math. Lett. **69**, 126–132 (2017)
10. Dzurina, J., Jadlovska, I., Stavroulakis, I.P.: Oscillatory results for second-order noncanonical delay differential equations. Opuscula Math. **39**(4), 483–495 (2019)
11. El-Morshedy, H.A., Attia, E.R.: New oscillation criterion for delay differential equations with non-monotone arguments. Appl. Math. Lett. **54**, 54–59 (2016)
12. Grace, S., Agarwal, R., Graef, J.: Oscillation theorems for fourth order functional differential equations. J. Appl. Math. Comput **30**, 75–88 (2009)
13. Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1977)
14. Li, T., Baculikova, B., Dzurina, J., Zhang, C.: Oscillation of fourth order neutral differential equations with p-Laplacian like operators. Bound. Value Probl. **56**, 41–58 (2014)
15. Moaaz, O., Dassios, I., Bazighifan, O., Muhib, A.: Oscillation theorems for nonlinear differential equations of fourth-order. Mathematics **8**, 520 (2020)
16. Moaaz, O., Dassios, I., Bin Jebreen, H., Muhib, A.: Criteria for the nonexistence of Kneser solutions of DDEs and their applications in oscillation theory. Appl. Sci. **11**, 425 (2021)
17. Moaaz, O., Dassios, I., Muhsin, W., Muhib, A.: Oscillation theory for non-linear neutral delay differential equations of third order. Appl. Sci. **10**, 4855 (2020)
18. Moaaz, O., Muhib, A.: New oscillation criteria for nonlinear delay differential equations of fourth-order. Appl. Math. Comput. **377**, 125192 (2020)
19. Moaaz, O., Park, C., Muhib, A., Bazighifan, O.: Oscillation criteria for a class of even-order neutral delay differential equations. J. Appl. Math. Comput. **63**, 607–617 (2020)
20. Muhib, A., Abdeljawad, T., Moaaz, O., Elabbasy, E.M.: Oscillatory properties of odd-order delay differential equations with distribution deviating arguments. Appl. Sci. **10**, 5952 (2020)
21. Parhi, N., Tripathy, A.: On oscillatory fourth order linear neutral differential equations-I. Math. Slovaca **54**, 389–410 (2004)
22. Philos, C.: On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. Arch. Math. (Basel) **36**, 168–178 (1981)
23. Zhang, C., Agarwal, R.P., Bohner, M., Li, T.: New results for oscillatory behavior of even-order half-linear delay differential equations. Appl. Math. Lett. **26**, 179–183 (2013)
24. Zhang, C., Li, T., Saker, S.H.: Oscillation of fourth-order delay differential equations. J. Math. Sci. **201**, 296–309 (2014)
25. Zhang, C., Li, T., Sun, B., Thandapani, E.: On the oscillation of higher-order half-linear delay differential equations. Appl. Math. Lett. **24**, 1618–1621 (2011)