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Study of $\Gamma-$Simulation Functions, $\mathcal{Z}_\Gamma-$Contractions and Revisiting the $\mathscr{L}-$Contractions

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Abstract. In this paper, we introduce the notions of $\mathcal{Z}_\Gamma-$contractions and Suzuki $\mathcal{Z}_\Gamma$-contractions via $\Gamma-$simulation functions. By using these new contractions, we extend and unify several existing fixed point results in the corresponding literature. We also show that the recently defined notion of $\mathscr{L}-$simulation function is an special case of $\mathcal{Z}_\Gamma-$contraction. In addition, some notable examples are given to illustrate and support the obtained results.

1. Introduction and Preliminaries

In 2000, Branciari [1] proposed to use the quadrilateral inequality instead of triangle inequality in the axioms standard metric. In this way, Branciari [1] supposed that this new distance brought a generalization of the standard metric that was why he called this new function as a "generalized metric". On the other hand, this chance brings a new topological structure that is not compatible with the topology of standard metric space [2]. In particular, it was noted that the observed distance is not necessarily continuous and open ball is not need to be open set see e.g. [3–8]. Throughout the manuscript, this new notion will be called Branciari distance space.

In [1], after defining this new structure, Branciari was able to get the analog of renowned fixed point theorem of Banach [9] with some gaps that was noted and easily removed in [4]. Since then a significant number of the authors have worked on this new abstract space and they have reported several interesting results dealing with the topology of Branciari distance space and concerning new fixed point results by using various contractions see e.g. [5–8, 16–18, 21, 24, 25, 27–29, 36, 38] and the related references therein.

For the sake of completeness, we recall necessary and fundamental definitions, notations as well as the basic results that are effectively employed in the sequel. Henceforward, the symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{R}_0^+$ and $\mathbb{N}_0$ are reserved to indicate the real numbers, positive real numbers, natural numbers, non-negative reals, and non-negative integers, respectively.

The following definition belongs to Branciari [1].

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Definition 1.1. *[1] For a non-empty set X, if a distance function $d : X \times X \to [0, \infty)$ satisfies*

(R1) $d(x, y) = 0$ *if and only if* $x = y$*;*

(R2) $d(x, y) = d(y, x)$ *for each* $x, y \in X$*;*

(R3) $d(x, z) \leq d(x, u) + d(u, v) + d(v, z)$ *for all* $x, z \in X$ *and all distinct points* $u, v \in X\backslash\{x, z\}$*,*

then d is called a Branciari distance or a rectangular/generalized metric on X and (X, d) is called a Branciari distance space or a rectangular/generalized metric space.

Khojasteh *et al.* [26] introduced an interesting notion, *simulation function*, in order to combine and unify several existing results in the literature of fixed point theory.

Definition 1.2. *A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ which satisfies in the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$,

(ζ_2) $\zeta(t, s) < s - t$ *for all* $t, s > 0$,

(ζ_3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

From now onward, the letter $\mathcal{Z}$ denotes the family of simulation functions.

Definition 1.3. *[26] A self-mapping T on a complete metric space (X, d) is called a $\mathcal{Z}-$contraction with respect to $\zeta \in \mathcal{Z}$ if there exists a simulation function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that*

$$\zeta(d(Tx, Ty), d(x, y) \geq 0 \quad \forall x, y \in X.$$

Theorem 1.4. *[26] If a self-mapping T on a complete metric space (X, d) forms a $\mathcal{Z}-$contraction with respect to $\zeta \in \mathcal{Z}$, then T has a unique fixed point.*

Recently, the notion of the simulation function and $\mathcal{Z}-$contractions have been extended and generalized in various way, see e.g. [11–15, 19, 22, 30–34, 37]. Among them we consider the notion of Ψ-*simulation function* [22] and we compare it with $\mathscr{L}-$simulation function [37]. We investigate the relationship between these concepts.

On the other hand, Jleli and Samet [23, 2014] introduced a notion of θ-contractions to generalize certain fixed point results in the framework of Branciari distance spaces by using the auxiliary function $\theta : (0, +\infty) \to (1, +\infty)$ with the following conditions:

(θ_1) θ is nondecreasing,

(θ_2) for all sequence $\{t_n\} \subset (0, +\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+,$$

(θ_3) there exists $r \in (0, 1)$ and $l \in (0, +\infty)$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l.$$

Herein after Θ represent the collection of all functions θ, and Θ_0 be the collection of all functions $\theta : (0, +\infty) \to (1, +\infty)$ such that (θ_1) and (θ_2) are held. Furthermore, we shall use the letter Ω to denote the collection of all continuous functions $\theta : (0, +\infty) \to (1, +\infty)$ such that (θ_1) and (θ_2) are satisfied. We note that Ahmad et al. [10] observed the analog of Jleli and Samet in the context of standard metric spaces by considering the continuity of Θ instead of (θ_3).

Definition 1.5. *Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. T is called $\Theta-$contraction, if there exists $\theta \in \Omega$ and a constant $k \in (0, 1)$ such that*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 1.6. *Every $\Theta-$contraction on a complete metric space has a unique fixed point.*

Theorem 1.7. *Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. If there exists $\theta \in \Omega$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{implies} \quad \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

then T has a unique fixed point.

Very recently, Cho [37] introduced the following class of functions as a new innovation and established a new fixed point theorem for such contraction mappings in Branciari distance spaces.

Definition 1.8. *[37] A mapping $\vartheta : [1, \infty) \times [1, \infty) \to \mathbb{R}$ is called $\mathscr{L}-$simulation function if it satisfies the following conditions:*

(ϑ_1) *$\vartheta(1, 1) = 1$;*

(ϑ_2) *$\vartheta(t, s) < \frac{s}{t}$ for all $t, s > 1$;*

(ϑ_3) *if $\{t_n\}, \{s_n\}$ are sequences in $(1, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$, and $t_n < s_n$ for all $n \in \mathbb{N}$, then*

$$\limsup_{n\to\infty} \vartheta(t_n, s_n) < 1. \tag{2}$$

Denote $\mathscr{L}$ as the collection of $\mathscr{L}-$simulation functions $\vartheta : [1, \infty) \times [1, \infty) \to \mathbb{R}$.

Definition 1.9. *Let (X, d) be a Branciari distance space, and let $T : X \to X$ be a mapping. T is called $\mathscr{L}-$contraction with respect to ϑ if there exists $\theta \in \Theta$ and $\vartheta \in \mathscr{L}$ such that,*

$$\vartheta(\theta(d(Tx, Ty)), \theta(d(x, y)) \geq 0,$$

for all $x, y \in X$.

Theorem 1.10. *[37, Theorem 4] Every $\mathscr{L}-$contraction on a complete Branciari distance spaces has a unique fixed point.*

In this paper, we show that the proof of [37, Theorem 4] is wrong and the continuity condition of θ is essential. In other words, we have to consider $\theta \in \Omega$.

Very recently, Heidary Joonaghany et al. [22] established a new generalization of simulation functions called Ψ-simulation function. The following notations and definitions have been taken from [22].
Denote $\Psi([0, +\infty))$, the set of all non-decreasing and continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.11. *[22] A function $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called Ψ-simulation if there exists $\psi \in \Psi([0, +\infty))$ such that*

(η1) *$\eta(t, s) < \psi(s) - \psi(t)$ for all $s, t > 0$,*
(η2) *if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \eta(t_n, s_n) < 0.$$

Example 1.12. *[22] Let $\psi \in \Psi([0,+\infty))$. The following models are some examples of $\Psi-$ simulation functions:*

(e_1) *For each $s,t \geq 0$, let $\eta(t,s) = \alpha\psi(s) - \psi(t)$, in which $\alpha \in [0,1)$.*

(e_2) *For each $s,t \geq 0$, let $\eta(t,s) = \varphi(\psi(s)) - \psi(t)$, in which $\varphi : [0,+\infty) \to [0,+\infty)$ is a function such that $\varphi(0) = 0$ and $0 < \varphi(s) < s$ for each $s > 0$, and $\limsup_{t\to s} \varphi(t) < s$. (For example, $\varphi(s) = \alpha s$ in which $0 \leq \alpha < 1$).*

Denote $\mathcal{Z}_\Psi$, the set of all Ψ-simulation functions. Note that every simulation function is obviously Ψ-simulation because ψ can be considered as identity function on $[0,\infty)$. However, a Ψ-simulation function is not necessary a simulation function (see [22, Example 2.4] for more detail).

The following results are acquired of [22]:

Theorem 1.13. *[22, Theorem 2.6] Let (X,d) be a complete metric space and let $T,S : X \to X$ be two mappings such that for all $x,y \in X$,*
$\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leq d(x,y)$ implies that

$$\eta(d(Tx,Sy), m(x,y)) \geq 0, \tag{3}$$

where $\eta \in \mathcal{Z}_\Psi$ and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T and S have a unique common fixed point.

Since the following lemma shorten the proofs of our result, we recollect it to here.

Lemma 1.14. *[20] Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. If $\{x_{2n}\}$ is not a Cauchy sequence then there exists $\epsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ such that, n_k is the smallest index for which $n_k > m_k > k$ and $d(x_{2m_k}, x_{2n_k}) > \epsilon$ and*

(1) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon$,

(2) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \epsilon$,

(3) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon$,

(4) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k+1}) = \epsilon$.

In this manuscript, we introduce new contractions that are based on the generalized simulation function, Γ−simulation function. We investigate the corresponding fixed results for these contractions in the context of complete metric spaces. We also bote that the $\mathscr{L}$-contraction is a special case of the the contractions generated by Ψ-simulation functions. The given results are supported with concrete examples.

2. Main Result

We, first, present a generalization of Ψ-simulation function that will be called Γ-simulation. Let $\Gamma([0,+\infty))$ denote the set of all non-decreasing functions $\gamma : [0,+\infty) \to [0,+\infty)$ such that $\gamma(t) = 0$ if and only if $t = 0$.

Definition 2.1. *A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called Γ-simulation, if there exists $\gamma \in \Gamma([0,+\infty))$ such that:*

(ζ_1) *$\zeta(t,s) < \gamma(s) - \gamma(t)$ for all $s,t > 0$,*

(ζ_2) *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called Γ_0-simulation, if there exists $\gamma \in \Gamma([0,+\infty))$ such that (ζ_1) and the following condition are satisfied:

(ζ_2') *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that for all $n \in \mathbb{N}$, $t_n \le s_n$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Let $\mathcal{Z}_\Gamma$ and $\mathcal{Z}_{\Gamma_0}$ denote the set of all Γ-simulation functions and Γ_0-simulation functions respectively. Every Γ-simulation function is a Γ_0-simulation function. Also, every Ψ-simulation function is obviously Γ-simulation function. But a Γ_0-simulation function is neither necessary a Ψ-simulation function nor a Γ-simulation function.

Example 2.2. *Define $\gamma : [0,+\infty) \to [0,+\infty)$ by*

$$\gamma(t) = \begin{cases} 2t & \text{if } 0 \le t < 1 \\ 3t & \text{if } 1 \le t. \end{cases}$$

Also, define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ as follows:

$$\zeta(t,s) = \frac{1}{2}\gamma(s) - \gamma(t).$$

One can easily verify that $\gamma \in \Gamma$ and $\zeta \in \mathcal{Z}_\Gamma$ with respect to γ. However, $\gamma \notin \Psi$ and $\zeta \notin \mathcal{Z}_\Psi$ with respect to γ.

Definition 2.3. *Let (X,d) be a metric space. We say that the mapping $T : X \to X$ is a $\mathcal{Z}_\Gamma$-contraction, if there exists $\zeta \in \mathcal{Z}_\Gamma$ such that for all $x, y \in X$,*

$$\zeta(d(Tx,Ty), d(x,y)) \ge 0. \tag{4}$$

Definition 2.4. *Let (X,d) be a metric space. A mapping $T : X \to X$ is said be a Suzuki $\mathcal{Z}_\Gamma$-contraction if there exists $\zeta \in \mathcal{Z}_\Gamma$ such that for all $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies that } \zeta(d(Tx,Ty), d(x,y)) \ge 0.$$

Definition 2.5. *Let (X,d) be a metric space. A mapping $T : X \to X$ is said be a $\mathcal{Z}_\Gamma$-weak contraction if there exists $\zeta \in \mathcal{Z}_\Gamma$ such that for all $x, y \in X$,*

$$\zeta(d(Tx,Ty), m_T(x,y)) \ge 0, \tag{5}$$

where

$$m_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Definition 2.6. *Let (X,d) be a metric space. A mapping $T : X \to X$ is said be a Suzuki $\mathcal{Z}_\Gamma$-weak contraction if there exists $\zeta \in \mathcal{Z}_\Gamma$ such that for all $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies that } \zeta(d(Tx,Ty), m_T(x,y)) \ge 0.$$

in which

$$m_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Remark 2.7. *It is clear that*

(1) $\Psi([0,+\infty)) \subseteq \Gamma([0,+\infty))$, *and so* $\mathcal{Z}_\Psi \subseteq \mathcal{Z}_\Gamma$,

(2) *every* $\mathcal{Z}_\Gamma$*-contraction is a Suzuki* $\mathcal{Z}_\Gamma$*-contraction,*

(3) *every* $\mathcal{Z}_\Gamma$*-weak contraction is a Suzuki* $\mathcal{Z}_\Gamma$*-weak contraction.*

The Example 2.2 shows that the converse of statement (1) *is not true. Also, the following example shows that the converse of statement* (2) *is not true.*

Example 2.8. *Let* $X = \{0,1,3,5\}$ *be endowed with the metric d defined by*

$$d(x,y) = |x-y|.$$

Clearly (X,d) *is a complete metric space. Let* $T : X \to X$ *be defined as follows:*

$$T(0) = 3 \quad \text{and} \quad T(1) = T(3) = T(5) = 5.$$

One can verify that T is not a $\mathcal{Z}_\Gamma$*-contraction. In fact, for any* $\zeta \in \mathcal{Z}_\Gamma$*, the map T dose not satisfy the condition (4) of Definition 2.3 at* $u = 0$ *and* $v = 1$*. Because, if* $\zeta \in \mathcal{Z}_\Gamma$ *be a* Γ*-simulation function with respect to the function* $\gamma \in \Gamma$ *then*

$$\begin{array}{rcl}\zeta(d(Tu,Tv),d(u,v)) & = & \zeta(2,1) \\ & < & \gamma(1)-\gamma(2) \\ & \leq & 0.\end{array}$$

On the other hand for $u = 0$ *and* $v = 1$ *we have*

$$\frac{1}{2}d(u,Tu) = \frac{1}{2}d(0,3) = \frac{3}{2}.$$

But $d(u,v) = d(0,1) = 1$*. So, we obtain that*

$$\frac{1}{2}d(u,Tu) \nleq d(u,v).$$

Also

$$\frac{1}{2}d(v,Tv) = \frac{1}{2}d(1,5) = 2.$$

But $d(u,v) = 1$*. So, we obtain that*

$$\frac{1}{2}d(v,Tv) \nleq d(u,v).$$

By choosing $\zeta(t,s) = \frac{s}{2} - t$*, it can be easily seen that, , for any* $u, v \in X$*,*

$$\frac{1}{2}d(u,Tu) \leq d(u,v) \Rightarrow \zeta(d(Tu,Tv),d(u,v)) \geq 0.$$

This means that T is a Suzuki $\mathcal{Z}_\Gamma$*-contraction.*

The next example indicates that the converse of statement (3) is not true.

Example 2.9. *Let* $X = \{(1,1),(1,5),(1,6),(5,1),(6,1),(5,6),(6,5)\}$ *be endowed with the metric d defined by*

$$d((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|\,.$$

It is easy to see that (X,d) *is a complete metric space.*

Suppose that $T : X \to X$ is defined as follows:

$$T(x,y) = \begin{cases} (\min\{x,y\}\ ,\ 1) & \text{if } x = 1 \text{ or } y = 1 \text{ or } x < y \\ (1\ ,\ \min\{x,y\}) & \text{otherwise.} \end{cases}$$

Now, we show that for any $\zeta \in \mathcal{Z}_\Gamma$ the map T dose not satisfy the condition (5) of Definition 2.5 at $u = (5,6)$ and $v = (6,5)$.

For this purpose, let $\zeta \in \mathcal{Z}_\Gamma$ be a Γ-simulation function with respect to the function $\gamma \in \Gamma$. Note that,

$$d(Tu,Tv) = d((6,1),(1,5)) = 9.$$

Also

$$\begin{aligned} m_T(u,v) &= \max\{d(u,v), d(u,Tu), d(v,Tv), \tfrac{d(u,Tv)+d(v,Tu)}{2}\} \\ &= \max\{d((5,6),(6,5)),\ d((5,6),(5,1)), d((6,5),(1,5)), \\ &\quad \tfrac{d((5,6),(1,5))+d((6,5),(6,1))}{2}\} \\ &= \max\{2,5,5,\tfrac{5+4}{2}\} \\ &= 5. \end{aligned}$$

So,

$$\begin{aligned} \zeta(d(Tu,Tv), m_T(u,v)) &= \zeta(9,5) \\ &< \gamma(5) - \gamma(9) \\ &\le 0. \end{aligned}$$

Thus T does not satisfy the condition (5). However, choosing $\zeta(t,s) = \frac{8}{9}s - t$, one can easily see that, for any $u, v \in X$,

$$\frac{1}{2}d(u,Tu) \le d(u,v) \Rightarrow \zeta(d(Tu,Tv), d(u,v)) \ge 0.$$

In fact, for $u = (5,6)$ and $v = (6,5)$ we have

$$\frac{1}{2}d(u,Tu) = \frac{1}{2}d((5,6),(6,1)) = 3.$$

But $d(u,v) = d((5,6),(6,5)) = 2$, and

$$\frac{1}{2}d(u,Tu) \nleq d(u,v).$$

Also,

$$\frac{1}{2}d(v,Tv) = \frac{1}{2}d((6,5),(1,5)) = \frac{5}{2}.$$

But $d(u,v) = 2$, so we obtain that

$$\frac{1}{2}d(v,Tv) \nleq d(u,v).$$

It is easily seen that for every two elements $x, y \in X$, if $\frac{1}{2}d(x,Tx) \le d(x,y)$ then

$$\zeta(d(Tu,Tv), d(u,v)) \ge 0.$$

For example, for $u = (5,6)$ and $z = (1,1)$, we have:

$$\begin{aligned} m_T(u,z) &= \max\{d(u,z), d(u,Tu), d(z,Tz), \tfrac{d(u,Tz)+d(z,Tu)}{2}\} \\ &= \max\{d((5,6),(1,1)),\ d((5,6),(6,1)), d((1,1),(1,1)), \\ &\quad \tfrac{d((5,6),(1,1))+d((1,1),(6,1))}{2}\} \\ &= \max\{9,6,0,\tfrac{9+5}{2}\} \\ &= 9. \end{aligned}$$

Also, we have

$$d(Tu,Tz) = d((6,1),(1,1))) = 5.$$

So, we get

$$\begin{aligned} \zeta(d(Tu,Tz), m_T(u,v)) &= \zeta(5,9) \\ &= \tfrac{8}{9}9 - 5 \\ &= 3 \\ &> 0. \end{aligned}$$

Again, for $u = (6,5)$ and $z = (1,1)$, we have:

$$\begin{aligned} m_T(u,z) &= \max\{d(u,z), d(u,Tu), d(z,Tz), \tfrac{d(u,Tz)+d(z,Tu)}{2}\} \\ &= \max\{d((6,5),(1,1)),\ d((6,5),(1,5)), d((1,1),(1,1)), \\ &\quad \tfrac{d((6,5),(1,1))+d((1,1),(1,5))}{2}\} \\ &= \max\{9,5,0,\tfrac{9+4}{2}\} \\ &= 9. \end{aligned}$$

Also, we have

$$d(Tu,Tz) = d((1,5),(1,1))) = 4.$$

So, we get

$$\begin{aligned} \zeta(d(Tu,Tz), m_T(u,v)) &= \zeta(4,9) \\ &= \tfrac{8}{9}9 - 4 \\ &= 4 \\ &> 0. \end{aligned}$$

The other cases can be verified analogously.

Consequently, T is a Suzuki $\mathcal{Z}_\Gamma$-weak contraction, however it is not a $\mathcal{Z}_\Gamma$-weak contraction.

Definition 2.10. *Let (X,d) be a metric space. We say that the mapping $T : X \to X$ is $\mathcal{Z}_{\Gamma_0}$-contraction, if there exists $\zeta \in \mathcal{Z}_{\Gamma_0}$ such that for all $x, y \in X$,*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0.$$

Now, we present our first main result.

Theorem 2.11. *Let (X, d) be a complete metric space, and let $T, S : X \to X$ be two mappings such that for all $x, y \in X$,*

$$\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \le d(x,y) \Rightarrow \eta(d(Tx,Sy), m(x,y)) \ge 0, \tag{6}$$

in which $\zeta \in \mathcal{Z}_\Gamma$ and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element. Define a sequence $\{x_n\}_{n\ge 0}$ by

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \text{ for each } n \ge 0.$$

If there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1}$, then we claim that $x_j = x_k$ for all $j \ge k$. To see this, suppose that k is an even number such that $x_k = x_{k+1}$. If $m(x_k, x_{k+1}) = 0$ then, by the definition of $m(x, y)$, we have $x_{k+1} = x_{k+2}$. So, one can suppose that $m(x_k, x_{k+1}) \ne 0$. Furthermore, one has

$$\begin{aligned}\frac{1}{2}\min\{d(x_k,Tx_k), d(x_{k+1},Sx_{k+1})\} &= \frac{1}{2}\min\{d(x_k,x_{k+1})\\ &\quad ,d(x_{k+1},x_{k+2})\}\\ &\le d(x_k,x_{k+1}).\end{aligned}$$

Hence, for each even number $k \in \mathbb{N}$ we get

$$\frac{1}{2}\min\{d(x_k,Tx_k), d(x_{k+1},Sx_{k+1})\} \le d(x_k,x_{k+1}). \tag{7}$$

Thus, from (6) and (ζ_1) we have

$$\gamma(d(x_{k+1},x_{k+2})) < \gamma(m(x_k,x_{k+1})).$$

So, since $\gamma \in \Gamma([0,+\infty))$, we have

$$d(x_{k+1},x_{k+2}) < m(x_k,x_{k+1}).$$

But,

$$\begin{aligned}m(x_k,x_{k+1}) &= \max\left\{d(x_k,x_{k+1}), d(x_k,Tx_k), d(x_{k+1},Sx_{k+1})\right.\\ &\qquad \left.,\frac{d(x_k,Sx_{k+1})+d(x_{k+1},Tx_k)}{2}\right\}\\ &= \max\left\{0, d(x_{k+1},x_{k+2}), \frac{d(x_k,x_{k+2})}{2}\right\}\\ &\le \max\left\{d(x_{k+1},x_{k+2}), \frac{d(x_k,x_{k+1})+d(x_{k+1},x_{k+2})}{2}\right\}\\ &= \max\left\{d(x_{k+1},x_{k+2}), \frac{0+d(x_{k+1},x_{k+2})}{2}\right\}\\ &= d(x_{k+1},x_{k+2}),\end{aligned}$$

which is a contradiction. So, $d(x_{k+1}, x_{k+2}) = 0$ or $m(x_k, x_{k+1}) = 0$, i.e., $x_{k+1} = x_{k+2}$. Hence, $x_k = x_{k+1} = x_{k+2}$. Similarly, if $k = 2n + 1$ for some $n \geq 0$, we can prove that $x_k = x_{k+1} = x_{k+2}$. Therefore, x_k is a common fixed point of T and S. So, for all $n \geq 0$, we suppose that $d(x_n, x_{n+1}) > 0$ and $m(x_n, x_{n+1}) \neq 0$.

Now, we intend to prove that $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$. To reach this goal, we claim that

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &\leq m(x_k, x_{k+1}) \\ &= d(x_k, x_{k+1}) \quad \forall k \in \mathbb{N}. \end{aligned} \tag{8}$$

To prove the claim, at first, suppose that k is an even number. We have

$$\begin{aligned} \frac{1}{2}\min\{d(x_k, Tx_k), d(x_{k+1}, Sx_{k+1})\} &= \frac{1}{2}\min\{d(x_k, x_{k+1}) \\ &\quad , d(x_{k+1}, x_{k+2})\} \\ &\leq d(x_k, x_{k+1}). \end{aligned}$$

So, from (6) and (ζ_1) we have:

$$\begin{aligned} \gamma(d(x_{k+2}, x_{k+1})) &= \gamma(d(Sx_{k+1}, Tx_k)) \\ &< \gamma(m(x_k, x_{k+1})), \end{aligned}$$

and by the fact that $\gamma \in \Gamma([0, +\infty))$ we have

$$d(x_{k+1}, x_{k+2}) < m(x_k, x_{k+1}). \tag{9}$$

On the other hand,

$$\begin{aligned} m(x_k, x_{k+1}) &= \max\Big\{d(x_k, x_{k+1}), d(x_k, Tx_k) \\ &\quad , d(x_{k+1}, Sx_{k+1}), \frac{d(x_k, Sx_{k+1}) + d(x_{k+1}, Tx_k)}{2}\Big\} \\ &= \max\left\{d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}), \frac{d(x_k, x_{k+2})}{2}\right\} \\ &\leq \max\Big\{d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}) \\ &\quad , \frac{d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2})}{2}\Big\} \\ &\leq \max\{d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2})\}. \end{aligned}$$

So, if $d(x_{k_0+1}, x_{k_0+2}) \geq d(x_{k_0}, x_{k_0+1})$ for some even number $k_0 \in \mathbb{N}$ we get

$$m(x_{k_0}, x_{k+1}) \leq d(x_{k_0+1}, x_{k_0+2}),$$

which is a contradiction by (9). Hence, for each even number $k \in \mathbb{N}$,

$$d(x_{k+1}, x_{k+2}) < d(x_k, x_{k+1}),$$

and so

$$m(x_k, x_{k+1}) \leq d(x_k, x_{k+1}).$$

Consequently, (8) is proved when $k \geq 0$ is an even number. By the same argument, one can verify that (8) holds when k is an odd number. Thus, the sequence $\{d(x_n, x_{n+1})\}_{n\geq 1}$ is non increasing and bounded below, so it converges to a real number $\ell \geq 0$. Hence,

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x_{n+1}) &= \lim_{n\to\infty} m(x_n, x_{n+1}) \\ &= \ell. \end{aligned} \tag{10}$$

We claim that $\ell = 0$.

Indeed, combining (7) and (6), we have

$$\zeta(d(Tx_k, Sx_{k+1}), m(x_k, x_{k+1})) \geq 0,$$

for each even number $k \in \mathbb{N}$. So,

$$\limsup_{n\to\infty} \zeta(d(x_{2n+1}, x_{2n+2}), m(x_{2n}, x_{2n+1})) \geq 0. \tag{11}$$

On the other hand, if we suppose that $\ell > 0$ then (10) implies that

$$\begin{aligned}\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) &= \lim_{n\to\infty} m(x_{2n}, x_{2n+1})\\ &= \ell\\ &> 0.\end{aligned}$$

So, using (ζ_2), it follows that

$$\limsup_{n\to\infty} \zeta(d(x_{2n+1}, x_{2n+2}), m(x_{2n}, x_{2n+1})) < 0,$$

which (11) cause a contradiction. So, the claim is completed and we obtain that

$$\begin{aligned}\lim_{n\to\infty} d(x_n, x_{n+1}) &\leq \lim_{n\to\infty} m(x_n, x_{n+1})\\ &= \lim_{n\to\infty} m(x_n, x_{n+1}).\end{aligned} \tag{12}$$

Now we intend to prove that $\{x_n\}$ is a Cauchy sequence.

In order to show that $\{x_n\}$ is a Cauchy sequence, using (12), it is enough to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then Lemma (1.14) shows there exist $\epsilon_0 > 0$ and subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and $d(x_{2m_k}, x_{2n_k)}) \geq \epsilon_0$ and

- (l_1) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon_0$,
- (l_2) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \epsilon_0$,
- (l_3) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon_0$,
- (l_4) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k+1}) = \epsilon_0$.

Therefore, from the definition of $m(x, y)$, we have:

$$\begin{aligned}\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) &= \lim_{k\to\infty} \max\Big\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1})\\ &\quad, d(x_{2m_k-1}, x_{2m_k})\\ &\quad, \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2n_k+1})}{2}\Big\}\\ &= \max\Big\{\epsilon_0, 0, 0, \frac{\epsilon_0 + \epsilon_0}{2}\Big\}\\ &= \epsilon_0.\end{aligned}$$

So,

$$\begin{aligned}\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) &= \lim_{k\to\infty} m(x_{2m_k-1,x_{2n_k}})\\ &= \epsilon_0\\ &> 0.\end{aligned}$$

Hence, (ζ_2) implies that

$$\limsup_{n\to\infty} \zeta(d(x_{2m_k}, x_{2n_k+1}), m(x_{2m_k-1,x_{2n_k}})) < 0. \tag{13}$$

On the other hand, we claim that for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$, then

$$\frac{1}{2}\min\{d(x_{2n_k}, Tx_{2n_k}), d(x_{2m_k-1}, Sx_{2m_k-1})\} \leq d(x_{2n_k}, x_{2m_k-1}). \tag{14}$$

Indeed, since $n_k > m_k$ and $\{d(x_n, x_{n+1})\}$ is non-increasing, we have

$$\begin{aligned}d(x_{2n_k}, Tx_{2n_k}) &= d(x_{2n_k}, x_{2n_k+1})\\ &\leq d(x_{2m_k+1}, x_{2m_k})\\ &\leq d(x_{2m_k}, x_{2m_k-1})\\ &= d(x_{2m_k-1}, Sx_{2m_k-1}).\end{aligned}$$

Hence, the left hand side of inequality (14) is equal to

$$\frac{1}{2}d(x_{2n_k}, Tx_{2n_k}) = \frac{1}{2}d(x_{2n_k}, x_{2n_k+1}).$$

Therefore, we first need to show that for sufficiently large $k \in \mathbb{N}$,
if $n_k > m_k > k$ then

$$d(x_{2n_k}, x_{2n_k+1}) \leq d(x_{2n_k}, x_{2m_k-1}).$$

According to (12), there exists $k_1 \in \mathbb{N}$ such that for any $k > k_1$,

$$d(x_{2n_k}, x_{2n_k+1}) < \frac{1}{2}\epsilon_0.$$

Also, there exists $k_2 \in \mathbb{N}$ such that for any $k > k_2$,

$$d(x_{2m_k-1}, x_{2m_k}) < \frac{1}{2}\epsilon_0.$$

Hence, for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$, we have

$$\begin{aligned}\epsilon_0 &\leq d(x_{2n_k}, x_{2m_k})\\ &\leq d(x_{2n_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k})\\ &\leq d(x_{2n_k}, x_{2m_k-1}) + \frac{\epsilon_0}{2}.\end{aligned}$$

So, one concludes that

$$\frac{\epsilon_0}{2} \leq d(x_{2n_k}, x_{2m_k-1}).$$

Thus, for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$, we have

$$\begin{aligned} d(x_{2n_k}, x_{2n_k+1}) &\leq \frac{\epsilon_0}{2} \\ &\leq d(x_{2n_k}, x_{2m_k-1}). \end{aligned}$$

So (14) is proved. Applying (14) and (6), we get

$$\zeta(d(Tx_{2n_k}, Sx_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})) \geq 0, \tag{15}$$

for sufficiently large $k \in \mathbb{N}$.
Taking (upper)limit on both side of (15), we obtain that

$$\limsup_{k\to\infty} \zeta(d(x_{2n_k+1}, x_{2m_k}), m(x_{2n_k}, x_{2m_k-1})) \geq 0, \tag{16}$$

which is a contradiction by (13). So, $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Now, we are going to show that u is a common fixed point of T and S.

Firstly, we prove that

$$\lim_{n\to\infty} m(u, x_{2n}) = d(Su, u). \tag{17}$$

Note that

$$\begin{aligned} d(u, Su) &\leq m(x_{2n}, u) \\ &= \max\Big\{d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, Su) \\ &\qquad , \tfrac{d(x_{2n}, Su) + d(u, x_{2n+1})}{2}\Big\}. \end{aligned} \tag{18}$$

Taking limit on both side of (18), we obtain that

$$\begin{aligned} d(u, Su) &\leq \lim_{n\to\infty} m(u, x_{2n}) \\ &\leq \max\left\{0, 0, d(u, Su), \frac{d(u, Su) + 0}{2}\right\} \\ &= d(u, Su). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} m(u, x_{2n}) = d(Su, u).$$

This completes the proof of (17). In the same manner, one can show that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(Tu, u). \tag{19}$$

Now, we claim that for each $n \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2} d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, u), \tag{20}$$

or

$$\frac{1}{2} d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, u). \tag{21}$$

On the contrary, if for some $n_0 \geq 0$ such that both of them be false, we get

$$\begin{aligned} d(x_{2n_0}, x_{2n_0+1}) &\leq d(x_{2n_0}, u) + d(u, x_{2n_0+1}) \\ &< \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2} d(x_{2n_0+1}, x_{2n_0+2}) \\ &\leq \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) \\ &= d(x_{2n_0}, x_{2n_0+1}), \end{aligned}$$

which is a contradiction and the claim is proved. So, one can consider the following two cases:

Case (1): The relation (20) is established for infinitely many $n \geq 0$.
In this case, for infinitely many $n \geq 0$ we have

$$\begin{aligned} \frac{1}{2} \min\{d(x_{2n}, Tx_{2n}), d(u, Su)\} &= \frac{1}{2} \min\{d(x_{2n}, x_{2n+1}), d(u, Su)\} \\ &\leq \frac{1}{2} d(x_{2n}, x_{2n+1}) \\ &\leq d(x_{2n}, u). \end{aligned}$$

Consequently, using (6), it follows that for infinitely many $n \geq 0$,

$$\zeta(d(Tx_{2n}, Su), m(x_{2n}, u)) \geq 0.$$

Therefore,

$$\limsup_{k\to\infty} \zeta(d(x_{2n+1}, Su), m(x_{2n}, u)) \geq 0. \tag{22}$$

Now, we show that $d(Su, u) = 0$. Suppose that $d(Su, u) > 0$. Then, since

$$\begin{aligned} \lim_{n\to\infty} d(Tx_{2n}, Su) &= \lim_{n\to\infty} m(u, x_{2n}) \\ &= d(u, Su) \\ &> 0, \end{aligned}$$

from (ζ2) we have

$$\limsup_{k\to\infty} \zeta(d(x_{2n+1}, Su), m(x_{2n}, u)) < 0,$$

which contradicts (22). So, $d(u, Su) = 0$, i.e., $Su = u$. On the other hand, we have

$$\begin{aligned} m(u, u) &= \max\left\{d(u, u), d(u, Tu), d(u, Su), \frac{d(u, Su) + d(u, Tu)}{2}\right\} \\ &= \max\left\{0, d(u, Tu), 0, \frac{d(u, Tu)}{2}\right\} \\ &= d(u, Tu). \end{aligned}$$

So,

$$m(u,u) = d(u,Tu). \tag{23}$$

Furthermore,

$$\begin{aligned}\frac{1}{2}\min\{d(u,Tu), d(u,Su)\} &= \frac{1}{2}\min\{d(u,Tu), 0\}\\ &= 0\\ &\leq d(u,u).\end{aligned}$$

Thus, if $d(Tu,u) > 0$ then (6) implies that

$$\eta(d(Tu,Su), m(u,u)) \geq 0.$$

So, from (ζ_1) one can observe that

$$d(Tu,Su) < m(u,u),$$

which contradicts (23). Hence, $d(Tu,u) = 0$, i.e., $Tu = u$. So $Tu = Su = u$.

Case (2): The relation (20) is established only for finitely many $n \geq 0$.

In this case there exists $n_0 \geq 0$ such that (21) is true for any $n \geq n_0$. Similar to Case (1), one can prove that, (21) leads us to a contradiction unless $Su = Tu = u$. So, in any case u is a common fixed point of T and S.

Finally, we show that the common fixed point of T and S is unique.
Suppose that u and v are two common fixed points of T and S. We have

$$\begin{aligned}\frac{1}{2}\min\{d(u,Tu), d(u,Su)\} &= \frac{1}{2}\min\{d(u,Tu), 0\}\\ &= 0\\ &= d(u,u).\end{aligned}$$

On contrary, if $d(u,v) \neq 0$ then $m(u,v) \neq 0$. So, (6) implies that

$$\begin{aligned}\zeta(d(u,v), m(u,v)) &= \zeta(d(Tu,Sv), m(u,v))\\ &\geq 0.\end{aligned}$$

So, from (ζ_1), one can conclude that

$$d(Tu,Sv) < m(u,v).$$

But

$$\begin{aligned}m(u,v) &= \max\left\{d(u,v), d(u,Tu), d(v,Sv), \frac{d(u,Sv)+d(v,Tu)}{2}\right\}\\ &= d(u,v),\end{aligned}$$

and it is a contradiction. So $d(u,v) = 0$ which completes the proof. □

The next result is an obvious consequence of Theorem 2.11.

Corollary 2.12. *[22, Theorem 2.6] Let (X,d) be a complete metric space, and let $T,S : X \to X$ be two mappings such that for all $x,y \in X$,*

$$\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \leq d(x,y) \Rightarrow \eta(d(Tx,Sy), m(x,y)) \geq 0, \tag{24}$$

where $\eta \in \mathcal{Z}_\Psi$ and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T and S have a unique common fixed point.

Proof. Taking into account the fact that $\mathcal{Z}_\Psi \subseteq \mathcal{Z}_\Gamma$, one can obtain desired result. □

Putting $S = T$ in the Theorem 2.11, we obtain:

Corollary 2.13. *Every Suzuki $\mathcal{Z}_\Gamma$-weak contraction on a complete metric space has a unique fixed point.*

The following results are some immediate consequences of Corollary 2.13:

Corollary 2.14. *Every $\mathcal{Z}_\Gamma$-weak contraction on a complete metric space has a unique fixed point.*

Corollary 2.15. *Every Suzuki $\mathcal{Z}_\Gamma$-contraction on a complete metric space has a unique fixed point.*

Corollary 2.16. *Every $\mathcal{Z}_\Gamma$-contraction on a complete metric space has a unique fixed point.*

Remark 2.17. *With due attention to this that every $\mathcal{Z}_\Gamma$-weak contraction is a Suzuki $\mathcal{Z}_\Gamma$-weak contraction, Corollary 2.13 is a generalization of the Corollary 2.14. The following example shows that Corollary 2.13 is a genuine generalization of the Corollary 2.14.*

Example 2.18. *In view of the Example 2.9, the mapping T is not a $\mathcal{Z}_\Gamma$-weak contraction. So T is not satisfied in the Corollary 2.14. But T is a Suzuki $\mathcal{Z}_\Gamma$-weak contraction and we can easily see that T is satisfied in all conditions of the Corollary 2.13, and $(1,1)$ is the unique fixed point of T.*

Following the proof of Theorem 2.11, if we replace "$\mathcal{Z}_\Gamma$-contraction" by "$\mathcal{Z}_{\Gamma_0}$-contraction", and metric space by Branciari distance space respectively, we can obtain the following result:

Theorem 2.19. *Every $\mathcal{Z}_{\Gamma_0}$-contraction on a complete Branciari distance space has a unique fixed point.*

Proof. Let (X,d) be a complete Branciari distance space, and $T : X \to X$ be a $\mathcal{Z}_{\Gamma_0}$-contraction. Then, there exists $\zeta \in \mathcal{Z}_{\Gamma_0}$ such that for all $x, y \in X$,

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0. \tag{25}$$

Since $\zeta \in \mathcal{Z}_{\Gamma_0}$, there exists $\gamma \in \Gamma_0$ such that (ζ_1), (ζ_2) and (ζ_2') of Definition 2.1 are satisfied.

Let $x_0 \in X$ be an arbitrary element. Define a sequence $\{x_n\}_{n\geq 0}$ by

$$x_{n+1} = Tx_n,$$

for each $n \geq 0$.

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}_0$ then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$, for each $n \in \mathbb{N}_0$.

From (25) we have:

$$\zeta(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})) \geq 0.$$

So, it follows from $(\zeta 1)$ that

$$\begin{aligned}\gamma(d(x_{n+1}, x_{n+2})) &= \gamma(d(Tx_n, Tx_{n+1})) \\ &< \gamma(d(x_n, x_{n+1})),\end{aligned}$$

and since ψ is a nondecreasing function, one conclude that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}),$$

for each $n \in \mathbb{N}$.
Thus, the sequence $\{d(x_n, x_{n+1})\}_{n\geq 1}$ is non increasing and bounded below, so it converges to a real number $\ell \geq 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \ell.$$

For that sake of convenience, suppose that $a_n = d(x_n, x_{n+1})$, for each $n \geq 0$. Then $a_{n+1} \leq a_n$, for each $n \geq 0$, and we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \ell.$$

we divide the rest of proof into five steps.

Step (1): We prove that $\ell = 0$.

Assume that $\ell \neq 0$. Then, (ζ_2') implies that

$$\limsup_{n\to\infty} \eta(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0.$$

But, (25) implies that for each $n \geq 0$

$$\begin{aligned} \zeta(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) &= \zeta(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})) \\ &\geq 0, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \zeta(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) \geq 0,$$

and this is a contradiction. So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 0. \tag{26}$$

Now, we claim that

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0.$$

Indeed, using (25) and $(\zeta 1)$, we obtain

$$\begin{array}{rcl} \gamma(d(x_{n-1}, x_{n+1})) - \gamma(d(x_n, x_{n+2})) & > & \zeta(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) \\ & = & \zeta(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})) \\ & \geq & 0, \end{array}$$

and since γ is a nondecreasing function, one conclude that

$$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1}),$$

for each $n \in \mathbb{N}$.
Thus, the sequence $\{d(x_{n-1}, x_{n+1})\}_{n\geq 1}$ is non increasing and bounded below. In the same manner to that which proved (26), one can show that

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0. \tag{27}$$

Step (2): We show that $\{x_n\}$ is a bounded sequence.

On contrary, assume that $\{x_n\}$ is not bounded. Then there exists a subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that for each $k \in \mathbb{N}$, n_k is the smallest index for which $d(x_{n_{k+1}}, x_{n_k}) \geq 1$ and $d(x_m, x_{n_k}) \leq 1$ for each $n_k \leq m \leq n_{(k+1)} - 1$.
Then, we have

$$\begin{aligned} 1 &\leq d(x_{n_{k+1}}, x_{n_k}) \\ &\leq d(x_{n_{(k+1)}}, x_{n_{(k+1)}-2}) + d(x_{n_{(k+1)}-2}, x_{n_{(k+1)-1}}) + d(x_{n_{(k+1)}-1}, x_{n_k}) \\ &\leq d(x_{n_{(k+1)}}, x_{n_k-2}) + d(x_{n_{(k+1)}-2}, x_{n_{(k+1)-1}}) + 1. \end{aligned}$$

Letting $n \to \infty$ and using (26) and (27), we have

$$\lim_{n\to\infty} d(x_{n_{(k+1)}}, x_{n_k}) = 0. \tag{28}$$

Now, using (26), (28) and $R3$, one can conclude that

$$\lim_{n\to\infty} d(x_{n_{(k+1)}-1}, x_{n_k-1}) = 0. \tag{29}$$

On So, using (25), ($\zeta 1$) and the fact that γ is a nondecreasing function, one conclude that

$$d(x_{n_{(k+1)}}, x_{n_k}) < d(x_{n_{(k+1)-1}}, x_{n_k-1}),$$

for each $n \in \mathbb{N}$.
Consequently, (ζ_2') implies that

$$\limsup_{n\to\infty} \eta(d(x_{n_{(k+1)}}, x_{n_k}), d(x_{n_{(k+1)}-1}, x_{n_k-1})) < 0.$$

But, it follows from (25) that, for each $k \geq 1$

$$\begin{aligned} \zeta(d(x_{n_{(k+1)}}, x_{n_k}), d(x_{n_{(k+1)-1}}, x_{n_k-1})) &= \zeta(d(Tx_{n_{(k+1)-1}}, Tx_{n_k-1}), d(x_{n_{(k+1)-1}}, x_{n_k-1})) \\ &\geq 0, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \zeta(d(x_{n_{(k+1)}}, x_{n_k}), d(x_{n_{(k+1)-1}}, x_{n_k-1})) \geq 0,$$

and this is a contradiction. So, $\{x_n\}$ is a bounded sequence.

Step (3): We going to prove that $\{x_n\}$ is a Cauchy sequence.

For this purpose, let

$$S_n = \sup d(x_i, x_j) : i.j \geq n.$$

Since $\{x_n\}$ is a bounded sequence, $S_n < \infty$ for all $n \in \mathbb{N}$. Furthermore, it is clear that the sequence $\{S_n\}$ is nondecreasing and bounded below. So, it converges to a real number $S \geq 0$. Assume that $S > 0$. It follows from the definition of S_n that for any $k \in \mathbb{N}$ there exists n_k and m_k such that $m_k > n_k \geq k$ and

$$S_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq S_k.$$

Thus,

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = S. \tag{30}$$

On the other hand, using (25), ($\zeta 1$) and the fact that γ is a nondecreasing function, one conclude that

$$d(x_{m_k}, x_{n_k}) < d(x_{m_k-1}, x_{n_k-1}),$$

for each $k \in \mathbb{N}$.
Hence, one has

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &< d(x_{m_k-1}, x_{n_k-1}) \\ &\leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \end{aligned}$$

Letting $n \to \infty$ and using (26) and (30), we get

$$\lim_{n\to\infty} d(x_{m_k-1}, x_{n_k-1}) = S. \tag{31}$$

Finally, (ζ_2') implies that

$$\limsup_{n\to\infty} \eta(d(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) < 0,$$

which contradicts (25). Thus, $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ such that $x_n \to u$, as $n \to \infty$.

Step (4): We prove that u is a fixed point of T.
Without losing of generality, one can suppose that $d(x_n, u) \neq 0$ for each $n \geq 0$. Using (25), ($\zeta 1$) and the fact that γ is a nondecreasing function, one conclude that

$$d(x_{n+1}, Tu) < d(x_n, u),$$

for each $n \in \mathbb{N}$.

So, for each $n \in \mathbb{N}$, one has

$$\begin{aligned} 0 &\leq d(u, Tu) \\ &\leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_n, u). \end{aligned}$$

Letting $n \to \infty$ and using (26), we get

$$d(u, Tu) = 0,$$

which implies that $Tu = u$.

Step (5): The fixed point of T is unique.
Suppose that u and v are two fixed points of T. We have $d(u, v) = d(Tu, Tv)$. If $d(u, v) \neq 0$ then $d(Tu, Tv) \neq 0$. So, (ζ_2') implies that

$$\limsup_{n\to\infty} \eta(d(u, v), d(u, v) = \limsup_{n\to\infty} \eta(d(u, v), d(Tu, Tv) < 0.$$

So, (ζ_1) implies that

$$\gamma(d(u, v)) < \gamma(d(u, v)),$$

and this is a contradiction. So, $d(u, v) \neq 0$.

This completes the proof of theorem. □

3. Results In Θ−Contractions and $\mathscr{L}$−Contractions In Metric Spaces

In the first part of this section we indicate that each Θ−contraction is really a $\mathcal{Z}_\Gamma$-contraction. We also show that the Theorem 1.6 and Theorem 1.7 are consequences of Corollaries 2.16 and 2.15 respectively.

Corollary 3.1. *(Theorem 1.6) Every Θ−contraction on a complete metric space has a unique fixed point.*

Proof. Let $T : X \to X$ be a Θ−contraction on a metric space (X, d). Then there exists $\theta \in \Omega$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$\theta(d(Tx, Ty)) \le [\theta(d(x, y))]^k. \tag{32}$$

Let us define the function $\zeta_\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ as follows;

$$\zeta_\theta(t,s) = \begin{cases} 0 & t = 0 \text{ or } s = 0, \\ kln(\theta(s)) - ln(\theta(t)) & \textit{otherwise.} \end{cases}$$

We prove that

(a_1) The function ζ_θ is a Γ-simulation function with respect to the following function

$$\gamma(t) = \begin{cases} 0 & t = 0 \\ ln(\theta(t)) & t > 0. \end{cases}$$

(a_2) T is a $\mathcal{Z}_\Gamma$-contraction with respect to the function ζ_θ.

It is clear that $\gamma \in \Gamma([0, +\infty))$. Furthermore, since $k < 1$, for each $s, t > 0$ we have

$$\begin{aligned} \zeta_\theta(t,s) &= kln(\theta(s)) - ln(\theta(t)) \\ &< ln(\theta(s)) - ln(\theta(t)) \\ &= \gamma(s) - \gamma(t), \end{aligned}$$

which proves (ζ_1) in Definition 2.3.

Now, let $\{t_n\} and \{s_n\}$ be sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell > 0$, then since $\theta \in \Omega$, by (θ_2) we have

$$\lim_{n\to\infty} \theta(t_n) = \lim_{n\to\infty} \theta(s_n) = \theta(\ell) > 1.$$

Therefore

$$\begin{aligned} \limsup_{n\to\infty} \zeta_\theta(t_n, s_n) &= \limsup_{n\to\infty} \Big(kln(\theta(s_n)) - ln(\theta(t_n))\Big) \\ &= \limsup_{n\to\infty} ln\frac{(\theta(s_n))^k}{\theta(t_n)} \\ &= ln\frac{(\theta(\ell))^k}{\theta(\ell)} \\ &< 0. \end{aligned}$$

This proves (ζ_2) in Definition 2.3. So, (a_2) is proved.

Finally, using (32), it follows that, for all $x, y \in X$ with $T(x) \neq T(y)$,

$$\begin{aligned} \zeta_\theta(d(Tx, Ty), d(x, y)) &= kln(\theta(d(x, y))) - ln(\theta(d(Tx, Ty))) \\ &= ln\frac{(\theta(d(x,y)))^k}{\theta(d(Tx,Ty))} \\ &\ge ln1 \\ &= 0, \end{aligned}$$

which means that T is a $\mathcal{Z}_\Gamma$-contraction, and then applying Corollary 2.16, we obtain desired result. □

Corollary 3.2. *(Theorem 1.7) Let (X, d) be a metric space and $T : X \to X$ be A self- mapping. If there exists $\theta \in \Omega$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$, $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies that*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

then T has a unique fixed point.

Proof. In the same manner as proof of Corollaries 3.1 one can see that T is a Suzuki $\mathcal{Z}_\Gamma$-contraction. So, by Corollary 2.15, T has a unique fixed point. □

We emphasize and underline that θ has not been assumed continuous in the [37, Theorem 4]. Under this observation, when we seek the proof of this theorem, we see that it is doubtful. Indeed, in the proof of [37, Theorem 4], the authors showed that

$$\lim_{k\to\infty} d(x_{n(k+1)}, x_{n(k)}) = 1 \quad \textit{and} \quad \lim_{k\to\infty} d(x_{n(k+1)-1}, x_{n(k)-1}) = 1,$$

and then they concluded that

$$\lim_{k\to\infty} \theta(d(x_{n(k+1)}, x_{n(k)})) > 1 \quad \textit{and} \quad \lim_{k\to\infty} \theta(d(x_{n(k+1)-1}, x_{n(k)-1})) > 1,$$

and after that they named

$$t_k = \theta(d(x_{n(k+1)}, x_{n(k)})) \quad \textit{and} \quad s_k = \theta(d(x_{n(k+1)-1}, x_{n(k)-1})),$$

and yielded immediately

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1!$$

It seems that the authors presumed the continuity of θ, although it is not assumed in their research. According to this fact, the answer of a question "why $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n$?" is unclear.
Notice that the authors used this unclear logic in the relation (52) of their proof too.

This is one of the main motivation of us to write a new proof for this theorem. Broadly translated our findings indicate that $\mathscr{L}-$contractions are special cases of $\mathcal{Z}_\Gamma$-contractions which we defined in this paper.

Theorem 3.3. *Every $\mathscr{L}-$contraction with respect to $\vartheta : [1, \infty) \times [1, \infty) \to [0, \infty)$ and $\theta \in \Omega$, on a complete metric space, is a $\mathcal{Z}_{\Gamma_0}$-contraction.*

Proof. Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathscr{L}-$contraction with respect to $\vartheta : [1, \infty) \times [1, \infty) \to [0, \infty)$ and $\theta \in \Omega$. For each $x, y \in X$, we have

$$\vartheta(\theta(d(Tx, Ty)), \theta(d(x, y)) \geq 1. \tag{33}$$

Now, we define a function $\zeta_\theta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta_\theta(t, s) = \begin{cases} 0 & t = 0 \text{ or } s = 0 \\ ln\Big(\vartheta(\theta(t), \theta(s))\Big) & \text{otherwise,} \end{cases}$$

and we prove that

(a_1) The function ζ_θ is a Γ_0-simulation function with respect to the following function

$$\gamma(t) = \begin{cases} 0 & t = 0 \\ ln(\theta(t)) & t > 0. \end{cases}$$

(a_2) T is a $\mathcal{Z}_{\Gamma_0}$-contraction with respect to the ζ_θ.

It is clear that $\gamma \in \Gamma([0, +\infty))$. Furthermore, for each $s, t > 0$, using (ϑ_2), we have

$$\begin{aligned} \zeta_\theta(t, s) &= ln\Big(\vartheta(\theta(t), \theta(s))\Big) \\ &< ln(\tfrac{\theta(s)}{\theta(t)}) \\ &= \gamma(s) - \gamma(t). \end{aligned}$$

Consequently (ζ_1) is satisfied.

Now, let $\{t_n\} and \{s_n\}$ be two sequences in $(0, \infty)$ such that for all $n \in \mathbb{N}$, $t_n \leq s_n$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then since $\theta \in \Omega$, by (θ_2) one has $\theta(t_n) \leq \theta(s_n)$ and

$$\lim_{n\to\infty} \theta(t_n) = \lim_{n\to\infty} \theta(s_n) = \theta(\ell) > 1.$$

Consequently (ϑ_3) implies that

$$\begin{aligned} \limsup_{n\to\infty} \zeta_\theta(t_n, s_n) &= \limsup_{n\to\infty} ln\Big(\vartheta(\theta(t_n), \theta(s_n))\Big) \\ &= \ln \limsup_{n\to\infty} \Big(\vartheta(\theta(t_n), \theta(s_n))\Big) \\ &< \ln 1 \\ &= 0. \end{aligned}$$

This proves (ζ_2') in Definition 2.10. So, (a_2) is proved which means that T is a $\mathcal{Z}_{\Gamma_0}$-contraction.

Finally, using (33), it follows that, for all $x, y \in X$ with $T(x) \neq T(y)$, we have $\theta(d(Tx, Ty)) > 1$ and

$$\begin{aligned} \zeta_\theta(d(Tx, Ty), d(x, y)) &= \ln\Big(\vartheta(\theta(d(Tx, Ty)), \theta(d(x, y)))\Big) \\ &\geq \ln 1 \\ &= 0, \end{aligned}$$

which means that T is a $\mathcal{Z}_{\Gamma_0}$-contraction, and then applying Theorem 2.19 , we obtain desired result. □

Corollary 3.4. *Every $\mathscr{L}$−contraction with respect to $\vartheta : [1, \infty) \times [1, \infty) \to [0, \infty)$ and $\theta \in \Omega$ on a complete metric space has a unique fixed point.*

Proof. By Theorem 3.3, T is a $\mathcal{Z}_{\Gamma_0}$-contraction, and then applying Theorem 2.19 , we obtain desired result. □

4. Conclusion and Future Directions

The purpose of the current study was to determine the Γ−simulation functions as a real generalization of Ψ−simuation mappings by which several known contractions. Also, we characterized the $\mathscr{L}$−contraction as a special case of Γ−contractions induced by Γ−simulation functions. Ultimately, we demonstrate that there is a gap in the proof of [37, Theorem]. In other words, the author have applied the continuity of θ in their results without assuming this fact and we change the assumption and present a new proof.

This research has thrown up many questions in need of further investigation. Taking into account that the Γ−sumulation mappings are the greater collection of classical ones and are more applicable, one can generalize the obtained results in metric-type spaces like b−metric space, ordered metric spaces and etc. Moreover, further research regarding the other single-valued and multi-valued contractions would be interesting, however working on multi-valued version of the current results seems to be more sophisticated.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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