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Research article

# Study of impulsive problems under Mittag-Leffler power law

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ABSTRACT

This article is fundamentally concerned with deriving the solution formula, existence, and uniqueness of solutions of two types of Cauchy problems for impulsive fractional differential equations involving Atangana-Baleanu-Caputo (ABC) fractional derivative which possesses nonsingular Mittag-Leffler kernel. Our investigation is based on nonlinear functional analysis and some fixed point techniques. Besides, some examples are given delineated to illustrate the effectiveness of our outcome.

## 1. Introduction

Fractional calculus has risen as a significant area of examination and the fertile area for research in light of the application of its tools in science and engineering majors. Fractional-order operators lead us to the emergence of more useful and concrete mathematical models as opposed to ordinary integer-order. It has been because of the non-local nature of the fractional operators, which describes the memory and it enables us to earn a closer look at the dynamics behavior and hereditary properties of the related phenomena. For the recent development of the theme, see the monographs [1, 2, 3] and the references referred to in that.

Different kinds of nonlocal fractional derivatives were suggested in the current literature to handle the reduction of classical derivatives operators. For example, the idea of fractional derivative in a Riemann-Liouville sense was introduced based on power-law. A new fractional derivative has proposed by Caputo-Fabrizio [4] relying on the exponential kernel. However, this operator it some troubles with regard to the locality of the kernel. Just recently, to get rid of Caputo-Fabrizio's trouble, Atangana and Baleanu (AB) in [5] have introduced a new modified version of a fractional derivative with the help of Mittag-Leffler function (MLF) as a nonsingular and nonlocal kernel. Due to generalized MLF is utilized as the nonlocal kernel and does not guarantee singularity, the ABC derivative provides an excellent memory description [6, 7, 8, 9].

The presented operator in [5] contains an accurate kernel that better describes the dynamics of systems with a memory effect. So this operator is advantageous to debate real-world problems and it also will have a big feature when utilizing the Laplace transform to solve some physical problems with the initial condition.

Lately, the authors [10, 11, 12, 13, 14] studied analytically and numerically for some fractional model by means of Atangana-Baleanu fractional derivative with a non-local smooth kernel.

Fractional differential equations (FDEs) with impulse effects acquired very great importance due to their applicability in modeling physical problems that experimenting with instantaneous changes. For modern papers on impulsive FDEs, we indicate to works [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and the references cited therein. Recently, few authors used this operator to study the qualitative properties of FDEs, for detail see [26, 27, 28, 29, 30, 31, 32, 33].

In the presented paper, we establish the existence and uniqueness results for solutions of the following impulsive FDEs with the initial and nonlocal conditions

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha}_{[\tau]}\varsigma(\tau) = f(\tau,\varsigma(\tau)), & \tau\in\Omega=[0,T], \quad \tau\neq\tau_{\ell}, \quad \ell=1,\ldots m, \\ \Delta\varsigma\,|_{\tau=\tau_{\ell}} = \mathcal{I}_{\ell}\varsigma(\tau_{\ell}^{-}), & \ell=1,\ldots m, \\ \varsigma(0)=\varsigma_{0}, \end{cases} \tag{1}$$

and

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$$\begin{cases} {}^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau) = f(\tau, \varsigma(\tau)), \quad \tau \in \Omega = [0, T], \quad \tau \neq \tau_{\ell}, \ \ell = 1, ...m, \\ \Delta\varsigma\mid_{\tau=\tau_{\ell}} = \mathcal{I}_{\ell}\varsigma(\tau_{\ell}^{-}), \quad \ell = 1, ...m, \\ \varsigma(0) + g(\varsigma) = \varsigma_0, \end{cases} \tag{2}$$

where $0 <\propto\leq 1$, ${}^{ABC}\mathbb{D}^{\propto}_{[\tau]}$ denotes the Atangana-Baleanu-Caputo (ABC) fractional derivative of order $\propto$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Further $f(0, \varsigma(0)) = 0$ and also it vanishes at impulsive points $\tau_{\ell}$, $\ell = 1, ...m$, $\mathcal{I}_{\ell} : \mathbb{R} \to \mathbb{R}$, $\ell = 1, ...m$, $\varsigma_0 \in \mathbb{R}$, $\tau_{\ell}$ satisfy $0 = \tau_0 < \tau_1 < ... < \tau_m < \tau_{m+1} = \tau$, $\Delta\varsigma\mid_{\tau=\tau_{\ell}} = \varsigma(\tau_{\ell}^{+}) - \varsigma(\tau_{\ell}^{-}) = \varsigma(\tau_{\ell}^{+}) - \varsigma(\tau_{\ell})$, $\varsigma(\tau_{\ell}^{+}) = \lim_{h\to 0^{+}} \varsigma(\tau_{\ell} + h)$, $\varsigma(\tau^{-}) = \lim_{h\to 0^{-}} \varsigma(\tau_{\ell} + h)$ represent the right and left limits of $\varsigma(\tau)$ at $\tau = \tau_{\ell}$, $\ell = 1, .., m$, and $g : \mathcal{PC}(\Omega, \mathbb{R}) \to \mathbb{R}$ is given function. Also, $[\tau] = \tau_{\ell}$ if $\tau \in (\tau_{\ell}, \tau_{\ell+1}]$, $\ell = 0, 1, ...$ and $\tau_0 = 0$.

We mention here that on the light of Theorem 3.11 in [34], we must always have the necessary condition $f(0, \varsigma(0) = 0$ to confirm the initial data for the solution. To our knowledge, there are no much studies on Cauchy problems for impulsive FDEs in the literature, especially those involving an ABC fractional operator. For instance, we mention [35, 36, 37] and the references therein.

The major aim of the article is to obtain the formula of solutions for two types of impulsive FDEs with ABC fractional operators. Moreover, we proved existence and uniqueness theorems by means of some fixed point theorems of Banach, Schaefer, and Kransosekliskii for proposed problems in the frame of ABC derivatives. We realized that the condition $f(0, \varsigma(0) = f(\tau_{\ell}, \varsigma(\tau_{\ell}) = 0$ $(\ell = 1, ...m)$ is necessary to guarantee a unique solution.

This paper is coordinated as follows. Section 1 deals with the introduction which contains a survey of the literature. Section 2 consists of some foundation preliminaries related the fractional calculus and nonlinear analysis. The formula of the solution for the proposed problems is presented in Section 3. The existence and uniqueness results on a Cauchy problem and nonlocal Cauchy problem are obtained in Sections 4 and 5. In Section 6, two examples are specified to the validation of our results.

## 2. Preliminaries

Consider the following space

$$\mathcal{PC}(\Omega, \mathbb{R}) = \Big\{\varsigma : \Omega \to \mathbb{R} : \varsigma \in C(\tau_{\ell}, \tau_{\ell+1}], \mathbb{R};\ \ell = 0, 1, ..., m+1$$
$$\text{and there exist } \varsigma(\tau_{\ell}^{+}) \text{ and } \varsigma(\tau_{\ell}^{-}),\ \ell = 1, ..., m,$$
$$\text{with } \varsigma(\tau_{\ell}^{-}) = \varsigma(\tau_{\ell}^{+})\Big\}$$

The space $\mathcal{PC}(\Omega, \mathbb{R})$ is a Banach space with the norm $\|\varsigma\|_{\mathcal{PC}} = \max_{\tau\in\Omega} |\varsigma(\tau)|$. Set $\Omega = [0, T]$ and $\Omega' := \Omega\backslash\{\tau_1, ..., \tau_m\}$.

**Definition 1.** [5, 38] Let $\propto\in [0, 1]$ and $\sigma \in H^1(a, b)$ $(a < b)$. Then the left AB-Caputo and AB-Riemann-Liouville fractional derivatives of order $\propto$ for a function $\sigma$ are described by

$${}^{ABC}\mathbb{D}^{\propto}_{a}\sigma(\tau) = \frac{\mathcal{N}(\propto)}{1-\propto}\int_{a}^{\tau} \mathbb{E}_{\propto}\left(\frac{-\propto}{\propto-1}(\tau-\theta)^{\propto}\right)\sigma'(\theta)d\theta, \ \tau > a,$$

and

$${}^{ABR}\mathbb{D}^{\propto}_{a}\sigma(\tau) = \frac{\mathcal{N}(\propto)}{1-\propto}\frac{d}{d\tau}\int_{a}^{\tau} \mathbb{E}_{\propto}\left(\frac{-\propto}{\propto-1}(\tau-\theta)^{\propto}\right)\sigma(\theta)d\theta, \ \tau > a,$$

respectively, where $\mathcal{N}(\propto)$ is the normalization function satisfies the result $\mathcal{N}(0) = \mathcal{N}(1) = 1$, and $\mathbb{E}_{\propto}$ is called the MLF defined by

$$\mathbb{E}_{\propto}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(j\propto+1)}, \ Re(\propto) > 0, \ z \in \mathbb{C}. \tag{3}$$

**Definition 2.** [5, 38] Let $\propto\in (0, 1]$ and $\sigma \in L^1(a, b)$. Then the left AB fractional integral of order $\propto$ for a function $\sigma$ is specified by

$${}^{AB}\mathbb{I}^{\propto}_{a}\sigma(\tau) = \frac{1-\propto}{\mathcal{N}(\propto)}\sigma(\tau) + \frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{a}^{\tau}(\tau-\theta)^{\propto-1}\sigma(\theta)d\theta, \ \tau > a.$$

**Definition 3.** [5] The Laplace transform of ABC fractional derivative of $\sigma(\tau)$ is specified by

$$\mathcal{L}\left[{}^{ABC}\mathbb{D}^{\propto}_{a}\sigma(\tau)\right] = \frac{\mathcal{N}(\propto)}{s^{\propto}(1-\propto)+\propto}\left[s^{\propto}\mathcal{L}[\sigma(\tau)] - s^{\propto-1}\sigma(a)\right],$$

where $\mathcal{L}$ is the Laplace transform starting from $a$ defined by

$$\mathcal{L}\{f(t)\}(s) = \int_{a}^{\infty} e^{-s(t-a)}f(t)dt.$$

**Lemma 1.** *(See Proposition 3 in [39]) Let $\sigma \in H^1(a, b)$, $b > a$, such that the ABC fractional derivative exists. Then we have*

$${}^{ABC}\mathbb{D}^{\propto}_{a}\,{}^{AB}\mathbb{I}^{\propto}_{a}\sigma(\tau) = \sigma(\tau),$$

*and*

$${}^{AB}\mathbb{I}^{\propto}_{a}\,{}^{ABC}\mathbb{D}^{\propto}_{a}\sigma(\tau) = \sigma(\tau) - \sigma(a),$$

*for $0 < \propto \leq 1$. Moreover, ${}^{ABC}\mathbb{D}^{\propto}_{a}\sigma(\tau) = 0$ if $\sigma(\tau)$ is constant function.*

**Lemma 2.** *[34, 39] For $\propto\in (0, 1]$, the solution of the following problem*

$$\begin{aligned} {}^{ABC}\mathbb{D}^{\propto}_{a}\sigma(\tau) &= \omega(\tau), \\ \sigma(a) &= \sigma_0 \end{aligned} \tag{4}$$

*is given by*

$$\sigma(\tau) = \sigma_0 + \frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau) + \frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{a}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta.$$

**Definition 4.** [40] Let $\aleph$ be a Banach space. Then the operator $\Pi : \aleph \to \aleph$ is a contraction if

$$\left\|\Pi\emptyset_1 - \Pi\emptyset_2\right\| \leq \kappa\left\|\emptyset_1 - \emptyset_2\right\|, \text{ for all } \emptyset_1, \emptyset_2 \in \aleph, \ 0 < \kappa < 1.$$

**Theorem 1.** *[40] Let $\aleph$ be a Banach space and $\mathcal{M}$ be a non-empty closed subset of $\aleph$. If $\Pi : \mathcal{M} \longrightarrow \mathcal{M}$ is a contraction, then there exists a unique fixed point of $\Pi$.*

**Theorem 2.** *[40] Let $\aleph$ be a Banach space and let $\Pi : \aleph \longrightarrow \aleph$ be a continuous and compact mapping (completely continuous mapping). Moreover, suppose*

$$S = \{x \in \aleph : x = \lambda\Pi x, \text{ for some } \lambda \in (0, 1)\}$$

*be a bounded set. Then $\Pi$ has at least one fixed point in $\aleph$.*

**Theorem 3.** *[40] Let $\mathbb{H}$ be a non-empty, closed, convex subset of a Banach space $\aleph$ and let $\mathbb{O}_1$, $\mathbb{O}_2$ be two operators such that (i) $\mathbb{O}_1 u + \mathbb{O}_2 v \in \mathbb{H}$, $\forall u, v \in \mathbb{H}$; (ii) $\mathbb{O}_1$ is compact and continuous; (iii) $\mathbb{O}_2$ is a contraction mapping. Then there exists $w \in \mathbb{H}$ such that $\mathbb{O}_1 w + \mathbb{O}_2 w = w$.*

## 3. Solution representation

**Definition 5.** A function $\varsigma \in \mathcal{PC}(\Omega, \mathbb{R})$ is a solution of (1) if $\varsigma$ satisfies the equation ${}^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau) = f(\tau, \varsigma(\tau))$ on $\Omega'$ and conditions $\Delta\varsigma\mid_{\tau=\tau_{\ell}} = \mathcal{I}_{\ell}\varsigma(\tau^{-})$, $\ell = 1, ...m$ and $\varsigma(0) = \varsigma_0$.

The following lemma is a direct consequence of Theorem 3.1 in [34].

**Lemma 3.** *Let $\propto\in(0,1]$ and let $\omega:\Omega\to\mathbb{R}$ be continuous with $\omega(0)=0$ and also vanishes at impulsive points $\tau_\ell$, for $\ell=1,2,...,m$. A function $\varsigma\in\mathcal{PC}(\Omega,\mathbb{R})$ is a solution of the fractional integral equation*

$$\varsigma(\tau)=\begin{cases}\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^\tau(\tau-\theta)^{\propto-1}\omega(\theta)d\theta, \text{ if } \tau\in[0,\tau_1],\\ \varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\sum_{i=1}^{\ell+1}\omega(\tau_i)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\sum_{i=1}^{\ell}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-\theta)^{\propto-1}\omega(\theta)d\theta\\ \quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta\\ \quad+\sum_{i=1}^{\ell}I_i\varsigma(\tau_i^-) \text{ if } \tau\in(\tau_\ell,\tau_{\ell+1}],\ \ell=1,...m,\end{cases}\tag{5}$$

*if and only if $\varsigma$ is a solution of the impulsive ABC-fractional FDE*

$$^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau)=\omega(\tau),\quad \tau\in\Omega':=\Omega\backslash\{\tau_1,...,\tau_m\},\tag{6}$$

$$\Delta\varsigma\mid_{\tau=\tau_\ell}=\mathcal{I}_\ell\varsigma(\tau_\ell^-),\quad \ell=1,...m,\tag{7}$$

$$\varsigma(0)=\varsigma_0,\tag{8}$$

*where $[\tau]=\tau_\ell$ if $\tau\in(\tau_\ell,\tau_{\ell+1}]$, $\ell=0,1,...$ and $\tau_0=0$.*

**Proof.** The proof is derived by using Lemma 1 repeatedly. Assume $\varsigma$ satisfies (6)-(8). If $\tau\in[0,\tau_1]$, then

$$^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau)=\omega(\tau),\ [\tau]=0.$$

Using Lemma 1, we get

$$\varsigma(\tau)=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^\tau(\tau-\theta)^{\propto-1}\omega(\theta)d\theta.$$

This means that

$$\varsigma(\tau_1^-)=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau_1)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta.$$

After the impulse $(\varsigma(\tau_1^-)=\varsigma(\tau_1^+)-\mathcal{I}_1\varsigma(\tau_1^-))$, we get

$$\varsigma(\tau_1^+)=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau_1)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-).$$

If $\tau\in(\tau_1,\tau_2]$, then $\omega$ vanishes at $\tau_1$ imply

$$\begin{aligned}\varsigma(\tau)&=\varsigma(\tau_1^+)+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta\\&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau)\right]+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta\\&\quad+\mathcal{I}_1\varsigma(\tau_1^-)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta.\end{aligned}$$

This means that

$$\begin{aligned}\varsigma(\tau_2^-)&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)\right]\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-)\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\omega(\theta)d\theta.\end{aligned}$$

After the impulse $(\varsigma(\tau_2^-)=\varsigma(\tau_2^+)-\mathcal{I}_2\varsigma(\tau_2^-))$, we get

$$\begin{aligned}\varsigma(\tau_2^+)&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)\right]+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-)+\mathcal{I}_2\varsigma(\tau_2^-).\end{aligned}$$

If $\tau\in(\tau_2,\tau_3]$, then $\omega$ vanishes at $\tau_2$ imply

$$\begin{aligned}\varsigma(\tau)&=\varsigma(\tau_2^+)+\frac{1-\propto}{\mathcal{N}(\propto)}\omega(\tau)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_2}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta\\&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)+\omega(\tau)\right]\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-)+\mathcal{I}_2\varsigma(\tau_2^-)\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_2}^{\tau}(\tau-\theta)^{\propto-1}\omega(\theta)d\theta.\end{aligned}$$

This means that

$$\begin{aligned}\varsigma(\tau_3^-)&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)+\omega(\tau_3)\right]\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-)+\mathcal{I}_2\varsigma(\tau_2^-)\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_2}^{\tau_3}(\tau_3-\theta)^{\propto-1}\omega(\theta)d\theta.\end{aligned}$$

After the impulse $(\varsigma(\tau_3^-)=\varsigma(\tau_3^+)-\mathcal{I}_3\varsigma(\tau_3^-))$, we get

$$\begin{aligned}\varsigma(\tau_3^+)&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)+\omega(\tau_3)\right]\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\omega(\theta)d\theta+\mathcal{I}_1\varsigma(\tau_1^-)+\mathcal{I}_2\varsigma(\tau_2^-)+\mathcal{I}_3\varsigma(\tau_3^-)\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_2}^{\tau_3}(\tau_3-\theta)^{\propto-1}\omega(\theta)d\theta.\end{aligned}$$

Assume that

$$\begin{aligned}\varsigma(\tau_\ell^+)&=\varsigma_0+\frac{1-\propto}{\mathcal{N}(\propto)}\left[\omega(\tau_1)+\omega(\tau_2)+\omega(\tau_3)+...+\omega(\tau_\ell)\right]\\&\quad+\mathcal{I}_1\varsigma(\tau_1^-)+\mathcal{I}_2\varsigma(\tau_2^-)+\mathcal{I}_3\varsigma(\tau_3^-)+....+\mathcal{I}_\ell\varsigma(\tau_\ell^-)\\&\quad+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\left[\int_0^{\tau_1}(\tau_1-\theta)^{\propto-1}\omega(\theta)d\theta+...+\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}\omega(\theta)d\theta\right].\end{aligned}$$

Then, inductively, for $\tau\in(\tau_\ell,\tau_{\ell+1}]$, $[\tau]=\tau_\ell$ and hence the solution becomes

$$\varsigma(\tau)=\varsigma(\tau_\ell^+)+\frac{1-\alpha}{\mathcal{N}(\alpha)}\omega(\tau)+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}\omega(\theta)d\theta$$

$$=\varsigma_0+\frac{1-\alpha}{\mathcal{N}(\alpha)}\sum_{i=1}^{\ell+1}\omega(\tau_i)+\sum_{i=1}^{\ell}\mathcal{I}_i\varsigma(\tau_i^-)$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{\ell}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-\theta)^{\alpha-1}\omega(\theta)d\theta$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}\omega(\theta)d\theta.$$

Thus (5) is satisfied.

Conversely, assume that $\varsigma$ satisfies the equation (1). If $\tau\in[0,\tau_1]$ and $\omega(0)=0$, then $\varsigma(0)=\varsigma_0$. Using the concept that ${}^{ABC}\mathbb{D}_a^\alpha$ is the left inverse of ${}^{AB}\mathbb{I}_a^\alpha$ and using Lemma 1, we find that

$${}^{ABC}\mathbb{D}_0^\alpha\varsigma(\tau)=\omega(\tau),\ \tau\in[0,\tau_1].$$

If $\tau\in[\tau_\ell,\tau_{\ell+1})$, $\ell=1,...,m$ and using the fact that ${}^{ABC}\mathbb{D}_{[\tau]}^\alpha\sigma(\cdot)=0$, where $\sigma(\cdot)$ is constant function, we obtain

$${}^{ABC}\mathbb{D}_{[\tau]}^\alpha\varsigma(\tau)=\omega(\tau),\ \text{for each } \tau\in[\tau_\ell,\tau_{\ell+1}).$$

As well, we can simply infer that

$$\varsigma(\tau_\ell^+)-\varsigma(\tau_\ell^-)=\mathcal{I}_\ell\varsigma(\tau_\ell^-),\ \ell=1,...,m.\quad\square$$

## 4. Cauchy problem (1)

Now, we introduce some hypotheses needed in our results:

**(H$_1$)** There exists a constant $L_f>0$ such that

$$\left|f(\tau,\varphi)-f(\tau,\varphi^*)\right|\le L_f\left|\varphi-\varphi^*\right|,\ \text{for each } \tau\in\Omega,\ \varphi,\varphi^*\in\mathbb{R}.$$

**(H$_2$)** The functions $I_\ell:\mathbb{R}\to\mathbb{R}$ are continuous and there exists a constant $L_I>0$ such that

$$\left|I_\ell(\varphi)-I_\ell(\varphi^*)\right|\le L_I\left|\varphi-\varphi^*\right|,\ \ell=1,...,m,\ \text{for } \varphi,\varphi^*\in\mathbb{R}.$$

**(H$_3$)** There exist $\mu\in C(\Omega,\mathbb{R})$ such that

$$|f(\tau,\varphi)|\le\mu(\tau),\ \text{for each } (\tau,\varsigma)\in\Omega\times\mathbb{R}.$$

**(H$_4$)** There exists $M>0$ such that

$$\left|I_\ell(\varphi)\right|\le M,\ \ell=1,...,m,\ \varphi\in\mathbb{R}.$$

Now, we are able to present our main results.

**Theorem 4.** *Assume $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous. If $(H_1)$ and $(H_2)$ hold with*

$$\gamma L_f+mL_I<1,\tag{9}$$

*then the impulsive ABC-fractional FDE (1) has a unique solution on $\Omega$, where*

$$\gamma:=\frac{(1-\alpha)m}{\mathcal{N}(\alpha)}+\frac{\tau^\alpha(m+1)}{\mathcal{N}(\alpha)\Gamma(\alpha)}.\tag{10}$$

**Proof.** Thanks to Lemma 3, we define the mapping $\mathcal{A}:\mathcal{PC}(\Omega,\mathbb{R})\to\mathcal{PC}(\Omega,\mathbb{R})$ by

$$\mathcal{A}\varsigma(\tau)=\varsigma_0+\frac{1-\alpha}{\mathcal{N}(\alpha)}\sum_{0<\tau_\ell<\tau}f(\tau_\ell,\varsigma(\tau_\ell))+\sum_{0<\tau_\ell<\tau}\mathcal{I}_\ell\varsigma(\tau_\ell^-)$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\alpha-1}f(\theta,\varsigma(\theta))d\theta$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}f(\theta,\varsigma(\theta))d\theta.\tag{11}$$

Let $\mathcal{B}_r=\{\varsigma\in\mathcal{PC}(\Omega,\mathbb{R}):\|\varsigma\|_{\mathcal{PC}}\le r\}$ with $r\ge\frac{|\varsigma_0|+\gamma\Lambda_f}{1-(\gamma L_f+mL_I)}$. Setting $\max_{\theta\in\Omega}|f(\theta,0)|=\Lambda_f$. From (H$_1$) we observe that

$$|f(\theta,\varsigma(\theta))|\le|f(\theta,\varsigma(\theta))-f(\theta,0)|+|f(\theta,0)|\le L_f|\varsigma|+\Lambda_f\le L_fr+\Lambda_f.$$

Now, we are required to prove that $\mathcal{A}$ has a fixed point. First we show that $\mathcal{A}\mathcal{B}_r\subset\mathcal{B}_r$.

For $\varsigma\in\mathcal{B}_r$, we have

$$\|\mathcal{A}\varsigma\|=\max_{\tau\in\Omega}|\mathcal{A}\varsigma(\tau)|\le|\varsigma_0|+\frac{1-\alpha}{\mathcal{N}(\alpha)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma(\tau_\ell))\right|$$

$$+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma(\tau_\ell^-)\right|$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\alpha-1}|f(\theta,\varsigma(\theta))|\,d\theta$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\max_{\tau\in\Omega}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}|f(\theta,\varsigma(\theta))|\,d\theta$$

$$\le|\varsigma_0|+\frac{1-\alpha}{\mathcal{N}(\alpha)}m\left(L_fr+\Lambda_f\right)$$

$$+mL_Ir+\frac{\tau^\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}(m+1)\left(L_fr+\Lambda_f\right)$$

$$=|\varsigma_0|+\gamma\Lambda_f+\left(\gamma L_f+mL_I\right)r$$

$$\le r\left[1-\left(\gamma L_f+mL_I\right)\right]+\left(\gamma L_f+mL_I\right)r$$

$$=r.$$

Thus, $\mathcal{A}$ maps $\mathcal{B}_r$ into itself. Next, we show that $\mathcal{A}$ is contraction on $\mathcal{PC}(\Omega,\mathbb{R})$. Let $\varsigma,\varsigma^*\in\mathcal{PC}(\Omega,\mathbb{R})$ and $\tau\in\Omega$. Then we obtain

$$\|\mathcal{A}\varsigma-\mathcal{A}\varsigma^*\|=\max_{\tau\in\Omega}|\mathcal{A}\varsigma(\tau)-\mathcal{A}\varsigma^*(\tau)|$$

$$\le\frac{1-\alpha}{\mathcal{N}(\alpha)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma(\tau_\ell)))-f(\tau_\ell,\varsigma^*(\tau_\ell))\right|$$

$$+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma(\tau_\ell^-)-\mathcal{I}_\ell\varsigma^*(\tau_\ell^-)\right|+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}$$

$$\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\alpha-1}\left|f(\theta,\varsigma(\theta))-f(\theta,\varsigma^*(\theta))\right|d\theta$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\max_{\tau\in\Omega}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}\left|f(\theta,\varsigma(\theta))-f(\theta,\varsigma^*(\theta))\right|d\theta$$

$$\le\frac{(1-\alpha)}{\mathcal{N}(\alpha)}\max_{\tau\in\Omega}mL_f\left|\varsigma(\tau)-\varsigma^*(\tau)\right|+\max_{\tau\in\Omega}mL_I\left|\varsigma(\tau)-\varsigma^*(\tau)\right|$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)}\frac{1}{\Gamma(\alpha)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\alpha-1}$$

$$L_f\left|\varsigma(\theta)-\varsigma^*(\theta)\right|d\theta$$

$$+\frac{\alpha}{\mathcal{N}(\alpha)\Gamma(\alpha)}\max_{\tau\in\Omega}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\alpha-1}L_f\left|\varsigma(\theta)-\varsigma^*(\theta)\right|d\theta$$

$$\le\frac{(1-\alpha)m}{\mathcal{N}(\alpha)}L_f\|\varsigma-\varsigma^*\|+mL_I\|\varsigma-\varsigma^*\|$$

$$+\frac{\tau^{\propto}(m+1)}{\mathcal{N}(\propto)\Gamma(\propto)}L_f\left\|\varsigma-\varsigma^*\right\|$$
$$=\left(\gamma L_f+mL_I\right)\left\|\varsigma-\varsigma^*\right\|.$$

The inequality (9) shows that $\mathcal{A}$ is contraction on $\mathcal{PC}(\Omega,\mathbb{R})$. It follows from Theorem 1 that the impulsive ABC-fractional FDE (1) has a unique solution. □

Now, we prove the existence result of (1) by applying Theorem 2.

**Theorem 5.** *Suppose $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and let the assumptions $(H_3)$ and $(H_4)$ hold. Then there exist at least one solution of the impulsive ABC-fractional FDE (1) on $\Omega$.*

**Proof. Step 1:** We can show that $\mathcal{A}:\mathcal{PC}(\Omega,\mathbb{R})\to\mathcal{PC}(\Omega,\mathbb{R})$ is compact.

Since $f$ and $I_\ell$ $(\ell=1,..,m)$ are continuous, it can be to checked that $\mathcal{A}$ is continuous.

Next, let $\mathcal{B}_{r_1}=\{\varsigma\in\mathcal{PC}(\Omega,\mathbb{R}):\|\varsigma\|_{\mathcal{PC}}\le r_1\}$ be a ball set with

$$r_1:=\left|\varsigma_0\right|+\gamma\mu_0+mM+1,$$

where $\mu_0:=\sup_{\tau\in\Omega}|\mu(\tau)|$ and $\gamma$ is given by (10). Then for $\varsigma\in\mathcal{B}_{r_1}$ and $\tau\in\Omega$, we have

$$\begin{aligned}\|\mathcal{A}\varsigma\|=\max_{\tau\in\Omega}|\mathcal{A}\varsigma(\tau)|&\le\left|\varsigma_0\right|+\frac{1-\propto}{\mathcal{N}(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma(\tau_\ell)\right|\\&+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma(\tau_\ell^-)\right|\\&+\frac{\propto}{\mathcal{N}(\propto)\Gamma(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))\right|d\theta\\&+\frac{\propto}{\mathcal{N}(\propto)\Gamma(\propto)}\max_{\tau\in\Omega}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))\right|d\theta\\&\le\left|\varsigma_0\right|+\frac{1-\propto}{\mathcal{N}(\propto)}m\mu_0+mM+\frac{\tau^\propto}{\mathcal{N}(\propto)\Gamma(\propto)}(m+1)\mu_0\\&=\left|\varsigma_0\right|+\gamma\mu_0+mM\le r_1,\end{aligned}$$

which means that $\mathcal{A}(\mathcal{B}_{r_1})$ is uniformly bounded. Finally, we prove that $\mathcal{A}$ maps bounded sets into equicontinuous set of $\mathcal{PC}(\Omega,\mathbb{R})$. In view of $(\mathrm{H}_3)$ we fix $\sup_{(\tau,\varsigma)\in\Omega\times\mathcal{B}_{r_1}}|f(\tau,\varsigma)|=f_0$. Hence, for $\tau_1,\tau_2\in\Omega$ such that $\tau_1<\tau_2$, we have

$$\begin{aligned}&\left|\mathcal{A}\varsigma(\tau_2)-\mathcal{A}\varsigma(\tau_1)\right|\\&\le\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\left|\int_{\tau_\ell}^{\tau_2}(\tau_2-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta-\int_{\tau_\ell}^{\tau_1}(\tau_1-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta\right|\\&\le\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))\right|d\theta\\&+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_\ell}^{\tau_1}\left[(\tau_1-\theta)^{\propto-1}-(\tau_2-\theta)^{\propto-1}\right]\left|f(\theta,\varsigma(\theta))\right|d\theta\\&\le\frac{\propto}{\mathcal{N}(\propto)}\frac{f_0}{\Gamma(\propto)}\left[2(\tau_2-\tau_1)^\propto+(\tau_1-\tau_\ell)^\propto-(\tau_2-\tau_\ell)^\propto\right]\\&\le\frac{2f_0}{\mathcal{N}(\propto)\Gamma(\propto)}(\tau_2-\tau_1)^\propto.\end{aligned}$$

As $\tau_1\to\tau_2$, $\left\|\mathcal{A}\varsigma(\tau_2)-\mathcal{A}\varsigma(\tau_1)\right\|\to0$. This confirms that $\mathcal{A}(\mathcal{B}_{r_1})$ is relatively compact for $\tau\in\Omega$. By Arzela-Ascoli's theorem, the operator $\mathcal{A}$ is compact on $\mathcal{B}_{r_1}$.

**Step 2:** We prove that the set

$$\mathcal{S}=\{\varsigma\in\mathcal{PC}(\Omega,\mathbb{R}):\ \varsigma=\lambda\mathcal{A}\varsigma\text{ for some }\lambda\in(0,1)\}$$

is bounded. Let $\varsigma\in\mathcal{S}$. Then $\varsigma=\lambda\mathcal{A}\varsigma$ for some $\lambda\in(0,1)$. Hence, for $\tau\in\Omega$ we obtain

$$\begin{aligned}|\varsigma(\tau)|&<|\mathcal{A}\varsigma(\tau)|\\&\le\left|\varsigma_0\right|+\frac{1-\propto}{\mathcal{N}(\propto)}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma(\tau_\ell)\right|\\&+\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma(\tau_\ell^-)\right|\\&+\frac{\propto}{\mathcal{N}(\propto)\Gamma(\propto)}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))\right|d\theta\\&+\frac{\propto}{\mathcal{N}(\propto)\Gamma(\propto)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))\right|d\theta\\&\le\left|\varsigma_0\right|+\frac{1-\propto}{\mathcal{N}(\propto)}m\mu_0+mM+\frac{\tau^\propto}{\mathcal{N}(\propto)\Gamma(\propto)}(m+1)\mu_0\\&=\left|\varsigma_0\right|+\gamma\mu_0+mM.\end{aligned}$$

Thus for every $\tau\in\Omega$, we have

$$\|\varsigma\|_{\mathcal{PC}}\le\left|\varsigma_0\right|+\gamma\mu_0+mM:=R.$$

This confirms that $\mathcal{S}$ is bounded. So, the deduction of Theorem 2 implies that $\mathcal{A}$ has a fixed point which is a solution of the impulsive ABC-fractional FDE (1). □

## 5. Nonlocal Cauchy problem (2)

Here we treatise the existence of solution for the impulsive ABC-fractional FDE (2). Suppose $g$ satisfies the following condition:

$(\mathrm{H}_5)$ $g:\mathcal{PC}(\Omega,\mathbb{R})\to\mathbb{R}$ is continuous and there exists a constant $0<L_g<1$ such that

$$\left|g(\varphi)-g(\varphi^*)\right|\le L_g\left|\varphi-\varphi^*\right|,\ \text{for all }\varphi,\varphi^*\in\mathcal{PC}(\Omega,\mathbb{R}).$$

**Theorem 6.** *Suppose $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and let the hypotheses $(H_1),(H_2)$ and $(H_5)$ hold. If*

$$\left(L_g+\gamma L_f+mL_I\right)<1,\tag{12}$$

*where $\gamma$ is given by (10), then the impulsive ABC-fractional FDE (2) has a unique solution on $\Omega$.*

**Proof.** Define the nonlinear mapping $\mathcal{A}^*:\mathcal{PC}(\Omega,\mathbb{R})\to\mathcal{PC}(\Omega,\mathbb{R})$ as follows

$$\begin{aligned}\mathcal{A}^*\varsigma(\tau)=\varsigma_0-g(\varsigma)+\frac{1-\propto}{\mathcal{N}(\propto)}\sum_{0<\tau_\ell<\tau}f(\tau_\ell,\varsigma(\tau_\ell))\\+\sum_{0<\tau_\ell<\tau}\mathcal{I}_\ell\varsigma(\tau_\ell^-)+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta\\+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta.\end{aligned}\tag{13}$$

Then, $\mathcal{A}^*$ has a fixed point if and only if the impulsive ABC-fractional FDE (2) possesses a solution. Choosing $r^*\ge\frac{|\varsigma_0|+|g(0)|+\gamma\Lambda_f}{1-\left(L_g+\gamma L_f+mL_I\right)}$. From the hypothesis $(\mathrm{H}_5)$, it is easy to find that $\mathcal{A}^*\mathcal{B}_{r^*}\subset\mathcal{B}_{r^*}$, where $\mathcal{B}_{r^*}=\{\varsigma\in\mathcal{PC}(\Omega,\mathbb{R}):\|\varsigma\|_{\mathcal{PC}}\le r^*\}$. We need to prove that $\mathcal{A}^*$ is contraction. Let $\varsigma,\varsigma^*\in\mathcal{PC}(\Omega,\mathbb{R})$ and $\tau\in\Omega$. Then we have

$$\begin{aligned}\|\mathcal{A}^*\varsigma-\mathcal{A}^*\varsigma^*\| &= \max_{\tau\in\Omega}\left|\mathcal{A}^*\varsigma(\tau)-\mathcal{A}^*\varsigma^*(\tau)\right| \le \max_{\tau\in\Omega}\left|g(\varsigma(\tau))-g(\varsigma^*(\tau))\right| \\ &+\frac{1-\propto}{\mathcal{N}(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma(\tau_\ell)-f(\tau_\ell,\varsigma^*(\tau_\ell)\right| \\ &+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma(\tau_\ell^-)-\mathcal{I}_\ell\varsigma^*(\tau_\ell^-)\right| \\ &+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau} \\ &\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))-f(\theta,\varsigma^*(\theta))\right|d\theta \\ &+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\max_{\tau\in\Omega}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}\left|f(\theta,\varsigma(\theta))-f(\theta,\varsigma^*(\theta))\right|d\theta \\ &\le L_g\|\varsigma-\varsigma^*\|+\frac{(1-\propto)m}{\mathcal{N}(\propto)}L_f\|\varsigma-\varsigma^*\|+mL_I\|\varsigma-\varsigma^*\| \\ &+\frac{\tau^{\propto}(m+1)}{\mathcal{N}(\propto)\Gamma(\propto)}L_f\|\varsigma-\varsigma^*\| \\ &=\left(L_g+\gamma L_f+mL_I\right)\|\varsigma-\varsigma^*\|.\end{aligned}$$

The inequality (12) shows that $\mathcal{A}^*$ is contraction on $\mathcal{PC}(\Omega,\mathbb{R})$. Then the impulsive ABC-fractional FDE (2) has a unique solution by the application of Theorem 1. □

The following theorem based on Theorem 3.

**Theorem 7.** *Suppose $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and let the assumptions $(H_1)-(H_5)$ hold. If*

$$\left(L_g+mL_I+\frac{(1-\propto)m}{\mathcal{N}(\propto)}L_f\right)<1, \tag{14}$$

*then the nonlocal impulsive ABC-FDE (2) has a solution on $\Omega$.*

**Proof.** Consider the operator $\mathcal{A}^*:\mathcal{PC}(\Omega,\mathbb{R})\to\mathcal{PC}(\Omega,\mathbb{R})$ defined by (13), and we define the operators $\mathcal{A}_1^*$ and $\mathcal{A}_1^*$ on $\mathcal{B}_r$ as

$$\mathcal{A}_1^*\varsigma(\tau)=\varsigma_0-g(\varsigma)+\sum_{0<\tau_\ell<\tau}\mathcal{I}_\ell\varsigma(\tau_\ell^-)+\frac{1-\propto}{\mathcal{N}(\propto)}\sum_{0<\tau_\ell<\tau}f(\tau_\ell,\varsigma(\tau_\ell)),$$

$$\begin{aligned}\mathcal{A}_2^*\varsigma(\tau)&=\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta \\ &+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}f(\theta,\varsigma(\theta))d\theta,\end{aligned}$$

where $\mathcal{A}^*=\mathcal{A}_1^*+\mathcal{A}_2^*$. Let the ball $\mathcal{B}_{r_0}=\{\varsigma\in\mathcal{PC}(\Omega,\mathbb{R}):\|\varsigma\|_{\mathcal{PC}}\le r_0\}$, and we select a suitable constant $r_0$ as

$$r_0\ge\frac{|\varsigma_0|+|g(0)|+\gamma\mu_0+mM}{1-L_g}.$$

For $\varsigma_1,\varsigma_2\in\mathcal{B}_{r_0}$, we obtain

$$\begin{aligned}\left\|\mathcal{A}_1^*\varsigma_1+\mathcal{A}_2^*\varsigma_2\right\| &\le\left\|\mathcal{A}_1^*\varsigma_1\right\|+\left\|\mathcal{A}_2^*\varsigma_2\right\| \\ &=\max_{\tau\in\Omega}\left|\mathcal{A}_1^*\varsigma_1(\tau)\right|+\max_{\tau\in\Omega}\left|\mathcal{A}_2^*\varsigma_2(\tau)\right| \\ &\le|\varsigma_0|+|g(0)|+L_g\max_{\tau\in\Omega}|\varsigma_1(\tau)|+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma_1(\tau_\ell^-)\right| \\ &+\frac{1-\propto}{\mathcal{N}(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma_1(\tau_\ell)\right| \\ &+\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\max_{\tau\in\Omega}\left(\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}\left|f(\theta,\varsigma_2(\theta))\right|d\theta\right. \\ &\left.+\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}\left|f(\theta,\varsigma_2(\theta))\right|d\theta\right) \\ &\le|\varsigma_0|+|g(0)|+L_gr_0+\frac{(1-\propto)m}{\mathcal{N}(\propto)}\mu_0+mM+\frac{\tau^{\propto}(m+1)}{\mathcal{N}(\propto)\Gamma(\propto)}\mu_0 \\ &=|\varsigma_0|+|g(0)|+L_gr_0+\gamma\mu_0+mM \\ &\le r_0.\end{aligned}$$

Thus, $\mathcal{A}_1^*\varsigma_1+\mathcal{A}_2^*\varsigma_2\in\mathcal{B}_{r_0}$.

Next, for any $\tau\in\Omega$ and $\varsigma_1,\varsigma_2\in\mathcal{PC}(\Omega,\mathbb{R})$ we have

$$\begin{aligned}\left\|\mathcal{A}_1^*\varsigma_1-\mathcal{A}_1^*\varsigma_2\right\| &=\max_{\tau\in\Omega}\left|\mathcal{A}_1^*\varsigma_1(\tau)-\mathcal{A}_1^*\varsigma_2(\tau)\right| \\ &\le\max_{\tau\in\Omega}\left|g(\varsigma_1(\tau))-g(\varsigma_2(\tau))\right| \\ &+\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|\mathcal{I}_\ell\varsigma_1(\tau_\ell^-)-\mathcal{I}_\ell\varsigma_2(\tau_\ell^-)\right| \\ &+\frac{1-\propto}{\mathcal{N}(\propto)}\max_{\tau\in\Omega}\sum_{0<\tau_\ell<\tau}\left|f(\tau_\ell,\varsigma_1(\tau_\ell)-f(\tau_\ell,\varsigma_2(\tau_\ell)\right| \\ &\le L_g\|\varsigma_1-\varsigma_2\|+mL_I\|\varsigma_1-\varsigma_2\|+\frac{(1-\propto)m}{\mathcal{N}(\propto)}L_f\|\varsigma_1-\varsigma_2\| \\ &=\left(L_g+mL_I+\frac{(1-\propto)m}{\mathcal{N}(\propto)}L_f\right)\|\varsigma_1-\varsigma_2\|.\end{aligned}$$

It follows from (14) that $\mathcal{A}_1^*$ is a contraction mapping.

Now, let us show that $\mathcal{A}_2^*$ is continuous and compact.

Continuity of $f$ implies that $\mathcal{A}_2^*$ is continuous. Also, $\mathcal{A}_2^*$ is uniformly bounded on $\mathcal{B}_{r_0}$ because, for $\varsigma\in\mathcal{B}_{r_0}$ and $\tau\in\Omega$, we get

$$\begin{aligned}\left\|\mathcal{A}_2^*\varsigma\right\| &=\max_{\tau\in\Omega}\left|\mathcal{A}_2^*\varsigma(\tau)\right| \\ &\le\frac{\propto}{\mathcal{N}(\propto)}\frac{1}{\Gamma(\propto)}\max_{\tau\in\Omega}\left(\sum_{0<\tau_\ell<\tau}\int_{\tau_{\ell-1}}^{\tau_\ell}(\tau_\ell-\theta)^{\propto-1}|f(\theta,\varsigma(\theta))|d\theta\right. \\ &\left.+\int_{\tau_\ell}^{\tau}(\tau-\theta)^{\propto-1}|f(\theta,\varsigma(\theta))|d\theta\right) \\ &\le\frac{\tau^{\propto}(m+1)}{\mathcal{N}(\propto)\Gamma(\propto)}\mu_0.\end{aligned}$$

Now, the operator $\mathcal{A}_2^*$ is compactness on $\mathcal{B}_{r_0}$, since $\mathcal{A}_2^*\subset\mathcal{A}$. Thus all assumptions in Theorem (3) are satisfied. Therefore, the nonlocal impulsive ABC-FDE (2) has a solution on $\Omega$. □

## 6. Examples

**Example 1.** For $\propto\in(0,1]$, we consider the following impulsive FDE with ABC fractional derivative

$$\begin{cases}{}^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau)=\frac{\tau(\tau-\frac{1}{2})}{9}\frac{|\varsigma(\tau)|}{1+|\varsigma(\tau)|}, & \tau\in\Omega=[0,1],\quad \tau\neq\frac{1}{2}, \\ \Delta\varsigma\,|_{\tau=\frac{1}{2}}=\frac{\left|\varsigma(\frac{1}{2}^-)\right|}{10+\left|\varsigma(\frac{1}{2}^-)\right|}, \\ \varsigma(0)=\varsigma_0,\end{cases} \tag{15}$$

Set $f(\tau,\varsigma)=\frac{\tau(\tau-\frac{1}{2})}{9}\frac{\varsigma}{1+\varsigma}$, for $\tau\in\Omega$, $\varsigma\in\mathbb{R}^+$, and $I_\ell(\varsigma)=\frac{\varsigma}{10+\varsigma}$, for $\varsigma\in\mathbb{R}^+$.

Clearly, $f(0,\varsigma(0))=f(\frac{1}{2},\varsigma(\frac{1}{2}))=0$. Let $\tau\in\Omega$ and $\varsigma,\overline{\varsigma}\in\mathbb{R}^+$. Then

$$\begin{aligned}\left|f(\tau,\varsigma)-f(\tau,\overline{\varsigma})\right| &=\frac{\tau(\tau-\frac{1}{2})}{9}\left|\frac{\varsigma}{1+\varsigma}-\frac{\overline{\varsigma}}{1+\overline{\varsigma}}\right| \\ &\le\frac{1}{18}\frac{|\varsigma-\overline{\varsigma}|}{(1+\varsigma)(1+\overline{\varsigma})}\le\frac{1}{18}|\varsigma-\overline{\varsigma}|,\end{aligned}$$

and

$$\left|I_\ell(\varsigma)-I_\ell(\overline{\varsigma})\right|=\left|\frac{\varsigma}{10+\varsigma}-\frac{\overline{\varsigma}}{10+\overline{\varsigma}}\right|\le\frac{10|\varsigma-\overline{\varsigma}|}{(10+\varsigma)(10+\overline{\varsigma})}\le\frac{1}{10}|\varsigma-\overline{\varsigma}|.$$

Therefore, the hypotheses $(H_1)$ and $(H_2)$ hold with $L_f = \frac{1}{18}$ and $L_I = \frac{1}{10}$. We shall examine that the condition (9) holds too with $\tau = 1$, $m = 1$, $\propto = \frac{1}{2}$ and $\mathcal{N}(\propto) = 1$. Indeed, first we have $\gamma = \frac{\sqrt{\pi}+4}{2\sqrt{\pi}}$ and $\gamma L_f + mL_I \approx 0.2 < 1$. Thus by Theorem 4, the problem (15) has a unique solution on $[0,1]$.

On the other hand, it's obvious that, for each $\varsigma \in \mathbb{R}^+$ and $\tau \in [0,1]$, we have $|f(\tau,\varsigma)| \leq \frac{\tau}{9}(\tau - \frac{1}{2})$ and $|I_\ell(\varsigma)| \leq \frac{1}{10}$. Hence the condition $(H_3)$ is satisfied with $\mu(\tau) = \frac{\tau}{9}(\tau - \frac{1}{2}) \in C([0,1],\mathbb{R}^+)$ and $M = \frac{1}{10}$. Thus all assumptions of Theorem 5 are satisfied. Consequently, Theorem 5 implies that the problem (15) has at least one solution on $[0,1]$.

**Example 2.** For $\propto \in (0,1]$, we consider the following nonlocal impulsive FDE with ABC fractional derivative

$$\begin{cases} {}^{ABC}\mathbb{D}^{\propto}_{[\tau]}\varsigma(\tau) = \frac{\tau(\tau-\frac{1}{3})}{9}\frac{|\varsigma(\tau)|}{1+|\varsigma(\tau)|}, \quad \tau \in \Omega = [0,1], \quad \tau \neq \frac{1}{3}, \\ \Delta\varsigma(\frac{1}{3}) = \frac{\left|\varsigma(\frac{1}{3}^-)\right|}{7+\left|\varsigma(\frac{1}{3}^-)\right|}, \\ \varsigma(0) + \sum_{i=1}^n \lambda_i \varsigma(\zeta_i) = \varsigma_0, \ 0 < \zeta_1 < \zeta_2 < \ldots \zeta_n < 1, \end{cases} \tag{16}$$

where $\lambda_i > 0$ for $i = 1,\ldots,n$. Set $f(\tau,\varsigma) = \frac{\tau(\tau-\frac{1}{3})}{9}\frac{\varsigma}{1+\varsigma}$, for $\tau \in \Omega$, $\varsigma \in \mathbb{R}^+$, $I_\ell(\varsigma) = \frac{\varsigma}{7+\varsigma}$, for $\varsigma \in \mathbb{R}^+$ and $g(\varsigma) = \sum_{i=1}^n \lambda_i \varsigma(\zeta_i)$ for $\varsigma \in \mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i < \frac{2}{9}$. Clearly, $f(0,\varsigma(0)) = f(\frac{1}{3},\varsigma(\frac{1}{3})) = 0$. Let $\tau \in \Omega$ and $\varsigma, \overline{\varsigma} \in \mathbb{R}^+$. Then

$$\begin{aligned} |f(\tau,\varsigma) - f(\tau,\overline{\varsigma})| &= \frac{\tau(\tau-\frac{1}{3})}{9}\left|\frac{\varsigma}{1+\varsigma} - \frac{\overline{\varsigma}}{1+\overline{\varsigma}}\right| \\ &\leq \frac{2}{27}\frac{|\varsigma-\overline{\varsigma}|}{(1+\varsigma)(1+\overline{\varsigma})} \leq \frac{2}{27}|\varsigma-\overline{\varsigma}|, \\ |I_\ell(\varsigma) - I_\ell(\overline{\varsigma})| &= \left|\frac{\varsigma}{7+\varsigma} - \frac{\overline{\varsigma}}{7+\overline{\varsigma}}\right| \leq \frac{7|\varsigma-\overline{\varsigma}|}{(7+\varsigma)(7+\overline{\varsigma})} \leq \frac{1}{7}|\varsigma-\overline{\varsigma}|, \end{aligned}$$

and

$$|g(\varsigma) - g(\overline{\varsigma})| \leq \sum_{i=1}^n \lambda_i |\varsigma(\zeta_i) - \overline{\varsigma}(\zeta_i)| \leq \frac{2}{9}|\varsigma-\overline{\varsigma}|.$$

Therefore, the hypotheses $(H_1)$ $(H_2)$ and $(H_5)$ hold with $L_f = \frac{2}{27}$, $L_I = \frac{1}{7}$, and $L_g = \frac{2}{9}$, We shall examine that the condition (12) holds too with $\tau = 1$, $m = 1$, $\propto = \frac{1}{2}$ and $\mathcal{N}(\propto) = 1$. Indeed, first we have $\gamma = \frac{\sqrt{\pi}+4}{2\sqrt{\pi}}$ and

$$L_g + \gamma L_f + mL_I \approx 0.5 < 1.$$

Thus by Theorem 6 the problem (16) has a unique solution on $[0,1]$.

Next, for all $\tau \in \Omega$, and $\varsigma \in \mathbb{R}^+$,

$$L_g + mL_I + \frac{(1-\propto)m}{\mathcal{N}(\propto)}L_f \approx 0.4 < 1.$$

Thus, all the assumptions in Theorem 7 are satisfied, our result can be applied to the nonlocal impulsive FDE (16).

## 7. Conclusion

The theory of fractional calculus with nonsingular kernels is new and there is a need to study qualitative properties of differential equations involving such operators. In this paper, we have studied two types of Cauchy problems for impulsive FDEs rely on ABC fractional derivative which contains Mittag-Leffler Power Law. Also, we have derived the formula for the solution and examine the existence and uniqueness of the results of the problems (1) and (2). The acquired results are extended to the Caputo impulsive fractional differential equation. The arguments are upon some fixed point theorems of Banach, Schaefer, and Krasnoselskii. The obtained results play a significant role in developing the theory of fractional analytical. In our fractional impulsive system, we consider the case when the starting point of the Caputo fractional derivative changes with respect to the impulse points. In fact, if the starting point of the used Caputo fractional derivative is fixed then the solution representation of the impulsive problem will be more complicated. This case could be of interest in future works.

## Declarations

### *Author contribution statement*

M.S. Abdo, T. Abdeljawad, K. Shah, F. Jarad: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.


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