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Research Article

Some Coincidence and Common Fixed-Point Results on Cone b_2-Metric Spaces over Banach Algebras with Applications to the Infinite System of Integral Equations

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In this article, common fixed-point theorems for self-mappings under different types of generalized contractions in the context of the cone b_2-metric space over the Banach algebra are discussed. The existence results obtained strengthen the ones mentioned previously in the literature. An example and an application to the infinite system of integral equations are also presented to validate the main results.

1. Introduction and Preliminaries

Gähler [1] proposed the definition of 2-metric spaces as a generalization of an ordinary metric space. He defined that $d(s, m, z)$ geometrically represents the area of a triangle with vertices $s, m, z \in \aleph$. 2-metric is not a continuous function of its variables. This was one of the key drawbacks of the 2-metric space while an ordinary metric is a continuous function.

Keeping these drawbacks in mind, Dhage [2], in his PhD thesis, proposed a concept of the D-metric space as a generalized version of the 2-metric space. He also defined an open ball in such spaces and studied other topological properties of the mentioned structure. According to him, $D(s, m, z)$ represented the perimeter of a triangle. He stated that the D-metric induced a Hausdorff topology and in the D-metric space, the family of all open balls forms a basis for such topology.

Later, Mustafa and Sims [3] illustrate that the topological structure of Dhage's D-metric is invalid. Then, they revised the D-metric and expanded the notion of a metric in which each triplet of an arbitrary set is given a real number called as the G-metric space [4].

In addition, the definition of the D^*-metric space is proposed by Sedghi et al. [5] as an updated version of Dhage's D-metric space. Later, they analyzed and found that G-metric and D^*-metric have shortcomings. Later, they proposed a new simplified sturcture called the S-metric space [6].

On the other hand, by swapping the real numbers with the ordered Banach space and established cone metric space, Huang and Zhang [7] generalized the notion of a metric space and demonstrated some fixed-point results of contractive maps using the normality condition in such spaces. Rezapour and Hamlbarani [8] subsequently ignored the normality assumption and obtained some generalizations of the Huang and Zhang [7] results. However, it should be noted that the equivalence between cone metric spaces and metric spaces has been developed in recent studies by some scholars in the context of the presence of fixed points in the mapping involved. Liu and Xu [9] proposed the concept of a cone metric space over the Banach algebra in order to solve these shortcomings by replacing the Banach space with the

Banach algebra. This became an interesting discovery in the study of fixed-point theory since it can be shown that cone metric spaces over the Banach algebra are not equal to metric spaces in terms of the presence of the fixed points of mappings. Among these generalizations, by generalizing the cone 2-metric spaces [10] over the Banach algebra and b_2-metric spaces [11], Fernandez et al. [12] examined cone b_2-metric spaces over the Banach algebra with the constant $b \geq 1$. In the setting of the new structure, they proved some fixed-point theorems under different types of contractive mappings and showed the existence and uniqueness of a solution to a class of system of integral equations as an application.

Recently, in 2020, Islam et al. [13] initiated the notion of the cone b_2-metric space over the Banach algebra with constant $b \succcurlyeq e$ which is a generalization of the definition of Fernandez et al. [12]. They proved some fixed-point theorems under α-admissible Hardy-Rogers contractions which generalize many of the results from the existence literature, and as an application, they proved results which guarantee the existence of solution of an infinite system of integral equations.

In 1973, Hardy and Rogers [14] proposed a new definition of mappings called the contraction of Hardy-Rogers that generalizes the theory of the Banach contraction and the theorem of Reich [15] in a metric space setting. For other related work about the concept of Hardy-Rogers contractions, see, for instance, [16, 17] and the references therein.

We recollect certain essential notes, definitions required, and primary results consistent with the literature.

Definition 1 (see [18]). Consider $\widehat{\mathcal{U}}$ the Banach algebra which is real, and the multiplication operation is defined under the below properties (for all $s, m, z \in \widehat{\mathcal{U}}, \rho \in \mathbb{R}$):

(a_1) $(sm)z = s(mz)$

(a_2) $s(m+z) = sm + sz$ and $(s+m)z = sz + mz$

(a_3) $\rho(sm) = (\rho s)m = s(\rho m)$

(a_4) $\|sm\| \leq \|s\|\|m\|$

Unless otherwise stated, we will assume in this article that $\widehat{\mathcal{U}}$ is a real Banach algebra. If $s \in \widehat{\mathcal{U}}$ occurs, we call e the unit of $\widehat{\mathcal{U}}$, so that $es = se = s$. We call $\widehat{\mathcal{U}}$ a unital in this case. If an inverse element $m \in \widehat{\mathcal{U}}$ exists, the element $s \in \widehat{\mathcal{U}}$ is said to be invertible, so that $sm = ms = e$. The inverse of s in such case is unique and is denoted by s^{-1}. We require the following propositions in the sequel.

Proposition 2 (see [18]). *Consider the unital Banach algebra $\widehat{\mathcal{U}}$ with unit e, and let $s \in \widehat{\mathcal{U}}$ be the arbitrary element. If the spectral radius $r(s) < 1$, i.e.,*

$$r(s) = \lim_{n \longrightarrow \infty} \|s^n\|^{1/n} = \inf \|s^n\|^{1/n} < 1, \tag{1}$$

then $e - s$ is invertible. In fact,

$$(e - s)^{-1} = \sum_{k=1}^{\infty} s^k. \tag{2}$$

Remark 3. We see from [18] that $r(s) \leq \|s\|$ for all $s \in \widehat{\mathcal{U}}$ with unit e.

Remark 4 (see [19]). In Proposition 2, by replacing "$r(s) < 1$" by $\|s\| \leq 1$, then the conclusion remains the same.

Remark 5 (see [19]). If $r(s) < 1$, then $\|s^n\| \longrightarrow 0$ as $(n \longrightarrow \infty)$.

Definition 6. Consider the Banach algebra $\widehat{\mathcal{U}}$ with unit element e, zero element $\theta_{\widehat{\mathcal{U}}}$, and $\mathscr{C}_{\widehat{\mathcal{U}}} \neq \varnothing$. Then, $\mathscr{C}_{\widehat{\mathcal{U}}} \subset \widehat{\mathcal{U}}$ is a cone in $\widehat{\mathcal{U}}$ if:

(b_1) $e \in \mathscr{C}_{\widehat{\mathcal{U}}}$

(b_2) $\mathscr{C}_{\widehat{\mathcal{U}}} + \mathscr{C}_{\widehat{\mathcal{U}}} \subset \mathscr{C}_{\widehat{\mathcal{U}}}$

(b_3) $\lambda \mathscr{C}_{\widehat{\mathcal{U}}} \subset \mathscr{C}_{\widehat{\mathcal{U}}}$ for all $\lambda \geq 0$

(b_4) $\mathscr{C}_{\widehat{\mathcal{U}}} \cdot \mathscr{C}_{\widehat{\mathcal{U}}} \subset \mathscr{C}_{\widehat{\mathcal{U}}}$

(b_5) $\mathscr{C}_{\widehat{\mathcal{U}}} \cap (-\mathscr{C}_{\widehat{\mathcal{U}}}) = \{\theta_{\widehat{\mathcal{U}}}\}$

Define the partial order relation $\preccurlyeq$ in $\widehat{\mathcal{U}}$ w.r.t $\mathscr{C}_{\widehat{\mathcal{U}}}$ by $s \preccurlyeq m$ if and only if $m - s \in \mathscr{C}_{\widehat{\mathcal{U}}}$; also, $s \prec m$ if $s \preccurlyeq m$ but $s \neq m$ while $s \ll m$ stands for $m - s \in \operatorname{int} \mathscr{C}_{\widehat{\mathcal{U}}}$, where $\operatorname{int} \mathscr{C}_{\widehat{\mathcal{U}}}$ is the interior of $\mathscr{C}_{\widehat{\mathcal{U}}}$. $\mathscr{C}_{\widehat{\mathcal{U}}}$ is solid if $\operatorname{int} \mathscr{C}_{\widehat{\mathcal{U}}} \neq \varnothing$.

If there is $M > 0$ such that for all $s, m \in \mathscr{C}_{\widehat{\mathcal{U}}}$, we have

$$\theta_{\widehat{\mathcal{U}}} \preccurlyeq s \preccurlyeq m \text{ implies } \|s\| \leq M\|m\|, \tag{3}$$

then $\mathscr{C}_{\widehat{\mathcal{U}}}$ is normal. If M is least and positive in the above, then it is the normal constant of $\mathscr{C}_{\widehat{\mathcal{U}}}$ [7].

Definition 7 (see [7, 9]). Let $d : \aleph \times \aleph \longrightarrow \widehat{\mathcal{U}}$ and $\aleph \neq \varnothing$ be the mapping:

(c_1) For all $s, m \in \aleph$, $d(s, m) \succcurlyeq \theta_{\widehat{\mathcal{U}}}$ and $d(s, m) = \theta_{\widehat{\mathcal{U}}}$ if and only if $s = m$

(c_2) For all $s, m \in \aleph$, $d(s, m) = d(m, s)$

(c_3) For all $s, m, z \in \aleph$, $d(s, z) \preccurlyeq d(s, m) + d(m, z)$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ with cone metric d is a cone metric space.

In [20], over the Banach algebra with constant $b \geq 1$, the cone b-metric space is introduced as a generalization of the cone metric space over the Banach algebra while in Mitrovic and Hussain [16], over the Banach algebra with parameter $b \succcurlyeq e$, the concept of cone b-metric spaces is introduced.

Definition 8 (see [16]). Let $d : \aleph \times \aleph \longrightarrow \widehat{\mathcal{U}}$ and $\aleph \neq \varnothing$ be the mapping:

(e_1) For all $s, m \in \aleph$, $\theta_{\widehat{\mathcal{U}}} \preccurlyeq d(s, m)$ and $d(s, m) = \theta_{\widehat{\mathcal{U}}}$ if and only if $s = m$

(e_2) For all $s, m \in \aleph$, $d(s, m) = d(m, s)$

(e_3) There is $b \in \mathscr{C}_{\widehat{\mathcal{U}}}$, $b \succcurlyeq e$, and for all $s, m, z \in \aleph$, $d(s, z) \preccurlyeq b[d(s, m) + d(m, z)]$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathcal{U}}$ with cone b-metric d is a cone b-metric space. Note that if we take $b = e$, then it reduces to the cone metric space over the Banach algebra $\widehat{\mathcal{U}}$.

Definition 9 (see [1]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \mathbb{R}^+$ and $\aleph \neq \varnothing$:

(f_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq 0$, if at least two of s, m, z are not equal

(f_2) $d(s, m, z) = 0$ if and only if at least two of s, m, z are equal

(f_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(f_4) $d(s, m, z) \leq d(s, m, t) + d(s, z, t) + d(m, z, t)$ for all $s, m, z, t \in \aleph$

Then, $(\aleph, d)$ is a 2-metric space with 2-metric d.

Definition 10 (see [12]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$, $\aleph \neq \varnothing$, and $b \geq 1$ be a real number:

(g_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq \theta_{\widehat{\mathscr{U}}}$ if at least two of s, m, z are not equal

(g_2) $d(s, m, z) = \theta_{\widehat{\mathscr{U}}}$ if and only if at least two of s, m, z are equal

(g_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(g_4) $d(s, m, z) \preccurlyeq b[d(s, m, t) + d(s, z, t) + d(m, z, t)]$ for all $s, m, z, t \in \aleph$

Then, $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ with parameter $b \geq 1$ is a cone b_2-metric space. By taking $b = 1$, it became a cone 2-metric space. We refer the reader to [21] for other details about the cone 2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Islam et al. [13] initiated the concept of the cone b_2-metric space over the Banach algebra with parameter $b \succcurlyeq e$.

Definition 11 (see [13]). Let $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ and $\aleph \neq \varnothing$:

(h_1) There is a point $z \in \aleph$ for $s, m \in \aleph$ such that $d(s, m, z) \neq \theta_{\widehat{\mathscr{U}}}$ if at least two of s, m, z are not equal

(h_2) $d(s, m, z) = \theta_{\widehat{\mathscr{U}}}$ iff at least two of s, m, z are equal

(h_3) $d(s, m, z) = d(P(s, m, z))$ for all $s, m, z \in \aleph$, where $P(s, m, z)$ stands for all permutations of s, m, z

(h_4) $d(s, m, z) \preccurlyeq b[d(s, m, t) + d(s, z, t) + d(m, z, t)]$ for all $s, m, z, t \in \aleph$ with $b \in \mathscr{C}_{\widehat{\mathscr{U}}}$, $b \succcurlyeq e$

Then, $(\aleph, d)$ over the Banach algebra with parameter $b \succcurlyeq e$ is a cone b_2-metric space. By taking $b = e$, it reduces to a cone 2-metric space.

Example 12 (see [13]). Let $\widehat{\mathscr{U}} = C^1_{\mathbb{R}}[0, 1]$. For each $h(t) \in \widehat{\mathscr{U}}$, $\|h(t)\| = \|h(t)\|_\infty + \|h'(t)\|_\infty$. Then, $\widehat{\mathscr{U}}$ is a Banach algebra with unit $e = 1$ as a constant function, and multiplication is defined pointwise. Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{h(t) \in \widehat{\mathscr{U}} \mid h(t) \geq 0, t \in [0, 1]\}$. Then, $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone in $\widehat{\mathscr{U}}$. Let $\aleph = \{(k, 0) \in \mathbb{R}^2 \mid \mid 0 \leq k \leq 1\} \cup \{(0, 1)\}$. For all $S, M, Z \in \aleph$, define $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ as

$$d(S, M, Z) = \begin{cases} d(P(S, M, Z)), & P \text{ denotes permutations}, \\ \Delta \cdot h(t), & \text{otherwise}, \end{cases} \tag{4}$$

where Δ is the square of the area of the triangle formed by S, M, Z and $h : [0, 1] \longrightarrow \mathbb{R}$ defined by $h(t) = e^t$. Consider

$$\begin{aligned} d((s, 0), (m, 0), (0, 1)) \cdot e^t \preccurlyeq & d((s, 0), (m, 0), (z, 0)) \cdot e^t \\ & + d((s, 0), (z, 0), (0, 1)) \cdot e^t \\ & + d((z, 0), (m, 0), (0, 1)) \cdot e^t. \end{aligned} \tag{5}$$

That is, $1/4(s - m)^2 \cdot e^t \preccurlyeq 1/4((s - z)^2 + (z - m)^2) \cdot e^t$, showing that d is not a cone 2-metric, because $-(3/16)e^t \in \mathscr{C}_{\widehat{\mathscr{U}}}$ for $0 \leq s, m, z \leq 1$ with $z = 1/2$, $m = 0$, and $s = 1$. But for $b \geq 2 \in \mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Definition 13 (see [13]). Consider that $(\aleph, d)$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$ with $b \succcurlyeq e$, and let $\{s_n\}$ be a sequence in $(\aleph, d)$; then,

(i_1) $\{s_n\}$ is said to converge to $s \in \aleph$ if for every $c \gg \theta_{\widehat{\mathscr{U}}}$ there is $N \in \mathbb{N}$ such that $d(s_n, s, a) \ll c$ for all $n \geq N$. That is,

$$\lim_{n \longrightarrow \infty} s_n = s \text{ (or) } s_n \longrightarrow s \, (n \longrightarrow \infty). \tag{6}$$

(i_2) If for every $c \gg \theta_{\widehat{\mathscr{U}}}$ there exists $N \in \mathbb{N}$ such that $d(s_n, s_m, a) \ll c$ for all $m, n \geq N$, then we say that $\{s_n\}$ is a Cauchy sequence

(i_3) $(\aleph, d)$ is complete if every Cauchy sequence is convergent in $\aleph$

Definition 14 (see [22]). Let a sequence $\{s_n\}$ be in $\widehat{\mathscr{U}}$; then, sequence $\{s_n\}$ is a c -sequence, if for each $c \gg \theta_{\widehat{\mathscr{U}}}$ there is $N \in \mathbb{N}$ such that $s_n \ll c$ for all $n > N$.

Lemma 15 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and $\operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Also, consider $\{s_n\}$ a c -sequence in $\widehat{\mathscr{U}}$ and $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ where k is arbitrary; then, $\{ks_n\}$ is a c -sequence.*

Lemma 16 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and $\operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Let $\{s_n\}$ and $\{z_n\}$ be c-sequences in $\widehat{\mathscr{U}}$. Then, for arbitrary $\eta, \zeta \in \mathscr{C}_{\widehat{\mathscr{U}}}$, we have $\{\eta s_n + \zeta z_n\}$ which is also a c-sequence.*

Lemma 17 (see [23]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and $\operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Let $\{s_n\} \subset \mathscr{C}_{\widehat{\mathscr{U}}}$ such that $\|s_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then, $\{s_n\}$ is a c-sequence.*

Lemma 18 (see [19]). *Let $\widehat{\mathscr{U}}$ be the Banach algebra, e be their unit element, and $m, s \in \widehat{\mathscr{U}}$. If m, s commutes, then*

(k_1) $r(s + m) \leq r(s) + r(m)$

(k_2) $r(sm) \leq r(s)r(m)$

Lemma 19 (see [19]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and $s \in \widehat{\mathscr{U}}$. If $0 \leq r(s) < 1$, then*

$$r\left((e - s)^{-1}\right) \leq (1 - r(s))^{-1}. \tag{7}$$

Lemma 20 (see [24]). *Consider the Banach algebra $\widehat{\mathscr{U}}$, e is their unit element, and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. Let $L \in \widehat{\mathscr{U}}$ and $s_n = L^n$. If $r(L) < 1$, then $\{s_n\}$ is a c-sequence.*

Lemma 21 (see [25]). *Let $\mathscr{C}_{\widehat{\mathscr{U}}} \subset \widehat{\mathscr{U}}$ be a cone.*

(l_1) If $s, m \in \widehat{\mathscr{U}}$, $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$, and $m \preccurlyeq s$, then $km \preccurlyeq ks$

(l_2) If $m, k \in \mathscr{C}_{\widehat{\mathscr{U}}}$, $r(k) < 1$, and $m \preccurlyeq km$, then $m = 0$

(l_3) For any $n \in \mathbb{N}$, $r(k^n) < 1$ with $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ and $r(k) < 1$

Lemma 22 (see [26]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$.

(n_1) If $m, s, z \in \widehat{\mathscr{U}}$ and $m \preccurlyeq s \ll z$, then $m \ll s$

(n^2) If $m \in \mathscr{C}_{\widehat{\mathscr{U}}}$ and $m \ll c$ for $c \gg \theta_{\widehat{\mathscr{U}}}$, then $m = \theta_{\widehat{\mathscr{U}}}$

Lemma 23 (see [21]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. *Let $m \in \widehat{\mathscr{U}}$, and suppose that $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ is an arbitrary given vector such that $m \ll c$ for any $\theta_{\widehat{\mathscr{U}}} \ll c$, then $km \ll c$ for any $\theta_{\widehat{\mathscr{U}}} \ll c$.*

Lemma 24 (see [27]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. *Let $\theta_{\widehat{\mathscr{U}}} \preccurlyeq m \ll c$ for each $\theta_{\widehat{\mathscr{U}}} \ll c$; then, $m = \theta_{\widehat{\mathscr{U}}}$.*

Lemma 25 (see [27]). *Consider the Banach algebra $\widehat{\mathscr{U}}$ and* int $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$. *If $\|s_n\| \longrightarrow 0$ as $(n \longrightarrow \infty)$, then for each $c \gg \theta_{\widehat{\mathscr{U}}}$, there is $N \in \mathbb{N}$ with $n > N$, such that $s_n \ll c$.*

Definition 26 (see [28]). Let g and g be self-maps of a set $\aleph$. If $m = gs = fs$ for some $s \in \aleph$, then for g and f, s is known as a coincidence point and m is known as a point of coincidence of g and f.

Definition 27 (see [29]). The mappings $g, f : \aleph \longrightarrow \aleph$ are said to be weakly compatible, whenever $fs = gs$ and $fgs = gfs$ for any $s \in \aleph$.

Lemma 28 (see [28]). *Let the mappings g and f be weakly compatible self-maps of a set $\aleph$. If g and f have a unique point of coincidence $m = fs = gs$, then m is the unique common fixed point of g and f.*

2. Main Results

In this section, in the framework of the cone b_2-metric space over Banach algebras with parameter $b \succcurlyeq e$, we prove some common fixed-point results.

Proposition 29. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be the complete cone b_2-metric space and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. If a sequence $\{s_n\}$ in $\aleph$ converges to $s \in \aleph$, then we have the following:*

(i) $\{d(s_n, s, a)\}$ is a c-sequence for all $a \in \aleph$

(ii) For any $\alpha \in \mathbb{N}$, $\{d(s_n, s_{n+\alpha}, a)\}$ is a c-sequence for all $a \in \aleph$

Proof. Since the proof is easy, so we left it.

Now, we here state and prove our first main results which generalize and extend many of the conclusions from the existence literature. □

Theorem 30. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$\begin{aligned} d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq & \vartheta_1 d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\ & + \vartheta_2 d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) \\ & + \vartheta_3 d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right) \\ & + \vartheta_4 d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) \\ & + \vartheta_5 d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right), \end{aligned} \tag{8}$$

where $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5 \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $\sum_{w=1}^{3} r(\vartheta_w) + 2r(\vartheta_4)r(b) + 2r(\vartheta_5)r(b) < 1$, $r(\vartheta_2)r(b) + r(\vartheta_5)r(b^2) < 1$, $r(\vartheta_3)r(b) + r(\vartheta_4)r(b^2) < 1$, and $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, b$ commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Set $E_i^{\eta_i} = S_{2i+1}$, $F_j^{\eta_j} = T_{2j+2}$, $G_k^{\eta_k} = I_{2k+3}$, and $H_l^{\eta_l} = J_{2l+2}$, $i, j, k, l \geq 1$. Then, by (8), we have

$$\begin{aligned} d\left(S_{2i+1}(s), T_{2j+2}(m), a\right) \preccurlyeq & \vartheta_1 d(I_{2k+3}(s), J_{2l+2}(m), a) \\ & + \vartheta_2 d(I_{2k+3}(s), S_{2i+1}(s), a) \\ & + \vartheta_3 d\left(J_{2l+2}(m), T_{2j+2}(m), a\right) \\ & + \vartheta_4 d\left(I_{2k+3}(s), T_{2j+2}(m), a\right) \\ & + \vartheta_5 d(J_{2l+2}(m), S_{2i+1}(s), a). \end{aligned} \tag{9}$$

Choose $s_0 \in \aleph$ to be arbitrary. Since $E_i(\aleph) \subseteq H_l(\aleph)$ and $F_j(\aleph) \subseteq G_k(\aleph)$ for each $i, j, k, l \geq 1$, there exists $s_1, s_2 \in \aleph$ such that $S_1(s_0) = J_2(s_1)$ and $T_2(s_1) = I_3(s_2)$. Continuing this process, we can define $\{s_n\}$ by $S_{2n+1}(s_{2n}) = J_{2n+2}(s_{2n+1})$ and $T_{2n+2}(s_{2n+1}) = I_{2n+3}(s_{2n+2})$.

Denote $\mu_{2n} = J_{2n+2}(s_{2n+1}) = S_{2n+1}(s_{2n})$ and $\mu_{2n+1} = I_{2n+3}(s_{2n+2}) = T_{2n+2}(s_{2n+1})$ for $n = 0, 1, 2, \cdots$. Now, we show that $\{\mu_n\}$ is a Cauchy sequence.

From (9), we know that

$$\begin{aligned} d(\mu_{2n},\mu_{2n+1},a) &= d(S_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),a) \\ &\preccurlyeq \vartheta_1 d(I_{2n+1}(s_{2n}),J_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_2 d(I_{2n+1}(s_{2n}),S_{2n+1}(s_{2n}),a) \\ &\quad + \vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_4 d(I_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(s_{2n}),a) \\ &= \vartheta_1 d(\mu_{2n-1},\mu_{2n},a) + \vartheta_2 d(\mu_{2n-1},\mu_{2n},a) \\ &\quad + \vartheta_3 d(\mu_{2n},\mu_{2n+1},a) + \vartheta_4 d(\mu_{2n-1},\mu_{2n+1},a) \\ &\quad + \vartheta_5 d(\mu_{2n},\mu_{2n},a) \preccurlyeq \vartheta_1 d(\mu_{2n-1},\mu_{2n},a) \\ &\quad + \vartheta_2 d(\mu_{2n-1},\mu_{2n},a) + \vartheta_3 d(\mu_{2n},\mu_{2n+1},a) \\ &\quad + \vartheta_4 b d(\mu_{2n-1},\mu_{2n+1},\mu_{2n}) \\ &\quad + \vartheta_4 b d(\mu_{2n-1},\mu_{2n},a) \\ &\quad + \vartheta_4 b d(\mu_{2n},\mu_{2n+1},a). \end{aligned} \tag{10}$$

$$\begin{aligned} d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}) &= d(S_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),\mu_{2n-1}) \\ &\leq \vartheta_1 d(I_{2n+1}(s_{2n}),J_{2n+2}(s_{2n+1}),\mu_{2n-1}) \\ &\quad + \vartheta_2 d(I_{2n+1}(s_{2n}),S_{2n+1}(s_{2n}),\mu_{2n-1}) \\ &\quad + \vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),\mu_{2n-1}) \\ &\quad + \vartheta_4 d(I_{2n+1}(s_{2n}),T_{2n+2}(s_{2n+1}),\mu_{2n-1}) \\ &\quad + \vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(s_{2n}),\mu_{2n-1}) \\ &= \vartheta_1 d(\mu_{2n-1},\mu_{2n},\mu_{2n-1}) \\ &\quad + \vartheta_2 d(\mu_{2n-1},\mu_{2n},\mu_{2n-1}) \\ &\quad + \vartheta_3 d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}) \\ &\quad + \vartheta_4 d(\mu_{2n-1},\mu_{2n+1},\mu_{2n-1}) \\ &\quad + \vartheta_5 \mathrm{d}(\mu_{2n},\mu_{2n},\mu_{2n-1}) \\ &\preccurlyeq \vartheta_3 d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}). \end{aligned} \tag{11}$$

It means that $d(\mu_{2n},\mu_{2n+1},\mu_{2n-1}) = \theta_{\widehat{\mathscr{U}}}$. Therefore, (10) becomes

$$(e - \vartheta_3 - \vartheta_4 b) d(\mu_{2n},\mu_{2n+1},a) \preccurlyeq (\vartheta_1 + \vartheta_2 + \vartheta_4 b) d(\mu_{2n-1},\mu_{2n},a), \tag{12}$$

that is,

$$d(\mu_{2n},\mu_{2n+1},a) \preccurlyeq L_1 d(\mu_{2n-1},\mu_{2n},a), \tag{13}$$

where $L_1 = (e - \vartheta_3 - \vartheta_4 b)^{-1}(\vartheta_1 + \vartheta_2 + \vartheta_4 b)$. Similarly, it is not difficult to show that

$$\begin{aligned} d(\mu_{2n+2},\mu_{2n+1},a) &= d(S_{2n+3}(s_{2n+2}),T_{2n+2}(s_{2n+1}),a) \\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(s_{2n+2}),J_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_2 d(I_{2n+3}(s_{2n+2}),S_{2n+3}(s_{2n+2}),a) \\ &\quad + \vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_4 d(I_{2n+3}(s_{2n+2}),T_{2n+2}(s_{2n+1}),a) \\ &\quad + \vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+3}(s_{2n+2}),a) \\ &= \vartheta_1 d(\mu_{2n+1},\mu_{2n},a) + \vartheta_2 d(\mu_{2n+1},\mu_{2n+2},a) \\ &\quad + \vartheta_3 d(\mu_{2n},\mu_{2n+1},a) + \vartheta_4 d(\mu_{2n+1},\mu_{2n+1},a) \\ &\quad + \vartheta_5 d(\mu_{2n},\mu_{2n+2},a) \preccurlyeq \vartheta_1 d(\mu_{2n+1},\mu_{2n},a) \\ &\quad + \vartheta_2 d(\mu_{2n+1},\mu_{2n+2},a) + \vartheta_3 d(\mu_{2n},\mu_{2n+1},a) \\ &\quad + \vartheta_5 b d(\mu_{2n},\mu_{2n+2},\mu_{2n+1}) \\ &\quad + \vartheta_5 b d(\mu_{2n},\mu_{2n+1},a) \\ &\quad + \vartheta_5 b d(\mu_{2n+1},\mu_{2n+2},a). \end{aligned} \tag{14}$$

As $d(\mu_{2n},\mu_{2n+2},\mu_{2n+1}) = \theta_{\widehat{\mathscr{U}}}$, therefore (14) becomes

$$(e - \vartheta_2 - \vartheta_5 b) d(\mu_{2n+2},\mu_{2n+1},a) \preccurlyeq (\vartheta_1 + \vartheta_3 + \vartheta_5 b) d(\mu_{2n},\mu_{2n+1},a), \tag{15}$$

that is,

$$d(\mu_{2n+2},\mu_{2n+1},a) \preccurlyeq L_2 d(\mu_{2n},\mu_{2n+1},a), \tag{16}$$

where $L_2 = (e - \vartheta_2 - \vartheta_5 b)^{-1}(\vartheta_1 + \vartheta_3 + \vartheta_5 b)$. Set $K = L_1 L_2$, and using inequalities (13) and (16), we deduce that

$$\begin{aligned} d(\mu_{2n+2},\mu_{2n+1},a) &\preccurlyeq L_2 d(\mu_{2\mathrm{n}+1},\mu_{2n},a) \\ &\preccurlyeq L_2 L_1 d(\mu_{2n},\mu_{2n-1},a) \\ &\preccurlyeq L_2 L_1 L_2 d(\mu_{2n-1},\mu_{2n-2},a) \\ &= L_2 K d(\mu_{2n-1},\mu_{2n-2},a) \\ &\preccurlyeq \cdots \preccurlyeq L_2 K^n d(\mu_1,\mu_0,a), \end{aligned} \tag{17}$$

$$\begin{aligned} d(\mu_{2n+3},\mu_{2n+2},a) &\preccurlyeq L_1 d(\mu_{2n+2},\mu_{2n+1},a) \\ &\preccurlyeq L_1 L_2 K^n d(\mu_1,\mu_0,a) \\ &= K^{n+1} d(\mu_1,\mu_0,a). \end{aligned} \tag{18}$$

In this case, for all $t < n$, similar to (17) and (18), we have

$$\begin{aligned} d(\mu_{2n+1},\mu_{2n},\mu_{2t+1}) &= L_2 K^{n-t-1} d(\mu_{2t+1},\mu_{2t+2},\mu_{2t+1}) = \theta_{\widehat{\mathscr{U}}}, \\ d(\mu_{2n-1},\mu_{2n},\mu_{2t+1}) &= K^{n-t-1} d(\mu_{2t+1},\mu_{2t+2},\mu_{2t+1}) = \theta_{\widehat{\mathscr{U}}}, \\ d(\mu_{2n+1},\mu_{2n},\mu_{2t}) &= K^{n-t} d(\mu_{2t+1},\mu_{2t},\mu_{2t}) = \theta_{\widehat{\mathscr{U}}}, \\ d(\mu_{2n-1},\mu_{2n},\mu_{2t}) &= L_2 K^{n-t-1} d(\mu_{2t+1},\mu_{2t},\mu_{2t}) = \theta_{\widehat{\mathscr{U}}}. \end{aligned} \tag{19}$$

Therefore, for all $m < n$, $a \in \aleph$, and using the above inequalities, we have

$$\begin{aligned}
d(\mu_{2n+1}, \mu_{2m+1}, a) \preccurlyeq{} & bd(\mu_{2n+1}, \mu_{2m+1}, \mu_{2n}) \\
& + bd(\mu_{2n+1}, \mu_{2n}, a) \\
& + bd(\mu_{2n}, \mu_{2m+1}, a) \\
={} & bd(\mu_{2n+1}, \mu_{2n}, a) + bd(\mu_{2n}, \mu_{2m+1}, a) \\
\preccurlyeq{} & bd(\mu_{2n+1}, \mu_{2n}, a) + b^2 d(\mu_{2n}, \mu_{2m+1}, \mu_{2n-1}) \\
& + b^2 d(\mu_{2n}, \mu_{2n-1}, a) \\
& + b^2 d(\mu_{2n-1}, \mu_{2m+1}, a) \\
={} & bd(\mu_{2n+1}, \mu_{2n}, a) + b^2 d(\mu_{2n}, \mu_{2n-1}, a) \\
& + b^2 d(\mu_{2n-1}, \mu_{2m+1}, a).
\end{aligned} \tag{20}$$

Continuing this process, we get

$$\begin{aligned}
d(\mu_{2n+1}, \mu_{2m+1}, a) \le{} & bd(\mu_{2n+1}, \mu_{2n}, a) + b^2 d(\mu_{2n}, \mu_{2n-1}, a) \\
& + b^3 d(\mu_{2n-1}, \mu_{2n-2}, a) \\
& + b^4 d(\mu_{2n-2}, \mu_{2n-3}, a) \\
& + \cdots + b^{n-m-4} d(\mu_{2m+5}, \mu_{2m+4}, a) \\
& + b^{n-m-3} d(\mu_{2m+4}, \mu_{2m+3}, a) \\
& + b^{n-m-2} d(\mu_{2m+3}, \mu_{2m+2}, a) \\
& + b^{n-m-1} d(\mu_{2m+2}, \mu_{2m+1}, a) \\
\le{} & bK^n d(\mu_1, \mu_0, a) + b^2 L_2 K^{n-1} d(\mu_1, \mu_0, a) \\
& + b^3 K^{n-1} d(\mu_1, \mu_0, a) \\
& + b^4 L_2 K^{n-2} d(\mu_1, \mu_0, a) \\
& + \cdots + b^{n-m-4} K^{m+2} d(\mu_1, \mu_0, a) \\
& + b^{n-m-3} L_2 K^{m+1} d(\mu_1, \mu_0, a) \\
& + b^{n-m-2} K^{m+1} d(\mu_1, \mu_0, a) \\
& + b^{n-m-1} L_2 K^m d(\mu_1, \mu_0, a) \\
\le{} & \left(bK^n + b^2 L_2 K^{n-1} + b^3 K^{n-1}\right. \\
& + b^4 L_2 K^{n-2} + \cdots + b^{n-m-4} K^{m+2} \\
& + b^{n-m-3} L_2 K^{m+1} + b^{n-m-2} K^{m+1} \\
& \left. + b^{n-m-1} L_2 K^m \right) d(\mu_1, \mu_0, a),
\end{aligned}$$

$$\begin{aligned}
d(\mu_{2n+1}, \mu_{2m+1}, a) ={} & \left(bK^n + b^3 K^{n-1} + \cdots + b^{n-m-4} K^{m+2}\right. \\
& + b^{n-m-2} K^{m+1} + b^2 L_2 K^{n-1} \\
& b^4 L_2 K^{n-2} + \cdots + b^{n-m-3} L_2 K^{m+1} \\
& \left. + b^{n-m-1} L_2 K^m \right) d(\mu_1, \mu_0, a) \\
={} & K^{m+1} b^{n-m-2} \left(e + b^{-2} K\right. \\
& \left. + \cdots + b^{m-n+3} K^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
& + L_2 K^m b^{n-m-1} \left(e + b^{-2} K\right. \\
& \left. + \cdots + b^{m-n+3} K^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
={} & \left(K^{m+1} b^{n-m-2} \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \right. \\
& \left. + L_2 K^m b^{n-m-1} \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \right) d(\mu_1, \mu_0, a).
\end{aligned} \tag{21}$$

That is,

$$\begin{aligned}
d(\mu_{2n+1}, \mu_{2m+1}, a) \preccurlyeq{} & \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \left(K^{m+1} b^{n-m-2}\right. \\
& \left. + L_2 K^m b^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
\preccurlyeq{} & \left(e - b^{-2} K\right)^{-1} K^m \left(K b^{n-m-2}\right. \\
& \left. + L_2 b^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
={} & MK^m d(\mu_1, \mu_0, a),
\end{aligned} \tag{22}$$

where $M = \left(e - b^{-2} K\right)^{-1} \left(K b^{n-m-2} + L_2 b^{n-m-1}\right)$.

Similarly, we have

$$\begin{aligned}
d(\mu_{2n}, \mu_{2m+1}, a) \le{} & bd(\mu_{2n}, \mu_{2n-1}, a) \\
& + b^2 d(\mu_{2n-1}, \mu_{2n-2}, a) \\
& + b^3 d(\mu_{2n-2}, \mu_{2n-3}, a) \\
& + b^4 d(\mu_{2n-3}, \mu_{2n-4}, a) \\
& + \cdots + b^{n-m-4} d(\mu_{2m+5}, \mu_{2m+4}, a) \\
& + b^{n-m-3} d(\mu_{2m+4}, \mu_{2m+3}, a) \\
& + b^{n-m-2} d(\mu_{2m+3}, \mu_{2m+2}, a) \\
& + b^{n-m-1} d(\mu_{2m+2}, \mu_{2m+1}, a) \\
\le{} & \left(bL_2 K^{n-1} + b^3 L_2 K^{n-2}\right. \\
& + \cdots + b^{n-m-4} L_2 K^{m+2} + b^{n-m-2} L_2 K^{m+1} \\
& + b^2 K^{n-1} + b^4 K^{n-2} + \cdots + b^{n-m-3} K^{m+2} \\
& \left. + b^{n-m-1} K^m \right) d(\mu_1, \mu_0, a) \\
\le{} & b^{n-m-2} L_2 K^{m+1} \left(e + b^{-2} K\right. \\
& \left. + \cdots + b^{m-n+3} K^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
& + b^{n-m-1} K^m \left(e + b^{-2} K\right. \\
& \left. + \cdots + b^{m-n+3} K^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
\le{} & \left(b^{n-m-2} L_2 K^{m+1} \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \right. \\
& \left. + b^{n-m-1} K^m \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \right) d(\mu_1, \mu_0, a) \\
\le{} & \sum_{r=0}^{\infty} \left(b^{-2} K\right)^r \left(b^{n-m-2} L_2 K^{m+1}\right. \\
& \left. + b^{n-m-1} K^m\right) d(\mu_1, \mu_0, a) \\
\le{} & \left(e - b^{-2} K\right)^{-1} K^m \left(b^{n-m-2} L_2 K\right. \\
& \left. + b^{n-m-1}\right) d(\mu_1, \mu_0, a) \\
={} & NK^m d(\mu_1, \mu_0, a),
\end{aligned} \tag{23}$$

where $N = \left(e - b^{-2} K\right)^{-1} \left(b^{n-m-2} L_2 K + b^{n-m-1}\right)$.

Similar to the above, one can easily get that

$$\begin{aligned} d(\mu_{2n}, \mu_{2m}, a) &\preccurlyeq OK^m d(\mu_1, \mu_0, a), \\ d(\mu_{2n+1}, \mu_{2m}, a) &\preccurlyeq PK^m d(\mu_1, \mu_0, a), \end{aligned} \tag{24}$$

where $O = (e - b^{-2}K)^{-1}(b^{n-m}L_2 + b^{n-m-1}K)$ and $P = (e - b^{-2}K)^{-1}(b^{n-m-1} + b^{n-m-2}L_2)$.

From Lemmas 18 and 19, we have that

$$\begin{aligned} r(K) &= r(L_1 L_2) \le r(L_1) \times r(L_2) \\ &= r\left[(e - \vartheta_3 - \vartheta_4 b)^{-1}(\vartheta_1 + \vartheta_2 + \vartheta_4 b)\right] \\ &\quad \times r\left[(e - \vartheta_2 - \vartheta_5 b)^{-1}(\vartheta_1 + \vartheta_3 + \vartheta_5 b)\right] \\ &\le r\left((e - \vartheta_3 - \vartheta_4 b)^{-1}\right) \times r(\vartheta_1 + \vartheta_2 + \vartheta_4 b) \\ &\quad \times r\left((e - \vartheta_2 - \vartheta_5 b)^{-1}\right) \times r(\vartheta_1 + \vartheta_3 + \vartheta_5 b) \\ &\le \frac{r(\vartheta_1) + r(\vartheta_2) + r(\vartheta_4) r(b)}{1 - r(\vartheta_3) - r(\vartheta_4) r(b)} \\ &\quad \times \frac{r(\vartheta_1) + r(\vartheta_3) + r(\vartheta_5) r(b)}{1 - r(\vartheta_2) - r(\vartheta_5) r(b)} < 1. \end{aligned} \tag{25}$$

Since $r(K) < 1$, therefore in the light of Remark 5 and Lemma 25, $\|K^m\| \longrightarrow 0$ as $(m \longrightarrow \infty)$, and so, for every $c \in \operatorname{int} \mathscr{C}_{\widehat{\mathscr{U}}}$, there exists $n_0 \in \mathbb{N}$ such that $K^m \ll c$ for all $n > n_0$; that is, the sequence $\{K^m\}$ is a c-sequence. By Lemma 15, the sequences $\{MK^m d(\mu_1, \mu_0, a)\}$, $\{NK^m d(\mu_1, \mu_0, a)\}$, $\{OK^m d(\mu_1, \mu_0, a)\}$, and $\{PK^m d(\mu_1, \mu_0, a)\}$ are also c-sequences. Therefore, for any $c \in \widehat{\mathscr{U}}$ with $\theta_{\widehat{\mathscr{U}}} \ll c$, there exists $n_1 \in \mathbb{N}$ such that, for any $n > m > n_1$, we have $d(\mu_n, \mu_m, a) \ll c$ for all $n > n_1$ and for all $a \in \aleph$. Thus, from Lemma 24, it means that $d(\mu_n, \mu_m, a) = \theta_{\widehat{\mathscr{U}}}$. This implies that $\{\mu_n\}$ is a Cauchy sequence in $\aleph$.

If $H_l(\aleph)$ is complete for each $l = 1, 2, 3, \cdots$, there exists $q \in H_l(\aleph)$ such that

$$\mu_{2n} = J_{2n+2}(s_{2n+1}) = S_{2n+1}(s_{2n}) \longrightarrow q \ (n \longrightarrow \infty). \tag{26}$$

So we can find a $p \in \aleph$ such that $J_{2n+2}(p) = q$ (if $E_i(\aleph)$ is complete for each $i = 1, 2, 3, \cdots$, there exists $q \in E_i(\aleph) \subseteq H_l(\aleph)$; then, the conclusion remains the same). Now, we show that $T_{2n+2}(p) = q$. By (9), we have

$$\begin{aligned} d(T_{2n+2}(p), q, a) &\preccurlyeq bd(T_{2n+2}(p), q, S_{2n+1}(s_{2n})) \\ &\quad + bd(T_{2n+2}(p), S_{2n+1}(s_{2n}), a) \\ &\quad + bd(S_{2n+1}(s_{2n}), q, a) \\ &\preccurlyeq bd(T_{2n+2}(p), q, S_{2n+1}(s_{2n})) \\ &\quad + bd(S_{2n+1}(s_{2n}), q, a) \\ &\quad + b\vartheta_1 d(I_{2n+1}(s_{2n}), J_{2n+2}(s_{2n+1}), a) \\ &\quad + b\vartheta_2 d(I_{2n+1}(s_{2n}), S_{2n+1}(s_{2n}), a) \\ &\quad + b\vartheta_3 d(J_{2n+2}(s_{2n+1}), T_{2n+2}(p), a) \\ &\quad + b\vartheta_4 d(I_{2n+1}(s_{2n}), T_{2n+2}(p), a) \\ &\quad + b\vartheta_5 d(J_{2n+2}(s_{2n+1}), S_{2n+1}(s_{2n}), a), \end{aligned}$$

$$\begin{aligned} d(T_{2n+2}(p), q, a) &= bd(T_{2n+2}(p), q, \mu_{2n}) + bd(\mu_{2n}, q, a) \\ &\quad + b\vartheta_1 d(\mu_{2n-1}, q, a) + b\vartheta_2 d(\mu_{2n-1}, \mu_{2n}, a) \\ &\quad + b\vartheta_3 d(q, T_{2n+2}(p), a) \\ &\quad + b\vartheta_4 d(\mu_{2n-1}, T_{2n+2}(p), a) \\ &\quad + b\vartheta_5 d(q, \mu_{2n}, a) \\ &\preccurlyeq bd(T_{2n+2}(p), q, \mu_{2n}) + bd(\mu_{2n}, q, a) \\ &\quad + b\vartheta_1 d(\mu_{2n-1}, q, a) + b\vartheta_2 d(\mu_{2n-1}, \mu_{2n}, a) \\ &\quad + b\vartheta_3 d(q, T_{2n+2}(p), a) \\ &\quad + b^2\vartheta_4 d(\mu_{2n-1}, T_{2n+2}(p), q) \\ &\quad + b^2\vartheta_4 d(\mu_{2n-1}, q, a) \\ &\quad + b^2\vartheta_4 d(q, T_{2n+2}(p), a) + b\vartheta_5 d(q, \mu_{2n}, a). \end{aligned} \tag{27}$$

That is,

$$\begin{aligned} &\left(e - b\vartheta_3 - b^2\vartheta_4\right) d(T_{2n+2}(p), q, a) \\ &\quad \preccurlyeq bd(\mu_{2n}, T_{2n+2}(p), a) + b^2\vartheta_4 d(\mu_{2n-1}, T_{2n+2}(p), q) \\ &\qquad + (b + b\vartheta_5) d(\mu_{2n}, q, a) + \left(b\vartheta_1 + b^2\vartheta_4\right) d(\mu_{2n-1}, q, a) \\ &\qquad + b\vartheta_2 d(\mu_{2n-1}, \mu_{2n}, a). \end{aligned} \tag{28}$$

Therefore, it follows from Proposition 29 and Lemmas 15 and 16 that

$$\left(e - b\vartheta_3 - b^2\vartheta_4\right) d(T_{2n+2}(p), q, a) \preccurlyeq z_n, \tag{29}$$

where $\{z_n\}$ is a c-sequence in $\mathscr{C}_{\widehat{\mathscr{U}}}$. In addition, from Proposition 2 and

$$r\left(b\vartheta_3 + b^2\vartheta_4\right) \le r(b) r(\vartheta_3) + r\left(b^2\right) r(\vartheta_4) < 1, \tag{30}$$

it means that $e - (b\vartheta_3 + b^2\vartheta_4)$ is invertible. In this case, we have

$$\left(e - b\vartheta_3 - b^2\vartheta_4\right) d(T_{2n+2}(p), q, a) \ll c, \tag{31}$$

for any $c \gg \theta_{\widehat{\mathscr{U}}}$, which together with Lemma 23 implies that $\theta_{\widehat{\mathscr{U}}} \preccurlyeq d(T_{2n+2}(p), q, a) \ll c$, for any $a \in \aleph$, $n \in \mathbb{N}$, and $c \gg \theta_{\widehat{\mathscr{U}}}$ as $(e - (b\vartheta_3 + b^2\vartheta_4))$ is invertible. Therefore, by Lemma 24, we have $d(T_{2n+2}(p), q, a) = \theta_{\widehat{\mathscr{U}}}$ for any $n \in \mathbb{N}$. Namely, $T_{2n+2}(p) = q$ for any $n \in \mathbb{N}$. That is, $T_{2n+2}(p) = q = J_{2n+2}(p)$.

At the same time, as $q = T_{2n+2}(p) \in F_j(\aleph) \subseteq G_k(\aleph)$, there exists $u \in \aleph$ such that $I_{2n+3}(u) = q$.

Now, we show that $S_{2n+1}(u)=q$. From (9), we have that

$$\begin{aligned} d(S_{2n+1}(u),q,a) &= d(S_{2n+1}(u),T_{2n+2}(p),a)\\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(u),J_{2n+2}(p),a)\\ &\quad + \vartheta_2 d(I_{2n+3}(u),S_{2n+1}(u),a)\\ &\quad + \vartheta_3 d(J_{2n+2}(p),T_{2n+2}(p),a)\\ &\quad + \vartheta_4 d(I_{2n+3}(u),T_{2n+2}(p),a)\\ &\quad + \vartheta_5 d(J_{2n+2}(p),S_{2n+1}(u),a)\\ &= \vartheta_1 d(q,q,a) + \vartheta_2 d(q,S_{2n+1}(u),a)\\ &\quad + \vartheta_3 d(q,q,a) + \vartheta_4 d(q,q,a)\\ &\quad + \vartheta_5 d(q,S_{2n+1}(u),a). \end{aligned} \tag{32}$$

That is,

$$d(S_{2n+1}(u),q,a) \preccurlyeq (\vartheta_2+\vartheta_5) d(S_{2n+1}(u),q,a). \tag{33}$$

Hence, by Lemma 20, we know that $d(S_{2n+1}(u),q,a)=\theta_{\widehat{\mathscr{U}}}$, and so $S_{2n+1}(u)=q$. Therefore, $S_{2n+1}(u)=I_{2n+3}(u)=q$ and $T_{2n+2}(p)=J_{2n+2}(p)=q$.

Next, if we assume $G_k(\aleph)$ is complete for each $k=1,2,3,\cdots$, there exists $q\in G_k(\aleph)$ such that

$$\mu_{2n+1} = I_{2n+3}(s_{2n+2}) = T_{2n+2}(s_{2n+1}) \longrightarrow q \text{ as } (n\longrightarrow\infty). \tag{34}$$

So, we can find $u\in\aleph$ such that $I_{2n+3}(u)=q$ (if $F_j(\aleph)$ is complete for each $j=1,2,3,\cdots$, there exists $q\in F_j(\aleph)\subseteq G_k(\aleph)$; then, the conclusion remains the same).

Now, we show that $S_{2n+1}(u)=q$. By (9), we get that

$$\begin{aligned} d(S_{2n+1}(u),q,a) &\preccurlyeq bd(S_{2n+1}(u),q,T_{2n+2}(s_{2n+1}))\\ &\quad + bd(S_{2n+1}(u),T_{2n+2}(s_{2n+1}),a)\\ &\quad + bd(T_{2n+2}(s_{2n+1}),q,a)\\ &\preccurlyeq bd(S_{2n+1}(u),q,T_{2n+2}(s_{2n+1}))\\ &\quad + bd(T_{2n+2}(s_{2n+1}),q,a)\\ &\quad + b\vartheta_1 d(I_{2n+3}(u),J_{2n+2}(s_{2n+1}),a)\\ &\quad + b\vartheta_2 d(I_{2n+3}(u),S_{2n+1}(u),a)\\ &\quad + b\vartheta_3 d(J_{2n+2}(s_{2n+1}),T_{2n+2}(s_{2n+1}),a)\\ &\quad + b\vartheta_4 d(I_{2n+3}(u),T_{2n+2}(s_{2n+1}),a)\\ &\quad + b\vartheta_5 d(J_{2n+2}(s_{2n+1}),S_{2n+1}(u),a)\\ &\preccurlyeq bd(S_{2n+1}(u),q,\mu_{2n+1}) + bd(\mu_{2n+1},q,a)\\ &\quad + b\vartheta_1 d(q,\mu_{2n},a) + b\vartheta_2 d(q,S_{2n+1}(u),a)\\ &\quad + b\vartheta_3 d(\mu_{2n},\mu_{2n+1},a) + b\vartheta_4 d(q,\mu_{2n+1},a)\\ &\quad + b^2\vartheta_5 d(\mu_{2n},S_{2n+1}(u),q)\\ &\quad + b^2\vartheta_5 d(\mu_{2n},q,a) + b^2\vartheta_5 d(q,S_{2n+1}(u),a). \end{aligned} \tag{35}$$

That is,

$$\begin{aligned} &\left(e-b\vartheta_3-b^2\vartheta_4\right) d(T_{2n+2}(p),q,a)\\ &\quad \le bd(\mu_{2n},T_{2n+2}(p),a) + b^2\vartheta_4 d(\mu_{2n-1},T_{2n+2}(p),q)\\ &\qquad + (b+b\vartheta_5) d(\mu_{2n},q,a) + \left(b\vartheta_1+b^2\vartheta_4\right) d(\mu_{2n-1},q,a)\\ &\qquad + b\vartheta_2 d(\mu_{2n-1},\mu_{2n},a). \end{aligned} \tag{36}$$

Therefore, it follows from Proposition 29 and Lemmas 15 and 16 that

$$\left(e-b\vartheta_2-b^2\vartheta_5\right) d(S_{2n+1}(u),q,a) \preccurlyeq z_n^*, \tag{37}$$

where $\{z_n^*\}$ is a c-sequence in $\mathscr{C}_{\widehat{\mathscr{U}}}$. In addition, from Proposition 2 and

$$r\left(b\vartheta_3+b^2\vartheta_4\right) \le r(b)r(\vartheta_3) + r\left(b^2\right)r(\vartheta_4) < 1, \tag{38}$$

it means that $e-\left(b\vartheta_2+b^2\vartheta_5\right)$ is invertible. In this case, we have

$$\left(e-b\vartheta_2-b^2\vartheta_5\right) d(S_{2n+1}(u),q,a) \ll c, \tag{39}$$

for any $c\gg\theta_{\widehat{\mathscr{U}}}$, which together with Lemma 23 implies that $\theta_{\widehat{\mathscr{U}}}\preccurlyeq d(S_{2n+1}(u),q,a)\ll c$ for any $a\in\aleph$, $n\in\mathbb{N}$, and $c\gg\theta_{\widehat{\mathscr{U}}}$ as $\left(e-\left(b\vartheta_2+b^2\vartheta_5\right)\right)$ is invertible. Therefore, by Lemma 24, we have $d(S_{2n+1}(u),q,a)=\theta_{\widehat{\mathscr{U}}}$ for any $n\in\mathbb{N}$. Namely, $S_{2n+1}(u)=q$ for any $n\in\mathbb{N}$. That is, $S_{2n+1}(u)=q=I_{2n+3}(u)$.

At the same time, as $q=S_{2n+1}(u)\in E_i(\aleph)\subseteq H_l(\aleph)$, there exists $p\in\aleph$ such that $J_{2n+2}(p)=q$. Now, we show that $T_{2n+2}(p)=q$. From (9), we have

$$\begin{aligned} d(T_{2n+2}(p),q,a) &= d(S_{2n+1}(u),T_{2n+2}(p),a)\\ &\preccurlyeq \vartheta_1 d(I_{2n+3}(u),J_{2n+2}(p),a)\\ &\quad + \vartheta_2 d(I_{2n+3}(u),S_{2n+1}(u),a)\\ &\quad + \vartheta_3 d(J_{2n+2}(p),T_{2n+2}(p),a)\\ &\quad + \vartheta_4 d(I_{2n+3}(u),T_{2n+2}(p),a)\\ &\quad + \vartheta_5 d(J_{2n+2}(p),S_{2n+1}(u),a)\\ &= \vartheta_1 d(q,q,a) + \vartheta_2 d(q,q,a)\\ &\quad + \vartheta_3 d(q,T_{2n+2}(p),a)\\ &\quad + \vartheta_4 d(q,T_{2n+2}(p),a) + \vartheta_5 d(q,q,a). \end{aligned} \tag{40}$$

That is,

$$d(T_{2n+2}(p),q,a) \preccurlyeq (\vartheta_3+\vartheta_4) d(T_{2n+2}(p),q,a). \tag{41}$$

Hence, by Lemma 20, we know that $d(T_{2n+2}(p),q,a)=\theta_{\widehat{\mathscr{U}}}$, and so $T_{2n+2}(p)=q$. Therefore, $T_{2n+2}(p)=J_{2n+2}(p)=q$ and $S_{2n+1}(u)=I_{2n+3}(u)=q$.

Finally, we show that S_{2i+1} and I_{2k+3}, T_{2j+2}, and J_{2l+2} have a unique point of coincidence in $\aleph$. Assume that there is another point $z\in\aleph$ such that $T_{2n+2}(x)=J_{2n+2}=z$; then,

$$
\begin{aligned}
d(q, z, a) &= d(S_{2n+1}(u), T_{2n+2}(x), a) \\
&\preccurlyeq \vartheta_1 d(I_{2n+3}(u), J_{2n+2}(x), a) \\
&\quad + \vartheta_2 d(I_{2n+3}(u), S_{2n+1}(u), a) \\
&\quad + \vartheta_3 d(J_{2n+2}(x), T_{2n+2}(x), a) \\
&\quad + \vartheta_4 d(I_{2n+3}(u), T_{2n+2}(x), a) \\
&\quad + \vartheta_5 d(J_{2n+2}(x), S_{2n+1}(u), a) \\
&= \vartheta_1 d(q, z, a) + \vartheta_2 d(q, q, a) + \vartheta_3 d(z, z, a) \\
&\quad + \vartheta_4 d(q, z, a) + \vartheta_5 d(z, q, a).
\end{aligned} \tag{42}
$$

That is,

$$
d(q, z, a) \preccurlyeq (\vartheta_1 + \vartheta_4 + \vartheta_5) d(q, z, a). \tag{43}
$$

Hence, by Lemma 20, we have that $d(q, z, a) = \theta_{\widehat{\mathscr{U}}}$, and so $q = z$; that is, q is the unique point of coincidence of T_{2j+2} and J_{2l+2}.

Similarly, we also have q which is the unique point of coincidence of S_{2i+1} and I_{2k+3} by induction.

So, according to Lemma 28, q is the unique common fixed point of $\{S_{2i+1}, I_{2k+3}\}$ and $\{T_{2j+2}, J_{2l+2}\}$ for each $i, j, k, l = 1, 2, 3, \cdots$. Therefore, q is the unique common fixed point of $S_{2i+1}, I_{2k+3}, T_{2j+2}$, and J_{2l+2}.

Now, it is left to show that q is the unique common fixed point of $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$.

As $q = S_{2n+1}(q) = E_n^{\eta_n}(q)$, so we have $E_n(q) = E_n(E_n^{\eta_n}(q)) = E_n^{\eta_n}(E_n(q)) = S_{2n+1}(q)$, that is, $S_{2n+1}(E_n(q)) = E_n(q)$. But $S_{2n+1}(q) = q$ is unique; therefore, $E_n(q) = q$ for $n = 1, 2, 3, \cdots$.

Also, as $q = T_{2n+2}(q) = F_n^{\eta_n}(q)$, so we have $F_n(q) = F_n(F_n^{\eta_n}(q)) = F_n^{\eta_n}(F_n(q)) = T_{2n+2}(q)$, that is, $T_{2n+2}(F_n(q)) = F_n(q)$. But $T_{2n+2}(q) = q$ is unique; therefore, $F_n(q) = q$ for $n = 1, 2, 3, \cdots$. Similarly, $G_n(q) = q$ and $H_n(q) = q$ for $n = 1, 2, 3, \cdots$. Thus, the four families of mappings $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point. □

Remark 31. Theorem 30 of this paper extends and improves Theorem 2.1 of [30] from cone metric spaces to cone b_2-metric spaces; also, it extends and improves Theorem 3.2 of [17] and Theorem 3.1 of [31] from one family and two families, respectively, to four families of mappings.

We obtain a series of new common fixed-point results using Theorem 30 for four families of mappings in the context of cone b_2-metric spaces over Banach algebras, which generalize and improve many known results from the existence literature.

Corollary 32. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \pm e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$
\begin{aligned}
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) &\preccurlyeq \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\
&\quad + \beta\left[d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) + d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right)\right] \\
&\quad + \gamma\left[d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) + d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right)\right],
\end{aligned} \tag{44}
$$

where $\alpha, \beta, \gamma \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(\alpha) + r(\beta) + 2r(\gamma)r(b) < 1$, $r(\beta)r(b) + r(\gamma)r(b^2) < 1$, and α, β, γ, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \alpha$, $\vartheta_2 = \vartheta_3 = \beta$, $\vartheta_4 = \vartheta_5 = \gamma$ in Theorem 30. □

Corollary 33. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \pm e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all $s, m, z \in \aleph$,*

$$
\begin{aligned}
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) &\preccurlyeq \vartheta_1 d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\
&\quad + \vartheta_2 d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) \\
&\quad + \vartheta_3 d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right),
\end{aligned} \tag{45}
$$

where $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(\vartheta_1) + r(\vartheta_2) + r(\vartheta_3) < 1$, $r(\vartheta_2)r(b) + r(\vartheta_3)r(b) < 1$, and $\vartheta_1, \vartheta_2, \vartheta_3, b$ commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Taking $\vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathscr{U}}}$ in Theorem 30, one can get the desired result. □

Remark 34. We can have Theorem 3.1 in [21], when $\{E_i\}_{i=1}^{\infty}$ and $\{F_j\}_{j=1}^{\infty}$ are the same mapping and $\{G_k\}_{k=1}^{\infty}$ and $\{H_l\}_{l=1}^{\infty}$ are the identity mappings. Therefore, Theorem 3.1 of [21] is a special case of Corollary 33. Also, Corollary 33 of this paper generalizes Theorem 2.1 of [10] from the cone 2-metric space to the cone b_2-metric space and extends Theorem 6.1 in [12].

Corollary 35. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2 -metric space with $b \pm e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all s, m, $z \in \aleph$,*

$$
\begin{aligned}
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq{} & \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\
& + \beta\left[d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right)\right. \\
& \left. + d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right)\right],
\end{aligned} \tag{46}
$$

where $\alpha, \beta \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(\alpha) + 2r(\beta) < 1$, $2r(\beta)r(b) < 1$, and α, β, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ is a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. One can the result taking $\vartheta_1 = \alpha$, $\vartheta_2 = \vartheta_3 = \beta$, and $\vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathscr{U}}}$ in Theorem 30. □

Remark 36. Corollary 35 of this paper extends Theorem 6 in [32]; therefore, Theorem 6 in [32] is a special case of Corollary 35.

Corollary 37. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all s, m, $z \in \aleph$,*

$$
\begin{aligned}
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq{} & \alpha d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right) \\
& + \beta d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) \\
& + \gamma d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right),
\end{aligned} \tag{47}
$$

where $\alpha, \beta, \gamma \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(\alpha) + 2r(\beta)r(b) + 2r(\gamma)r(b) < 1$, $r(\beta)r(b^2) + r(\gamma)r(b^2) < 1$, and α, β, γ, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. One can get the result taking $\vartheta_1 = \alpha$, $\vartheta_2 = \vartheta_3 = \theta_{\widehat{\mathscr{U}}}$ and $\vartheta_4 = \beta$, $\vartheta_5 = \gamma$ in Theorem 30. □

Corollary 38. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all s, m, $z \in \aleph$,*

$$
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), E_i^{\eta_i}(s), a\right) + l d\left(H_l^{\eta_l}(m), F_j^{\eta_j}(m), a\right), \tag{48}
$$

where $k, l \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(k) + r(l) < 1$, $r(k)r(b) + r(l)r(b) < 1$, and k, l, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathscr{U}}}$, $\vartheta_2 = k$, $\vartheta_3 = l$ in Theorem 30. □

Corollary 39. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2-metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all s, m, $z \in \aleph$,*

$$
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), F_j^{\eta_j}(m), a\right) + l d\left(H_l^{\eta_l}(m), E_i^{\eta_i}(s), a\right), \tag{49}
$$

where $k, l \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $2r(k)r(b) + 2r(l)r(b) < 1$, $r(k)r(b^2) + r(l)r(b^2) < 1$, and k, l, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = \vartheta_2 = \vartheta_3 = \theta_{\widehat{\mathscr{U}}}$, $\vartheta_4 = k$, $\vartheta_5 = l$ in Theorem 30. □

Corollary 40. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a cone b_2 -metric space with $b \succcurlyeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ be four families of self-mappings on $\aleph$. For all $i, j, k, l \in \mathbb{N}$, if a sequence $\{\eta_n\}_{n=1}^{\infty}$ exists of nonnegative integers, such that for all s, m, $z \in \aleph$,*

$$
d\left(E_i^{\eta_i}(s), F_j^{\eta_j}(m), a\right) \preccurlyeq k d\left(G_k^{\eta_k}(s), H_l^{\eta_l}(m), a\right), \tag{50}
$$

where $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(k) < 1$ and k, b commute. If $E_i(\aleph) \subseteq H_l(\aleph)$, $F_j(\aleph) \subseteq G_k(\aleph)$, and one of $E_i(\aleph)$, $G_k(\aleph)$, $H_l(\aleph)$, and $F_j(\aleph)$ are a complete subspace of $\aleph$ for each $i, j, k, l \geq 1$, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique point of coincidence in $\aleph$. Moreover, if $\{F_j, H_l\}$ and $\{E_i, G_k\}$ are weakly compatible, respectively, then $\{E_i\}_{i=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$, $\{G_k\}_{k=1}^{\infty}$, and $\{H_l\}_{l=1}^{\infty}$ have a unique common fixed point.

Proof. Let $\vartheta_1 = k$, $\vartheta_2 = \vartheta_3 = \vartheta_4 = \vartheta_5 = \theta_{\widehat{\mathscr{U}}}$ in Theorem 30. □

From the above corollary, we obtain the following.

Corollary 41. *Let $(\aleph, d)$ over the Banach algebra $\widehat{\mathscr{U}}$ be a complete cone b_2-metric space with $b \succeq e$ and $\mathscr{C}_{\widehat{\mathscr{U}}} \neq \varnothing$ be a cone in $\widehat{\mathscr{U}}$. Let $\{E_i\}_{i=1}^{\infty}$ be the family of self-mapping on $\aleph$. For all $i \in \mathbb{N}$ and for all $s, m, z \in \aleph$,*

$$d(E_i(s), E_i(m), a) \preccurlyeq k d(s, m, a), \tag{51}$$

where $k \in \mathscr{C}_{\widehat{\mathscr{U}}}$ with $r(k) < 1$ and k, b commute. Then, $\{E_i\}_{i=1}^{\infty}$ have a unique common fixed point.

Proof. Taking $\eta_n = 1$, $E_i = F_j$, and G_k, H_l which are identity mappings in Corollary 40, then we can obtain the required result. □

We finish this section with an example that will demonstrate the consequence of Theorem 30.

Example 42. Let $\widehat{\mathscr{U}} = \mathbb{R}^2$. For each $(s_1, s_2) \in \widehat{\mathscr{U}}$, $\|(s_1, s_2)\| = |s_1| + |s_2|$. The multiplication is defined by $sm = (s_1, s_2)(m_1, m_2) = (s_1 m_1, s_1 m_2 + s_2 m_1)$. Then, $\widehat{\mathscr{U}}$ is a Banach algebra with unit element $e = (1, 0)$. Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{(s_1, s_2) \in \mathbb{R}^2 \mid s_1, s_2 \geq 0\}$. Then, $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a cone in $\widehat{\mathscr{U}}$.

Let $\aleph = \{(s, 0) \in \mathbb{R}^2 \mid s \geq 0\} \cup \{(0, 2)\} \subset \mathbb{R}^2$ and define $d : \aleph \times \aleph \times \aleph \longrightarrow \widehat{\mathscr{U}}$ as follows:

$$d(S, M, Z) = \begin{cases} (0, 0), & \text{if atleast two of } S, M, Z \text{ are equal}, \\ d(P(S, M, Z)), & P \text{ denotes permutations}, \\ (\Delta, \Delta), & \text{otherwise}, \end{cases} \tag{52}$$

where Δ is the square of the area of the triangle S, M, Z. We have

$$\begin{aligned} d((s,0),(m,0),(0,2)) \preccurlyeq\, & d((s,0),(m,0),(z,0)) \\ & + d((s,0),(z,0),(0,2)) \\ & + d((z,0),(m,0),(0,2)). \end{aligned} \tag{53}$$

That is, $(s-m)^2 \preccurlyeq (s-z)^2 + (z-m)^2$, which shows that d is not a cone 2-metric, because $(-9/2, -9/2) \in \mathscr{C}_{\widehat{\mathscr{U}}}$ for $s, m, z \geq 0$ with $s = 5$, $m = 0$, and $z = 1/2$. But for the parameter $b = (2, 0) \succeq e$ is a cone b_2-metric space over the Banach algebra $\widehat{\mathscr{U}}$.

Now, we define mappings $E_i : \aleph \longrightarrow \aleph (i = 1, 2, 3, \cdots)$ by

$$\begin{aligned} E_i((s,0)) &= \left(\left(\frac{1}{6}\right)^{1/(2i-1)} \left(\frac{3}{2}\right)^{1/(2i-1)} \frac{s}{s^{2i-2}}, 0\right), \\ E_i((0,2)) &= (0,0). \end{aligned} \tag{54}$$

We have

$$\begin{aligned} E_i^{2i-1}((s,0)) &= E_i^{2i-2}(E_i((s,0))) \\ &= E_i^{2i-2}\left(\left(\frac{1}{6}\right)^{1/(2i-1)} \left(\frac{3}{2}\right)^{1/(2i-1)} \frac{s}{s^{2i-2}}, 0\right) \\ &= E_i^{2i-3}\left(\left(\frac{1}{6}\right)^{2/(2i-1)} \left(\frac{3}{2}\right)^{2/(2i-1)} \frac{s^2}{s^{2i-2}}, 0\right) \\ &= E_i^{2i-4}\left(\left(\frac{1}{6}\right)^{3/(2i-1)} \left(\frac{3}{2}\right)^{3/(2i-1)} \frac{s^3}{s^{2i-2}}, 0\right) \\ &= \cdots \cdots \\ &= E_i\left(\left(\frac{1}{6}\right)^{(2i-2)/(2i-1)} \left(\frac{3}{2}\right)^{(2i-2)/(2i-1)} \frac{s^{2i-2}}{s^{2i-2}}, 0\right) \\ &= \left(\left(\frac{1}{6}\right)^{(2i-1)/(2i-1)} \left(\frac{3}{2}\right)^{(2i-1)/(2i-1)} \frac{s^{2i-1}}{s^{2i-2}}, 0\right) \\ &= \left(\frac{1}{4}s, 0\right). \end{aligned} \tag{55}$$

We define mappings $G_k : \aleph \longrightarrow \aleph$ $(k = 1, 2, 3, \cdots)$ by

$$\begin{aligned} G_k((s,0)) &= \left(\left(\frac{1}{3}\right)^{1/(2k-1)} \left(\frac{1}{2}\right)^{-1/(2k-1)} \frac{s}{s^{2k-2}}, 0\right), \\ G_k((0,2)) &= (0,0). \end{aligned} \tag{56}$$

We have

$$\begin{aligned} G_k^{2k-1}((s,0)) &= G_k^{2k-2}(G_k((s,0))) \\ &= G_k^{2k-2}\left(\left(\frac{1}{3}\right)^{1/(2k-1)} \left(\frac{1}{2}\right)^{-1/(2k-1)} \frac{s}{s^{2k-2}}, 0\right) \\ &= G_k^{2k-3}\left(\left(\frac{1}{3}\right)^{2/(2i-1)} \left(\frac{1}{2}\right)^{-2/(2k-1)} \frac{s^2}{s^{2k-2}}, 0\right) \\ &= G_k^{2k-4}\left(\left(\frac{1}{3}\right)^{3/(2k-1)} \left(\frac{1}{2}\right)^{-3/(2k-1)} \frac{s^3}{s^{2k-2}}, 0\right) \\ &= \cdots \cdots \\ &= G_k\left(\left(\frac{1}{3}\right)^{(2k-2)/(2k-1)} \left(\frac{1}{2}\right)^{(-2k-2)/(2k-1)} \frac{s^{2k-2}}{s^{2k-2}}, 0\right), \end{aligned}$$

$$\begin{aligned} G_k^{2k-1}((s,0)) &= \left(\left(\frac{1}{3}\right)^{(2k-1)/(2k-1)} \left(\frac{1}{2}\right)^{(-2k+1)/(2k-1)} \frac{s^{2k-1}}{s^{2k-2}}, 0\right) \\ &= \left(\frac{2}{3}s, 0\right). \end{aligned} \tag{57}$$

Similarly, we define mappings $F_j, H_l : \aleph \longrightarrow \aleph$ $(j, l = 1, 2, 3, \cdots)$ by

$$\begin{gathered} F_j((s,0)) = \left(\left(\frac{2}{3}\right)^{1/(2j-1)}\left(\frac{3}{10}\right)^{1/(2j-1)}\frac{s}{s^{2j-2}},0\right), \\ F_j((0,2)) = (0,0), \\ H_l((s,0)) = \left(\left(\frac{2}{3}\right)^{1/(2l-1)}\left(\frac{1}{2}\right)^{-1/(2l-1)}\frac{s}{s^{2l-2}},0\right), \\ H_l((0,2)) = (0,0). \end{gathered} \tag{58}$$

Then, it is not difficult to show that $F_j^{2j-1}((s,0)) = ((1/5)s,0)$ and $H_l^{2l-1}((s,0)) = ((1/3)s,0)$. Choose $\vartheta_1 = (1/10,0)$, $\vartheta_2 = \vartheta_3 = (1/8,0)$, and $\vartheta_4 = \vartheta_5 = (1/16,0)$. Clearly,

$$\begin{aligned} &\sum_{w=1}^{3} r(\vartheta_w) + 2r(\vartheta_4)r(b) + 2r(\vartheta_5)r(b) \\ &\quad = \frac{1}{10} + \frac{1}{8} + \frac{1}{8} + 2\left(\frac{1}{16}\right)2 + 2\left(\frac{1}{16}\right)2 = \frac{34}{40} < 1, \end{aligned} \tag{59}$$

also $r(\vartheta_2)r(b) + r(\vartheta_5)r(b^2) = 2(1/8) + 4(1/16) = 1/2 < 1$ and $r(\vartheta_3)r(b) + r(\vartheta_4)r(b^2) = 2(1/8) + 4(1/16) = 1/2 < 1$.

Now, considering the contractive condition (8), we have

$$\begin{aligned} d\left(\left(\frac{s}{4},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \preccurlyeq & \left(\frac{1}{10},0\right) d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{3},0\right),(0,2)\right) \\ & + \left(\frac{1}{8},0\right) d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{4},0\right),(0,2)\right) \\ & + \left(\frac{1}{8},0\right) d\left(\left(\frac{s}{3},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \\ & + \left(\frac{1}{16},0\right) d\left(\left(\frac{2s}{3},0\right),\left(\frac{s}{5},0\right),(0,2)\right) \\ & + \left(\frac{1}{16},0\right) d\left(\left(\frac{s}{3},0\right),\left(\frac{s}{4},0\right),(0,2)\right), \end{aligned} \tag{60}$$

that is,

$$\begin{aligned} \left(\left(\frac{s}{4}-\frac{s}{5}\right)^2,\left(\frac{s}{4}-\frac{s}{5}\right)^2\right) \preccurlyeq & \left(\frac{1}{10},0\right)\left(\left(\frac{2s}{3}-\frac{s}{3}\right)^2,\left(\frac{2s}{3}-\frac{s}{3}\right)^2\right) \\ & + \left(\frac{1}{8},0\right)\left(\left(\frac{2s}{3}-\frac{s}{4}\right)^2,\left(\frac{2s}{3}-\frac{s}{4}\right)^2\right) \\ & + \left(\frac{1}{8},0\right)\left(\left(\frac{s}{3}-\frac{s}{5}\right)^2,\left(\frac{s}{3}-\frac{s}{5}\right)^2\right) \\ & + \left(\frac{1}{16},0\right)\left(\left(\frac{2s}{3}-\frac{s}{5}\right)^2,\left(\frac{2s}{3}-\frac{s}{5}\right)^2\right) \\ & + \left(\frac{1}{16},0\right) + \left(\left(\frac{s}{3}-\frac{s}{4}\right)^2,\left(\frac{s}{3}-\frac{s}{4}\right)^2\right) \\ = & \left(\frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2,\frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2\right) \\ & + \left(\frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2,\frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2\right) \\ & + \left(\frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2,\frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2\right) \\ & + \left(\frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2,\frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2\right) \\ & + \left(\frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2,\frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2\right), \end{aligned} \tag{61}$$

which means that

$$\begin{aligned} \left(\frac{s}{4}-\frac{s}{5}\right)^2 \preccurlyeq & \frac{1}{10}\left(\frac{2s}{3}-\frac{s}{3}\right)^2 + \frac{1}{8}\left(\frac{2s}{3}-\frac{s}{4}\right)^2 \\ & + \frac{1}{8}\left(\frac{s}{3}-\frac{s}{5}\right)^2 + \frac{1}{16}\left(\frac{2s}{3}-\frac{s}{5}\right)^2 \\ & + \frac{1}{16}\left(\frac{s}{3}-\frac{s}{4}\right)^2, \left(\frac{s}{20}\right)^{2\circ}\frac{1}{10}\left(\frac{s}{3}\right)^2 \\ & + \frac{1}{8}\left(\frac{5s}{12}\right)^2 + \frac{1}{8}\left(\frac{2s}{15}\right)^2 \\ & + \frac{1}{16}\left(\frac{7s}{15}\right)^2 + \frac{1}{16}\left(\frac{s}{12}\right)^2, \end{aligned} \tag{62}$$

that is,

$$\frac{s^2}{400} \preccurlyeq \frac{s^2}{90} + \frac{25s^2}{1152} + \frac{4s^2}{1800} + \frac{49s^2}{3600} + \frac{s^2}{2304}, \tag{63}$$

which shows that $s^2/400 \preccurlyeq 2827s^2/57600$, and so $(2827s^2/57600) - (s^2/400) \in \mathscr{C}_{\widehat{\mathscr{U}}}$, which is true for all $s \geq 0$. Hence, condition (8) is true for all $s, m, a \in \aleph$ and $i, j, k, l \geq 1$, where $\eta_i = 2i-1$, $\eta_j = 2j-1$, $\eta_k = 2k-1$, and $\eta_l = 2l-1$. All other conditions of Theorem 30 are satisfied. By Theorem 30, E_i, F_j, G_k, and H_l have a unique common fixed point $(0,0)$ for all $i, j, k, l \geq 1$.

3. Application to the Infinite System of Integral Equations

We give here a couple of auxiliary facts that will be needed in our further considerations. Let $\widehat{\mathscr{U}} = \mathbb{R}^2$ with norm $\|.\|_{\widehat{\mathscr{U}}}$ be a real Banach algebra. Let $I = [0,T]$, and denote by $C(I,\widehat{\mathscr{U}})$ the space consisting of all continuous functions defined on interval I with values in the Banach algebra $\widehat{\mathscr{U}}$. The space $C(I,\widehat{\mathscr{U}})$ will be equipped with $\|s\| = \max\{\|s(a)\|_{\widehat{\mathscr{U}}} : a \in I\}$.

Let $\aleph = C(I,\widehat{\mathscr{U}})$ and define $d : \aleph^3 \longrightarrow \widehat{\mathscr{U}}$ by

$$d(s(t),m(t),z) = [\min\{|s(t)-m(t)|,|m(t)-z|,|s(t)-z|\}]^p, \tag{64}$$

where $p \geq 1$ and for all $s(t), m(t), z \in \aleph$. Then, $(\aleph, d)$ is a complete cone b_2-metric space over the Banach algebra.

We consider the infinite system of integral equations of the form

$$s_i(t) = g_i(t) + \int_0^T M_i(t,w) f_i(w,s(w)) dw, \tag{65}$$

where $i = 1, 2, 3, \cdots$. Let $E_i : \aleph \longrightarrow \aleph$. We redefine the above infinite system of integral equations as

$$E_i(s_i(t)) = g_i(t) + \int_0^T M_i(t, w) f_i(w, s(w)) dw, \tag{66}$$

for all $s_i(t) \in \aleph$ and $t, w \in I$. Clearly, by using Corollary 41, the existence of solution to (65) is equivalent to the existence of a common fixed point of E_i.

We assume that

(i) $g_i : I \longrightarrow \mathbb{R}$ are continuous for each $i = 1, 2, 3, \cdots$

(ii) $M_i : I \times \mathbb{R} \longrightarrow [0,+\infty)$ are continuous and $\int_0^T M_i(t, w) dw \leq 1$ for each $i = 1, 2, 3, \cdots$

(iii) $f_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous for each $i = 1, 2, 3, \cdots$ such that

$$|f_i(w, s(w)) - f_i(w, m(w))| \leq v^{1/p} [\min \{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}], \tag{67}$$

for all $s(w), m(w), z \in \aleph$ and $0 \leq v < 1$.

Theorem 43. *Under the assumptions (i)–(iii), the infinite system of integral equations (65) has a unique solution in* $\aleph$.

Proof. Take $\widehat{\mathscr{U}} = \mathbb{R}^2$ with norm $\|s\| = \|(s_1, s_2)\| = |s_1| + |s_2|$, and multiplication is defined by the following way:

$$sm = (s_1, s_2)(m_1, m_2) = (s_1 m_1, s_1 m_2 + s_2 m_1). \tag{68}$$

Let $\mathscr{C}_{\widehat{\mathscr{U}}} = \{(s_1, s_2) \in \widehat{\mathscr{U}} : s_1, s_2 \geq 0\}$. It is clear that $\mathscr{C}_{\widehat{\mathscr{U}}}$ is a normal cone and $\widehat{\mathscr{U}}$ is a Banach algebra with unit element $e = (1, 0)$.

Consider the family of mapping $E_i : \aleph \longrightarrow \aleph$ defined by (66). Let $s_i(t), m_i(t), z \in \aleph$. From (64), we deduce that

$$\begin{aligned} d(E_i(s_i(t)), E_i(m_i(t)), z) &= \max_{a \in [0,T]} [\min \{|E_i(s_i(t)) - E_i(m_i(t))|, |E_i(s_i(t)) - z|, |E_i(m_i(t)) - z|\}]^p \\ &\preccurlyeq \left(\max_{a \in [0,T]} |E_i(s_i(t)) - E_i(m_i(t))| \right)^p \\ &= \left(\max_{a \in [0,T]} \left| \int_0^T M_i(t, w) f_i(w, s(w)) dw - \int_0^T M_i(t, w) f_i(w, m(w)) dw \right| \right)^p \\ &= \left(\max_{a \in [0,T]} \left| \int_0^T M_i(t, w) [f_i(w, s(w)) - f_i(w, m(w))] dw \right| \right)^p \\ &\preccurlyeq \left(\max_{a \in [0,T]} \int_0^T M_i(t, w) |f_i(w, s(w)) - f_i(w, m(w))| dw \right)^p \\ &\preccurlyeq \left(\max_{a \in [0,T]} \int_0^T M_i(t, w) v^{1/p} [\min \{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}] dw \right)^p \\ &\preccurlyeq \left(\int_0^T \left(\max_{a \in [0,T]} M_i(t, w) \right) v^{1/p} \cdot \left(\max_{a \in [0,T]} [\min \{|s(w) - m(w)|, |m(w) - z|, |s(w) - z|\}]^p \right)^{1/p} dw \right)^p \\ &\leq \left(\int_0^T \left(\max_{a \in [0,T]} M_i(t, w) \right) v^{1/p} \cdot (d(s_i(t), m_i(t), z))^{1/p} dw \right)^p. \end{aligned} \tag{69}$$

Therefore,

$$d(E_i(s_i(t)), E_i(m_i(t)), z) \preccurlyeq v d(s_i(t), m_i(t), z). \tag{70}$$

Now, all the assumptions of Corollary 41 are fulfilled and the family of mapping E_i has a unique common fixed point in $\aleph$, which means that the infinite system of integral equations (65) has a unique solution in $\aleph$. ☐

Data Availability

Data are available upon request or included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors make equal contributions and read and supported the last original copy.

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