The general bilinear techniques for studying the propagation of mixed-type periodic and lump-type solutions in a homogenous-dispersive medium

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ABSTRACT

This paper aims to construct new mixed-type periodic and lump-type solutions via dependent variable transformation and Hirota's bilinear form (general bilinear techniques). This study considers the (3 + 1)-dimensional generalized B-type Kadomtsev–Petviashvili equation, which describes the weakly dispersive waves in a homogeneous medium in fluid dynamics. The obtained solutions contain abundant physical structures. Consequently, the dynamical behaviors of these solutions are graphically discussed for different choices of the free parameters through 3D plots.

I. INTRODUCTION

Nonlinear phenomena are investigated in many disciplines of science, such as marine engineering, fluid dynamics, plasma physics, chemistry, applied mathematics, and so on.[1–19] With the development of nonlinear dynamics, the research of nonlinear partial differential equations (NPDEs) becomes more and more important. To further understand these phenomena, solving NPDEs plays a significant role in nonlinear sciences.[20–35] In the past few decades, many efficient and powerful techniques have been introduced to obtain the analytical solutions of these equations.[36–51] More recently, soliton wave solutions have attracted a growing amount of attention, and many theoretical and experimental studies of these waves are investigated.[1,3,6] The importance of solitons is due to their presence in a variety of nonlinear differential equations portraying many complex nonlinear phenomena, including acoustics, nonlinear optics, telecommunication industry, plasma physics, condensed matter, and solid-state physics. There are different structures or types of solitons such as lump waves (usually called rogue waves in the presence of certain conditions), which are algebraically localized waves that decay in all space directions and exist in all time, breather waves, and mixed waves, which describe the interaction between two different types of soliton waves.

In this paper, a (3 + 1)-dimensional generalized B-type Kadomtsev–Petviashvili (BKP) equation is considered as follows:[52]

$$u_{yt} + 3\,u_{xz} - 3u_x\,u_{xy} - 3\,u_y\,u_{xx} - u_{xxxy} = 0. \tag{1}$$

Equation (1) is an exceedingly useful model for assaying the dynamics of nonlinear waves and solitons in various fields of science especially in plasma physics, weakly dispersive environment, and fluid dynamics. Multiple-soliton solutions are generated and discussed by Ma.[52] Ma and Zhu[53] derived multiple wave solutions by using the multiple exp-function algorithm. Tang[54] obtained new analytical solutions that contain different wave structures such as periodic soliton, kinky periodic solitary, and periodic soliton solutions by using the extended homoclinic test approach. By employing the improved (G'/G)-expansion method with the aid of symbolic computations, Chen and Ma [55] obtain new soliton solutions of Eq. (1).

This paper is organized as follows: Sec. II gives the new mixed-type periodic solutions for the (3 + 1)-dimensional generalized BKP equation based on the dependent variable transformation and Hirota's bilinear form. Section III presents the lump-type solutions and illustrates the dynamical behaviors of the obtained solutions through 3D plots. Section IV makes the conclusions.

II. NEW MIXED-TYPE PERIODIC SOLUTIONS

Substituting the transformation $u = 2[\ln \xi(x, y, z, t)]_x$ into Eq. (1), we have the following Hirota's bilinear form:[56–61]

$$(D_tD_y - D_x^3D_y + 3D_xD_z)f\cdot f = 0. \tag{2}$$

Equivalently, we have

$$-\xi_t\xi_y + \xi_{xxx}\xi_y - 3\xi_z\xi_x - 3\xi_{xy}\xi_{xx} + 3\xi_x\xi_{xxy} + \xi(\xi_{yt} + 3\xi_{xz} - \xi_{xxxy}) = 0. \tag{3}$$

In order to obtain the new mixed-type periodic solutions, a direct test function is written as

$$\xi = k_1\,e^{\zeta_1} + e^{-\zeta_1} + k_2\,\tan(\zeta_2) + k_3\,\tanh(\zeta_3), \tag{4}$$

where $\zeta_i = \eta_i x + \mu_i y + \gamma_i z + \nu_i t, i = 1, 2, 3$ and η_i, μ_i, γ_i, and ν_i are unknown constants. Substituting Eq. (4) into Eq. (3), we have the following:

Case (1)

$$k_2 = \mu_1 = \gamma_1 = \eta_3 = \nu_3 = 0, \nu_1 = \frac{\eta_1^3\mu_3 - 3\eta_1\gamma_3}{\mu_3}, \tag{5}$$

where η_1, γ_3, μ_3, k_1, and k_3 are arbitrary constants. Then,

$$\xi = e^{x\eta_1 + \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}}k_1 + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}} + k_3\,\tanh(y\mu_3 + z\gamma_3). \tag{6}$$

Substituting Eq. (6) into $u = 2[\ln \xi]_x$, the first mixed-type periodic solution reads as

$$u_1 = \frac{2[e^{x\eta_1 + \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}}k_1\eta_1 - e^{-x\eta_1 - \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}}\eta_1]}{e^{x\eta_1 + \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}}k_1 + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_3 - 3\eta_1\gamma_3)}{\mu_3}} + k_3\,\tanh(y\mu_3 + z\gamma_3)}. \tag{7}$$

The physical structure of Eq. (7) is exhibited in Fig. 1.

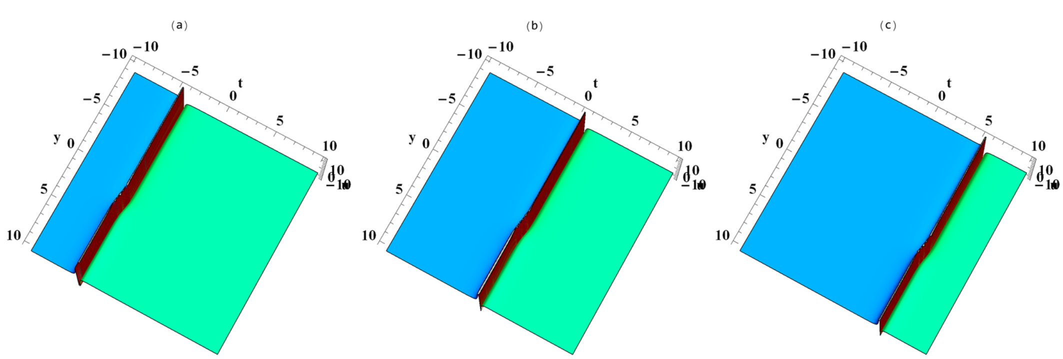

FIG. 1. Solution (7) at $k_1 = k_3 = -0.5$, $\eta_1 = -1$, $\mu_3 = 1\gamma_3 = 1$, $z = -2$, (a) $x = -10$, (b) $x = 0$, (c) $x = 10$.

Case (2)

$$k_3 = \mu_1 = \gamma_1 = \eta_2 = \nu_2 = 0, \nu_1 = \frac{\eta_1^3\mu_2 - 3\eta_1\gamma_2}{\mu_2}, \tag{8}$$

where η_1, γ_2, μ_2, k_1, and k_2 are free real constants. Then,

$$\xi = e^{x\eta_1 + \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} k_1 + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} + k_2 \tan(y\mu_2 + z\gamma_2). \tag{9}$$

Substituting Eq. (9) into $u = 2[\ln \xi]_x$, the second mixed-type periodic solution is given by

$$u_2 = \frac{2\left[e^{x\eta_1 + \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} k_1\eta_1 - e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} \eta_1\right]}{e^{x\eta_1 + \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} k_1 + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} + k_2 \tan(y\mu_2 + z\gamma_2)}. \tag{10}$$

Case (3)

$$k_1 = \mu_1 = \gamma_1 = \eta_2 = \nu_2 = \eta_3 = \nu_3 = 0, \quad \nu_1 = \frac{\eta_1^3\mu_2 - 3\eta_1\gamma_2}{\mu_2}, \quad \gamma_3 = \frac{\mu_3\gamma_2}{\mu_2}, \tag{11}$$

where η_1, γ_2, μ_2, μ_3, k_2, and k_3 are free real constants. Then,

$$\xi = k_2 \tan(y\mu_2 + z\gamma_2) + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_2\mu_3}{\mu_2}\right). \tag{12}$$

Substituting Eq. (12) into $u = 2[\ln \xi]_x$, the third mixed-type periodic solution has the form

$$u_3 = \frac{-2e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} \eta_1}{k_2 \tan(y\mu_2 + z\gamma_2) + e^{-x\eta_1 - \frac{t(\eta_1^3\mu_2 - 3\eta_1\gamma_2)}{\mu_2}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_2\mu_3}{\mu_2}\right)}. \tag{13}$$

The physical structure of Eq. (13) is revealed in Figs. 2 and 3.

Case (4)

$$k_1 = \eta_2 = \nu_2 = \eta_3 = \nu_3 = 0, \quad \nu_1 = \frac{\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \quad \gamma_2 = \frac{\mu_2\gamma_1}{\mu_1}, \quad \gamma_3 = \frac{\mu_3\gamma_1}{\mu_1}, \tag{14}$$

where η_1, μ_1, γ_1, μ_2, μ_3, k_2 and k_3 are free real constants. Then,

$$\xi = k_2 \tan\left(y\mu_2 + \frac{z\gamma_1\mu_2}{\mu_1}\right) + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_1\mu_3}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}. \tag{15}$$

Substituting Eq. (15) into $u = 2[\ln \xi]_x$, the fourth mixed-type periodic solution takes the following type:

$$u_4 = \left[-2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} \eta_1\right] \Big/ \left[k_2 \tan\left(y\mu_2 + \frac{z\gamma_1\mu_2}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_1\mu_3}{\mu_1}\right)\right]. \tag{16}$$

The physical structure for the solution in Eq. (16) is similar to that one given by Eq. (13).

Case (5)

$$k_2 = k_3 = 0, \nu_1 = \frac{4\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \tag{17}$$

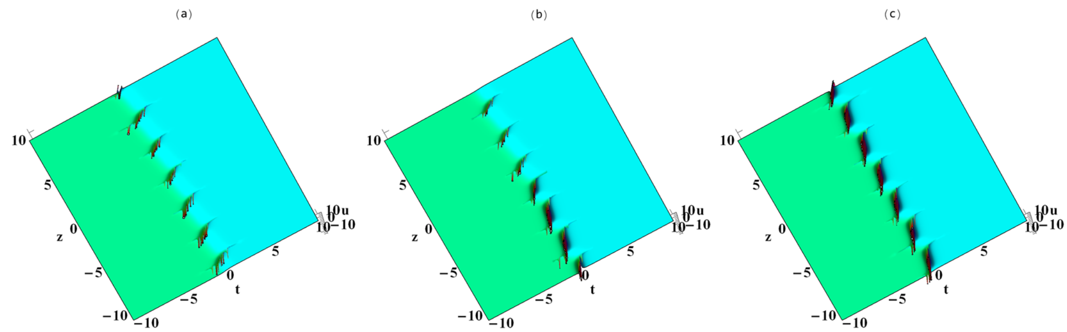

FIG. 2. Solution (13) at $k_3 = k_2 = -0.5$, $\eta_1 = \mu_2 = \mu_3 = 1$, $\gamma_2 = -1$, $x = 0$, (a) $y = -20$, (b) $y = 0$, (c) $y = 20$.

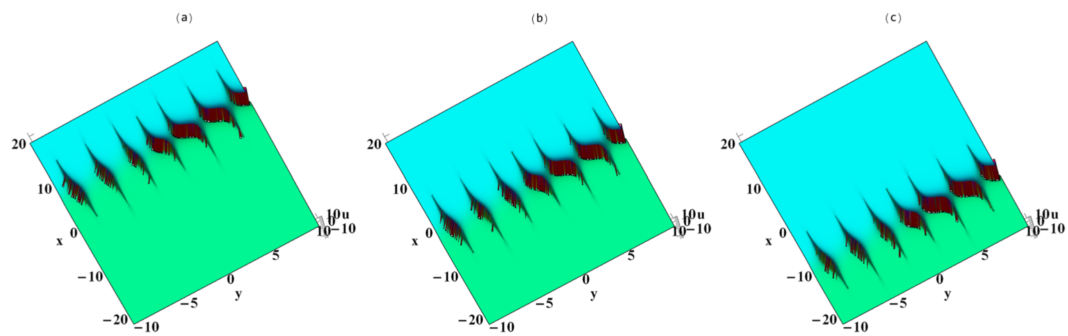

FIG. 3. Solution (13) at $k_3 = k_2 = -0.5$, $\eta_1 = \mu_2 = \mu_3 = 1$, $\gamma_2 = -1$, $z = 0$, (a) $t = -2$, (b) $t = 0$, (c) $t = 2$.

where η_1, γ_1, μ_1 and k_1 are free real constants. Substituting these results into (4), we have

$$\xi = e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} k_1 + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}. \tag{18}$$

Substituting Eq. (18) into $u = 2[\ln \xi]_x$, the fifth mixed-type periodic solution reads as

$$u_5 = 2\left[e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} k_1\eta_1 - e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} \eta_1\right] \Big/ \left[e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}} k_1 + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(4\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}\right]. \tag{19}$$

The physical structure of Eq. (19) is demonstrated in Fig. 4.

Case (6)

$$k_1 = k_3 = \eta_2 = \mu_2 = 0, \quad \nu_1 = \frac{\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \quad \nu_2 = -\frac{3\eta_1\gamma_2}{\mu_1}, \tag{20}$$

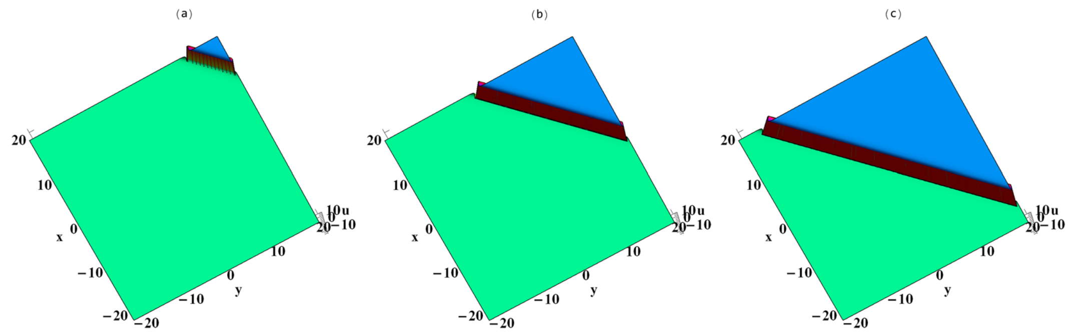

FIG. 4. Solution (19) at $\gamma_1 = -1$, $\eta_1 = \mu_1 = 1$, $k_1 = -2$, $z = 20$, (a) $t = -2$, (b) $t = 0$, (c) $t = 2$.

where $\eta_1, \gamma_1, \gamma_2, \mu_1$, and k_2 are free real constants. Then

$$\xi = k_2 \tan\left(z\gamma_2 - \frac{3t\eta_1\gamma_2}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}. \tag{21}$$

Substituting Eq. (21) into $u = 2[\ln\xi]_x$, the sixth mixed-type periodic solution is given by

$$u_6 = -\frac{2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}\eta_1}{k_2 \tan\left(z\gamma_2 - \frac{3t\eta_1\gamma_2}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}}. \tag{22}$$

The physical structure of Eq. (22) is shown in Fig. 5.

Case (7)

$$k_1 = k_2 = \eta_3 = \mu_3 = 0, \quad \nu_1 = \frac{\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \quad \nu_3 = -\frac{3\eta_1\gamma_3}{\mu_1}, \tag{23}$$

where $\eta_1, \gamma_1, \gamma_3, \mu_1$, and k_3 are free real constants. Then,

$$\xi = k_3 \tanh\left(z\gamma_3 - \frac{3t\eta_1\gamma_3}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}. \tag{24}$$

Substituting Eq. (24) into $u = 2[\ln\xi]_x$, the seventh mixed-type periodic solution is obtained by

$$u_7 = -\frac{2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}\eta_1}{k_3 \tanh\left(z\gamma_3 - \frac{3t\eta_1\gamma_3}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t(\eta_1^3\mu_1 - 3\eta_1\gamma_1)}{\mu_1}}}. \tag{25}$$

The physical structure of Eq. (25) is listed in Fig. 6.

III. LUMP-TYPE SOLUTIONS

To derive the lump-type solutions of Eq. (1), we have

$$\xi = (t\hbar_4 + x\hbar_1 + y\hbar_2 + z\hbar_3 + \hbar_5)^2 + (t\hbar_9 + x\hbar_6 + y\hbar_7 + z\hbar_8 + \hbar_{10})^2 + \hbar_{11} + \kappa_1 e^{t\Xi_4 + \Xi_5 + \Xi_1 x + \Xi_2 y + \Xi_3 z} + \kappa_2 e^{-t\Xi_4 - \Xi_5 - \Xi_1 x - \Xi_2 y - \Xi_3 z}, \tag{26}$$

where $\hbar_i (i = 1, \ldots, 11)$, $\kappa_i (i = 1, 2)$, and $\Xi_i (i = 1, \ldots, 5)$ are unknown constants. Substituting Eq. (26) into Eq. (3), the values of the unknown parameters in Eq. (26) are obtained as follows:

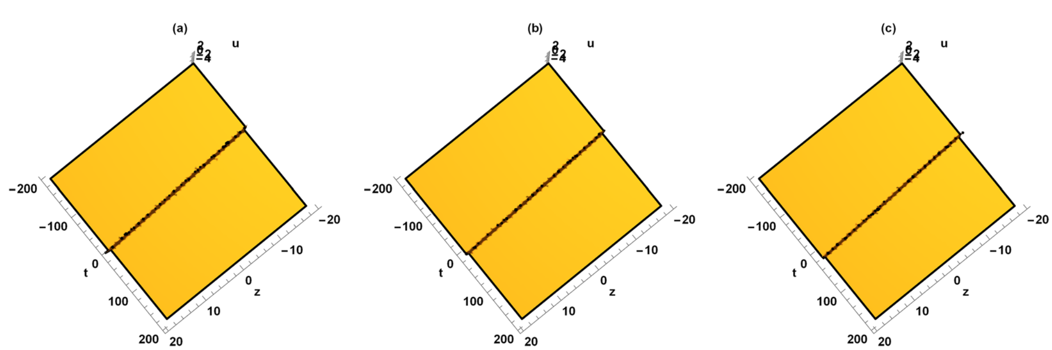

FIG. 5. Solution (22) at $\eta_1 = \mu_1 = 1$, $\gamma_1 = 1$, $\gamma_2 = k_2 = 2$, $y = 2$, (a) $x = -20$, (b) $x = 0$, (c) $x = 20$.

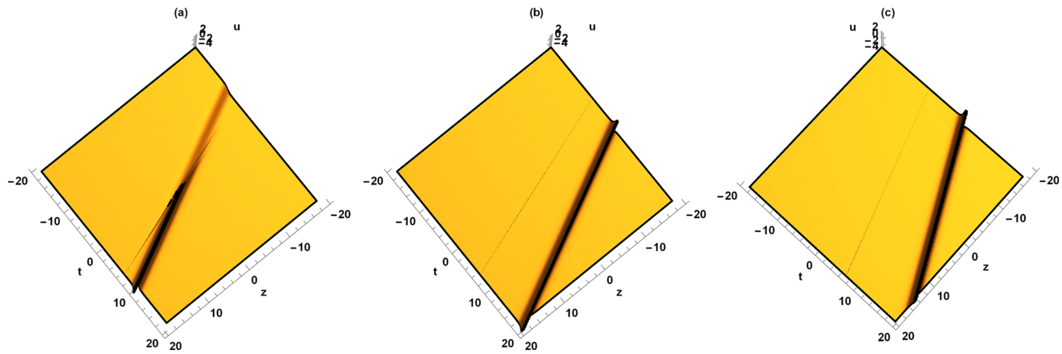

FIG. 6. Solution (25) at $\eta_1 = \mu_1 = 1$, $\gamma_1 = 1$, $\gamma_3 = k_3 = 2$, $x = 20$, (a) $y = -20$, (b) $y = 0$, (c) $y = 20$.

$$(I): \hbar_7 = \frac{\hbar_2\hbar_6}{\hbar_1}, \quad \hbar_8 = \frac{\hbar_3\hbar_6}{\hbar_1}, \quad \Xi_3 = \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1},$$
$$\kappa_2 = \frac{\left(\hbar_1^2 + \hbar_6^2\right)^2}{\kappa_1\Xi_1^4}, \quad \Xi_4 = -\frac{\Xi_1^3}{2} - \frac{3\Xi_1\hbar_3}{\hbar_2}, \quad \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2}, \tag{27}$$
$$\hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \quad \Xi_2 = -\frac{\Xi_1\hbar_2}{\hbar_1},$$

with $\hbar_1 \neq 0$, $\hbar_2 \neq 0$, $\Xi_1 \neq 0$, and $\kappa_1 \neq 0$.

$$(II): \hbar_7 = -\frac{\hbar_1\hbar_2}{\hbar_6}, \quad \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \quad \Xi_1 = \Xi_4 = 0,$$
$$\hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \quad \Xi_3 = \frac{\Xi_2\hbar_3}{\hbar_2}, \quad \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2}, \tag{28}$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$.

$$(III): \hbar_8 = \frac{3\hbar_1\hbar_2\hbar_3 + 3\hbar_6\hbar_7\hbar_3 + \hbar_4\left(\hbar_2^2 + \hbar_7^2\right)}{3\hbar_2\hbar_6 - 3\hbar_1\hbar_7}, \quad \Xi_2 = \frac{\Xi_1\epsilon_1\sqrt{\hbar_2^2 + \hbar_7^2}}{\sqrt{\hbar_1^2 + \hbar_6^2}}, \quad \hbar_9 = \frac{3\hbar_3\hbar_1^2 + \hbar_2\hbar_4\hbar_1 + \hbar_6(3\hbar_3\hbar_6 + \hbar_4\hbar_7)}{\hbar_1\hbar_7 - \hbar_2\hbar_6},$$
$$\Xi_3 = \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1}, \quad \hbar_{11} = \frac{3\kappa_1\kappa_2\Xi_2\Xi_1^3 + 3\left(\hbar_1^2 + \hbar_6^2\right)(\hbar_1\hbar_2 + \hbar_6\hbar_7)}{3\hbar_1\hbar_3 + \hbar_2\hbar_4},$$
$$\Xi_4 = \frac{\dfrac{3\Xi_1^4\left(\Xi_1\hbar_2\left(\hbar_2^2 + \hbar_7^2\right) + \Xi_2\left(\hbar_1\left(\hbar_7^2 - \hbar_2^2\right) - 2\hbar_2\hbar_6\hbar_7\right)\right)}{\Xi_1^2\left(\hbar_2^2 + \hbar_7^2\right) - 2\Xi_2\Xi_1(\hbar_1\hbar_2 + \hbar_6\hbar_7) + \Xi_2^2\left(\hbar_1^2 + \hbar_6^2\right)} - 2\Xi_1^3\hbar_2 - 3\Xi_1\hbar_3}{\hbar_2}, \tag{29}$$
$$\hbar_4 = \frac{3\left(\dfrac{\Xi_1^3\Xi_2(\hbar_2\hbar_6 - \hbar_1\hbar_7)^2}{\Xi_1^2\left(\hbar_2^2 + \hbar_7^2\right) - 2\Xi_2\Xi_1(\hbar_1\hbar_2 + \hbar_6\hbar_7) + \Xi_2^2\left(\hbar_1^2 + \hbar_6^2\right)} - \hbar_1\hbar_3\right)}{\hbar_2},$$

with $\hbar_2 \neq 0$, $\Xi_1 \neq 0$, $3\hbar_1\hbar_3 + \hbar_2\hbar_4 \neq 0$, $\epsilon_1 = \pm 1$, $\hbar_2\hbar_6 - \hbar_1\hbar_7 \neq 0$, and $\hbar_1^2 + \hbar_6^2 \neq 0$.

$$(IV): \hbar_7 = -\frac{\hbar_1\hbar_2}{\hbar_6}, \quad \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6},$$
$$\hbar_4 = \frac{3\Xi_1^3\Xi_2\hbar_2\left(\hbar_1^2 + \hbar_6^2\right)}{\Xi_1^2\hbar_2^2 + \Xi_2^2\hbar_6^2} - \frac{3\hbar_1\hbar_3}{\hbar_2},$$
$$\Xi_3 = \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1}, \quad \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \quad \hbar_{11} = \frac{2\kappa_1\kappa_2\Xi_1^2}{\hbar_1^2 + \hbar_6^2},$$
$$\Xi_4 = -2\Xi_1^3 + \frac{3\Xi_1^4\hbar_2(\Xi_1\hbar_2 + \Xi_2\hbar_1)}{\Xi_1^2\hbar_2^2 + \Xi_2^2\hbar_6^2} - \frac{3\Xi_1\hbar_3}{\hbar_2}, \quad \Xi_2 = \frac{\epsilon_2\Xi_1\hbar_2}{\hbar_6}, \tag{30}$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$, $\hbar_1^2 + \hbar_6^2 \neq 0$, and $\epsilon_2 = \pm 1$.

$$(V): \hbar_7 = -\frac{\hbar_1\hbar_2}{\hbar_6}, \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \quad \Xi_2 = \Xi_3 = 0, \quad \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2},$$
$$\Xi_4 = \Xi_1^3 - \frac{3\Xi_1\hbar_3}{\hbar_2}, \quad \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2}, \tag{31}$$

with $\hbar_2 \neq 0$ and $\hbar_6 \neq 0$.

$$(VI): \hbar_7 = -\frac{\hbar_1\hbar_2}{\hbar_6}, \quad \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \quad \Xi_3 = 0, \quad \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2},$$
$$\Xi_4 = \Xi_1^3 - \frac{3\Xi_1\hbar_3}{\hbar_2}, \quad \hbar_4 = -\frac{3\hbar_3\hbar_6(\hbar_1 + \hbar_6)}{\hbar_2(\hbar_6 - \hbar_1)},$$
$$\Xi_2 = \frac{\Xi_1\hbar_2\hbar_3}{\Xi_1^2\hbar_2(\hbar_1 - \hbar_6) - \hbar_3\hbar_6},$$
$$\Xi_1 = \frac{\epsilon_3\sqrt{2}\sqrt{\hbar_1^2 + \hbar_6^2}\sqrt{\dfrac{\hbar_3^2\hbar_6^2\left(\hbar_1^2 + \hbar_6^2\right)}{(\hbar_1 - \hbar_6)^2}}}{\sqrt{\dfrac{\hbar_2\hbar_3\hbar_6\left(\hbar_1^2 + \hbar_6^2\right)^2}{\hbar_1 - \hbar_6}}}, \tag{32}$$

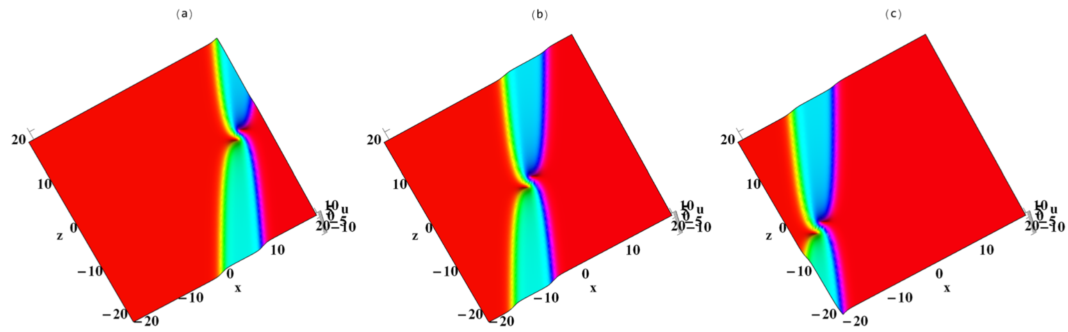

FIG. 7. Dynamical behaviors for solution (33) with $y = 0$, (a) $t = -2$, (b) $t = 0$, (c) $t = 2$.

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$, $\hbar_1 \neq \hbar_6$, and $\epsilon_3 = \pm 1$. Substituting Eqs. (27)–(32) into the variable substitution $u = 2[\ln\xi]_x$, six lump-type solutions can be derived.

As an example, substituting Eq. (26) and Eq. (27) into the variable substitution $u = 2[\ln\xi]_x$, the lump-type solution of Eq. (1) can be written as follows:

$$
\begin{aligned}
u = {} & \Bigg[\Bigg[2\Bigg[-\frac{\left(\hbar_1^2+\hbar_6^2\right)^2 \exp\left[\Xi_1\left(\frac{3t\hbar_3}{\hbar_2}-x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)+\frac{\Xi_1^3(t\hbar_1+z\hbar_2)}{2\hbar_1}-\Xi_5\right]}{\kappa_1\Xi_1^3}+\kappa_1\Xi_1\exp\left[\Xi_1\left(-\frac{3t\hbar_3}{\hbar_2}+x-\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)-\frac{\Xi_1^3(t\hbar_1+z\hbar_2)}{2\hbar_1}+\Xi_5\right] \\
& +2\hbar_1\left[\hbar_1\left(x-\frac{3t\hbar_3}{\hbar_2}\right)+y\hbar_2+z\hbar_3+\hbar_5\right]+2\hbar_6\left[\hbar_{10}+\hbar_6\left(-\frac{3t\hbar_3}{\hbar_2}+x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)\right]\Bigg]\Bigg]\Bigg]\Bigg/\Bigg[\hbar_{11}+\kappa_1\exp\Bigg[\Xi_1\left(-\frac{3t\hbar_3}{\hbar_2}+x-\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right) \\
& -\frac{\Xi_1^3(t\hbar_1+z\hbar_2)}{2\hbar_1}+\Xi_5\Bigg]+\left[\left(\hbar_1^2+\hbar_6^2\right)^2\exp\left[\Xi_1\left(\frac{3t\hbar_3}{\hbar_2}-x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)+\frac{\Xi_1^3(t\hbar_1+z\hbar_2)}{2\hbar_1}-\Xi_5\right]\right]\Big/\left(\kappa_1\Xi_1^4\right) \\
& +\left[\hbar_1\left(x-\frac{3t\hbar_3}{\hbar_2}\right)+y\hbar_2+z\hbar_3+\hbar_5\right]^2+\left[\hbar_6\left(-\frac{3t\hbar_3}{\hbar_2}+x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)+\hbar_{10}\right]^2\Bigg],
\end{aligned}
\tag{33}
$$

with the constraint $\hbar_1 \neq 0$, $\hbar_2 \neq 0$, $\Xi_1 \neq 0$, and $\kappa_1 \neq 0$.

To analyze the dynamical behaviors for solution (33), the values of parameters are selected as follows:

$$\hbar_1 = \hbar_5 = 2, \quad \hbar_2 = \hbar_6 = -1, \quad \hbar_3 = 3,$$

$$\hbar_{10} = \hbar_{11} = \Xi_5 = 0, \quad \Xi_1 = -2, \quad \kappa_1 = 1. \tag{34}$$

Substituting Eq. (34) into Eq. (33), the dynamical behaviors for solution (33) are shown in Figs. 7 and 8.

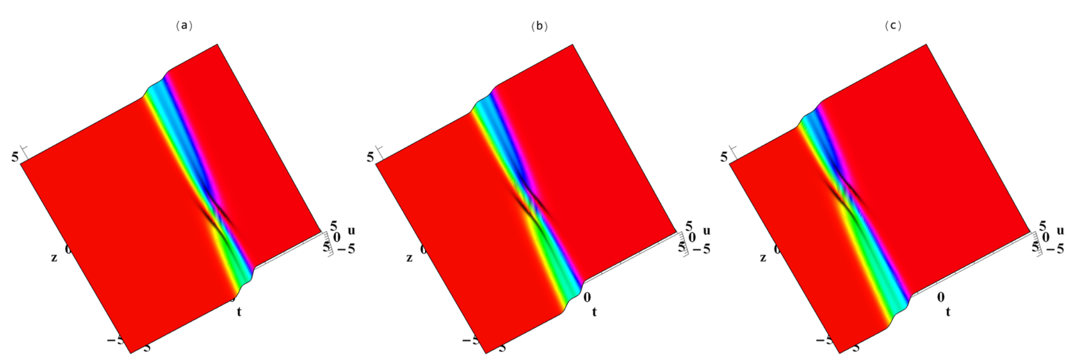

FIG. 8. Dynamical behaviors for solution (33) with $y = 0$, (a) $x = -10$, (b) $x = 0$, (c) $x = 10$.

In Fig. 7, the interaction behavior between two solitary waves and a lump wave can be found with t = -2, 0, and 2 on the $x-z$ plane. The interaction solutions reveal the characteristic of "elastic collision," that is, two solitary waves and lump wave keep their shape and velocity invariant in the process of transmission. Figure 8 demonstrates the interaction behavior between two solitary waves and a lump wave with $x = -10$, 0, and 10 on the $t-z$ plane.

IV. CONCLUSION

Based on the dependent variable transformation and Hirota's bilinear form, new mixed-type and lump-type solutions of the (3 + 1)-dimensional generalized BKP equation are presented. Moreover, Figs. 1–6 show the dynamical behaviors for the mixed-type periodic solution. Figure 7 demonstrates the interaction behavior between two solitary waves and a lump wave on the $x-z$ plane, which describes the characteristic of "elastic collision." Figure 8 reveals the interaction behavior between two solitary waves and a lump wave on the $t-z$ plane. As can be seen from the above solution process, the direct test function is very effective for solving the mixed-type periodic solutions of NPDEs. Our results show that the structures of the obtained wave solutions are multifarious in the nonlinear dynamic system. Furthermore, they not only show the efficiency of the dependent variable transformation and Hirota's bilinear method but also establish that the solutions are enriched with new mixed-type and lump-type features. We hope our research will be of great help in exploring the complex natural world. In the near future, we will modify the scheme presented here to deal with different NPDEs when their coefficients are variables for expressing nonautonomous multi-soliton, breather, and rogue wave solutions.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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