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ARTICLE

Solution of Modified Bergman Minimal Blood Glucose-Insulin Model Using Caputo-Fabrizio Fractional Derivative

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ABSTRACT

Diabetes is a burning issue in the whole world. It is the imbalance between body glucose and insulin. The study of this imbalance is very much needed from a research point of view. For this reason, Bergman gave an important model named-Bergman minimal model. In the present work, using Caputo-Fabrizio (CF) fractional derivative, we generalize Bergman's minimal blood glucose-insulin model. Further, we modify the old model by including one more component known as diet $D(t)$, which is also essential for the blood glucose model. We solve the modified model with the help of Sumudu transform and fixed-point iteration procedures. Also, using the fixed point theorem, we examine the existence and uniqueness of the results along with their numerical and graphical representation. Furthermore, the comparison between the values of parameters obtained by calculating different values of t with experimental data is also studied. Finally, we draw the graphs of $G(t)$, $X(t)$, $I(t)$, and $D(t)$ for different values of τ. It is also clear from the obtained results and their graphical representation that the obtained results of modified Bergman's minimal model are better than Bergman's model.

KEYWORDS

Bergman minimal model; blood glucose; Caputo-Fabrizio fractional derivative; uniqueness and existence; fractional calculus

1 Introduction

These days, Mathematical modeling is turning out to be a vital tool in mathematical science. Since it translates real-world problems into mathematical language and after applying the necessary methods, we again translate the results into real-world languages to forecast the objective. In every field, modeling is being used to achieve or predict the future prospective. In recent years, modeling is being used with another important but less interactive field of mathematics—Fractional Calculus (see [1–8]). We have studied several problems and their solutions by ordinary calculus methods, but sometimes fractional calculus gives us better results to describe the model than the old one. Fractional calculus has several real-world applications. A few of them are Fractional conservation of mass, Groundwater flow problem, Time-space fractional diffusion equation models, Acoustical wave equations, Fractional Schrdinger equation in quantum theory, etc. (see [9–22]).

Nowadays, diabetes is turning out to be a fatal disease. The imbalance between body glucose and insulin causes diabetes. It is categorized into two types. Type-I diabetes is more severe than Type-II. According to a survey, Type-I and Type-II diabetes sufferers are one ratio nine. So, the study of this imbalance was very much needed from a research point of view. For this reason, Bergman gave an important model named-Bergman minimal model (see [23–27]).

Riemann-Liouville and Caputo introduced the initial concept of fractional calculus. But they use a singular kernel in their definition. However, Caputo and Fabrizio recently found specific systems related to material heterogeneities that cannot be well connected with Riemann-Liouville or Caputo derivative. Because of that, Caputo and Fabrizio introduced a new fractional derivative operator involving a non-singular kernel. Due to its memory and non-singularity property, this operator is used to study different models of engineering, science, and bio-mathematical fields. In general, we get more reliable results while using this operator than others for more details of applications of Caputo-Fabrizio derivative (see [28–45]).

Apart from changing the model to fractional-order, we also include the Diet factor in the model that describes the effect of meals on the glucose level. The generalize of Bergman's minimal blood glucose-insulin model gives us a more precise and detailed prediction about the problem.

1.1 The Caputo-Fabrizio Fractional Differential Operator [46–48]

Definition 1: Suppose $h \in H^1(a_1, b_1)$, $b_1 > a_1$, $\beta \in [0, 1]$ hence the Caputo-Fabrizio differential coefficient of fractional order:

$$D_t^{\beta}\,(\mathrm{h(t)}) = \frac{M(\beta)}{(1-\beta)} \int_a^t h'(x)\, e^{\left[-\beta \frac{t-x}{1-\beta}\right]} dx, \tag{1}$$

$M(\beta)$ is function of normalization such as $M(0) = M(1) = 1$.

If, $h \notin H^1(a_1, b_1)$, then the derivative is

$$D_t^{\beta}\,(h(t)) = \frac{N(\sigma)}{\sigma} \int_a^t h'(x)\, e^{\left[-\frac{t-x}{\sigma}\right]} dx, \quad N(0) = N(\infty) = 1, \tag{2}$$

also,

$$\lim_{\sigma \to 0} \frac{1}{\sigma} e^{\left[-\frac{t-x}{1-\beta}\right]} = \delta(x-t). \tag{3}$$

One thing is to be noted here that the Caputo-Fabrizio operator has an exponent differential coefficient. We know that the exponential function has no singularity which, means the derivative exists at every point of the territory. So CF operator gives better results in its domain, see [49–56].

Definition 2: The fractional integral of the function h(t) of order β $(0 < \beta < 1)$, is

$$I_{\beta}^{t}\,(h(t)) = \frac{2\,(1-\beta)}{(2-\beta)\,M\,(\beta)} h\,(t) + \frac{2\beta}{(2-\beta)\,M\,(\beta)} \int_0^t h(s)\, ds, \quad t \geq 0. \tag{4}$$

Remark 1: From the above equation, we conclude that the fractional integral of order β $(0 < \beta < 1)$, is the mean of h and its anti-derivative. So Nieto et al. gave this condition,

$$\frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)} = 1. \tag{5}$$

$$M(\beta) = \frac{2}{(2-\beta)}, \quad 0 < \beta < 1.$$

Based on the above relation, Nieto and Losada defined the following operator;

$${}_{0}^{CF}D_t^{\beta}(h(t)) = \frac{1}{(1-\beta)} \int_{a_1}^{t} h'(x) e^{\left[-\beta \frac{t-x}{1-\beta}\right]} dx. \tag{6}$$

1.2 Sumudu Transform of Caputo-Fabrizio Operator [57,58]

Suppose $f(t)$ be a function whose Caputo-Fabrizio derivative occurs, hence the Sumudu Transform of Caputo-Fabrizio fractional differential coefficient of f(t) is defined below:

$$ST\left({}_{0}^{CF}D_t^{\beta}\right)(f(t)) = M(\beta)\left[\frac{ST(f(t)) - f(0)}{1-\beta+\beta u}\right] \tag{7}$$

The whole paper is divided into six sections. Section first introduces fractional calculus and mathematical modeling with its brief history. The second section describes the Bergman model's fractional exponent and an overview of the fractional modified minimal model. The third section discusses the existence and uniqueness of the modified mathematical model with the help of the Caputo-Fabrizio operator. Section 4 is devoted to the solution of the model by using the Sumudu Transform operator. Section 5 deals with numerical solutions along with graphical representations of the modified Bergman model. In section 6, we have concluded our findings.

2 Mathematical Model

2.1 Bergman Model of Fractional Order

Here, we are going to discuss the Bergman model of fractional order. In this model, we consider a glucose chamber and plasma insulin which is inherent to carry out across the isolated chamber to control the final glucose intake. This model, asserts that this is good to satisfy specific validation criteria with minimum parameters. This system is efficient in describing the gesture of interactivity of blood sugar and insulin. In the presented model, we suppose that $G(t)$ is the imbalance of plasma glucose cluster and $I(t)$ is the free plasma insulin cluster, from their initial values. The model was

$$\frac{d^{\alpha}G(t)}{dt^{\alpha}} = -(p_1 + X(t))G(t) + p_1 G_b, \quad 0 < \alpha < 1 \tag{8}$$

$$\frac{d^{\beta}X(t)}{dt^{\beta}} = -p_2 X(t) + p_3(I(t) - I_b), \quad 0 < \beta < 1 \tag{9}$$

$$\frac{d^{\gamma}I(t)}{dt^{\gamma}} = p_6(G(t) - p_5)^{+} t - p_4(I(t) - I_b), \quad 0 < \gamma < 1 \tag{10}$$

with starting conditions $G(0) = G_0$, $X(0) = X_0$, and $I(0) = I_0$.

2.2 Overview of Fractional Modified Minimal Model

Here, we redefined the structure of the previous model. We made some changes in the model and added few parameters. As we have, 'Diet' having significant effect on the glucose level. Now the changed structured Bergman modified system has been described in the following equations-

$$ {}_0^{CF}D_t^{\tau} G(t) = -(p_1 + X(t))G(t) + D(t) - X(t)G_b, \tag{11} $$

$$ {}_0^{CF}D_t^{\tau} X(t) = -p_2 X(t) + p_3 I(t), \tag{12} $$

$$ {}_0^{CF}D_t^{\tau} I(t) = -n[I(t) + I_b] + \frac{u(t)}{V_1}, \tag{13} $$

$$ {}_0^{CF}D_t^{\tau} D(t) = -kD(t). \tag{14} $$

Here one thing is to be noted that $0 < \tau \le 1$ and initial restrictions are, $G(0) = G_0$, $X(0) = X_0$, $I(0) = I_0$ and $D(0) = D_0$.

This model can be a tool in search of an artificial pancreas. But, unfortunately, it also adopts the problems with the glucose-minimal model. Here, $G(t)$—blood glucose cluster, $X(t)$—aftermath of effective insulin, $I(t)$—blood insulin cluster, $D(t)$—infusion of exogenous glucose, $u(t)$—insulin distribution function, G_b—initial blood glucose cluster, I_b—initial blood insulin cluster, V_1—insulin issuance volume, n—fractional fading rate of insulin, p_1—insulin free glucose dispensation rate, p_2—active insulin dispensation rate and p_3—the increase in absorption capacity caused by insulin.

3 Existence of the Solutions of Described Model

3.1 Theorem 1

Define K_1, K_2, K_3, K_4 and their relations with variables.

Proof: Since the system is

$$ {}_0^{CF}D_t^{\tau} G(t) = -(p_1 + X(t))G(t) + D(t) - X(t)G_b, \tag{15} $$

$$ {}_0^{CF}D_t^{\tau} X(t) = -p_2 X(t) + p_3 I(t), \tag{16} $$

$$ {}_0^{CF}D_t^{\tau} I(t) = -n[I(t) + I_b] + \frac{u(t)}{V_1}, \tag{17} $$

$$ {}_0^{CF}D_t^{\tau} D(t) = -kD(t). \tag{18} $$

Here one thing is to be noted that $0 < \tau \le 1$. Then converting the above system into a system of integral equations

$$ G(t) - G(0) = {}_0^{CF}I_t^{\tau}[-(p_1 + X(t))G(t) + D(t) - X(t)G_b], \tag{19} $$

$$ X(t) - X(0) = {}_0^{CF}I_t^{\tau}[-p_2 X(t) + p_3 I(t)], \tag{20} $$

$$ I(t) - I(0) = {}_0^{CF}I_t^{\tau}\left[-n[I(t) + I_b] + \frac{u(t)}{V_1}\right], \tag{21} $$

$$ D(t) - D(0) = {}_0^{CF}I_t^{\tau}[-kD(t)]. \tag{22} $$

Then by definition stated by Nieto, we get

$$G(t) = G(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}[-(p_1 + X(t))G(t) + D(t) - X(t)G_b]$$
$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t [-(p_1 + X(s))G(s) + D(s) - X(s)G_b]\, ds, \tag{23}$$

$$X(t) = X(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}[-p_2 X(t) + p_3 I(t)] + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t [-p_2 X(s) + p_3 I(s)]\, ds, \tag{24}$$

$$I(t) = I(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}[-n[I(t) + I_b] + \frac{u(t)}{V_1}] + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left[-n[I(s) + I_b] + \frac{u(s)}{V_1}\right] ds, \tag{25}$$

$$D(t) = D(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}[-kD(t)] + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t [-kD(s)]\, ds. \tag{26}$$

Now let us suppose the kernels are given as

$$K_1(t, G) = -(p_1 + X(t))G(t) + D(t) - X(t)G_b,$$
$$K_2(t, X) = -p_2 X(t) + p_3 I(t),$$
$$K_3(t, I) = -n[I(t) + I_b] + \frac{u(t)}{V_1},$$
$$K_4(t, D) = -kD(t).$$

3.2 Theorem 2

Show that K_1, K_2, K_3 and K_4 satisfies the Lipchitz condition.

Proof: At first, we shall show this for K_1. Suppose G and G_1 are any functions, so

$$\|K_1(t, G) - K_1(t, G_1)\| = \|-(p_1 + X(t))G(t) + (p_1 + X(t))G_1(t)\|$$
$$= \|p_1 + X(t)\| \, \|G(t) - G_1(t)\|$$
$$\|K_1(t, G) - K_1(t, G_1)\| \le H \, \|G(t) - G_1(t)\|, \quad \textit{where} \;\; \|p_1 + X(t)\| \le H < 1.$$

Similarly, we can have

$$\|K_2(t, X) - K_2(t, X_1)\| \le H_1 \, \|X(t) - X_1(t)\|, \quad \textit{where} \;\; \|p_2\| \le H_1 < 1.$$
$$\|K_3(t, I) - K_3(t, I_1)\| \le H_2 \, \|I(t) - I_1(t)\|, \quad \textit{where} \;\; \|n\| \le H_2 < 1.$$

and

$$\|K_4(t, D) - K_4(t, D_1)\| \le H_3 \, \|D(t) - D_1(t)\| \quad \textit{where} \;\; \|k\| \le H_3 < 1.$$

Now consider the recursive formula

$$G_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-1}) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s, G_{n-1})\, ds, \tag{27}$$

$$X_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_2(t, X_{n-1}) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_2(s, X_{n-1})\, ds, \tag{28}$$

$$I_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_3(t, I_{n-1}) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_3(s, I_{n-1})\, ds, \tag{29}$$

$$D_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_4(t, D_{n-1}) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_4(s, D_{n-1})\, ds, \tag{30}$$

Now suppose that the deviation amid two successive terms is

$$\begin{aligned} U_n(t) &= G_n(t) - G_{n-1}(t) \\ &= \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-1}) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s, G_{n-1})\, ds \\ &\quad - \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-2}) - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s, G_{n-2})\, ds, \end{aligned} \tag{31}$$

$$\begin{aligned} U_n(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-1}) - \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-2}) \\ &\quad + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \{K_1(s, G_{n-1}) - K_1(s, G_{n-2})\}\, ds. \end{aligned} \tag{32}$$

Now

$$\begin{aligned} \|U_n(t)\| &= \|G_n(t) - G_{n-1}(t)\| \\ &= \left\| \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-1}) - \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t, G_{n-2}) \right. \\ &\quad \left. + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \{K_1(s, G_{n-1}) - K_1(s, G_{n-2})\}\, ds \right\|, \\ &\le \frac{2(1-\tau)}{(2-\tau)M(\tau)} \|K_1(t, G_{n-1}) - K_1(t, G_{n-2})\| \\ &\quad + \frac{2\tau}{(2-\tau)M(\tau)} \left\| \int_0^t \{K_1(s, G_{n-1}) - K_1(s, G_{n-2})\}\, ds \right\|. \end{aligned} \tag{33}$$

But K_1 satisfies Lipchitz condition so,

$$\|U_n(t)\| \le \frac{2(1-\tau)}{(2-\tau)M(\tau)} H \|G_{n-1} - G_{n-2}\| + \frac{2\tau}{(2-\tau)M(\tau)} K \int_0^t \|G_{n-1} - G_{n-2}\|\, ds. \tag{34}$$

Similarly, we have

$$\|V_n(t)\| \le \frac{2(1-\tau)}{(2-\tau)M(\tau)} H_1 \|X_{n-1} - X_{n-2}\| + \frac{2\tau}{(2-\tau)M(\tau)} J_1 \int_0^t \|X_{n-1} - X_{n-2}\|\, ds, \tag{35}$$

$$\|W_n(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}H_2 \|I_{n-1} - I_{n-2}\| + \frac{2\tau}{(2-\tau)M(\tau)}J_2 \int_0^t \|I_{n-1} - I_{n-2}\| \, ds, \tag{36}$$

$$\|T_n(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}H_3 \|D_{n-1} - D_{n-2}\| + \frac{2\tau}{(2-\tau)M(\tau)}J_3 \int_0^t \|D_{n-1} - D_{n-2}\| \, ds. \tag{37}$$

3.3 Theorem 3

Establish that Bergman Minimal Model with Fractional-order is a minimum system of sugar insulin dynamics.

Proof: Using the recursive technique, we have

$$\|U_n(t)\| \leq \|G(0)\| + \left\{\frac{2(1-\tau)H}{(2-\tau)M(\tau)}\right\}^n + \left\{\frac{2\tau Kt}{(2-\tau)M(\tau)}\right\}^n \tag{38}$$

$$\|V_n(t)\| \leq \|X(0)\| + \left\{\frac{2(1-\tau)H_1}{(2-\tau)M(\tau)}\right\}^n + \left\{\frac{2\tau J_1 t}{(2-\tau)M(\tau)}\right\}^n \tag{39}$$

$$\|W_n(t)\| \leq \|I(0)\| + \left\{\frac{2(1-\tau)H_2}{(2-\tau)M(\tau)}\right\}^n + \left\{\frac{2\tau J_2 t}{(2-\tau)M(\tau)}\right\}^n \tag{40}$$

$$\|T_n(t)\| \leq \|D(0)\| + \left\{\frac{2(1-\tau)H_3}{(2-\tau)M(\tau)}\right\}^n + \left\{\frac{2\tau J_3 t}{(2-\tau)M(\tau)}\right\}^n \tag{41}$$

Hence the existence of results is verified, which is continuous too. Now we get

$G(t) = G_n(t) + P_n(t)$,

$X(t) = X_n(t) + Q_n(t)$,

$I(t) = I_n(t) + R_n(t)$,

$D(t) = D_n(t) + S_n(t)$,

where P_n, Q_n, R_n and S_n are residues of series solution.

So,

$$G(t) - G_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}K_1(t, G_n) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s, G_n) \, ds,$$

$$G(t) - G_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}K_1(t, G - P_n(t)) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s, G - P_n(s)) \, ds.$$

Similarly, we have

$$X(t) - X_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}K_2(t, X - Q_n(t)) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_2(s, X - Q_n(s)) \, ds,$$

$$I(t) - I_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}K_3(t, I - R_n(t)) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_3(s, I - R_n(s)) \, ds,$$

$$D(t) - D_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_4(t,\, D - S_n(t)) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_4(s,\, D - S_n(s))\, ds.$$

From the above, it is clear that,

$$G(t) - G_n(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} K_1(t,\, G - P_n(t)) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s,\, G - P_n(s))\, ds,$$

$$G(t) - G(0) - \frac{2(1-\tau)K_1(t,\, G)}{(2-\tau)M(\tau)} - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s,\, G)\, ds$$
$$= P_n(t) + \frac{2(1-\tau)K_1(t,\, G - P_n(t))}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s,\, G - P_n(s))\, ds.$$

Now,

$$\left\| G(t) - \frac{2(1-\tau)K_1(t,\, G)}{(2-\tau)M(\tau)} - G(0) - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s,\, G) ds \right\|$$
$$\leq \| P_n(t) \| + \left\{ \frac{2(1-\tau)H}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} Kt \right\} \| P_n(t) \|, \tag{42}$$

$$\left\| X(t) - \frac{2(1-\tau)K_2(t,\, X)}{(2-\tau)M(\tau)} - X(0) - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_2(s,\, X) ds \right\|$$
$$\leq \| Q_n(t) \| + \left\{ \frac{2(1-\tau)H_1}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} J_1 t \right\} \| Q_n(t) \|, \tag{43}$$

$$\left\| I(t) - \frac{2(1-\tau)K_3(t, I)}{(2-\tau)M(\tau)} - I(0) - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_3(s,\, I) ds \right\|$$
$$\leq \| R_n(t) \| + \left\{ \frac{2(1-\tau)H_2}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} J_2 t \right\} \| R_n(t) \|, \tag{44}$$

$$\left\| D(t) - \frac{2(1-\tau)K_4(t,\, D)}{(2-\tau)M(\tau)} - D(0) - \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_4(s,\, D) ds \right\|$$
$$\leq \| T_n(t) \| + \left\{ \frac{2(1-\tau)H_3}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} J_3 t \right\} \| T_n(t) \|. \tag{45}$$

Now taking $n \to \infty$ we have

$$G(t) = G(0) + \frac{2(1-\tau)K_1(t,\, G)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_1(s,\, G)\, ds, \tag{46}$$

$$X(t) = X(0) + \frac{2(1-\tau)K_2(t,\, X)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_2(s,\, X)\, ds, \tag{47}$$

$$I(t) = I(0) + \frac{2(1-\tau)K_3(t, I)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_3(s, I)\, ds, \tag{48}$$

$$D(t) = D(0) + \frac{2(1-\tau)K_4(t, D)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t K_4(s, D)\, ds. \tag{49}$$

On behalf of the above equations, we can state that the solution of the system exists.

3.4 The Uniqueness of Result

To show the uniqueness of results, we suppose that other sets of results exist for the system specified from Eqs. (46) to (49) such that

$$G(t) - G_1(t) = \frac{2(1-\tau)}{M(\tau)(2-\tau)} [K_1(t, G) - K_1(t, G_1)] + \frac{2(\tau)}{M(\tau)(2-\tau)} \int_0^t [K_1(s, G) - K_1(s, G_1)]\, ds, \tag{50}$$

taking norm both sides, we get

$$\|G - G_1\| = \frac{2(1-\tau)}{M(\tau)(2-\tau)} [\|K_1(t, G) - K_1(t, G_1)\|] + \frac{2(\tau)}{M(\tau)(2-\tau)} \int_0^t [\|K_1(s, G) - K_1(s, G_1)\|]\, ds, \tag{51}$$

using Lipchitz condition, we obtain

$$\|G - G_1\| < \frac{2(1-\tau)}{M(\tau)(2-\tau)} HD + \left(\frac{2(\tau)}{M(\tau)(2-\tau)} J_1 Dt\right)^n, \tag{52}$$

which is valid for all n, so

$$G = G_1, \tag{53}$$

similarly

$$X = X_1, \quad I = I_1, \quad and \quad D = D_1. \tag{54}$$

Hence, it claims uniqueness of the system.

4 Result of the Model by Using Sumudu Transform

Considering the system has various equations so it can be challenging to find the exact results. For this, we will adopt the iterative technique together with the Sumudu Transform. Now take Sumudu Transform either sides, side of Eq. (15)

$$ST({}_0^{CF}D_t^{\tau})(G(t)) = ST\{-(p_1 + X(t))G(t) + D(t) - X(t)G_b\},$$

or

$$ST(G(t)) = G(0) + \frac{(1-\tau+\tau u)}{M(\tau)} ST\{-(p_1 + X(t))G(t) + D(t) - X(t)G_b\}.$$

At this moment, take inverse Sumudu Transform on both ends, we have-

$$G(t) = G(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-(p_1+X(t))G(t)+D(t)-X(t)G_b\right\}\right] \tag{55}$$

Similarly, we have from Eqs. (16)–(18)

$$X(t) = X(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-p_2X(t)+p_3I(t)\right\}\right] \tag{56}$$

$$I(t) = I(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-n(I(t)+I_b)+\frac{u(t)}{V_1}\right\}\right] \tag{57}$$

$$D(t) = D(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-kD(t)\right\}\right] \tag{58}$$

Then we get the following recurrent form from the above

$$G_{n+1}(t) = G_n(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-\left(p_1+X_n(t)\right)G_n(t)+D_n(t)-X_n(t)G_b\right\}\right] \tag{59}$$

$$X_{n+1}(t) = X_n(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-p_2X_n(t)+p_3I_n(t)\right\}\right] \tag{60}$$

$$I_{n+1}(t) = I_n(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-n\left(I_n(t)+I_b\right)+\frac{u(t)}{V_1}\right\}\right] \tag{61}$$

$$D_{n+1}(t) = D_n(0) + ST^{-1}\left[\frac{(1-\tau+\tau u)}{M(\tau)} ST\left\{-kD_n(t)\right\}\right] \tag{62}$$

The result is obtained by

$G(t) = \lim_{n\to\infty} G_n(t)$,

$X(t) = \lim_{n\to\infty} X_n(t)$,

$I(t) = \lim_{n\to\infty} I_n(t)$,

and

$D(t) = \lim_{n\to\infty} D_n(t)$.

5 Numerical Solution

For numerical solution, we use the values given below by the experiment defined in [59,60].

Table 1: Table with initial value and parameters

Parameter	Value
G_0	287 mg/dL

(Continued)

Table 1 (continued)

Parameter	Value
X_0	0
I_0	241 mu/L
D_0	9 mg/dL
G_b	92 mg/dL
I_b	7.3 mu/L
p_1	0.01572
p_2	0.01301
p_3	0.4031×10^{-5}
n	0.3606/min.
k	0.05

Table 2: Comparison between the values of G for different values of t with experimental data

Time	G(tau = 1)	G(tau = 0.95)	G(tau = 0.7)	Exp.	|Exp. − G(tau = 1)|	|Exp. − G(tau = 0.95)|	|Exp. − G(tau = 0.7)|
0	287	287	287	92	195	195	195
4	279.35	279.35	279.36	287	7.65	7.65	7.64
6	275.64	275.71	275.72	251	24.64	24.71	24.72
12	264.71	265.09	265.03	211	53.71	54.09	54.03
18	253.51	253.61	253.41	193	60.51	60.61	60.41
30	224.75	209.89	210.07	149	75.75	60.89	61.07
36	201.98	154.15	156.87	133	68.98	21.15	23.87
38	192.10	124.69	129.23	130	62.1	5.31	0.77

Table 3: Comparison between the values of I for different values of t with experimental data

Time	I(tau = 1)	I(tau = 0.95)	I(tau = 0.7)	Exp.	|Exp. − I(tau = 1)|	|Exp. − I(tau = 0.95)|	|Exp. − I(tau = 0.7)|
0	241	238.74	225.17	11	230	227.74	214.17
4	117.17	117.06	121.96	130	12.83	12.94	8.04
6	81.70	81.97	89.76	85	3.30	3.03	4.76
10	39.72	40.20	48.66	49	9.27	8.80	0.34
14	19.44	19.91	27.11	41	21.56	21.09	13.89
18	11.38	12.36	21.19	32	20.62	19.64	10.81
19	11.40	12.83	23.42	30	18.59	17.17	6.56
20	12.74	14.88	28.19	30	17.26	15.12	1.81

By using the above-defined values, we can easily determine the numerical results for the defined model. In that analysis, we found the numerical result by using the sumudu transform. We found the result for some different τ values and formed a table that shows the comparison of the obtained result with the experimental data (see Tab. 1). In Tab. 2, we discussed G, and in Tab. 3, we discussed I and found from these analyses that the fractional model gives less error than the integer model. We also drew four figures to show the numerical outcomes. In Fig. 1 we showed the values of G at different values of τ, i.e., $\tau = 1$, $\tau = 0.95$, $\tau = 0.9$, $\tau = 0.8$, and $\tau = 0.7$. In Fig. 2, we showed the values of I at different values of τ. In the same way, we drew the Figs. 3

and 4 to show the numerical results of X and D at different values of τ. From the numerical outcome, it is clear that the diet is an essential component of glucose level.

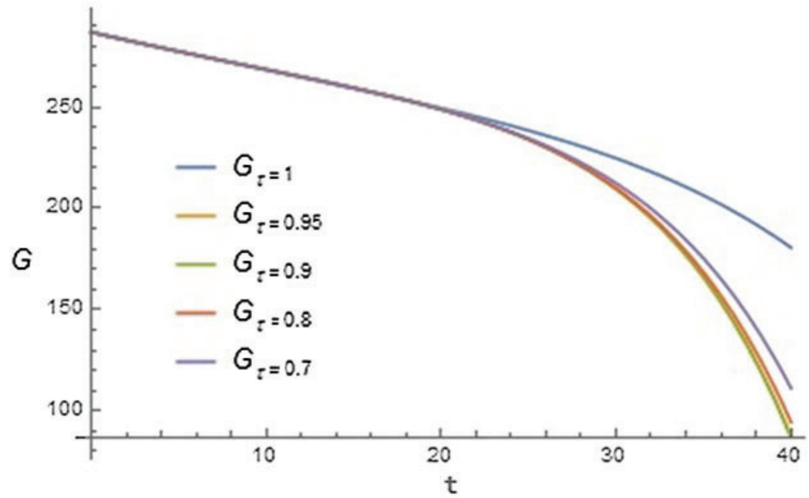

Figure 1: Graph of blood glucose cluster (G) with respect to time t for different values of τ

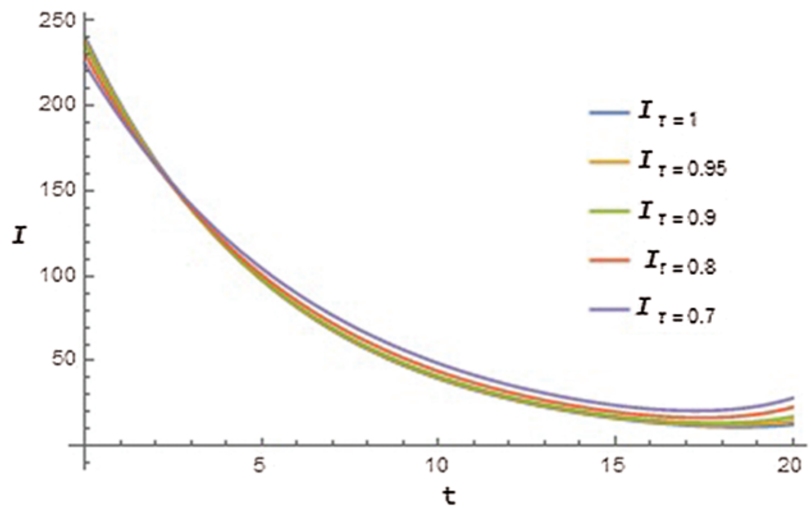

Figure 2: Graph of blood insulin cluster (I) with respect to time t for different values of τ

Furthermore, it has been observed from Tabs. 2 and 3, that the values obtained for G and I with linear order have more error than fractional order. So we can see that the fractional model with Caputo-Fabrizio operator gives better results in contrast to linear model, and our model defined the real-world problem in a better manner.

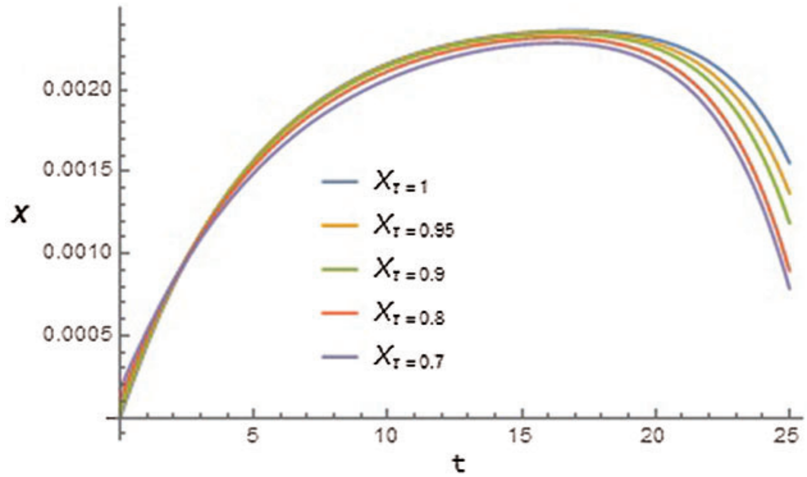

Figure 3: Graph of aftermath of effective insulin (X) with respect to time t for different values of τ

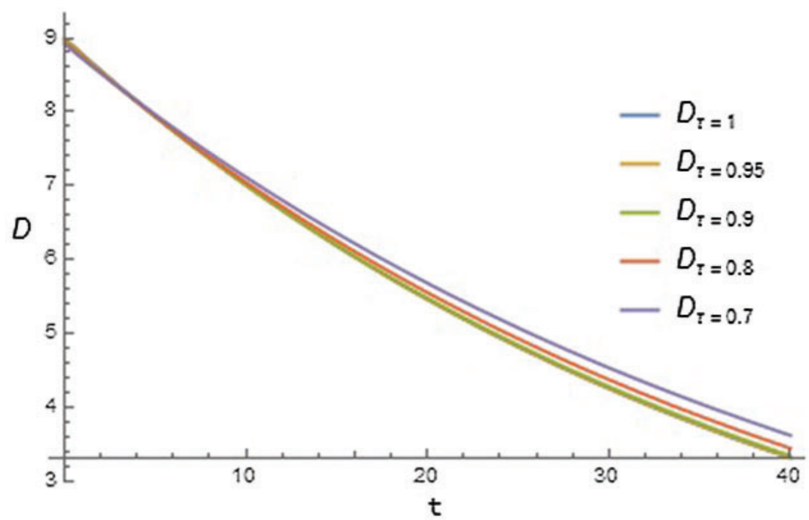

Figure 4: Graph of infusion of exogenous glucose (D) with respect to time t for different values of τ

6 Conclusion

The presented work strives to explain the presence and oneness of the Modified Bergman Minimal Model that is stretched out by Caputo-Fabrizio fractional differential coefficient in the frame of reference of sugar and insulin quantity in blood. From the obtained results, it is clear that the fractional model error reduces compared to integer order. Thus, we get nearby results of a set-up that displays the aftermath of time on the concentrations $G(t)$, $X(t)$, $I(t)$, and $D(t)$. As in future work, we can generalize the model or compare the results using various differential operators.

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