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Structure preserving numerical scheme for spatio-temporal epidemic model of plant disease dynamics

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ABSTRACT

In this article, an implicit numerical design is formulated for finding the numerical solution of spatiotemporal nonlinear dynamical system with advection. Such type of problems arise in many fields of life sciences, mathematics, physics and engineering. The epidemic model describes the population densities that have some special types of features. These features should be maintained by the numerical design. The proposed scheme, not only solves the nonlinear physical system but also preserves the structure of the state variables. Von-Neumann criteria, M-matrix theory and Taylor's expansion are used for proving some standard results. Basic reproduction number is evaluated and its key role in deciding the stability at the equilibrium points is also investigated. Graphical solutions are also presented against the test problem.

Introduction

The plants play very important role in this real world, as they are the basis for the survival of all types of creatures for instance human beings, animals and micro-organisms. However, it is a fact that many plant diseases disturb the ratio of the plants to other living creatures. Like the human viruses, there are many types of viruses which effect the health and population of the plants, for example cucumber virus, broad beam with virus, Curly top beat virus and streak virus that effect the maize [1,2]. The plant diseases are not new, the potato disease destructed most of the potatoes of the Irish in 1845–1846. This disease caused a starvation in the country. Actually, the plant diseases were taken into consideration very late. By the end of 18th century, many scientists and researchers started investigation about plant diseases. For instance, Marthien Tillet showed practically that wheat bunt disease spread occurs due to a black powder, which is a type of fungus. Adlof Mayer investigated that tobacco mosaic infection is transmitted by the sap of infected leafs. Similarly, many other plant diseases have been discovered. Also, it is recognized that many infectious plant diseases have a close relation with the insect vectors such as aphids, leaf and plant happens etc. [3,4]. At present moment, vector-born plant diseases have fascinated the many researchers and they have started the research on these grounds. In this regard, Bosc and Jeger have investigated plant infection transmission features and population dynamics [5,6]. They also discussed the effect of insect's different modes of communication and migration on the diffusion of virus in plant diseases [7]. Grill described the plant virus' disease rate in connection to the effect of the timings of insect moderators feeding [8]. Jager et al. [9] investigated some central approaches and showed that these approaches impart key roles in pest administration. Cunniffe and Gilligan [10] addressed the influence of biological management on soil borne plant microorganisms. They also considered a contestant in the system for controlling the plant diseases. They got intrusion principal for host, microorganisms and contestant. The research related to the plant infections is also tempting to the epidemiologists. They are required to investigate an appropriate method to secure the susceptible hosts, permitting coexistence of virus and host plants, which is in line with the empirical data of infection in the population of plant. The dynamics of such type of models is furnished by compartmental epidemic models [11,12].

Ever since the revolutionary work of Kermack and Kendrick in the 1930's [13–15], the mathematical modeling of transmission dynamics of plant diseases caused by different agents is an effective approach

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to understand the transmission of disease in plants. On this basis, to prevent infection in plants some effective measures can be posed. These models are often of the form of systems of non-linear differential equations, whose exact solutions are not easily obtainable. Demanding the use of numerical method to get their approximate solutions. Typically used numerical methods which are easy to apply such as forward Euler method, upwind finite difference method and explicit Runge–Kutta methods. But sometimes these methods may show false behaviors, when certain values of the related discretization and model parameters are used in the simulations which are not the features of the continuous approximated models. It is shown in literature that many standard methods of discretization can lead to negative solutions or converges to some point outside the system [16,17]. While these schemes are constructed with the aim that discrete model exhibits the same behavior as the continuous model. Discrete models must contain some very important properties like consistency, convergence, stability, positivity. In this present work, an advective SIRXY dynamical model of a vector borne disease in plants is developed, which address positivity as well as all basic traits of continual model.

Mathematical model

In this section the mathematical model of vector-borne plant SIRXY model is introduced which is firstly discussed by Shi and Zhao in [1] and then by Rafiq in [3] as follows:

$$\begin{aligned} S' &= dI + \mu(k - S) - \left(\frac{\beta_p Y}{1+\alpha_p Y} + \frac{\beta_S I}{1+\alpha_S I}\right)S,\\ I' &= \left(\frac{\beta_p Y}{1+\alpha_p Y} + \frac{\beta_S I}{1+\alpha_S I}\right)S - (d+\mu+\gamma)I,\\ R' &= \gamma I - \mu R,\\ X' &= A - \frac{\beta_1 I X}{1+\alpha_1 I} - mX,\\ Y' &= \frac{\beta_1 I X}{1+\alpha_1 I} - mY. \end{aligned} \tag{1}$$

Where $N = X + Y$. We can reduce model as

$$\begin{aligned} S' &= dI + \mu(K - S) - \left(\frac{\beta_p Y}{1+\alpha_p Y} + \frac{\beta_S I}{1+\alpha_S I}\right)S,\\ I' &= \left(\frac{\beta_p Y}{1+\alpha_p Y} + \frac{\beta_S I}{1+\alpha_S I}\right)S - \omega I,\\ Y' &= \frac{\beta_1 I}{1+\alpha_1 I}\left(\frac{A}{m} - Y\right) - mY. \end{aligned} \tag{2}$$

Where $K = S + I + R$, $\omega = d + \mu + \gamma$.

During this study, we let $\mathbb{L}, \mathbb{T}$ are nonnegative constants and $\Omega = (0,\mathbb{L}) \times (0,\mathbb{T}) \subseteq \Re^2$. The PDE's including advection terms are as follow,

$$\frac{\partial s}{\partial t} + a_1\frac{\partial s}{\partial x} = di + \mu(K - s) - \left(\frac{\beta_p y}{1+\alpha_p y} + \frac{\beta_s i}{1+\alpha_s i}\right)s, \quad \forall (x,t) \in \Omega, \tag{3}$$

$$\frac{\partial i}{\partial t} + a_2\frac{\partial i}{\partial x} = \left(\frac{\beta_p y}{1+\alpha_p y} + \frac{\beta_s i}{1+\alpha_s i}\right)s - \omega i, \quad \forall (x,t) \in \Omega, \tag{4}$$

$$\frac{\partial y}{\partial t} + a_3\frac{\partial y}{\partial x} = \frac{\beta_1 i}{1+\alpha_1 i}\left(\frac{A}{m} - y\right) - my, \quad \forall (x,t) \in \Omega. \tag{5}$$

Initial and boundary conditions will be:

$$\begin{aligned} s &= s(x,0) = g_1, \quad \forall x \in (0,\mathbb{L}),\\ i &= i(x,0) = g_2, \quad \forall x \in (0,\mathbb{L}),\\ y &= y(x,0) = g_3, \quad \forall x \in (0,\mathbb{L}),\\ \frac{\partial s}{\partial x} &= \frac{\partial i}{\partial x} = \frac{\partial y}{\partial x} = 0, \quad \forall (x,t) \in \Omega. \end{aligned} \tag{6}$$

Let us define functions $s = s(x,t), i = i(x,t), y = y(x,t)$ as smooth real functions on Ω and $g_1, g_2, g_3 : (0,\mathbb{L}) \to \Re$ are continuously differentiable functions. In this work s represents suspectable plants at x point and time t, i represents the accounting for number of involved subjects. y represents quantity of septic insects at time t. Certainly due to biological reasons, s, i and y has to be non-negative function. As a consequence g_1, g_2, g_3 are also non-negative functions. In addition K is the sum of total plants population, where N is total quantity of insects. X is quantity of insects influenceable at time t.

The parameters a_1, a_2, a_3 are advective rates for each of s, i and y respectively. In SIRXY model β_1 is infectivity ratio of septic host and susceptible-vector. β_p is biting ratio from septic vector to influenceable host plants. β_S is infectivity relation between septic and susceptible hosts. α_1, α_p, and α_S are level of septicity saturates. γ is the rate of conversion from septic host to recovered host. μ is the rate of natural biological expiry of hosts, A is insect-vectors birth or immigration and m is natural biological expiry of insect-vectors. d is disease caused mortality of infected hosts. All of the constants mentioned above are positive. For more details see [18–23].

The DFE of the system (3)–(5) is $E_0 = (K, 0, 0)$. Meanwhile, the EE of the system is $E_n = (S*, I*, Y*)$, where

$$\begin{aligned} S^* &= K - \left(1 + \frac{\gamma}{\mu}\right)I^*,\\ I^* &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\\ a &= (\gamma+\mu)(\beta_1\beta_p\alpha_S A + m\beta_1\beta_S + m^2\alpha_1\beta_S + \beta_1\beta_S\alpha_p A) + \omega\mu(m\beta_1\alpha_S\\ &\quad + m^2\alpha_1\alpha_S + \beta_1\alpha_p\alpha_S A),\\ b &= (\gamma+\mu)(\beta_1\beta_p A + +m^2\beta_S) - \mu K(\beta_1\beta_p\alpha_S A + m\beta_1\beta_S + m^2\beta_S\alpha_1\\ &\quad +\beta_1\beta_S\alpha_p A)\\ &\quad +\omega\mu(m\beta_1 + m^2\alpha_1 + \beta_1\alpha_p A + m^2\alpha_S),\\ c &= \mu m^2\omega(1 - R_0),\\ Y^* &= \mu\frac{\beta_1 I^* A}{m\beta_1 I^* + m^2(1+\alpha_1 I^*)}. \end{aligned} \tag{7}$$

And reproductive number is,

$$R_0 = \frac{\beta_S K}{\omega} + \frac{\beta_1\beta_p A K}{m^2\omega}.$$

While R_0 is a reproductive number which firstly introduced by Kermack and Mckendric [13–15]. It helps in deciding that weather disease will die out or spread.

Lemma 1 (*[1]*). *If $R_0 < 1$ then system will have DFE and if $R_0 > 1$ then it will experience EE.*

It is important to mention that our under study model is modified version of the model considered in [1]. The global stability analysis and numerical simulations were provided in that work. In this study our purpose is to investigate all the problems caused when advection terms are embedded in SIRXY model. As well as, we make sure that important physical and the computational features of the discretized model are not only fulfilled but also converges to continuous system. Particularly, our goal is to study the effect of advection operands on the thru act of types contained in given model.

Numerical modeling

We are introducing a finite difference approximation to get approximate solutions of model (3)–(5) with the initial and boundary data (11). For comparison purpose upwind-like system will be introduced here too. For convenience, let $I_n = 1, 2, 3, \ldots\ldots, n$ and $\bar{I}_n = 0, 1, 2, 3, \ldots\ldots, n$, $n \in \mathbb{N}$. In this study we suppose $M, N \in \mathbb{N}$, and describe nonnegative $\mathfrak{h} = \frac{\mathbb{L}}{M}$ and $\mathfrak{l} = \frac{\mathbb{T}}{N}$. By fixing uniform partitions of $[0,\mathbb{L}]$, $[0,\mathbb{T}]$ with partition norms equal to $\mathfrak{h}$ and $\mathfrak{l}$, respectively, and let $x_q = qh$ and $t_p = pl$, for each $q \in \bar{I}_M$ and $p \in \bar{I}_N$ [24]. Moreover, we will suppose that S_q^p, I_q^p and Y_q^p numerical representations of the exact value $s(x_q,t_p), i(x_q,t_p)$ and $y(x_q,t_p)$ for each $(q,p) \in \bar{I}_M \times \bar{I}_N$. In addition , if Q represents S, I or Y, than we will say that

$$Q = (Q_0^p, Q_1^p, \cdots, Q_M^p), \quad \forall p \in \bar{I}_N. \tag{8}$$

Definition 2. Let us assume the symbol $\mathfrak{R}_\mathfrak{h}$ represents the mesh grid $x_q \in \mathfrak{R} : q \in \bar{I_M}$, and the vector space of real functions on $\mathfrak{R}_\mathfrak{h}$ for $q \in \bar{I}_M$ is denoted by $\mathfrak{V}_h$. If $(Q^p)_{p\in\bar{I}_N}$ is any finite sequence in $\mathfrak{V}_h$, the discrete linear operators are

$$\delta_t Q_q^{p+1} = \frac{Q_q^{p+1} - Q_q^p}{l}, \quad \forall (q,p) \in \bar{I}_M \times \bar{I}_{N-1}, \tag{9}$$

$$\delta_x Q_q^p = \frac{Q_q^p - Q_{q-1}^p}{h}, \quad \forall (q,p) \in I_M \times \bar{I}_N. \tag{10}$$

(9) presents the first order approximation of the partial derivative Q with respect to time at points (x_q, t_p) and (x_q, t_{p+1}), and (10) provides approximation of the partial derivative of Q with respect to space x.

Applying above assumptions, the model (3)–(5) will be discretized as follow,

$$\delta_t S_q^{p+1} + a_1 \delta_x S_q^{p+1} = d I_q^p + \mu(K - S_q^{p+1}) - (\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p}) S_q^{p+1}, \tag{11}$$

$$\delta_t I_q^{p+1} + a_2 \delta_x I_q^{p+1} = (\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p}) S_q^p - \omega I_q^{p+1}, \tag{12}$$

$$\delta_t Y_q^{p+1} + a_3 \delta_x Y_q^{p+1} = \frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}(\frac{A}{m} - Y_q^{p+1}) - m Y_q^{p+1}, \tag{13}$$

for each $(q,p) \in I_{M-1} \times \bar{I}_{N-1}$. Our proposed scheme is an implicit two step NS finite difference method. After some algebraic calculations, the discretized model (11)–(13) can be presented as follow,

$$S_q^{p+1}\left(1 + r_1 + \mathfrak{l}\mu + l(\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p})\right) - r_1 S_{q-1}^{p+1} = S_q^p + \mathfrak{l} d I_q^p + \mathfrak{l}\mu K, \tag{14}$$

$$I_q^{p+1}\left(1 + r_2 + \mathfrak{l}\omega\right) - r_2 I_{q-1}^{p+1} = I_q^p(1 + \mathfrak{l}(\frac{\beta_S}{1+\alpha_S I_q^p}) S_q^p) + l(\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p}) S_q^p, \tag{15}$$

$$Y_q^{p+1}\left(1 + r_3 + \mathfrak{l}m + l\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}\right) - r_3 Y_{q-1}^{p+1} = Y_q^p + l\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}(\frac{A}{m}). \tag{16}$$

For each $(q,p) \in I_{M-1} \times \bar{I}_{N-1}$. Here we agree that $r_1 = a_1 \mathfrak{l}/\mathfrak{h}$, $r_2 = a_2\mathfrak{l}/\mathfrak{h}$, $r_3 = a_3\mathfrak{l}/\mathfrak{h}$. As initial data we will set

$$\begin{aligned} S_q^0 &= g_1(x_q), \quad \forall q \in I_{M-1}, \\ I_q^0 &= g_2(x_q), \quad \forall q \in I_{M-1}, \\ Y_q^0 &= g_3(x_q), \quad \forall q \in I_{M-1}, \\ \delta_x S_1^p &= \delta_x I_1^p = \delta_x Y_1^p = 0, \quad \forall p \in \bar{I}_N, \\ \delta_x S_M^p &= \delta_x I_M^p = \delta_x Y_M^p = 0, \quad \forall p \in \bar{I}_N. \end{aligned} \tag{17}$$

Definition 3. We define the operator $*$: $\mathfrak{V}_h \times \mathfrak{V}_h \to \mathfrak{V}_h$ such as $(Q * P)_Q = Q_q P_q$, for each $P, Q \in \mathfrak{V}_h, q \in \bar{I}_M$. The identity represents component-wise multiplication.

The Vector form of the finite difference system (14)–(16) can be written as,

$$U S^{p+1} = S^p + \mathfrak{l} d I^p + l\mu K, \quad \forall p \in \bar{I}_{N-1} \tag{18}$$

$$V I^{p+1} = I^p + l(\frac{\beta_p Y^p}{1+\alpha_q Y^p} + \frac{\beta_S I^p}{1+\alpha_S I^p}) S^p, \quad \forall p \in \bar{I}_{N-1} \tag{19}$$

$$W Y^{p+1} = Y^p + l\frac{\beta_1 I^p}{1+\alpha_1 I^p}(\frac{A}{m}), \quad \forall p \in \bar{I}_{N-1}. \tag{20}$$

Here real matrices of size $(M+1)\times(M+1)$ are U, V, W. Let us suppose the general form of U, V or W, can be expressed by J, i,e.

$$J = \begin{pmatrix} j_2 & j_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ j_1 & j_2 & 0 & \ddots & & & & \vdots \\ 0 & j_1 & j_2 & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & j_1 & j_2 & 0 & 0 \\ \vdots & & & & \ddots & j_1 & j_2 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & j_1 & j_2 \end{pmatrix}, \tag{21}$$

where $u_1 = -r_1$, $v_1 = -r_2$, $w_1 = -r_3$, and

$$u_2^p = 1 + r_1 + \mathfrak{l}\mu + l(\frac{\beta_p Y_p^q}{1+\alpha_p Y_p^q} + \frac{\beta_S I_p^q}{1+\alpha_S I_p^q}), \quad \forall q \in \bar{I}_N, \tag{22}$$

$$v_2^p = 1 + r_2 + \mathfrak{l}\omega, \quad \forall p \in \bar{I}_N, \tag{23}$$

$$w_2^p = 1 + r_3 + \mathfrak{l}m + l\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}, \quad \forall p \in \bar{I}_N. \tag{24}$$

Next for comparison purpose, we will consider upwind-like system.

$$\delta_t S_q^{p+1} + a_1 \delta_x S_q^{p+1} = d I_q^p + \mu(K - S_q^p) - (\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p}) S_q^p, \tag{25}$$

$$\delta_t I_q^{p+1} + a_2 \delta_x I_q^{p+1} = (\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p}) S_q^p - \omega I_q^p, \tag{26}$$

$$\delta_t Y_q^{p+1} + a_3 \delta_x Y_q^{p+1} = \frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}(\frac{A}{m} - Y_q^p) - m Y_q^p, \tag{27}$$

with $(q,p) \in I_{M-1} \times \bar{I}_{N-1}$. After some calculations and rearranging the equations, the discrete system (25)–(27) can be written into linear identities as

$$(1+r_1)S_q^{p+1} - r_1 S_{q-1}^{p+1} = S_q^p\left(1 - \mathfrak{l}\mu - \mathfrak{l}(\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p})\right) + \mathfrak{l} d I_q^p + \mathfrak{l}\mu K, \tag{28}$$

$$(1+r_2)I_q^{p+1} - r_2 I_{q-1}^{p+1} = I_q^p\left(1 - \mathfrak{l}\omega + \mathfrak{l}\frac{\beta_S S_q^p}{1+\alpha_S I_q^p}\right) + (\frac{\beta_q Y_q^p}{1+\alpha_q Y_q^p}) S_q^p, \tag{29}$$

$$(1+r_3)Y_q^{p+1} - r_3 Y_{q-1}^{p+1} = Y_q^p\left(1 - \mathfrak{l}m - l\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}\right) + \mathfrak{l}\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}(\frac{A}{m}), \tag{30}$$

for each $(q,p) \in I_{M-1} \times \bar{I}_{N-1}$.

Structural properties

In following section, we are interested in establishing main structural and numerical properties of the proposed system (11)–(13) subjected to (17), including positivity of solutions, consistency, order of accuracy, stability and the existence of unique constant solutions. first of all, we will prove that our proposed NSFD scheme contain positivity. For this purpose , we will use a standard definitions of Z-matrix and M-matrix, from literature [25–27] in computational techniques.

Definition 4. We say that A is Z-matrix if all the off-diagonal entries of a real matrix A are non-positive.

Definition 5. A is an M-matrix if the following conditions hold:

1. A is a Z-matrix,
2. all the diagonal constituent parts of A are positive, and
3. A is strictly diagonally dominant.

M-matrices are invertible and their inverses are positive matrices [25]. Then, we can prove the following theorem by this property. Here, the fact is being used that U, V, W described in Section "Numerical modeling" are m-matrices, therefore they are strictly diagonally dominant i,e., under suitable postulates on the initial conditions, their inverses exists and are positive matrices.

Theorem 6 (*Positivity*). *If g_1, g_2, g_3 are non-negative functions on $(0, \mathbb{L})$ then the system (11)–(13) subjected to (17) is capable of being solved for all $\mathfrak{h}, \mathfrak{l} > 0$, and corresponding results are also nonnegative.*

Proof. To prove this hypothesis we will use mathematical induction. If we observe the initial data, it grantees that S^0, I^0 and Y^0 are vectors with position components. Let us suppose it is true for S^p, I^p, Y^p, for some $p \in \bar{I}_{N-1}$. By using m-matrix theory, we can say that U, V, W are non-singular, their inverse matrices exist and they contain positive values. Also by hypothesis, note that the right hand side vectors of the identities (18)–(20) contain all positive values . Under this influence, we can illustrate

$$S^{q+1} = U^{-1}(S^q + \mathfrak{l} d I^q + \mathfrak{l}\mu K), \tag{31}$$

$$I^{q+1} = V^{-1}(I^q + \mathfrak{l}(\frac{\beta_p Y^q}{1+\alpha_p Y^q} + \frac{\beta_S I^q}{1+\alpha_S I^q})S^q), \tag{32}$$

$$Y^{q+1} = W^{-1}(Y^q + \mathfrak{l}\frac{\beta_1 I^q}{1+\alpha_1 I^q}(\frac{A}{m})). \tag{33}$$

are the positive vectors, whence we conclude that approximate solutions are positive and solvable of the system (11)–(13) subjected to (17).

Theorem 7 (*Positive Constant Solutions*). *The points $E_0 = (K, 0, 0)$ and E_n are constant Positive solutions of the system (11)–(13).*

Proof. Assume that $S_q^p = K, I_q^p = Y_q^p = 0$, for $p = 0$ and $\forall q \in I_M$. By using mathematical induction, suppose that those identities are hold for the $q \in \bar{I}_{M-1}$. We obtain $VI^{q+1} = WY^{q+1} = 0$ by utilizing (20) and (21) by non-singularity of V and W. Therefore, we reach $I^{p+1} = Y^{p+1} = 0$. We get

$$\begin{aligned} S_0^{p+1} - S_1^{p+1} &= 0, \\ r_1 S_0^{p+1} + S_1^{p+1}\left(1 + r_1 + \mathfrak{l}\mu\right) &= K(1 + \mathfrak{l}\mu). \end{aligned} \tag{34}$$

by utilizing the boundary condition at $q = 0$ and (14). Where $S_0^{p+1} = S_1^{p+1} = K$ is the solution of our proposed system. Assuming now that $S_{q-1}^{p+1} = K$ and using (14) we have $S_q^{p+1} = K$. Now induction grantees that the $(K, 0, 0)$ is a constant solution of discretized system (11)–(13). Similarly, after tedious calculations we can proof that E_n is also a constant solution of under study system.

Definition 8. We describe the Euclidean norm and infinity norm on $\mathfrak{V}_h$ by the $\|\cdot\|, \|\cdot\|_\infty : \mathfrak{V}_h \to \mathfrak{R}$,

$$\|Q\| = \sqrt{\sum_{q=0}^{M} |Q_q|^2}, \quad \forall Q \in \mathfrak{V}_h, \tag{35}$$

$$\|Q\|_\infty = \max |Q_q| : q \in \bar{I}_M, \quad \forall Q \in \mathfrak{V}_h. \tag{36}$$

Moreover. if $Q = (Q^p)_{p\in\bar{I}_N}$ in $\mathfrak{V}_h$ represents a sequence, then defining

$$|\,\|Q|_\infty = \max\{\|Q^p\|_\infty : p \in \bar{I}_N\}. \tag{37}$$

Next we deal with the consistency of the proposed method (11)–(13). For this purpose, we introduce the differential operators

$$\mathcal{G} = \frac{\partial s}{\partial t} + a_1\frac{\partial s}{\partial x} - di - \mu(K - s) + (\frac{\beta_p y}{1+\alpha_p y} + \frac{\beta_s i}{1+\alpha_s i})s, \quad \forall (x,t) \in \Omega, \tag{38}$$

$$\mathcal{K} = \frac{\partial i}{\partial t} + a_2\frac{\partial i}{\partial x} - (\frac{\beta_p y}{1+\alpha_p y} + \frac{\beta_s i}{1+\alpha_s i})s + \omega i, \quad \forall (x,t) \in \Omega, \tag{39}$$

$$\mathcal{L} = \frac{\partial y}{\partial t} + a_3\frac{\partial y}{\partial x} - \frac{\beta_1 i}{1+\alpha_1 i}(\frac{A}{m} - y) + my, \quad \forall (x,t) \in \Omega \tag{40}$$

If $\mathcal{F}$ is any of the operators $\mathcal{G}, \mathcal{K}$ or $\mathcal{L}$, then we will agree that $\mathcal{F} = \mathcal{F}(x_q, t_p)$, for each $(q,p) \in \bar{I}_M \times \bar{I}_N$. For each $p \in \bar{I}_N$, let us define $\mathcal{F}^p = (\mathcal{F}^p)_0, \mathcal{F}_1^p, \cdots, \mathcal{F}_M^p)$. Obviously, $\mathcal{F}^p \in \mathfrak{V}_h$. Finally, $\mathcal{F} = (\mathcal{F}^p)_{p\in\bar{I}_N}$. On the other hand, by considering $s_q^p = s(x_q, t_p), i_q^p = i(x_q, t_p)$ and $y_q^p = y(x_q, t_p)$. For each $(q,p) \in \bar{I}_M \times \bar{I}_N$. Let us assume difference operators as

$$\begin{aligned} G^{p+1} &= \delta_t s_q^{p+1} + a_1\delta_x s_q^{p+1} - d i_q^p - \mu(K - s_q^{p+1}) \\ &\quad + (\frac{\beta_p y_q^p}{1+\alpha_p y_q^p} + \frac{\beta_s i_q^p}{1+\alpha_s i_q^p})s_q^{p+1}, \end{aligned} \tag{41}$$

$$K^{p+1} = \delta_t i_q^{p+1} - i_q^p + a_2\delta_x i_q^{p+1} - (\frac{\beta_p y_q^p}{1+\alpha_q y_q^p} + \frac{\beta_s i_q^p}{1+\alpha_s i_q^p})s_q^p + \omega i_q^{p+1}, \tag{42}$$

$$L^{p+1} = \delta_t y_q^{p+1} + a_3\delta_x y_q^{p+1} - \frac{\beta_1 i_q^p}{1+\alpha_1 i_q^p}(\frac{A}{m} - i_q^{p+1}) + m y_q^{p+1}. \tag{43}$$

Next we define general form for continuous operators i.e, if M represents any of G, K, L, by assuming $F^p = (F_0^p, F_1^p, \cdots, F_M^p) \in \mathfrak{V}_h$, for each $p \in \bar{I}_{N-1}$.

Theorem 9. *If $s, i, y \in C_{x,t}^{2,2}(\bar{\Omega})$ then we will have a positive constant $Z \geq 0$, independent of $\mathfrak{h}, \mathfrak{l}$, such that*

$$\max\{|\,\|\mathcal{G} - G|_\infty, |\|\mathcal{K} - K|_\infty, |\|\mathcal{L} - L|_\infty\} \leq Z(\mathfrak{h} + \mathfrak{l}). \tag{44}$$

Proof. Since $s \in C_{x,t}^{2,2}(\bar{\Omega})$ implies function s is a smooth function. Then we have to show there exists a non-negative constant $Z \geq 0$, independent of $\mathfrak{h}, \mathfrak{l}$. For this let us take $\mathcal{G}-G$ and apply Taylor's theorem. There exist non-negative constants $Z_1^s, Z_2^s, Z_3^s, Z_4^s$ and Z_5^s, such that for each $(q,p) \in \bar{I}_{M-1} \times \bar{I}_{N-1}$:

$$\left|\frac{\partial s(x_q, t_{p+1})}{\partial t} - \delta_t s_q^{p+1}\right| \leq Z_1^s \mathfrak{l}, \tag{45}$$

$$\left|\frac{\partial s(x_q, t_{p+1})}{\partial x} - \delta_x s_q^{p+1}\right| \leq Z_2^s \mathfrak{h}, \tag{46}$$

$$\left|s(x_q, t_p) - s_q^{p+1}\right| \leq Z_3^s \mathfrak{l}, \tag{47}$$

$$\left|\frac{\beta_p y(x_q, t_p)}{1+\alpha_p y(x_q, t_p)}s(x_q, t_p) - \frac{\beta_p y_q^p}{1+\alpha_p y_q^p}s_q^{p+1}\right| \leq Z_4^s \mathfrak{l}, \tag{48}$$

$$\left|\frac{\beta_s i(x_q, t_p)}{1+\alpha_s i(x_q, t_p)}s(x_q, t_p) - \frac{\beta_s i_q^p}{1+\alpha_s i_q^p}s_q^{p+1}\right| \leq Z_5^s \mathfrak{l}. \tag{49}$$

As a consequence, $|\,\|\mathcal{G} - G|_\infty \leq Z^s(\mathfrak{h} + \mathfrak{l})$ holds if we define the constant $Z^s = Z_1^s + a_1 Z_2^s + \mu Z_3^s + (\frac{\beta_p y_q^p}{1+\alpha_q y_q^p})Z_4^s + \frac{\beta_s i_q^p}{1+\alpha_s i_q^p}Z_5^s$.

Similarly, we have here non-negative constants Z^i and Z^y. These are independent of $\mathfrak{h}$ and $\mathfrak{l}$. Thus, we have $|\,\|\mathcal{K} - K|_\infty \leq Z^i(\mathfrak{h} + \mathfrak{l})$ and $|\,\|\mathcal{L} - L|_\infty \leq Z^y(\mathfrak{h} + \mathfrak{l})$. The conclusion follows straight away, when we define the non-negative constants

$$Z = Z^s \vee Z^i \vee Z^y. \tag{50}$$

At the present stage,linear stability of the numerical scheme (11)–(13) will be established by using von-Neumann criteria. For this purpose, the constants are defined as

$$A_q^p = \left(\mathfrak{l}\mu + \mathfrak{l}(\frac{\beta_p Y_q^p}{1+\alpha_p Y_q^p} + \frac{\beta_S I_q^p}{1+\alpha_S I_q^p})\right), \tag{51}$$

$$B = \mathfrak{l}\omega, \tag{52}$$

$$D_q^p = (1 + \mathfrak{l}(\frac{\beta_S}{1+\alpha_S I_q^p})S_q^p), \tag{53}$$

$$E_q^p = \left(\mathfrak{l}m + \mathfrak{l}\frac{\beta_1 I_q^p}{1+\alpha_1 I_q^p}\right), \tag{54}$$

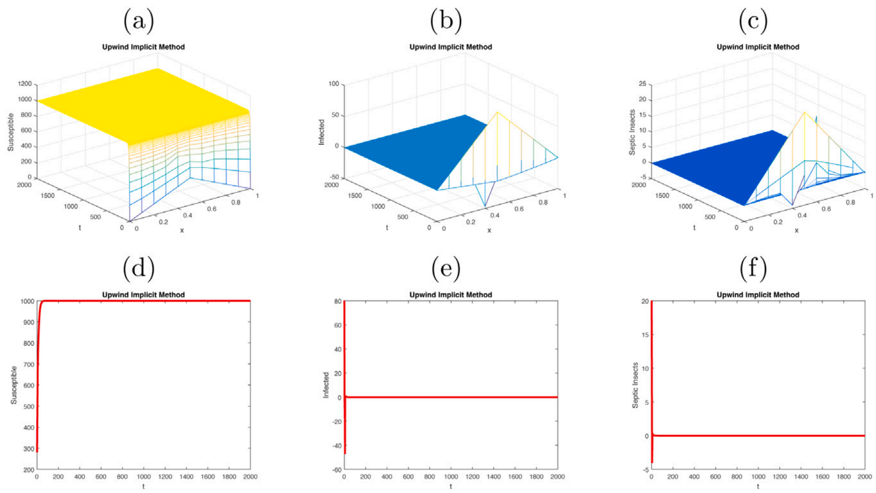

Fig. 1. Graphical representation of the numerical simulation of model (1)–(3) by using upwind like scheme (25)–(27). The parameters employed are $\beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_S = 0.0001$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$, $R_0 = 0.89444 < 1$. $E_0 = (K, 0, 0) = (1000, 0, 0)$, $\mathfrak{h} = 0.1$ and $r_1 = r_2 = r_3 = 0.2$. For plot graphs, we take $x = 0.4$.

for each $(p, q) \in \bar{I}_{M-1} \times \bar{I}_{N-1}$. Here dependence of A, D, E on q and p is prevented for simplification purposes. All of these constants, by considering the presumption of Theorem 6 are positive. Implicitly this fact is being used to proof following theorem.

Theorem 10 (*von-Neumann Stability*). *If g_1, g_2 and g_3 are positive functions then NSFD scheme (11)–(13) is von Neumann stable.*

Proof. By using the von-Neumann approach [24,28,29], assume ϕ, θ, φ are non-negative numbers and let χ_S^p, χ_I^p and χ_Y^p be real values functions. Let us assume

$$S_q^p = \chi_S^p e^{i\theta q\mathfrak{h}}, \quad \forall (q,p) \in \bar{I}_{M-1} \times \bar{I}_{N-1}, \tag{55}$$

$$I_q^p = \chi_I^p e^{i\phi q\mathfrak{h}}, \quad \forall (q,p) \in \bar{I}_{M-1} \times \bar{I}_{N-1}, \tag{56}$$

$$Y_q^p = \chi_Y^p e^{i\varphi q\mathfrak{h}}, \quad \forall (q,p) \in \bar{I}_{M-1} \times \bar{I}_{N-1}. \tag{57}$$

Substituting (55), (56), (57) respectively into discretized model (11)–(13). After linearizing and simplifications, we reach the system

$$\chi_S(1 + r_1 + A_q^p - r_1 e^{-i\theta\mathfrak{h}}) = 1, \tag{58}$$

$$\chi_I(1 + r_2 + B - r_2 e^{-i\phi\mathfrak{h}}) = 1 + D_q^p, \tag{59}$$

$$\chi_Y(1 + r_3 + E_q^p - r_3 e^{-i\varphi\mathfrak{h}}) = 1. \tag{60}$$

Taking complex norm on both sides of (58)–(60) and bounding them from above, straight away we have

$$|\chi_S| = \frac{1}{(1 + 2r_1 + A_q^p)} < 1, \tag{61}$$

$$|\chi_I| = \frac{1 + D_q^p}{(1 + 2r_2 + B)} < 1, \tag{62}$$

$$|\chi_Y| = \frac{1}{(1 + 2r_3 + E_q^p)} < 1, \tag{63}$$

Hence proposed NSFD is stable in von Neumann sense.

Computational results

Two examples are presented in this section to check our proposed NSFD method (11)–(13) developed in Section "Numerical modeling", is stable, reliable and efficient or not. The following data is used:

$\mathbb{L} = 1$, $\mathbb{T} = 200$. The level of septicity saturates are $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$. $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$. The sum of total plants population $K = 1000$. $A = 5$. $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$, $\mathfrak{h} = 0.1$ and $\mathfrak{l} = 2$. $g_1, g_2, g_3 : (0, \mathbb{L}) \to \Re$ be continuously differentiable functions.

$$g_1(x) = \begin{cases} 70x, & \text{if } x \in (0, 0.5), \\ 70(1-x), & \text{if } x \in [0.5, 1), \end{cases} \tag{64}$$

$$g_2(x) = \begin{cases} 10x, & \text{if } x \in (0, 0.5), \\ 10(1-x), & \text{if } x \in [0.5, 1), \end{cases} \tag{65}$$

$$g_3(x) = \begin{cases} 10x, & \text{if } x \in (0, 0.5), \\ 10(1-x), & \text{if } x \in [0.5, 1). \end{cases} \tag{66}$$

Example 1

Disease free equilibrium

For the simulations we let $\beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_S = 0.0001$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$. By doing simple calculations we have $R_0 = 0.89444 < 1$. By using Lemma 1, we established that $E_0 = (K, 0, 0) = (1000, 0, 0)$ is DFE point of the system (1)–(3).

We used upwind implicit scheme(25))–(27) to plot graphs of Fig. 1, and proposed non-standard finite difference scheme (11)–(13) to plot the graphs of (Fig. 2). Each of three column in these graphs provide approximate solutions of s, i, y respectively. Graphs show that upwind like scheme is not converging to disease free equilibrium. It also failed to preserve positivity. While as compare to this scheme, our proposed scheme not only converges to disease free equilibrium, it also preserve positivity. In this sense, our proposed scheme is more efficient and reliable to this problem.

Example 2

Endemic-equilibrium

If we fix all the parameters values as $\beta_1 = 0.01$, $\beta_p = 0.02$, $\beta_S = 0.01$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$. Then $R_0 = 26.2222 > 1$. Figs. 3–4 show the approximate solution of (2.1–2.3).

More precisely we used upwind implicit scheme (25))–(27) to plot graphs of Fig. 3 and nonstandard finite difference scheme (11)–(13) to plot graphs of (Fig. 4). Each of three column in these graphs provide approximate solutions of s, i, y respectively. These graphs show that upwind like scheme is incapable of converging to endemic equilibrium point. It is also incapable of proving solutions which preserve positivity.

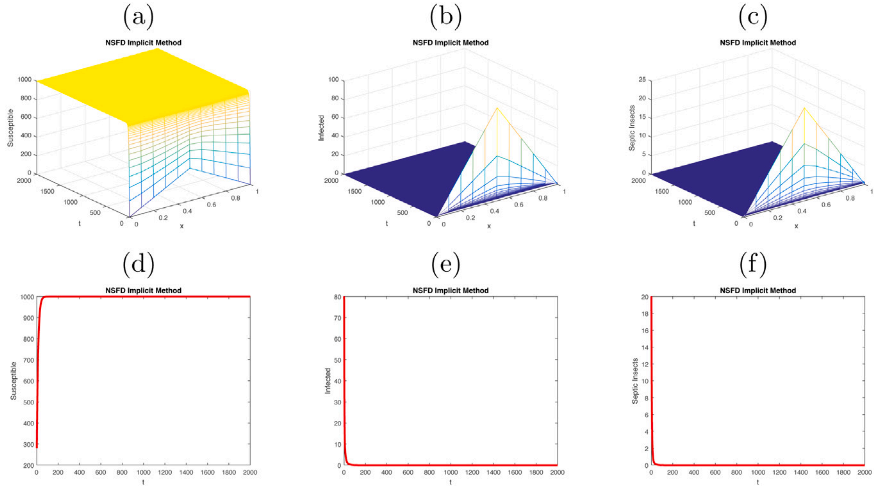

Fig. 2. Graphical representation of the numerical simulation of the approximate solutions of (1)–(3) by using nonstandard finite difference scheme (11)–(13). The parameters employed are $\beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_S = 0.0001$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$, $R_0 = 0.89444 < 1$. $E_0 = (K, 0, 0) = (1000, 0, 0)$, $\mathfrak{h} = 0.1$ and $r_1 = r_2 = r_3 = 0.2$. For plot graphs, we take $x = 0.4$.

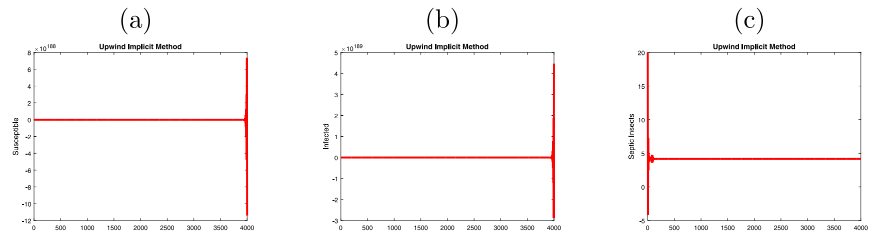

Fig. 3. Numerical simulation of the approximate solutions of (1)–(3) by using upwind like scheme (25)–(27). The parameters employed are $\beta_1 = 0.01$, $\beta_p = 0.02$, $\beta_S = 0.01$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$. $R_0 = 26.2222 > 1$, $\mathfrak{h} = 0.1$ and $r_1 = r_2 = r_3 = 0.2$. For plot graphs, we take $x = 0.4$.

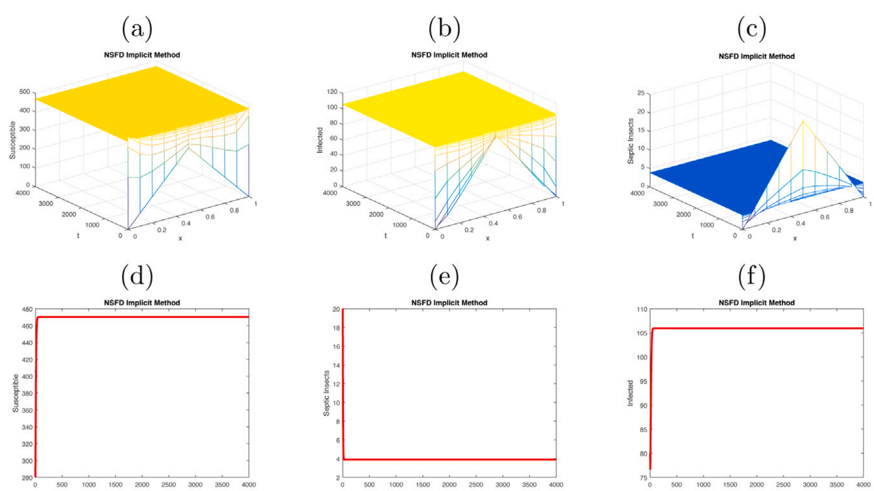

Fig. 4. Numerical simulations of the approximate solutions of (1)–(3) by NSFD (11)–(13). The parameters employed are $\beta_1 = 0.01$, $\beta_p = 0.02$, $\beta_S = 0.01$, $\alpha_1 = 0.1$, $\alpha_p = 0.2$, $\alpha_S = 0.2$, $\gamma = 0.4$, $\mu = 0.1$, $m = 0.3$, $d = 0.1$, $K = 1000$, $A = 5$, $a_1 = a_2 = a_3 = 0.001$, $r_1 = r_2 = r_3 = 0.2$. $R_0 = 26.2222 > 1$, $\mathfrak{h} = 0.1$ and $r_1 = r_2 = r_3 = 0.2$. For plot graphs, we take $x = 0.4$.

As opposed to upwind like method, our proposed scheme is in Fig. 4 is stable, converging to equilibria and also contain positivity. It yields our proposed scheme is more reliable and accurate discrete model for under investigated problem.

Conclusion

In this paper, we proposed an advective SIRXY plant disease model. This describes how the diseases caused by vectors transmit in plants. The main object of this work was to investigate the effect of advection term in the dynamics of the plant diseases caused by vectors. We developed a non-standard FD scheme to solve system (2.1–2.3). We showed that the system have two steady-state points, which are also happened to be constant solutions of the system by using mathematical induction. We also proved in our computational simulations that the proposed nonstandard finite difference scheme converges to both endemic and the DF equilibria. We used m-matrix theory to check positivity of our proposed scheme. We used Taylor's theorem to check our proposed

NSFD scheme is consistent and first order accurate in space as well as in time. To check stability, von Neumann stability analysis applied to our proposed NSFD scheme which run successfully. For comparison purposes, upwind like scheme was proposed. Our graphs show that the up-wind like scheme failed to converge at all step-sizes, as well as, it also failed in preserving positivity of the solutions. Numerical simulation also exhibits the fact that our proposed discrete model is von Neumann stable. It also preserves important structural properties of the continuous system. We also noticed that R_0 plays an important role in all these simulations. Whence our proposed scheme is more accurate and effective and reliable for under study problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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