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Symmetry reduction, conservation laws and acoustic wave solutions for the extended Zakharov–Kuznetsov dynamical model arising in a dust plasma

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ABSTRACT

In this article, we consider the extended Zakharov–Kuznetsov (EZK) equation, which describes the nonlinear plasma dust acoustic waves (DAWs) in a magnetized dusty plasma. Dusty plasmas consist of three components: electrons, highly negatively charged dust grains, and two-temperature ions (low-temperature ions and high-temperature ions). We study the Lie symmetries, reductions, conservation laws and new exact solutions of EZK equations. Conservation laws for EZK equation is derived by applying the new conservation theorem of Ibragimov. Similarity solution for EZK equation will be obtained using Lie symmetry method. We find the Lie symmetries group of EZK equation, using similarity variables, get reduction equation, solving the reduction equations and then get the similarity solution. Solitary wave solutions of the EZK equation are derived from the reduction equation. Thus, some new exact explicit solutions of the EZK equation are obtained.

Introduction

There are two types of acoustic waves in dusty plasma essential; a high-frequency dust ion acoustic waves (DIAWs) that include static dust grains and mobile ions, and low-frequency dust acoustic waves (DAWs) involving mobile dust grains. These modes have been studied experimentally and theoretically (and references cited in [1–6]).

The Zakharov–Kuznetsov equation (ZK) is an isotropic nonlinear evolution equation first derived for weak ionic nonlinear sound waves in a plasma magnetized by a lossless power in two dimensions [7]. Generally, the ZK can be found many areas of mathematical physics, plasma physics, and engineering [8]. Especially, The ZK equation can be found in an electron–ion quantum magnetic plasma troubled by periodic external influences. ZK type equations can be used to describe nonlinear waves as solitons, which have been noted in astrophysical plasmas and in high-intensity laser irradiated plasma such as in Earths auroral zone and solar corona and oceans (see [9]).

There are several authors who have studied ZK-type equation associated shock and solitary waves in physics of plasma. Seadawy [10] studied the stability analysis of the ZK equation for weakly nonlinear ion acoustic waves in plasmas. Moslem et al. [11] have studied the nonlinear DIA shock waves in a magnetized dusty plasma. Das et al. [12] were studied effect of dust ion collision on dust ion acoustic waves in the framework of damped ZK equation in presence of external periodic force. Roychoudhury and Sahu [13] studied the ZK equation for ion acoustic waves with superthermal electrons in cylindrical geometry. Idir and Tribeche [14] have studied alternative DIAWs in a magnetized charge varying dusty plasma with nonthermal electrons having a vortex-like velocity distribution. In Ref. [15] they studied the solution of the SZKB equation in a dusty plasma with non-thermal electrons having a vortex-like velocity distribution. Researchers recently studied a nonlinear ZK-type equation in dusty plasma and quantum physics [16–26].

Partial differential equations (PDEs) are commonly used to model various phenomena in nonlinear sciences, ranging from physics to mechanics, biology, chemistry, meteorology, oceans, etc. Several powerful methods have been identified to construct the solitons, solitary wave and shock wave solutions, such as inverse scattering method, Bäcklund transform method, Painlevé analysis, Hirota's bilinear transform, the direct algebraic method, tanh and extended tanh method, auxiliary equation method, $exp(-\phi(\eta))$-expansion method, elliptic function method, rational expansion method, extended mapping method [27–37]. In recent times, several scientists have achieved great success

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and many attempts to construct various kinds of variational principles in different fields such as fluid dynamics, plasma physics, solid state physics, meteorology, optics, mathematical biology, etc. Lie symmetries analysis and conservation laws and can be used to study the properties of the existence of solutions and their uniqueness and stability [38–54]. The aim of this paper is to obtain a conservation laws for the EZK equation are obtained by using the Ibragimov theorem. In addition, symmetry reduction and exact solutions of the EZK equation are obtained.

The paper is organized as follows: The introduction is presented in first section. In second section, Basic equations and problem formulations are presented and the EZK equation is considered. The symmetry group of EZK equation is presented in third section. Symmetry reduction and closed-form solutions of EZK equation are obtained in fourth section. In fifth section, conservation Laws for EZK equation are obtained by using the Ibragimov theorem. Finally, the conclusions of this paper are presents in last section.

Problem formulations

Let us consider the dust acoustic wave (DAW) in a two ion temperature magnetized and non-collisional dusty plasma with single-sized dust grains. Assume that the waves propagate in the x-direction and the static external magnetic field is directed along the z-axis, i.e. $\mathbf{B} = B_0\mathbf{k}$, where $\mathbf{k}$ and B_0 were defined in [21]. Dusty plasmas consist of three components: highly negatively charged dust grains, two-temperature ions, and electrons. The dust particles are very larger and so massive than electrons or ions. At equilibrium, the charge neutrality condition requires that, $n_{ih0} + n_{il0} = n_{e0} + Z_{d0}n_{d0}$, where n_{ih0}, n_{il0}, n_{e0} and n_{d0} are high temperature ion, the number densities of unperturbed low temperature ion, dust grains, electron, respectively. Z_{d0} is the number of non-perturbed charges present on the dust grains measured in unit electron charge. The nonlinear three dimensional DAW can be described by the following system [21]:

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla.\left(n_d\vec{u}_d\right) = 0,\\
&\frac{\partial \vec{u}_d}{\partial t} + \left(\vec{u}_d.\nabla\right)\vec{u}_d = \frac{Z_d}{m_d}\left[\nabla\phi - \omega_{cd}\left(\vec{u}_d \times \mathbf{k}\right)\right],\\
&\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = Z_d n_d + n_e - n_{il} - n_{ih},
\end{aligned} \tag{1}$$

where $\vec{u}_d = u_d\underline{\mathbf{i}} + v_d\underline{\mathbf{j}} + w_d\underline{\mathbf{k}}, m_d$, and n_d point to the dimensionless velocity, mass of dust grain, and number density, respectively. ϕ is the dimensionless electrostatic potential. Z_d is the dust grain's charge. The distribution of ions and electrons is assumed with Maxwell Boltzmann distribution functions, so the relevant dimensionless number densities for electrons, high temperature ions and low temperature ions are: $n_e = \nu exp(s\beta_1\phi)$, $n_{ih} = \mu_h exp(-s\beta_2\phi)$, $n_{il} = \mu_1 exp(-s\phi)$, where $s = T_{eff}/T_{il}$, $\nu = n_{e0}/(Z_{d0}n_{d0})$, $\beta_1 = T_{il}/T_e$, $\beta_2 = T_{il}/T_{ih}$, $\mu_h = n_{ih0}/(Z_{d0}n_{d0})$, and $\mu_l = n_{il0}/(Z_{d0}n_{d0})$. The measured quantities are given as follows. T_{eff} is The inverse of effective temperature given by $T_{eff} = n_{d0}Z_{d0}\left(\frac{n_{e0}}{T_e} + \frac{n_{il0}}{T_{il}} + \frac{n_{ih0}}{T_{ih}}\right)^{-1}$, T_{il} and T_{ih} are the temperatures of lower- and higher-temperature ions, and T_e is the temperature of electrons. Z_d is normalized by Z_{d0}, the dust density is normalized by n_{d0}. The velocity $\vec{u}_d$, the space coordinates x and y, time t, m_d and the electrostatic potential ϕ are normalized by the effective Debye length $\lambda_{Dd} = (T_{eff}/4\pi e^2 n_{d0}Z_{d0})^{1/2}$, the effective dust plasma frequency's inverse $\omega_{pd}^{-1} = (m_{d0}/4\pi e^2 Z_{d0}^2 n_{d0})^{1/2}$, the effective dust acoustic speed $C_d = (Z_{d0}T_{eff}/m_{d0})^{1/2}$, T_{eff}/e, and m_{d0}. $\omega_{cd} = (\frac{Z_{d0}}{m_{d0}}eB_0)/\omega_{pd}$ is the dust cyclotron frequency normalized to ω_{pd}. Dong-Ning Gao et al. [21] derived the Extended Zakharov–Kuznetsov (EZK) equation as

$$u_t - u_x + Auu_x + Bu^2u_x + Cu_{xxx} + D\left(u_{xyy} + u_{xzz}\right) = 0, \tag{2}$$

where A, B, C and D be defined in [21].

Lie symmetry analysis for the EZK equation

Lie group analysis plays a basic role in building exact explicit solutions for N-LEEs and obtaining the invariant solutions. First, let us consider a one-parameter Lie group of point transformation:

$$\begin{aligned}
&\tilde{x} = x + \epsilon\ \xi(x, y, z, t, u) + O(\epsilon^2),\\
&\tilde{y} = y + \epsilon\ \zeta(x, y, z, t, u) + O(\epsilon^2),\\
&\tilde{z} = z + \epsilon\ \gamma(x, y, z, t, u) + O(\epsilon^2),\\
&\tilde{t} = t + \epsilon\ \tau(x, y, z, t, u) + O(\epsilon^2),\\
&\tilde{u} = u + \epsilon\ \eta(x, y, z, t, u) + O(\epsilon^2),
\end{aligned}$$

with a small parameter $\epsilon \ll 1$. The vector field related with the above group of transformation can be written as

$$\begin{aligned}
\mathbf{V} = {}& \xi(x, y, z, t, u)\frac{\partial}{\partial x} + \zeta(x, y, z, t, u)\frac{\partial}{\partial y} + \gamma(x, y, z, t, u)\frac{\partial}{\partial z}\\
&+ \tau(x, y, z, t, u)\frac{\partial}{\partial t} + \eta(x, y, z, t, u)\frac{\partial}{\partial u}.
\end{aligned} \tag{3}$$

This vector field equation (3) generates a symmetry of Eq. (2), and $\mathbf{V}$ must satisfy Lie symmetry conditions

$$pr^{(3)}\mathbf{V}(\Delta)|_\Delta = 0, \tag{4}$$

where $pr^{(3)}\mathbf{V}$ is the third prolongation of $\mathbf{V}$ and Δ is equation (2). Applying the third prolongation $pr^{(3)}\mathbf{V}$ to Eq. (2), the invariant conditions given by

$$\eta^t - \eta^x + Au\eta^x + Au_x\eta + Bu^2\eta^x + 2Buu_x\eta + C\eta^{xxx} + D\left(\eta^{xyy} + \eta^{xzz}\right) = 0, \tag{5}$$

where η^t, η^x, η^{xxx}, η^{xyy} and η^{xzz} being given by

$$\begin{aligned}
\eta^t &= D_t(\eta) - u_x D_t(\xi) - u_y D_t(\zeta) - u_z D_t(\gamma) - u_t D_t(\tau),\\
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_y D_x(\zeta) - u_z D_x(\gamma) - u_t D_x(\tau),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{xxy} D_x(\zeta) - u_{xxz} D_x(\gamma) - u_{xxt} D_x(\tau),\\
\eta^{xyy} &= D_y(\eta^{xy}) - u_{xyx} D_y(\xi) - u_{xyy} D_y(\zeta) - u_{xyz} D_y(\gamma) - u_{xyt} D_y(\tau),\\
\eta^{xzz} &= D_z(\eta^{xz}) - u_{xzx} D_z(\xi) - u_{xzy} D_z(\zeta) - u_{xzz} D_z(\gamma) - u_{xzt} D_z(\tau),
\end{aligned} \tag{6}$$

where D_t, D_x, D_y and D_z are total derivatives with respect to t, x, y and z, respectively. Substituting Eq. (6) into Eq. (5) and solving the system, then infinitesimal symmetries are given as follows

$$\begin{aligned}
&\xi = \frac{x}{3}\alpha_1 + \alpha_6, \qquad \zeta = \frac{y}{3}\alpha_1 + \alpha_3 z + \alpha_4, \qquad \gamma = \frac{z}{3}\alpha_1 - \alpha_3 y + \alpha_5,\\
&\tau = \alpha_1 t + \alpha_2, \qquad \eta = -\frac{2}{3}\frac{Bu^2 + Au - 1}{A + 2Bu}\alpha_1,
\end{aligned}$$

where α_i, $i = 1, \dots, 6$ are all arbitrary constants. Therefore, Lie algebra of infinitesimal symmetries of Eq. (2) is extended by the following vector field

$$\begin{aligned}
v_1 &= \frac{x}{3}\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + \frac{z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{Bu^2 + Au - 1}{A + 2Bu}\frac{\partial}{\partial u} \quad \text{scaling},\\
v_2 &= \frac{\partial}{\partial t} \quad \text{time translation},\\
v_3 &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \quad \text{rotation translation},\\
v_4 &= \frac{\partial}{\partial y} \quad \text{space translation},\\
v_5 &= \frac{\partial}{\partial z} \quad \text{space translation},\\
v_6 &= \frac{\partial}{\partial x} \quad \text{space translation}.
\end{aligned} \tag{7}$$

The commutation relations of Lie algebra obtained by v_i, $1 \le i \le 6$ are shown in Table 1 and we observe that v_i is closed under the Lie bracket.

Symmetry group of EZK equation

The main purpose of this section is to find some exact solutions from known ones, so we should find the Lie symmetry groups from the concerning symmetries. For this reason, the one parameter group g_i:

$$g_i : (x, y, z, t, u) \to (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}), \tag{8}$$

Table 1
Commutation table of Lie algebra.

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	$-v_2$	0	$-\frac{1}{3}v_4$	$-\frac{1}{3}v_5$	$-\frac{1}{3}v_6$
v_2	v_2	0	0	0	0	0
v_3	0	0	0	v_5	$-v_4$	0
v_4	$\frac{1}{3}v_4$	0	$-v_5$	0	0	0
v_5	$\frac{1}{3}v_5$	0	v_4	0	0	0
v_6	$-\frac{1}{3}v_6$	0	0	0	0	0

which is created by the vector fields v_i for $1 \le i \le 6$, is formed. For this purpose, we solve following system of ODEs

$$\frac{d}{d\epsilon}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) = \sigma(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}), \tag{9}$$

$$(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u}) \mid_{\epsilon=0} = (x, y, z, t, u), \tag{10}$$

where ϵ is an arbitrary real parameter and

$$\sigma = \xi u_x + \zeta u_y + \gamma u_z + \tau u_t + \eta u. \tag{11}$$

For the infinitesimal generator $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 + \alpha_5 v_5 + \alpha_6 v_6$, we will take the following different values to get the corresponding infinitesimal generators:

Case 1. $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$, the infinitesimal generator is $v_1 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + \frac{z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{Bu^2+Au-1}{A+2Bu}\frac{\partial}{\partial u}$.

Case 2. $\alpha_2 = 1$, $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$, the infinitesimal generator is $v_2 = \frac{\partial}{\partial t}$.

Case 3. $\alpha_3 = 1$, $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = 0$, the infinitesimal generator is $v_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$.

Case 4. $\alpha_4 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$, the infinitesimal generator is $v_4 = \frac{\partial}{\partial y}$.

Case 5. $\alpha_6 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, the infinitesimal generator is $v_5 = \frac{\partial}{\partial x}$.

Case 6. $\alpha_2 = \alpha_4 = 1$, $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_6 = 0$, the infinitesimal generator is $v_6 = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}$.

The Lie symmetry group $g : (x, y, z, t, u) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u})$ of above corresponding the infinitesimal generators are given as follows:

$$\begin{aligned}
&g_1 : (x, y, z, t, u) \rightarrow (x + \frac{\epsilon x}{3},\ y + \frac{\epsilon y}{3},\ z + \frac{\epsilon z}{3},\ t + \epsilon t,\ u - \frac{2}{3}\frac{Bu^2 + Au - 1}{A + 2Bu}\epsilon),\\
&g_2 : (x, y, z, t, u) \rightarrow (x,\ y,\ z,\ t + \epsilon,\ u),\\
&g_3 : (x, y, z, t, u) \rightarrow (x,\ y + \epsilon z,\ z - \epsilon y,\ t,\ u),\\
&g_4 : (x, y, z, t, u) \rightarrow (x,\ y + \epsilon,\ z,\ t,\ u),\\
&g_5 : (x, y, z, t, u) \rightarrow (x,\ y,\ z + \epsilon,\ t,\ u),\\
&g_6 : (x, y, z, t, u) \rightarrow (x + \epsilon,\ y,\ z,\ t,\ u).
\end{aligned} \tag{12}$$

If $u = f(x, y, z, t)$ is known solution of Eq. (2), then using the above groups $g_i (i = 1, \ldots, 6)$, the corresponding new solutions $u^{(i)} (i = 1, \ldots, 6)$ can be obtained as follows:

$$\begin{aligned}
&u^{(1)} = (x - \frac{\epsilon x}{3},\ y - \frac{\epsilon y}{3},\ z - \frac{\epsilon z}{3},\ t - \epsilon t,\ u + \frac{\epsilon}{3}(u + 1)),\\
&\quad \text{under condition } A = -2,\ B = -1,\\
&u^{(2)} = f_2(x,\ y,\ z,\ t - \epsilon),\\
&u^{(3)} = f_3(x,\ y - \epsilon z,\ z + \epsilon y,\ t),\\
&u^{(4)} = f_4(x,\ y - \epsilon,\ z,\ t),\\
&u^{(5)} = f_5(x,\ y,\ z - \epsilon,\ t),\\
&u^{(6)} = f_6(x - \epsilon,\ y,\ z,\ t).
\end{aligned} \tag{13}$$

By choosing the arbitrary functions f_i, $i = 1, \ldots, 6$, we can obtain many new solutions using different method like Bäcklund transform method, inverse scattering method, the extended tanh-function method, Hirota method, and Lie group analysis. Moreover new solution from those known using g_2 can be obtained as

$$u_1 = -\frac{A + 4\sqrt{-6B\left[Cc_1^2 + D\left(c_2^2 + c_3^2\right)\right]}\coth\left[2(c_1 x + c_2 y + c_3 z + c_4(t - \epsilon) + c_5)\right]}{2B}. \tag{14}$$

Also, using g_3 another new solution can be found

$$u_2 = -\frac{A}{B} + \sqrt{\frac{-6D\left(c_2^2 + c_3^2\right)\left(\epsilon^2 + 1\right) - 6Cc_1^2}{B}}\ \mathrm{csch}(2\varphi), \tag{15}$$

where $\varphi = c_1 x + c_2(y - \epsilon z) + c_3(z + \epsilon y) + c_4 t + c_5$. By identifying the arbitrary constants, one can get many new solutions. Thus, we get the invariant solutions of Eq. (2) using the corresponding Lagrange system given below:

$$\frac{dx}{\xi(x, y, z, t)} = \frac{dy}{\zeta(x, y, z, t)} = \frac{dz}{\gamma(x, y, z, t)} = \frac{dt}{\tau(x, y, z, t)} = \frac{du}{\eta(x, y, z, t)}.$$

Symmetry reductions and exact solutions of the EZK equation

In this section, we reduce the Lagrange equations associated with the vector fields obtained in the previous section to obtain the reduction equations and get the similarity solution.

Case 1.

In this case, the symmetry algebra is given as $\upsilon_1 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + \frac{z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{u+1}{3}\frac{\partial}{\partial u}$, has the associated Lagrange system is given by

$$\frac{dx}{\frac{x}{3}} = \frac{dy}{\frac{y}{3}} = \frac{dz}{\frac{z}{3}} = \frac{dt}{t} = \frac{du}{-\frac{u+1}{3}}, \tag{16}$$

under condition $A = -2$ and $B = -1$. The group invariant form is $u = t^{-\frac{1}{3}}H(X, Y, Z) - 1$ where similarity variables are $X = xt^{-\frac{1}{3}}$, $Y = yt^{-\frac{1}{3}}$ and $Z = zt^{-\frac{1}{3}}$. Substituting the group invariant solution into Eq. (2), we get the following reduced (2+1) non-linear PDE equation

$$3CH_{XXX} + 3D(H_{XYY} + H_{XZZ}) - 3H^2 H_X - H - XH_X - YH_Y - ZH_Z = 0. \tag{17}$$

To solve equation (17), we do again symbolic computation to obtain overdetermined equations. Thus, the following infinitesimal symmetries can be derived as

$$\xi_X = 0, \quad \zeta_Y = \beta_1 Z, \quad \gamma_Z = -\beta_1 Y, \quad \eta_H = 0, \tag{18}$$

where β_1 is an arbitrary constant. From Eq. (18), corresponding characteristic equation is given by

$$\frac{dX}{0} = \frac{dY}{Z} = \frac{dZ}{-Y} = \frac{dH}{0}. \tag{19}$$

By solving equation (19) we obtain similarity form as

$$H = S(p, r), \quad \text{where} \quad p = X, \quad r = Y^2 + Z^2. \tag{20}$$

The reduced (1+1) PDE is given as

$$-pS_p - S(3SS_p + 1) + 12DS_{pr} - 2r(S_r - 6DS_{prr}) + 3CS_{ppp} = 0. \tag{21}$$

Eq. (21) is a non-linear PDE. It is clear that it is difficult to obtain a general solution to Eq. (21). However, one particular solutions can be obtained as:

$$S(p, r) = \frac{\mu}{\sqrt{r}}.$$

Therefore, invariant solution of Eq. (2) is given as

$$u_{11} = \frac{\mu}{\sqrt{y^2 + z^2}} - 1, \tag{22}$$

where μ is an arbitrary constant.

Case 2.

For the infinitesimal generator $v_2 = \frac{\partial}{\partial t}$, solving the invariant surface condition

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{du}{0}, \tag{23}$$

yields the group invariant solution is

$$u = H(X, Y, Z), \text{where} \quad X = x, \quad Y = y, \quad Z = z. \tag{24}$$

Substituting the group invariant solution into Eq. (2), we get the following reduced (2+1) non-linear PDE equation

$$-H_X - 2HH_X - H^2H_X + CH_{XXX} + D(H_{XYY} + H_{XZZ}) = 0. \tag{25}$$

Looking for more solutions for EZK equation, the Lie symmetry method will be applied to get infinitesimals of Eq. (25) shown below:

$$\xi_X = \beta_1 X + \beta_5, \quad \zeta_Y = \beta_1 Y + \beta_2 Z + \beta_3, \quad \gamma_Z = \beta_1 Z - \beta_2 Y + \beta_4,$$
$$\eta_H = -\beta_1(H+1), \tag{26}$$

where β_1, β_2, β_3, β_4 and β_5 are arbitrary constants. By determining the appropriate values of β_i, $1 \le i \le 5$ in Eq. (26), we get the following characteristic equation:

$$\frac{dX}{\beta_1 X + \beta_5} = \frac{dY}{\beta_1 Y + \beta_2 Z + \beta_3} = \frac{dZ}{\beta_1 Z - \beta_2 Y + \beta_4} = \frac{dH}{-\beta_1(H+1)}. \tag{27}$$

Reduced equations and invariant solutions for Eq. (2) for subcases of vector field v_2 are shown as follow

Subcase 2.1. $\beta_2 = 0$.

Substituting $\beta_2 = 0$ in Eq. (27), we get reduced Lagrange's system given as:

$$\frac{dX}{\beta_1 X + \beta_5} = \frac{dY}{\beta_1 Y + \beta_3} = \frac{dZ}{\beta_1 Z + \beta_4} = \frac{dH}{-\beta_1(H+1)}. \tag{28}$$

Taking $\beta_1 \neq 0$, Eq. (28) becomes

$$\frac{dX}{X + d_1} = \frac{dY}{Y + d_2} = \frac{dZ}{Z + d_3} = \frac{dH}{-(H+1)}, \tag{29}$$

where $d_1 = \frac{\beta_5}{\beta_1}$, $d_2 = \frac{\beta_3}{\beta_1}$ and $d_3 = \frac{\beta_4}{\beta_1}$. Similarity form is

$$H(X, Y, Z) = \frac{1}{Z + d_3} S(p, r) - 1 \quad \text{where} \quad p = \frac{X + d_1}{Z + d_3}, \quad r = \frac{Y + d_2}{Z + d_3}. \tag{30}$$

Substituting Eq. (30) in Eq. (25), we can get the reduced non-linear PDE as

$$\left(C + Dp^2\right)S_{ppp} + D\left[2p\left(3S_{pp} + rS_{ppr}\right) + 6rS_{pr} + \left(r^2+1\right)S_{prr}\right] + \left(6D - S^2\right)S_p = 0. \tag{31}$$

Infinitesimals of Eq. (31) are

$$\delta_p = pr, \quad \delta_r = r^2 + 1, \quad \eta_S = -rS. \tag{32}$$

Hence, using Eq. (32), we get associated characteristic equations

$$\frac{dp}{pr} = \frac{dr}{r^2+1} = \frac{dS}{-rS} \tag{33}$$

By solving equation (33), we get a similarity form

$$S = \frac{1}{\sqrt{1+r^2}} Q(w), \quad \text{where} \quad w = \frac{p}{\sqrt{1+r^2}}. \tag{34}$$

Eq. (31) reduces to ODE

$$(4D - Q^2)Q' + 5\ D\ w\ Q'' + (C + D\ w^2)Q''' = 0. \tag{35}$$

After solving equation (35), we can get two particular solutions as

$$Q(w) = \pm\kappa_1, \qquad Q(w) = \pm\frac{\sqrt{6C}}{w}, \tag{36}$$

where κ_1 is an arbitrary constant. Then comprising Eqs. (24), (34), and (36), we get the similarity solutions as

$$u_{25} = \frac{\pm\kappa_1}{(d_1 + y)^2 + (d_3 + z)^2} - 1, \tag{37}$$

$$u_{26} = \frac{\pm\sqrt{6C}}{x + d_1} - 1. \tag{38}$$

Subcase 2.2. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = 0$.

For this subcase the invariant is given as $H = S(p, r)$ where $p = X$ and $r = Y^2 + Z^2$, inserting the value of H into Eq. (25), we obtain

$$S_p + 2SS_p + S^2S_p - 4D(S_{pr} + rS_{prr}) - CS_{ppp} = 0. \tag{39}$$

One can find new set of generators

$$\delta_p = \frac{c_1}{2}p + c_2, \qquad \delta_r = c_1 r, \qquad \eta_S = -\frac{c_1}{2}(S+1). \tag{40}$$

For $c_2 = 0$, the invariant is given as

$$S = r^{-\frac{1}{2}}Q(w) - 1, \qquad \text{where} \qquad w = pr^{-\frac{1}{2}}, \tag{41}$$

inserting the value of S from Eq. (41) in Eq. (39), we get

$$(4D - Q^2)Q' + 5DwQ'' + (C + Dw^2)Q''' = 0. \tag{42}$$

Solving equation (42), we obtain the group invariant solution as

$$u_{27} = \frac{\sqrt{6C}}{x} - 1. \tag{43}$$

Subcase 2.3. $\beta_1 = \beta_2 = \beta_4 = \beta_5 = 0$.

The following form of H can be written as

$$H = S(p, r), \qquad \text{where} \quad p = X \quad \text{and} \quad r = Z, \tag{44}$$

substituting Eq. (44) into Eq. (25), we get

$$-S_p - 2SS_p - S^2S_p + CS_{ppp} + DS_{prr} = 0. \tag{45}$$

Infinitesimals of Eq. (45) are

$$\delta_p = c_1 p + c_3, \qquad \delta_r = c_1 r + c_2, \qquad \eta_S = -c_1(S+1). \tag{46}$$

For $c_2 = c_3 = 0$, the invariant is given as

$$S = \frac{Q(w)}{r} - 1, \qquad \text{where} \qquad w = \frac{p}{r}, \tag{47}$$

inserting the value of S from Eq. (47) in Eq. (45), we get

$$(6D - Q^2)Q' + 6DwQ'' + (C + Dw^2)Q''' = 0. \tag{48}$$

After solving equation (48), it give the same solution in (43).

Subcase 2.4. $\beta_1 = \beta_2 = \beta_3 = \beta_5 = 0$.

The group invariant solution of this subcase is given as

$$H = S(p, r), \qquad \text{where} \quad p = X \quad \text{and} \quad r = Y, \tag{49}$$

substituting the group invariant solution (49) into Eq. (25), we get

$$-S_p - 2SS_p - S^2S_p + CS_{ppp} + DS_{prr} = 0. \tag{50}$$

Case 3.

For the infinitesimal generator $v_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, The associated Lagrange system is

$$\frac{dx}{0} = \frac{dy}{z} = \frac{dz}{-y} = \frac{dt}{0} = \frac{du}{0}. \tag{51}$$

The group invariant form is

$u = H(X, Y, T)$ where similarity variables are

$$X = x, \quad Y = y^2 + z^2, \quad T = t. \tag{52}$$

The reduced equation is given as

$$H_T - H_X + -2HH_X - H^2H_X + CH_{XXX} + 4D(H_{XY} + YH_{XYY}) = 0. \tag{53}$$

We again apply the Lie symmetry method to get infinitesimals of Eq. (53) shown below:

$$\xi_X = \frac{X}{3}\beta_1 + \beta_3, \quad \zeta_Y = \frac{2}{3}Y\beta_1, \quad \tau_T = \beta_1 T + \beta_2, \quad \eta_H = -\frac{\beta_1}{3}(H+1), \tag{54}$$

Table 2
Reduced equations and invariant solutions of the EZK equation for case 4.

Subcase	Similarity variables	Reduced equations	Invariant solutions
$\beta_2=\beta_3=\beta_4=0$	$p=XT^{-\frac{1}{3}},\ \ r=ZT^{-\frac{1}{3}}$, $H=T^{-\frac{1}{3}}S(p,r)-1$,	$3CS_{ppp}+3DS_{prr}$ $-(3SS_p+1)S$ $-(rS_r+pS_p)=0$	One particular solutions can be obtained as $u_{42}=\frac{1}{z}-1$
$\beta_1=\beta_3=\beta_4=0$	$p=X,\ \ r=Z$, $H=S(p,r)$	$-S_p+ASS_p+BS^2S_p$ $+CS_{ppp}+DS_{prr}=0$	Same solution in (43) .
$\beta_1=\beta_2=\beta_4=0$	$p=X,\ \ r=T$, $H=S(p,r)$	$S_r-S_p-2SS_p$ $-S^2S_p+CS_{ppp}=0$	$Q+(w+3Q^2)Q'-3CQ'''=0$.
$\beta_1=\beta_2=\beta_3=0$	$p=Z,\ \ r=T$, $H=S(p,r)$	$S_r=0$	$u_{44}=f(z)$

where β_1, β_2 and β_3 are arbitrary constants. By determining the suitable values of $\beta_i, 1\le i\le 3$ in Eq. (54), we get reduced equations and many invariant solutions as follow

Subcase 3.1. $\beta_2=\beta_3=0$.

For this subcase, the group invariant is given as

$$H=T^{\frac{-1}{3}}S(p,r)-1 \quad\text{where}\quad p=XT^{\frac{-1}{3}} \quad\text{and}\quad r=YT^{\frac{-2}{3}}, \tag{55}$$

substituting Eq. (55) into Eq. (53), we get the following reduction equation

$$3CS_{ppp}+12DS_{pr}-2r(S_r-6DS_{prr})-S(3SS_p+1)-pS_p=0. \tag{56}$$

One particular solution can be obtained as

$$u_{31}=\frac{1}{\sqrt{y^2+z^2}}-1. \tag{57}$$

Subcase 3.2. $\beta_1=\beta_3=0$.

The group invariant is given as $H = S(p,r)$ with the similarity variables $p = X$ and $r = Y$, substituting the group invariant solution into Eq. (53), we get

$$S_p+2SS_p+S^2S_p-CS_{ppp}-4D(S_{pr}+rS_{prr})=0. \tag{58}$$

Subcase 3.3. $\beta_1=\beta_2=0$.

The following form of H can be written as $H=S(p,r)$ where $p=Y$ and $r = T$, substituting the group invariant solution into Eq. (53), we get the reduced equation as $S_r = 0$. The general solution of EZK equation (2) can be expressed as

$$u_{33}=f(y^2+z^2). \tag{59}$$

Case 4.

The associated Lagrange system for $v_4=\frac{\partial}{\partial y}$ is given by

$$\frac{dx}{0}=\frac{dy}{1}=\frac{dz}{0}=\frac{dt}{0}=\frac{du}{0}. \tag{60}$$

The group invariant form is

$$u=H(X,Z,T) \quad\text{where}\quad X=x,\quad Z=z,\quad T=t. \tag{61}$$

Inserting the value of u from Eq. (61) into (2), we obtain the following equation

$$H_T-H_X-2HH_X-H^2H_X+CH_{XXX}+DH_{XZZ}=0. \tag{62}$$

Solving equation (62) by using tanh-method, we get the following similarity solution

$$u_{41}=\frac{-2+4\sqrt{6(c_2^2D+Cc_1^2)}\coth(2(c_1x+c_2y+c_3z+c_4))}{2}. \tag{63}$$

A new set of infinitesimal generators can be obtained for Eq. (62) as the following:

$$\xi_X=\frac{X}{3}\beta_1+\beta_4,\quad \gamma_z=\frac{Z}{3}\beta_1+\beta_3,\quad \tau_T=\beta_1T+\beta_2,\quad \eta_H=-\frac{\beta_1}{3}(H+1), \tag{64}$$

where β_1, β_2, β_3 and β_4 are arbitrary constants. By setting the suitable values of $\beta_i, 1\le i\le 4$ in Eq. (54), we get reduced equations and many invariant solutions as shown in Table 2.

Case 5.

The associated Lagrange system for $v_6=\frac{\partial}{\partial x}$ is given as

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dz}{0}=\frac{dt}{0}=\frac{du}{0}. \tag{65}$$

After solving equation (65), we get similarity form as

$$u=H(Y,Z,T),\quad\text{where}\quad Y=y,\quad Z=z,\quad T=t. \tag{66}$$

Substituting the value of u from Eq. (66) into Eq. (2), we get the reduced equation as $H_T=0$. The general solution of reduced equation is given as $H = f(Y,Z)$. therefore, general solution of EZK equation is given as

$$u_{51}(x,y,z,t)=f(y,z). \tag{67}$$

Case 6.

For the infinitesimal generator $v_6 = v_2+v_4=\frac{\partial}{\partial y}+\frac{\partial}{\partial t}$. Solving the invariant surface condition

$$\frac{dx}{0}=\frac{dy}{1}=\frac{dz}{0}=\frac{dt}{1}=\frac{du}{0}, \tag{68}$$

yields

$u=H(X,Y,Z)$, with the similarity variables

$$X=x,\quad Y=y-t,\quad Z=z. \tag{69}$$

Substituting the value of u from Eq. (69) into Eq. (2), we get the following reduced (2+1) non-linear PDE equation as

$$-H_Y-H_X-2HH_X-BH^2H_X+CH_{XXX}+D(H_{XYY}+H_{XZZ})=0. \tag{70}$$

After apply again the similarity transformation method on (70), we obtain the reduced (1+1)-dimensional PDE as

$$S_r+S_p+2SS_p+S^2S_p-CS_{ppp}-D(S_{ppp}+S_{prr})=0, \tag{71}$$

with similarity form as

$$H=S(p,r)\quad\text{where}\quad p=X-Z,\quad r=Y. \tag{72}$$

Inserting the value of S from Eq. (73) in Eq. (71), we get

$$(2+Q)QQ'-(C+2D)Q'''=0. \tag{73}$$

Conservation laws of Eq. (2)

In this section, we will construct conservation laws for (2) using the new conservation theorem [55]. The formal Lagrangian for the EZK equation

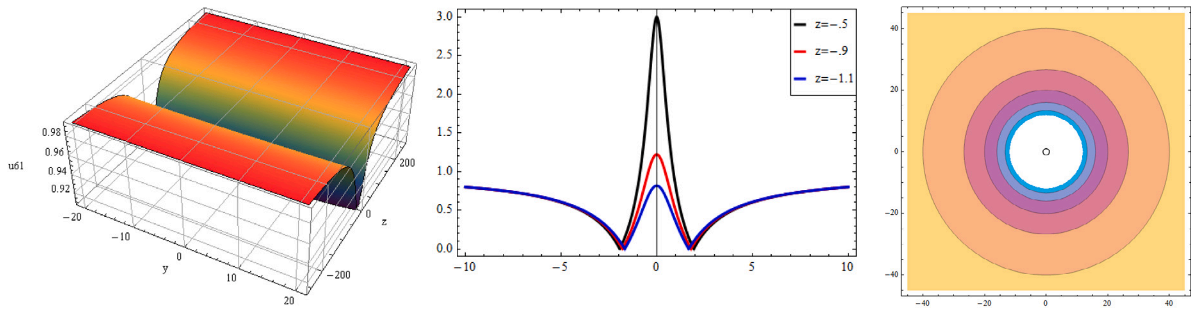
$$F\equiv u_t-u_x+Auu_x+Bu^2u_x+Cu_{xxx}+D\big(u_{xyy}+u_{xzz}\big)=0, \tag{74}$$

is defined by

$$L=\vartheta\big[u_t-u_x+Auu_x+Bu^2u_x+Cu_{xxx}+D\big(u_{xyy}+u_{xzz}\big)\big], \tag{75}$$

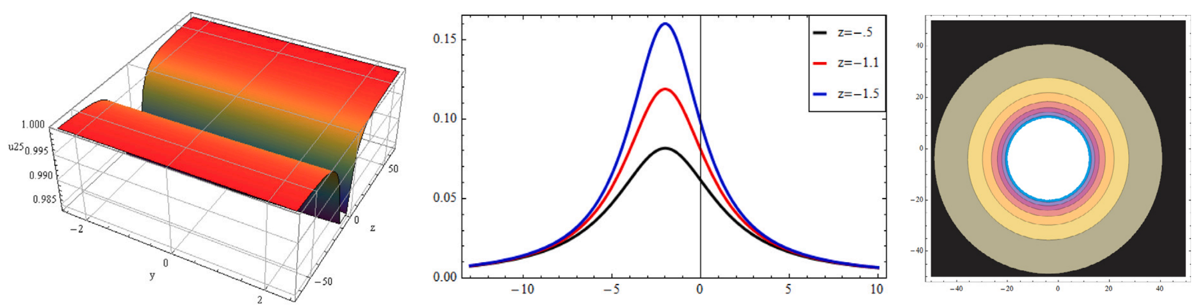
which can be reduced to the second-order Lagrangian:

$$L=\vartheta\big[u_t-u_x+Auu_x+Bu^2u_x\big]-\vartheta_x\big[Cu_{xx}+D\big(u_{yy}+u_{zz}\big)\big], \tag{76}$$



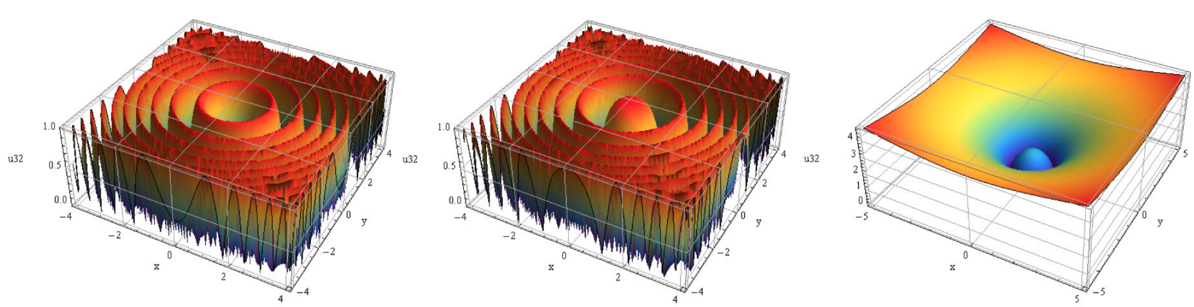
(a) (b) (c)

Fig. 1. Solitary wave profile of Eq. (22) for $\mu = 2$. (a) Single soliton. (b) The wave propagation pattern of the wave along the y-axis with $z = -.5$ (black), $z = -.9$ (red) and $z = -1.1$ (blue). (c) Contour plot.



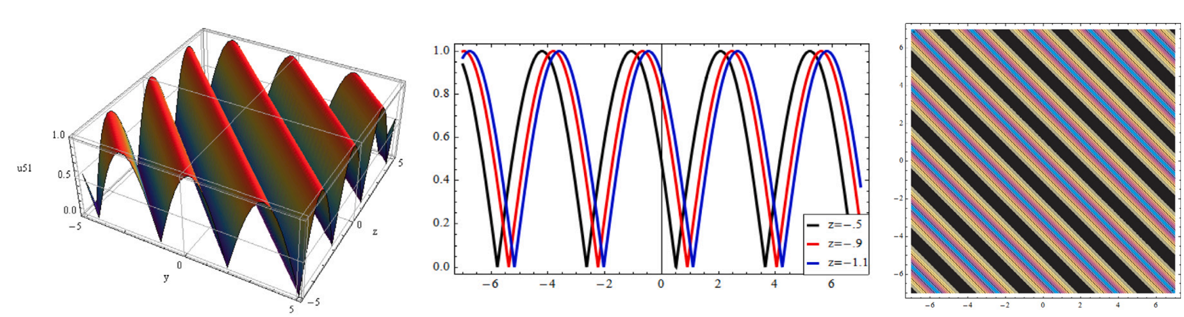
(a) (b) (c)

Fig. 2. Solitary wave profile of Eq. (37) for $c_1 = 1$, $d_1 = 2$ and $d_3 = 4$. (a) Single soliton. (b) The wave propagation pattern of the wave along the y-axis with $z = -.5$ (black), $z = -1.1$ (red) and $z = -1.5$ (blue). (c) Contour plot.



(a) (b) (c)

Fig. 3. Solution u_{33} (59) at (a) $f(y,z) = \sin\left(y^2 + z^2\right)$, (b) $f(y,z) = \cos\left(y^2 + z^2\right)$, (c) $f(y,z) = \operatorname{arccosh}\left(y^2 + z^2\right)$.



(a) (b) (c)

Fig. 4. Solitary wave profile of Eq. (67). (a) Multi-soliton soliton. (b) The wave propagation pattern of the wave along the y-axis with $z = -.5$ (black), $z = -.9$ (red) and $z = -1.1$ (blue). (c) Contour plot.

here ϑ is a new dependent variable. Consequently the adjoint equation to (76) has the form

$$F^* \equiv \frac{\delta L}{\delta u} = 0, \tag{77}$$

the variational derivatives of the Lagrangian defined by

$$\frac{\delta L}{\delta u} = L_u - D_t\left(L_{u_t}\right) - D_x\left(L_{u_x}\right) - D_x^3\left(L_{u_{xxx}}\right) - D_x D_y^2\left(L_{u_{xyy}}\right) - D_x D_z^2\left(L_{u_{xzz}}\right) + \cdots, \tag{78}$$

then, we obtain the adjoint equation of (74)

$$F^* \equiv \vartheta_t - \vartheta_x + Au\vartheta_x + Bu^2\vartheta_x + C\vartheta_{xxx} + D\left(\vartheta_{xyy} + \vartheta_{xzz}\right) = 0, \tag{79}$$

from Eq. (7), we can obtain the following six cases:

Case 1. Firstly, we consider the Lie point symmetry $v_1 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + \frac{z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{Bu^2+Au-1}{A+2Bu}\frac{\partial}{\partial u}$ of (2). Corresponding to this symmetry, the Lie characteristic functions are $W^1 = -\frac{2(Au+Bu^2-1)}{3(2Bu+A)} - tu_t - \frac{x}{3}u_x - \frac{y}{3}u_y - \frac{z}{3}u_z$ and $W^2 = -\frac{2(Au+Bu^2-1)}{3(2Bu+A)} - t\vartheta_t - \frac{x}{3}\vartheta_x - \frac{y}{3}\vartheta_y - \frac{z}{3}\vartheta_z$. Thus, using the Ibragimov theorem [47]

$$\begin{aligned}
C_2^t &= t\vartheta\left[u_t - u_x + Auu_x + Bu^2u_x\right] - t\vartheta_x\left[Cu_{xx} + D\left(u_{yy} + u_{zz}\right)\right] \\
&\quad - \vartheta\left[tu_t + \frac{x}{3}u_x + \frac{y}{3}u_y + \frac{z}{3}u_z + \frac{2\left(Au + Bu^2 - 1\right)}{3(2Bu + A)}\right],
\end{aligned}$$

$$\begin{aligned}
C_2^x &= \frac{x}{3}\vartheta\left[u_t - u_x + Auu_x + Bu^2u_x\right] - \frac{x}{3}\vartheta_x\left[Cu_{xx} + D\left(u_{yy} + u_{zz}\right)\right] \\
&\quad - \left[\vartheta(Au + Bu^2 - 1) + C\vartheta_{xx}\right]\left[tu_t + \frac{x}{3}u_x + \frac{y}{3}u_y + \frac{z}{3}u_z + \frac{2\left(Au + Bu^2 - 1\right)}{3(2Bu + A)}\right] \\
&\quad + \frac{C\vartheta_x}{3(2Bu + A)^2}\left[u_x\left(8Bu(Bu + A) + 3A^2 + 4B\right) + \left(3tu_{xt} + zu_{xz} + yu_{xy} + xu_{xx}\right)(2Bu + A)^2\right] \\
&\quad + \left[Cu_{xx} + Du_{yy} + Du_{zz}\right]\left[t\vartheta_t + \frac{x}{3}\vartheta_x + \frac{y}{3}\vartheta_y + \frac{z}{3}\vartheta_z + \frac{2\left(Au + Bu^2 - 1\right)}{3(2Bu + A)}\right],
\end{aligned}$$

$$\begin{aligned}
C_2^y &= \frac{y}{3}\vartheta\left[u_t - u_x + Auu_x + Bu^2u_x\right] - \frac{y}{3}\vartheta_x\left[Cu_{xx} + D\left(u_{yy} + u_{zz}\right)\right] \\
&\quad - D\vartheta_{xy}\left[tu_t + \frac{x}{3}u_x + \frac{y}{3}u_y + \frac{z}{3}u_z + \frac{2\left(Au + Bu^2 - 1\right)}{3(2Bu + A)}\right] \\
&\quad + \frac{D\vartheta_x}{3(2Bu + A)^2}\left[u_y\left(8Bu(Bu + A) + 3A^2 + 4B\right) + \left(3tu_{yt} + zu_{yz} + yu_{yy} + xu_{xy}\right)(2Bu + A)^2\right],
\end{aligned}$$

$$\begin{aligned}
C_2^z &= \frac{z}{3}\vartheta\left[u_t - u_x + Auu_x + Bu^2u_x\right] - \frac{z}{3}\vartheta_x\left[Cu_{xx} + D\left(u_{yy} + u_{zz}\right)\right] \\
&\quad - D\vartheta_{xz}\left[tu_t + \frac{x}{3}u_x + \frac{y}{3}u_y + \frac{z}{3}u_z + \frac{2\left(Au + Bu^2 - 1\right)}{3(2Bu + A)}\right] \\
&\quad + \frac{D\vartheta_x}{3(2Bu + A)^2}\left[u_z\left(8Bu(Bu + A) + 3A^2 + 4B\right) + \left(3tu_{zt} + zu_{zz} + yu_{yz} + xu_{xz}\right)(2Bu + A)^2\right].
\end{aligned}$$

Case 2. The Lie point symmetry $v_2 = \partial_t$ has the Lie characteristic functions $W^1 = -u_t$ and $W^2 = -\vartheta_t$. We obtain the conserved vector, whose components are:

$$\begin{aligned}
C_2^t &= \vartheta\left[-u_x + Auu_x + Bu^2u_x\right] - \vartheta_x\left[Cu_{xx} + Du_{yy} + Du_{zz}\right], \\
C_2^x &= -u_t\left[\vartheta(Au + Bu^2 - 1) + C\vartheta_{xx}\right] + C\vartheta_x u_{xt} + \vartheta_t\left[Cu_{xx} + Du_{yy} + Du_{zz}\right], \\
C_2^y &= -D\vartheta_{xy}u_t + Du_{yt}\vartheta_x, \\
C_2^z &= -D\vartheta_{xz}u_t + Du_{zt}\vartheta_x.
\end{aligned}$$

Case 3. The Lie point symmetry $v_3 = z\partial_y - y\partial_z$ has the Lie characteristic functions $W^1 = -zu_y + yu_z$ and $W^2 = -z\vartheta_y + y\vartheta_z$, the components of the conserved vector are:

$$\begin{aligned}
C_3^t &= \vartheta\left[-zu_y + yu_z\right], \\
C_3^x &= (-zu_y + yu_z)\left[\vartheta(Au + Bu^2 - 1) + C\vartheta_{xx}\right] + (z\vartheta_y - y\vartheta_z) \\
&\quad \left[Cu_{xx} + Du_{yy} + Du_{zz}\right] - C\vartheta_x\left[-zu_{xy} + yu_{xz}\right], \\
C_3^y &= z\vartheta\left[-u_x + Auu_x + Bu^2u_x\right] - z\vartheta_x\left[Cu_{xx} + Du_{yy} + Du_{zz}\right] \\
&\quad + D\vartheta_{xy}[-zu_y + yu_z] - \vartheta_x\left[-zu_{yy} + yu_{yz} + u_z\right], \\
C_3^z &= -y\vartheta\left[-u_x + Auu_x + Bu^2u_x\right] + y\vartheta_x\left[Cu_{xx} + Du_{yy} + Du_{zz}\right] \\
&\quad + D\vartheta_{xz}[-zu_y + yu_z] - \vartheta_x\left[-zu_{yz} + yu_{zz} - u_y\right].
\end{aligned}$$

Case 4. The Lie point symmetry $v_4 = \partial_y$ has the Lie characteristic functions $W^1 = -u_y$ and $W^2 = -\vartheta_y$, the components of the conserved vector are:

$$\begin{aligned}
C_4^t &= -\vartheta u_y, \\
C_4^x &= -u_y\left[\vartheta(Au + Bu^2 - 1) + C\vartheta_{xx}\right] + C\vartheta_x u_{xy} + \vartheta_y\left[Cu_{xx} + Du_{yy} + Du_{zz}\right], \\
C_4^y &= \vartheta\left[-u_x + Auu_x + Bu^2u_x\right] - \vartheta_x\left[Cu_{xx} + Du_{yy} + Du_{zz}\right] \\
&\quad - Du_y\vartheta_{xy} + D\vartheta_x u_{yy}, \\
C_4^z &= -D\vartheta_{xz}u_y + Du_{yz}\vartheta_x.
\end{aligned}$$

Case 5. The Lie point symmetry $v_5 = \partial_z$ has the Lie characteristic functions $W^1 = -u_z$ and $W^2 = -\vartheta_z$, the components of the conserved vector are:

$$\begin{aligned}
C_5^t &= -\vartheta u_z, \\
C_5^x &= -u_y\left[\vartheta(Au + Bu^2 - 1) + C\vartheta_{xx}\right] + C\vartheta_x u_{xz} + \vartheta_z\left[Cu_{xx} + Du_{yy} + Du_{zz}\right], \\
C_5^y &= -D\vartheta_{xy}u_y + Du_{zz}\vartheta_x, \\
C_5^z &= \vartheta\left[-u_x + Auu_x + Bu^2u_x\right] - \vartheta_x\left[Cu_{xx} + Du_{yy} + Du_{zz}\right] \\
&\quad - Du_z\vartheta_{xy} + D\vartheta_x u_{yz}.
\end{aligned}$$

Case 6. Finally, the Lie point symmetry $v_6 = \partial_x$ has the Lie characteristic functions $W^1 = -u_x$ and $W^2 = -\vartheta_x$, the components of the conserved vector are:

$$\begin{aligned}
C_6^t &= -\vartheta u_x, \\
C_6^x &= \vartheta u_t + C\vartheta_x u_{xx} - Cu_x\vartheta_{xx}, \\
C_6^y &= -D\vartheta_{xy}u_x + Du_{xy}\vartheta_x, \\
C_6^z &= -D\vartheta_{xy}u_x + Du_{xy}\vartheta_x.
\end{aligned}$$

Conclusion

In this paper, we have considered the EZK equation (2). Conservation laws for the EZK equation are constructed for the first time using the new conservation theorem of Ibragimov. Moreover, The Lie point symmetry generators of the underlying equation (2) are derived and used it to get similarity solution for EZK equation (2). We used similarity variables to reduced the EZK equation (2) into a new partial differential equation with less number of independent variables, and again using Lie group symmetry method, the new partial differential equation is reduced into an ODE. In addition to that, solitary wave solutions of the EZK equation are obtained from the reduction equation. A number of exact solutions for the EZK equation (2) have been obtained. The geometric representation of the solutions was analyzed as follows, Figs. 1 and 2 shows solitary wave profile, propagating waves for various values of y and contour plot for Eqs. (22) and (37) with certain values of parameters as well as Fig. 4 demonstrates multisoliton and solitary wave solutions profile of Eq. (67) for free choice of function $f(y, z) = \sin(y + z)$. To our knowledge, the solutions obtained in this paper have not been applied in previous literature so, these solutions are new solutions for (2) (see Fig. 3).

CRediT authorship contribution statement

Shrouk Wael: Writing and Editing. **Aly R. Seadawy:** Visualization, Investigation. **O.H. EL-Kalaawy:** Conceptualization, Methodology, Software, Supervision. **S.M. Maowad:** Data curation, Writing - original draft, Software, Validation. **Dumitru Baleanu:** Reviewing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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