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Research article

The inequalities for the analysis of a class of ternary refinement schemes

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Abstract: The ternary refinement schemes are the generalized version of the binary refinement schemes. This class of the schemes produce the smooth curves with the less number of refinement steps as compared to the binary class of schemes. In this paper, we present the inequalities for the analysis of a class of ternary refinement schemes. There are three simple algebraic expressions in each inequality. Further these algebraic expressions contain only the coefficients used in the refinement rules of the ternary schemes.

Keywords: refinement scheme; binary; ternary; analysis; continuity; inequalities
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1. Introduction

The refinement schemes can be viewed as a class of iterative algorithms. The main feature of these schemes is that they use initially the rough sketch of any curve called polygon to finally produce smooth sketch also called refined sketch. Initially, a refinement scheme was introduced by de Rahm [3] in 1956. Later on, Chaikin [1] introduced the scheme for curve generation in 1974. Deslauriers and Dubuc presented the interpolating schemes for curve modeling in 1989. This triggered off the modern era for the investigation of the refinement schemes. These refinement schemes have been used for curve and surface modeling for a long time by designers, engineers, computer scientists and mathematicians. The details of these applications are presented by Farin [7].

Initially, the binary refinement schemes were introduced. These schemes have two rules to refine each edge of the polygon. These rules are same for each edge of the polygon except the initial and final edges. These rules are classified as even and odd rules. These schemes take initial sketch as an input and return the refined sketch as an output. In the next step, these schemes take previously refined sketch as an input and produce more refined sketch. Repeated application of this procedure give the smooth curve/shape. Nowadays, there are many generalizations of the binary schemes. The class of ternary refinement schemes is one of the generalization of the class of binary schemes. In this class of the schemes, three rules are used instead of two rules to refine each edge of the polygon.

The analysis of the schemes is the crucial issue. Initially, Dyn et al. [4], introduced the divided difference (DD) technique to analyze the schemes in 1987. Later on, Dyn et al. [5], extended this technique for the analysis of the class of binary schemes in 1991. The study of the limiting curves produced by the binary refinement schemes was also presented by Micchelli and Prautzch [11] in 1989. Sabin [15] in 2010, gave further insight in this technique. The further extension of Dyn et al.'s work was done by Mustafa and Zahid [12] in 2013. The cross-differences of directional DD techniques was introduced by Qu [14] in 1991. The further generalization of the DD technique has not been done so for.

Another technique for the smoothness analysis of the schemes was introduced by Dyn [6] in 2002. This technique is known as Laurent polynomial technique. This technique was used by Hassan and Dodgson [8], Hassan et al. [9], Mustafa et al. [13]. Khan and Mustafa [10], Siddiqi and Rehan [16], Zheng et al. [17] and many others to analyze their binary and ternary refinement schemes. Nowadays many authors are also using this technique. However, this technique also has some limitations and need improvement. In this technique, first a sequence of coefficients used in the refinement rules of the schemes is converted into the polynomial. Secondly, the polynomial has been factorized. In this technique, multiplication, division, factorization of the polynomials are involved. Further, the computation of inequalities and the comparison of the terms are also involved. The technique, we are going to introduce is the generalization of DD technique. In this technique, the computation of inequalities and the comparison of three terms are involved. There are three simple algebraic expressions in each inequality. Further these algebraic expressions contain only the coefficients used in the refinement rules of the schemes.

The rest of this paper is structured as follows. In Section 2, a general ternary refinement scheme and its DD schemes are introduced. Their convergence is also presented in this section. In Section 3, the deviation between successive levels of polygons produced by the ternary and its DD refinement schemes are presented. The inequalities for the analysis of ternary schemes are presented in Section 4. Applications of these inequalities are also presented in this section. Summary of the work and comparison are presented in Sections 5 and 6 respectively.

2. Ternary and its DD refinement schemes

We first introduce the ternary and its first, second and third order divided differences (DD) refinement schemes. Then we present their convergence in this section.

2.1. Ternary refinement scheme (TRS)

Let $f^k = \{\left(i/3^k, f_i^k\right) \in \mathbb{R}^N,\ i \in \mathbb{Z},\ N \geq 2,\ k \geq 0\}$, be a polygon at kth refinement level then the $(k+1)$th polygon f^{k+1}, can be obtained by the following three refinement rules

$$\begin{cases} f_{3i}^{k+1} = \sum\limits_{j=0}^{m} \rho_j f_{i+j}^k, \\ f_{3i+1}^{k+1} = \sum\limits_{j=0}^{m} \varrho_j f_{i+j}^k, \\ f_{3i+2}^{k+1} = \sum\limits_{j=0}^{m} \sigma_j f_{i+j}^k. \end{cases} \tag{2.1}$$

whereas $m > 0$ while if $t_i^k = i/3^k$ then $\left(t_{3i}^{k+1} = t_i^k,\ f_{3i}^{k+1}\right)$, $\left(t_{3i+1}^{k+1} = \frac{1}{3}(2t_i^k + t_{i+1}^k),\ f_{3i+1}^{k+1}\right)$ and $\left(t_{3i+2}^{k+1} = \frac{1}{3}(t_i^k + 2t_{i+1}^k),\ f_{3i+2}^{k+1}\right)$ are the points of the polygon f^{k+1}.

The above rules are the affine combinations of f_i^k's so

$$\sum_{\nu=0}^{m} \rho_\nu = \sum_{\nu=0}^{m} \varrho_\nu = \sum_{\nu=0}^{m} \sigma_\nu = 1. \tag{2.2}$$

The one step of the refinement procedure is shown in Figure 1. The repeated application of these rules is known as ternary refinement scheme.

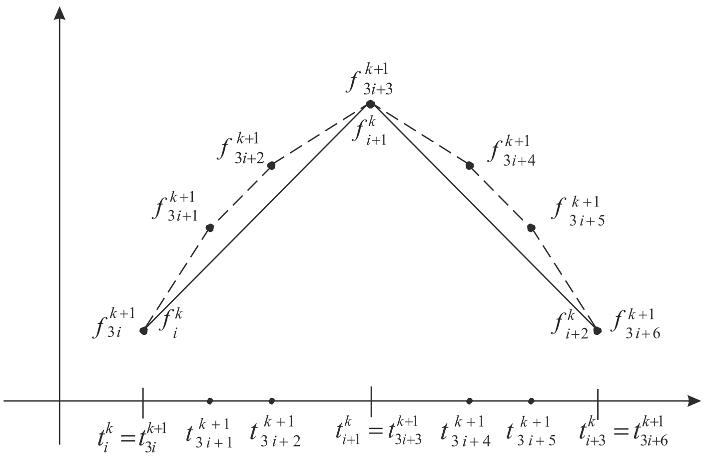

Figure 1. The part of polygons at kth and $(k+1)$th levels are shown by solid and dash lines respectively.

2.2. First to third order DD refinement schemes

The first, second and third order divided difference (DD) ternary refinement schemes (TRS) are presented in this section.

Lemma 2.1. *Let* $f^k = \{\left(i/3^k, f_i^k\right)\}$ *be a polygon of TRS at kth refinement level and* $d_j^{[1]k} = 3^k(f_{j+1}^k - f_j^k)$ *be the first divided difference then the first order DD TRS is defined as*

$$\begin{cases} d_{3i}^{[1](k+1)} = 3\sum_{\mu=0}^{m-1}\left\{\sum_{\nu=0}^{\mu}(\rho_{\mu-\nu} - \varrho_{\mu-\nu})d_{i+\mu}^{[1]k}\right\}, \\ d_{3i+1}^{[1](k+1)} = 3\sum_{\mu=0}^{m-1}\left\{\sum_{\nu=0}^{\mu}(\varrho_{\mu-\nu} - \sigma_{\mu-\nu})d_{i+\mu}^{[1]k}\right\}, \\ d_{3i+2}^{[1](k+1)} = 3\sum_{\mu=0}^{m}\left\{\sum_{\nu=0}^{\mu}(\sigma_{\mu-\nu} - \rho_{\mu-\nu-1})d_{i+\mu}^{[1]k}\right\}. \end{cases} \tag{2.3}$$

Proof. Since the first order DD is defined as

$$d_\nu^{[1]k} = 3^k(f_{\nu+1}^k - f_\nu^k), \tag{2.4}$$

therefore by (2.1), we get

$$\begin{aligned} d_{3i}^{[1](k+1)} &= 3^{k+1}\left\{(\varrho_0 - \rho_0)\, f_i^k + (\varrho_1 - \rho_1)\, f_{i+1}^k + (\varrho_2 - \rho_2)\, f_{i+2}^k + \dots \right. \\ &\quad \left. + (\varrho_{m-1} - \rho_{m-1})\, f_{i+m-1}^k + (\varrho_m - \rho_m)\, f_{i+m}^k\right\}. \end{aligned} \tag{2.5}$$

Now consider the linear combination

$$d_{3i}^{[1](k+1)} = y_0 d_i^{[1]k} + y_1 d_{i+1}^{[1]k} + y_2 d_{i+2}^{[1]k} + \dots + y_{m-1} d_{i+(m-1)}^{[1]k}. \tag{2.6}$$

The goal is to find the unknown $y_0, \cdots, y_{m-1}$. It can be written as

$$\begin{aligned} d_{3i}^{[1](k+1)} &= 3^k\left\{-y_0 f_i^k + (y_0 - y_1) f_{i+1}^k + (y_1 - y_2) f_{i+2}^k + \dots + (y_{m-3} - y_{m-2}) f_{m-2}^k \right. \\ &\quad \left. + (y_{m-2} - y_{m-1}) f_{i+(m-1)}^k + y_{m-1} f_{i+m}^k\right\}. \end{aligned} \tag{2.7}$$

Comparing (2.5) and (2.7), we get

$$\begin{aligned} y_0 &= 3(\rho_0 - \varrho_0), \\ y_1 &= 3(\rho_0 - \varrho_0) + 3(\rho_1 - \varrho_1), \\ y_2 &= 3(\rho_0 - \varrho_0) + 3(\rho_1 - \varrho_1) + 3(\rho_2 - \varrho_2), \\ &\vdots \\ y_{m-1} &= 3(\rho_0 - \varrho_0) + 3(\rho_1 - \varrho_1) + \cdots + 3(\rho_{m-1} - \varrho_{m-1}). \end{aligned}$$

Substituting in (2.6), we get

$$\begin{aligned} d_{3i}^{[1](k+1)} &= 3\left[(\rho_0 - \varrho_0)\, d_i^{[1]k} + \{(\rho_0 - \varrho_0) + (\rho_1 - \varrho_1)\}\, d_{i+1}^{[1]k} + \{(\rho_0 - \varrho_0) + (\rho_1 - \right. \\ &\quad \varrho_1) + (\rho_2 - \varrho_2)\}\, d_{i+2}^{[1]k} + \dots + \{(\rho_0 - \varrho_0) + (\rho_1 - \varrho_1) + \dots \\ &\quad \left. + (\rho_{m-1} - \varrho_{m-1})\}\, d_{i+(m-1)}^{[1]k}\right]. \end{aligned}$$

This implies

$$d_{3i}^{[1](k+1)} = 3\sum_{\mu=0}^{m-1}\left\{\sum_{\nu=0}^{\mu}(\rho_{\mu-\nu} - \varrho_{\mu-\nu})d_{i+\mu}^{[1]k}\right\}.$$

Similarly, we get

$$d_{3i+1}^{[1](k+1)} = 3\sum_{\mu=0}^{m-1}\left\{\sum_{\nu=0}^{\mu}(\varrho_{\mu-\nu} - \sigma_{\mu-\nu})d_{i+\mu}^{[1]k}\right\},$$

and

$$d_{3i+2}^{[1](k+1)} = 3\sum_{\mu=0}^{m}\left\{\sum_{\nu=0}^{\mu}(\sigma_{\mu-\nu} - \rho_{\mu-\nu-1})d_{i+\mu}^{[1]k}\right\}.$$

This completes the proof. □

By adopting the similar procedure, we get the following results.

Lemma 2.2. *Let $f^k = \{\left(i/3^k, f_i^k\right)\}$ be a polygon of TRS at kth refinement level and*

$$d_j^{[2]k} = 3^{2k}(2!)^{-1}\left(f_{j-1}^k - 2f_j^k + f_{j+1}^k\right)$$

be the second order divided difference then the second order DD TRS is defined as

$$\begin{cases} d_{3i}^{[2](k+1)} = 3^2\sum\limits_{\mu=0}^{m-1}\left[\sum\limits_{\nu=0}^{\mu}\left\{(\nu+1)\sigma_{\mu-\nu} + \nu\varrho_{\mu-\nu} - 2\nu\rho_{\mu-\nu}\right\}d_{i+\mu}^{[2]k}\right], \\ d_{3i+1}^{[2](k+1)} = 3^2\sum\limits_{\mu=0}^{m-2}\left[\sum\limits_{\nu=0}^{\mu}\left\{(\nu+1)(\rho_{\mu-\nu} + \sigma_{\mu-\nu} - 2\varrho_{\mu-\nu})\right\}d_{i+\mu+1}^{[2]k}\right], \\ d_{3i+2}^{[2](k+1)} = 3^2\sum\limits_{\mu=0}^{m-1}\left[\sum\limits_{\nu=0}^{\mu}\left\{\nu\rho_{\mu-\nu} + (\nu+1)\varrho_{\mu-\nu} - (2\nu+2)\sigma_{\mu-\nu}\right\}d_{i+\mu+1}^{[2]k}\right]. \end{cases} \tag{2.8}$$

Lemma 2.3. *Let $f^k = \{\left(i/3^k, f_i^k\right)\}$ be a polygon of TRS at kth refinement level and*

$$d_j^{[3]k)} = 3^{3k}(3!)^{-1}\left(-f_{j-1}^k + 3f_j^k - 3f_{j+1}^k + f_{j+2}^k\right)$$

be the third order divided difference then the third order DD TRS is defined as

$$\begin{cases} d_{3i}^{[3](k+1)} = 3^3\sum\limits_{\mu=0}^{m-2}\left[\sum\limits_{\nu=0}^{\mu}\left\{(\nu+1)\sigma_{\mu-\nu} - \frac{3\nu(\nu+1)}{2}\left(\rho_{\mu-\nu} - \varrho_{\mu-\nu}\right)\right\}d_{i+\mu}^{[3]k}\right], \\ d_{3i+1}^{[3](k+1)} = 3^3\sum\limits_{\mu=0}^{m-2}\left[\sum\limits_{\nu=0}^{\mu}\left\{(\nu+1)\rho_{\mu-\nu} - \frac{3(\nu+1)(\nu+2)}{2}(\varrho_{\mu-\nu} - \sigma_{\mu-\nu})\right\}d_{i+\mu+1}^{[3]k}\right], \\ d_{3i+2}^{[3](k+1)} = 3^3\sum\limits_{\mu=0}^{m-2}\left[\sum\limits_{\nu=0}^{\mu}\left\{\frac{3\nu(\nu+1)}{2}\rho_{\mu-\nu} + (\nu+1)\varrho_{\mu-\nu} - \frac{3(\nu+1)(\nu+2)}{2}\sigma_{\mu-\nu}\right\}d_{i+\mu+1}^{[3]k}\right]. \end{cases} \tag{2.9}$$

2.3. The convergence of ternary and its DD refinement schemes

The convergence of first, second and third order divided difference (DD) ternary refinement schemes (TRS) is presented in this section. Throughout the paper, $\pi[0,n]$ denotes the set of continuous functions on the closed and bounded interval $[0,n]$.

Lemma 2.4. *Let $f^k = \left\{\left(i/3^k, f_i^k\right)\right\}$ be a polygon of TRS at kth refinement level and $d^{[1]k} = \left\{\left(i/3^k, d_i^{[1]k}\right)\right\}$ be a polygon of first order DD scheme. If $lim_{k\to\infty} d^{[1]k} = d^{[1]} \in \pi[0,n]$, and f is the limiting curve/shape produced by TRS then $d^{[1]} = f'$.*

Proof. Bernstein polynomial for $x \in [0,n]$ with data values f_i^k is

$$B_k(x) = \sum_{i=0}^{s} \binom{s}{i} \left(\frac{x}{n}\right)^i \left(1-\frac{x}{n}\right)^{s-i} f_i^k.$$

Its first derivative is

$$\begin{aligned} B_k'(x) &= \sum_{i=0}^{s} \binom{s}{i} \frac{i}{n} \left(\frac{x}{n}\right)^{i-1} \left(1-\frac{x}{n}\right)^{s-i} f_i^k \\ &\quad + \sum_{i=0}^{s} \binom{s}{i} \left(\frac{x}{n}\right)^i (s-i) \left(\frac{-1}{n}\right) \left(1-\frac{x}{n}\right)^{s-i-1} f_i^k. \end{aligned} \tag{2.10}$$

This implies

$$B_k'(x) = \frac{s}{n} \sum_{i=0}^{s-1} \binom{s-1}{i} \left(\frac{x}{n}\right)^i \left(1-\frac{x}{n}\right)^{s-i-1} \left(f_{i+1}^k - f_i^k\right).$$

For $s = 3^k n$, we get

$$B_k'(x) = \sum_{i=0}^{s-1} \binom{s-1}{i} \left(\frac{x}{n}\right)^i \left(1-\frac{x}{n}\right)^{s-i-1} 3^k \left(f_{i+1}^k - f_i^k\right).$$

This again implies

$$B_k'(x) = \sum_{i=0}^{s-1} \binom{s-1}{i} \left(\frac{x}{n}\right)^i \left(1-\frac{x}{n}\right)^{s-i-1} d_i^{[1]k}.$$

Since the Bernstien polynomials are uniformly convergent therefore $lim_{k\to\infty} B_k = f$ and $lim_{k\to\infty} B_k' = d^{[1]}$ on $[0,n]$. So $f' = d^{[1]} \in \pi[0,n]$. Hence the proof. □

Lemma 2.5. *Let $f^k = \left\{\left(i/3^k, f_i^k\right)\right\}$ and $d^{[2]k} = \left\{\left(i/3^k, d_i^{[2]k}\right)\right\}$ be the polygons of TRS and its second order DD schemes respectively. If $lim_{k\to\infty} d^{[2]k} = d^{[2]} \in \pi[0,n]$, and f is the limiting curve produced by TRS then $d^{[2]} = f''$.*

Proof. Take the derivative of (2.10)

$$B_k''(x) = \frac{1}{n^2}\left\{s(s-1)\sum_{i=2}^{s}\frac{(s-2)!}{(i-2)!(s-i)!}\left(\frac{x}{n}\right)^{i-2}\left(1-\frac{x}{n}\right)^{s-i} f_i^k\right.$$
$$-2s(s-1)\sum_{i=1}^{s-1}\frac{(s-2)!}{(i-1)!(s-i-1)!}\left(\frac{x}{n}\right)^{i-1}\left(1-\frac{x}{n}\right)^{s-i-1} f_i^k$$
$$\left.+\ s(s-1)\sum_{i=0}^{s-2}\frac{(s-2)!}{i!(s-i-2)!}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-2} f_i^k\right\}. \quad (2.11)$$

This implies

$$B_k''(x) = \frac{s(s-1)}{n^2}\left\{\sum_{i=0}^{s-2}\frac{(s-2)!}{(i)!(s-i-2)!}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-2}\times\right.$$
$$\left.\left(f_{i+2}^k - 2f_{i+1} + f_i^k\right)\right\}.$$

Since $s = 3^k n$ therefore $\frac{s(s-1)}{n^2} = 3^{2k}2^{-1}$, so we get

$$B_k''(x) = 3^{2k}2^{-1}\left\{\sum_{i=0}^{s-2}\frac{(s-2)!}{(i)!(s-i-2)!}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-2}\times\right.$$
$$\left.\left(f_{i+2}^k - 2f_{i+1} + f_i^k\right)\right\}.$$

This implies

$$B_k''(x) = \sum_{i=0}^{s-2}\left\{\frac{(s-2)!}{(i)!(s-i-2)!}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-2}\right\} d_{i+1}^{[2]k},$$

where $d_{i+1}^{[2]k} = 3^{2k}(2!)^{-1}(f_{i+2}^k - 2f_{i+1}^k + f_i^k)$.

Again the uniform convergence of the Bernstien polynomials implies $lim_{k\to\infty}B_k'' = d^{[2]}$ and $lim_{k\to\infty}B_k = f$. Therefore when $k \to \infty$ then $d^{[2]}$ converges uniformly to f'' on $[0,n]$. This concludes $d^{[2]} = f'' \in \pi[0,n]$. Hence the proof. □

Lemma 2.6. *Let $f^k = \{(i/3^k, f_i^k)\}$ and $d^{[3]k)} = \{(i/3^k, d_i^{[3]k})\}$ be the polygons of TRS and its third order DD schemes respectively. If $lim_{k\to\infty}d^{[3]k} = d^{[3]} \in \pi[0,n]$ then $d^{[3]} = f'''$.*

Proof. Now take the derivative of (2.11), then

$$\begin{aligned} B_k'''(x) &= \frac{1}{n^3}\sum_{i=3}^{s}\frac{s(s-1)(s-2)(s-3)!}{(i-3)!(s-i)!}\left(\frac{x}{n}\right)^{i-3}\left(1-\frac{x}{n}\right)^{s-i} f_i^k \\ &-\frac{3}{n^3}\sum_{i=2}^{s-1}\frac{s(s-1)(s-2)(s-3)!}{(i-2)!(s-i-1)!}\left(\frac{x}{n}\right)^{i-2}\left(1-\frac{x}{n}\right)^{s-i-1} f_i^k \\ &+\frac{3}{n^3}\sum_{i=1}^{s-2}\frac{s(s-1)(s-2)(s-3)!}{(i-1)!(s-i-2)!}\left(\frac{x}{n}\right)^{i-1}\left(1-\frac{x}{n}\right)^{s-i-2} f_i^k \\ &-\frac{1}{n^3}\sum_{i=0}^{s-3}\frac{s(s-1)(s-2)(s-3)!}{i!(s-i-3)!}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-3} f_i^k. \end{aligned}$$

Since, $s = 3^k n$, then

$$\begin{aligned} B_k'''(x) &= 3^{3k}(3!)^{-1}\sum_{i=0}^{s-3}\binom{s-3}{i}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-3}\times \\ &\quad \left(-f_i^k + 3f_{i+1}^k - 3f_{i+2}^k + f_{i+3}^k\right). \end{aligned}$$

This implies

$$B_k'''(x) = \sum_{i=0}^{s-3}\binom{s-3}{i}\left(\frac{x}{n}\right)^{i}\left(1-\frac{x}{n}\right)^{s-i-3} d_{i+1}^{[3]k},$$

where

$$d_{i+1}^{[3]k} = 3^{3k}(3!)^{-1}\left(-f_i^k + 3f_{i+1}^k - 3f_{i+2}^k + f_{i+3}^k\right).$$

This implies $B_k''' \to d^{[3]}$ and $B_k \to f$. This implies that for $k \to \infty$ the $d^{[3]}$ converges uniformly to f''' on $[0, n]$. This concludes that $d^{[3]} = f''' \in \pi[0, n]$. Hence the proof. □

3. The deviation of ternary and its DD refinement schemes

We first introduce the inequalities to compute the deviation between two consecutive points at the same refinement level then we introduce the inequalities to compute the deviation between successive levels of polygons produced by ternary and its DD refinement schemes.

Lemma 3.1. *If $f^0 = \{(i/3^0, f_i^0)\}$ is the initial polygon and $f^{k+1} = \{(i/3^{k+1}, f_i^{k+1})\}$ is the polygon obtained by TRS at $(k+1)$th refinement level. If $\varpi < 1$ then the deviation between two consecutive points at $(k+1)$th level is*

$$\max_i \left\| f_{i+1}^{k+1} - f_i^{k+1} \right\| \le (\varpi)^{k+1} \max_i \left\| f_{i+1}^0 - f_i^0 \right\|, \tag{3.1}$$

where

$$\varpi = \max\{\varpi_1, \varpi_2, \varpi_3\} < 1, \tag{3.2}$$

$$\begin{cases} \varpi_1 = \sum\limits_{\mu=0}^{m-1} \left| \sum\limits_{\nu=0}^{\mu} \left(\rho_{\mu-\nu} - \varrho_{\mu-\nu}\right) \right|, \\ \varpi_2 = \sum\limits_{\mu=0}^{m-1} \left| \sum\limits_{\nu=0}^{\mu} (\varrho_{\mu-\nu} - \sigma_{\mu-\nu}) \right|, \\ \varpi_3 = \sum\limits_{\mu=0}^{m} \left| \sum\limits_{\nu=0}^{\mu} (\sigma_{\mu-\nu} - \rho_{\mu-\nu-1}) \right|. \end{cases} \tag{3.3}$$

Proof. From (2.1), we get

$$f_{3i+1}^{k+1} - f_{3i}^{k+1} = \sum_{\nu=0}^{m} \varrho_\nu f_{i+\nu}^k - \sum_{\nu=0}^{m} \rho_\nu f_{i+\nu}^k = \sum_{\nu=0}^{m} (\varrho_\nu - \rho_\nu) f_{i+\nu}^k.$$

This further implies

$$\begin{aligned} f_{3i+1}^{k+1} - f_{3i}^{k+1} &= \Big[(\rho_0 - \varrho_0)\left(f_{i+1}^k - f_i^k\right) + \{(\rho_0 - \varrho_0) + (\rho_1 - \varrho_1)\}\left(f_{i+2}^k - f_{i+1}^k\right) \\ &\quad + \{(\rho_0 - \varrho_0) + (\rho_1 - \varrho_1) + (\rho_2 - \varrho_2)\}\left(f_{i+3}^k - f_{i+2}^k\right) + \ldots \\ &\quad + \{(\rho_0 - \varrho_0) + (\rho_1 - \varrho_1) + \ldots + (\rho_{m-1} - \varrho_{m-1})\}\left(f_{i+m}^k - f_{i+m-1}^k\right) \\ &\quad + \{(\varrho_0 - a_0) + (\varrho_1 - \rho_1) + \ldots + (\varrho_m - \rho_m)\} f_{i+m}^k\Big]. \end{aligned}$$

This leads to

$$f_{3i+1}^{k+1} - f_{3i}^{k+1} = \sum_{\mu=0}^{m-1} \left\{ \sum_{\nu=0}^{\mu} \left(\rho_{\mu-\nu} - \varrho_{\mu-\nu}\right) \right\} \left(f_{i+\mu+1}^k - f_{i+\mu}^k\right) + \sum_{\nu=0}^{m} (\varrho_\nu - \rho_\nu) f_{i+m}^k.$$

Since by (2.2), $\sum\limits_{\nu=0}^{m} (\varrho_\nu - \rho_\nu) = 0$, then

$$f_{3i+1}^{k+1} - f_{3i}^{k+1} = \sum_{\mu=0}^{m-1} \left\{ \sum_{\nu=0}^{\mu} (\rho_{\mu-\nu} - \varrho_{\mu-\nu}) \left(f_{i+\mu+1}^k - f_{i+\mu}^k\right) \right\}.$$

By taking maximum norm, we get

$$\left\| f_{3i+1}^{k+1} - f_{3i}^{k+1} \right\| \leq \varpi_1 \max_i \left\| f_{i+1}^k - f_i^k \right\|, \tag{3.4}$$

where

$$\varpi_1 = \sum_{\mu=0}^{m-1} \left| \sum_{\nu=0}^{\mu} \left(\rho_{\mu-\nu} - \varrho_{\mu-\nu}\right) \right|.$$

Using (2.1) and similar procedure as above, we get

$$\left\| f_{3i+2}^{k+1} - f_{3i+1}^{k+1} \right\| \leq \varpi_2 \max_i \left\| f_{i+1}^k - f_i^k \right\|, \tag{3.5}$$

where

$$\varpi_2 = \sum_{\mu=0}^{m-1} \left| \sum_{\nu=0}^{\mu} (\varrho_{\mu-\nu} - \sigma_{\mu-\nu}) \right|.$$

and

$$\left\| f_{3i+3}^{k+1} - f_{3i+2}^{k+1} \right\| \leq \varpi_3 \max_i \left\| f_{i+1}^{k} - f_i^{k} \right\|, \tag{3.6}$$

where

$$\varpi_3 = \sum_{\mu=0}^{m} \left| \sum_{\nu=0}^{\mu} (\sigma_{\mu-\nu} - \rho_{\mu-\nu-1}) \right|.$$

Combining (3.4), (3.5) and (3.6), it proceeds as

$$\max_i \left\| f_{i+1}^{k+1} - f_i^{k+1} \right\| \leq \varpi \max_i \left\| f_{i+1}^{k} - f_i^{k} \right\|,$$

where

$$\varpi = \max\{\varpi_1, \varpi_2, \varpi_3\}.$$

This concludes

$$\max_i \left\| f_{i+1}^{k+1} - f_i^{k+1} \right\| \leq (\varpi)^{k+1} \max_i \left\| f_{i+1}^{0} - f_i^{0} \right\|.$$

This completes the proof. □

Theorem 3.2. *If $f^0 = \{\left(i/3^0, f_i^0\right)\}$ is the initial polygon and $f^{k+1} = \{\left(i/3^{k+1}, f_i^{k+1}\right)\}$ is the polygon obtained by TRS at $(k+1)$th refinement level. If $\varpi^* < 1$ then the deviation between two successive polygons at kth and $(k+1)$th levels is*

$$\left\| f^{k+1} - f^{k} \right\|_\infty \leq \varpi^* \max_i \left\| f_{i+1}^{k} - f_i^{k} \right\|, \tag{3.7}$$

where

$$\varpi^* = \max\{\varpi_4, \varpi_5, \varpi_6\}, \tag{3.8}$$

$$\begin{cases} \varpi_4 = \sum\limits_{\mu=0}^{m-1} \left| 1 - \sum\limits_{\nu=0}^{\mu} \rho_{\mu-\nu} \right|, \\ \varpi_5 = \left| \frac{2}{3} - \varrho_0 \right| + \sum\limits_{\mu=1}^{m-1} \left| 1 - \sum\limits_{\nu=0}^{\mu} \varrho_{\mu-\nu} \right|, \\ \varpi_6 = \left| \frac{1}{3} - \sigma_0 \right| + \sum\limits_{\mu=1}^{m-1} \left| 1 - \sum\limits_{\nu=0}^{\mu} \sigma_{\mu-\nu} \right|. \end{cases} \tag{3.9}$$

Proof. Since the maximum deviation between f^{k+1} and f^k occurs at the diadic values $t_{3i}^{k+1} = t_i^k$, $t_{3i+1}^{k+1} = \frac{1}{3}(2t_i^k + t_{i+1}^k)$ and $t_{3i+2}^{k+1} = \frac{1}{3}(t_i^k + 2t_{i+1}^k)$ respectively, therefore

$$\left\| f^{k+1} - f^{k} \right\|_\infty \leq \max\{Z_k^1, Z_k^2, Z_k^3\}, \tag{3.10}$$

where

$$\begin{cases} Z_k^1 = \max_i \left\| f_{3i}^{k+1} - f_i^k \right\|, \\ Z_k^2 = \max_i \left\| f_{3i+1}^{k+1} - \frac{1}{3}\left(2f_i^k + f_{i+1}^k\right) \right\|, \\ Z_k^3 = \max_i \left\| f_{3i+2}^{k+1} - \frac{1}{3}\left(f_i^k + 2f_{i+1}^k\right) \right\|. \end{cases} \tag{3.11}$$

From (2.1), we obtain

$$f_{3i}^{k+1} - f_i^k = \sum_{\nu=0}^{m} \rho_\nu f_{i+\nu}^k - f_i^k.$$

Since by (2.1), $\sum_{\nu=0}^{m} \rho_\nu - 1 = 0$, then

$$f_{3i}^{k+1} - f_i^k = \sum_{\mu=0}^{m-1} \left(1 - \sum_{\nu=0}^{\mu} \rho_{\mu-\nu}\right)\left(f_{i+\mu+1}^k - f_{i+\mu}^k\right).$$

Taking norm, we get

$$\left\| f_{3i}^{k+1} - f_i^k \right\| \leq \varpi_4 \max_i \left\| f_{i+1}^k - f_i^k \right\|, \tag{3.12}$$

where

$$\varpi_4 = \sum_{\mu=0}^{m-1} \left| 1 - \sum_{\nu=0}^{\mu} \rho_{\mu-\nu} \right|.$$

Using (2.1) and similar procedure as above, we get

$$\left\| f_{3i+1}^{k+1} - \left(\frac{2}{3} f_i^k + \frac{1}{3} f_{i+1}^k\right) \right\| \leq \varpi_5 \max_i \left\| f_{i+1}^k - f_i^k \right\|, \tag{3.13}$$

where

$$\varpi_5 = \left| \frac{2}{3} - \varrho_0 \right| + \sum_{\mu=1}^{m-1} \left| 1 - \sum_{\nu=0}^{\mu} \varrho_{\mu-\nu} \right|.$$

$$\left\| f_{3i+2}^{k+1} - \left(\frac{1}{3} f_i^k + \frac{2}{3} f_{i+1}^k\right) \right\| \leq \varpi_6 \max_i \left\| f_{i+1}^k - f_i^k \right\|, \tag{3.14}$$

where

$$\varpi_6 = \left| \frac{1}{3} - \sigma_0 \right| + \sum_{\mu=1}^{m-1} \left| 1 - \sum_{\nu=0}^{\mu} \sigma_{\mu-\nu} \right|.$$

From (3.10)–(3.14), we get

$$\left\| f^{k+1} - f^k \right\|_\infty \leq \varpi^* \max_i \left| f_{i+1}^k - f_i^k \right|,$$

where

$$\varpi^* = \max\left\{\varpi_4, \varpi_5, \varpi_6\right\}.$$

Hence the proof. □

Lemma 3.3. *If* $d^{[1]0} = \{(i/3^0, d_i^{[1]0})\}$ *is the initial polygon and* $d^{[1](k+1)} = \{(i/3^{k+1}, d_i^{[1](k+1)})\}$ *is the polygon obtained by first order DD TRS at* $(k+1)$*th refinement level. If* $\varpi^{**} < 1$ *and*

$$\begin{cases} c_1 = \sum\limits_{\nu=0}^{m-1}(m-\nu)(2\varrho_\nu - \sigma_\nu - \rho_\nu) = 0, \\ c_2 = \sum\limits_{\nu=0}^{m}\{(2\nu+1)\sigma_{m-\nu} - (m-\nu)(\rho_\nu + \varrho_\nu)\} = 0, \\ c_3 = \sum\limits_{\nu=0}^{m}\{2\nu\rho_{m-\nu} - \nu\varrho_{m-\nu} - (\nu+1)\sigma_{m-\nu}\} = 0, \end{cases} \tag{3.15}$$

then the deviation between two consecutive points at $(k+1)$*th level is*

$$\max_i \left\| d_{i+1}^{[1](k+1)} - d_i^{[1](k+1)} \right\| \leq (\varpi^{**})^{k+1} \max_i \left\| d_{i+1}^{[1]0} - d_i^{[1]0} \right\|, \tag{3.16}$$

where

$$\varpi^{**} = \max\{\varpi_1^*, \varpi_2^*, \varpi_3^*\}, \tag{3.17}$$

$$\begin{cases} \varpi_1^* = 3\sum\limits_{\mu=0}^{m-2}\left|\sum\limits_{\nu=0}^{\mu}(\nu+1)(\rho_{\mu-\nu} - 2\varrho_{\mu-\nu} + \sigma_{\mu-\nu})\right|, \\ \varpi_2^* = 3\sum\limits_{\mu=0}^{m-1}\left|\sum\limits_{\nu=0}^{\mu}\left\{(\nu+1)\varrho_{\mu-\nu} - (2\nu+2)\sigma_{\mu-\nu} + \nu\rho_{\mu-\nu}\right\}\right|, \\ \varpi_3^* = 3\sum\limits_{\mu=0}^{m-1}\left|\sum\limits_{\nu=0}^{\mu}\{(\nu+1)\sigma_{\mu-\nu} - 2\nu\rho_{\mu-\nu} + \nu\varrho_{\mu-\nu}\}\right|. \end{cases} \tag{3.18}$$

Proof. Using (2.3) and proceeding similarly as in Lemma 3.1, we get

If, $c_1 = \sum\limits_{\nu=0}^{m-1}(m-\nu)(2\varrho_\nu - \sigma_\nu - \rho_\nu) = 0$, then

$$\left\| d_{3i+1}^{[1](k+1)} - d_{3i}^{[1](k+1)} \right\| \leq \varpi_1^* \max_i \left\| d_{i+1}^{[1]k} - d_i^{[1]k} \right\|, \tag{3.19}$$

where

$$\varpi_1^* = 3\sum_{\mu=0}^{m-2}\left|\sum_{\nu=0}^{\mu}(\nu+1)(\rho_{\mu-\nu} - 2\varrho_{\mu-\nu} + \sigma_{\mu-\nu})\right|.$$

Using (2.3) and similar procedure as above, we get $c_2 = \sum\limits_{\nu=0}^{m}\{(2\nu+1)\sigma_{m-\nu} - (m-\nu)(\rho_\nu + \varrho_\nu)\} = 0$,

$$\left\| d_{3i+2}^{[1](k+1)} - d_{3i+1}^{[1](k+1)} \right\| \leq \varpi_2^* \max_i \left\| d_{i+1}^{[1]k} - d_i^{[1]k} \right\|. \tag{3.20}$$

where

$$\varpi_2^* = 3\sum_{\mu=0}^{m-1}\left|\sum_{\nu=0}^{\mu}\left\{(\nu+1)\varrho_{\mu-\nu} - (2\nu+2)\sigma_{\mu-\nu} + \nu\rho_{\mu-\nu}\right\}\right|.$$

And
$c_3 = \sum_{\nu=0}^{m} \{2\nu\rho_{m-\nu} - \nu\varrho_{m-\nu} - (\nu+1)\sigma_{m-\nu}\} = 0,$

$$\left\|d_{3i+3}^{[1](k+1)} - d_{3i+2}^{[1](k+1)}\right\| \leq \varpi_3^* \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|, \tag{3.21}$$

where

$$\varpi_3^* = 3\sum_{\mu=0}^{m-1}\left|\sum_{\nu=0}^{\mu}\left\{(\nu+1)\sigma_{\mu-\nu} - 2\nu\rho_{\mu-\nu} + \nu\varrho_{\mu-\nu}\right\}\right|.$$

Combining Eqs (3.19), (3.20) and (3.21), we get

$$\max_i \left\|d_{i+1}^{[1](k+1)} - d_i^{[1](k+1)}\right\| \leq \varpi^{**} \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|,$$

whereas

$$\varpi^{**} = \max\{\varpi_1^*, \varpi_2^*, \varpi_3^*\}.$$

This implies

$$\max_i \left\|d_{i+1}^{[1](k+1)} - d_i^{[1](k+1)}\right\| \leq (\varpi^{**})^{k+1} \max_i \left\|d_{i+1}^{[1]0} - d_i^{[1]0}\right\|.$$

This completes the proof. □

Theorem 3.4. *If $d^{[1]0} = \left\{\left(i/3^0, d_i^{[1]0}\right)\right\}$ is the initial polygon and $d^{[1](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[1](k+1)}\right)\right\}$ is the polygon obtained by first order DD TRS at $(k+1)$th refinement level. If $\varpi^{***} < 1$ and*

$$\begin{cases} c_4 = -\frac{1}{3} + \sum_{\nu=1}^{m} \nu(\rho_{m-\nu} - \varrho_{m-\nu}) = 0, \\ c_5 = -\frac{1}{3} + \sum_{\nu=1}^{m} \nu(\varrho_{m-\nu} - \sigma_{m-\nu}) = 0, \\ c_6 = -\frac{1}{3} + \sum_{\nu=0}^{m} \{(\nu+1)\sigma_{m-\nu} - \nu\rho_{m-\nu}\} = 0, \end{cases} \tag{3.22}$$

then the deviation between two successive polygons at kth and $(k+1)$th levels is

$$\left\|d^{[1](k+1)} - d^{[1]k}\right\|_\infty \leq \varpi^{***} \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|, \tag{3.23}$$

where

$$\varpi^{***} = \max\{\varpi_4^*, \varpi_5^*, \varpi_6^*\}, \tag{3.24}$$

$$\begin{cases} \varpi_4^* = 3\sum_{\mu=0}^{m-2}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}\left\{(\nu+1)\left(\varrho_{\mu-\nu} - \rho_{\mu-\nu}\right)\right\}\right|, \\ \varpi_5^* = 3\left|\frac{2}{3^2} - \varrho_0 + \sigma_0\right| + 3\sum_{\mu=1}^{m-2}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}\{(\nu+1)(\sigma_{\mu-\nu} - \varrho_{\mu-\nu})\}\right|, \\ \varpi_6^* = 3\left|\frac{1}{3^2} - \sigma_0\right| + 3\sum_{\mu=1}^{m-1}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}(\nu\rho_{\mu-\nu} - (\nu+1)\sigma_{\mu-\nu})\right|. \end{cases} \tag{3.25}$$

Proof. Since the maximum deviation between $d^{[1](k+1)}$ and $d^{[1]k}$ occurs at the diadic values $t_{3i}^{k+1} = t_i^k$, $t_{3i+1}^{k+1} = \frac{1}{3}(2t_i^k + t_{i+1}^k)$ and $t_{3i+2}^{k+1} = \frac{1}{3}(t_i^k + 2t_{i+1}^k)$ respectively, therefore

$$\left\|d^{[1](k+1)} - d^{[1]k}\right\|_\infty \leq \max\{Z_k^4, Z_k^5, Z_k^6\}, \tag{3.26}$$

where

$$\begin{cases} Z_k^4 = \max_i \left\|d_{3i}^{[1](k+1)} - d_i^{[1]k}\right\|, \\ Z_k^5 = \max_i \left\|d_{3i+1}^{[1](k+1)} - \frac{1}{3}(2d_i^{[1]k} + d_{i+1}^{[1]k})\right\|, \\ Z_k^6 = \max_i \left\|d_{3i+2}^{[1](k+1)} - \frac{1}{3}(d_i^{[1]k} + 2d_{i+1}^{[1]k})\right\|. \end{cases} \tag{3.27}$$

By (2.3), we get

$$d_{3i}^{[1](k+1)} - d_i^{[1]k} = 3\sum_{\mu=0}^{m-1}\left\{\sum_{\nu=0}^{\mu}(\rho_{\mu-\nu} - \varrho_{\mu-\nu})d_{i+\mu}^{[1]k}\right\} - d_i^{[1]k}.$$

This leads to

$$\begin{aligned} d_{3i}^{[1](k+1)} - d_i^{[1]k} &= 3\left[\sum_{\mu=0}^{m-2}\left\{\frac{1}{3} + \sum_{\nu=0}^{\mu}(\nu+1)(\varrho_{\mu-\nu} - \rho_{\mu-\nu})(d_{i+\mu+1}^{[1]k} - d_{i+\mu+1}^{[1]k})\right\}\right. \\ &\quad \left. + \left\{-\frac{1}{3} + \sum_{\nu=1}^{m}\nu(\rho_{m-\nu} - \varrho_{m-\nu})\right\}d_{i+m-1}^{[1]k}\right]. \end{aligned}$$

If, $c_4 = -\frac{1}{3} + \sum_{\nu=1}^{m}\nu(\rho_{m-\nu} - \varrho_{m-\nu}) = 0$, taking norm we get

$$\left\|d_{3i}^{[1](k+1)} - d_i^{[1]k}\right\| \leq \varpi_4^* \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|, \tag{3.28}$$

where,

$$\varpi_4^* = 3\sum_{\mu=0}^{m-2}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}\left\{(\nu+1)(\varrho_{\mu-\nu} - \rho_{\mu-\nu})\right\}\right|.$$

Using (2.3) and similar procedure as above, we get
$c_5 = -\frac{1}{3} + \sum_{\nu=1}^{m}\nu(\varrho_{m-\nu} - \sigma_{m-\nu}) = 0$,

$$\left\|d_{3i+1}^{[1](k+1)} - \left(\frac{2}{3}d_i^{[1]k} + \frac{1}{3}d_{i+1}^{[1]k}\right)\right\| \leq \varpi_5^* \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|, \tag{3.29}$$

where

$$\varpi_5^* = 3\left|\frac{2}{3^2} - \varrho_0 + \sigma_0\right| + 3\sum_{\mu=1}^{m-2}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}\{(\nu+1)(\sigma_{\mu-\nu} - \varrho_{\mu-\nu})\}\right|.$$

And
$c_6 = -\frac{1}{3} + \sum_{\nu=0}^{m}\{(\nu+1)\sigma_{m-\nu} - \nu\rho_{m-\nu}\} = 0$,

$$\max_i \left\|\hat{d}_{3i+2}^{k+1} - \left(\frac{1}{3}\hat{d}_i^k + \frac{2}{3}\hat{d}_{i+1}^k\right)\right\| \leq \varpi_6^* \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|, \tag{3.30}$$

where

$$\varpi_6^* = 3\left|\frac{1}{3^2} - \sigma_0\right| + 3\sum_{\mu=1}^{m-1}\left|\frac{1}{3} + \sum_{\nu=0}^{\mu}(\nu\varrho_{\mu-\nu} - (\nu+1)\sigma_{\mu-\nu})\right|.$$

From (3.26)–(3.30)

$$\left\|d^{[1](k+1)} - d^{[1]k}\right\|_\infty \leq \varpi^{***} \max_i \left\|d_{i+1}^{[1]k} - d_i^{[1]k}\right\|,$$

whereas

$$\varpi^{***} = \max\left\{\varpi_4^*, \varpi_5^*, \varpi_6^*\right\}.$$

Hence the proof. □

Lemma 3.5. *If $d^{[2]0} = \left\{\left(i/3^0, d_i^{[2]0}\right)\right\}$ is the initial polygon and $d^{[2](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[2](k+1)}\right)\right\}$ is the polygon obtained by second order DD TRS at $(k+1)$th refinement level. If $\vartheta < 1$ and*

$$\begin{cases} \chi_1 = \sum_{\nu=0}^{m-1}\left\{\frac{3\nu(\nu+1)}{2}(\rho_{m-\nu-1} - \varrho_{m-\nu-1}) - (\nu+1)\sigma_{m-\nu-1}\right\} = 0, \\ \chi_2 = \sum_{\nu=0}^{m-1}\left\{\frac{(\nu+1)(3\nu+2)}{2}\varrho_{m-1-\nu} - \frac{(\nu+1)(3\nu+4)}{2}\sigma_{m-1-\nu}\right\} = 0, \\ \chi_3 = \sum_{\nu=0}^{m-1}\left\{\frac{3(\nu+1)(\nu+2)}{2}\sigma_{m-1-\nu} - (\nu+1)\varrho_{m-1-\nu} - \frac{3\nu(\nu+1)}{2}\rho_{m-1-\nu}\right\} = 0, \end{cases} \tag{3.31}$$

then the deviation between two consecutive points at $(k+1)$th level is

$$\max_i \left\|d_{i+1}^{[2](k+1)} - d_i^{[2](k+1)}\right\| \leq (\vartheta)^{k+1} \max_i \left\|d_{i+1}^{[2]0} - d_i^{[2]0}\right\|, \tag{3.32}$$

where

$$\vartheta = \max\{\vartheta_1, \vartheta_2, \vartheta_3\}, \tag{3.33}$$

$$\begin{cases} \vartheta_1 = 3^2\sum_{\mu=0}^{m-2}\left|\sum_{\nu=0}^{\mu}\left\{(\nu+1)\sigma_{\mu-\nu} + \frac{3\nu(\nu+1)}{2}(\varrho_{\mu-\nu} - \rho_{\mu-\nu})\right\}\right|, \\ \vartheta_2 = 3^2\sum_{\mu=0}^{m-2}\left|\sum_{\nu=0}^{\mu}\left\{(\nu+1)\rho_{\mu-\nu} + \frac{3(\nu+1)(\nu+2)}{2}(\sigma_{\mu-\nu} - \varrho_{\mu-\nu})\right\}\right|, \\ \vartheta_3 = 3^2\sum_{\mu=0}^{m-2}\left|\sum_{\nu=0}^{\mu}\left\{\frac{3\nu(\nu+1)}{2}\rho_{\mu-\nu} + (\nu+1)\varrho_{\mu-\nu} - \frac{3(\nu+1)(\nu+2)}{2}\sigma_{\mu-\nu}\right\}\right|. \end{cases} \tag{3.34}$$

Theorem 3.6. *If $d^{[2]0} = \left\{\left(i/3^0, d_i^{[2]0}\right)\right\}$ is the initial polygon and $d^{[2](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[2](k+1)}\right)\right\}$ is the polygon obtained by second order DD TRS at $(k+1)$th refinement level. If $\vartheta^* < 1$ and*

$$\begin{cases} \chi_4 = -\frac{1}{3^2} + \sum_{\nu=0}^{m-1}\left[\frac{(\nu+1)}{2}\left\{\nu\varrho_{m-1-\nu} + (\nu+2)\sigma_{m-1-\nu} - 2\nu\rho_{m-1-\nu}\right\}\right] = 0, \\ \chi_5 = -\frac{1}{3^2} + \sum_{\nu=1}^{m-1}\left[\frac{\nu(\nu+1)}{2}\left\{\rho_{m-\nu-1} + \sigma_{m-\nu-1} - 2\varrho_{m-\nu-1}\right\}\right] = 0, \\ \chi_6 = -\frac{1}{3^2} + \sum_{\nu=1}^{m}\left\{\frac{\nu(\nu+1)}{2}\varrho_{m-\nu} - \nu(\nu+1)\sigma_{m-\nu} + \frac{\nu(\nu-1)}{2}\rho_{m-\nu}\right\} = 0, \end{cases} \tag{3.35}$$

then the deviation between two successive polygons at kth and $(k+1)$*th levels is*

$$\left\|d^{[2](k+1)} - d^{[2]k}\right\|_\infty \leq \vartheta^* \max_i \left\|d_{i+1}^{[2]k} - d_i^{[2]k}\right\|, \tag{3.36}$$

where

$$\vartheta^* = \max\{\vartheta_4, \vartheta_5, \vartheta_6\}, \tag{3.37}$$

$$\begin{cases} \vartheta_4 = 3^2 \sum\limits_{\mu=0}^{m-2} \left| \frac{1}{3^2} + \sum\limits_{\nu=0}^{\mu} \left\{ \nu(\nu+1)\rho_{\mu-\nu} - \frac{\nu(\nu+1)}{2}\varrho_{\mu-\nu} - \frac{(\nu+1)(\nu+2)}{2}\sigma_{\mu-\nu} \right\} \right|, \\ \vartheta_5 = 3^2 \left|\frac{2}{3^3}\right| + 3^2 \sum\limits_{\mu=1}^{m-2} \left| \frac{1}{9} + \sum\limits_{\nu=1}^{\mu} \left\{ \frac{\nu(\nu+1)}{2}(2\varrho_{\mu-\nu} - \rho_{\mu-\nu} - \sigma_{\mu-\nu}) \right\} \right|, \\ \vartheta_6 = 3^2 \left|\frac{1}{3^3}\right| + 3^2 \sum\limits_{\mu=1}^{m-1} \left| \frac{1}{3^2} + \sum\limits_{\nu=1}^{\mu} \left\{ \nu(\nu+1)\sigma_{\mu-\nu} - \frac{\nu(\nu+1)}{2}\varrho_{\mu-\nu} - \frac{\nu(\nu-1)}{2}\rho_{\mu-\nu} \right\} \right|. \end{cases} \tag{3.38}$$

Lemma 3.7. *If* $d^{[3]0} = \left\{\left(i/3^0, d_i^{[3]0}\right)\right\}$ *is the initial polygon and* $d^{[3](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[3](k+1)}\right)\right\}$ *is the polygon obtained by third order DD TRS at* $(k+1)$*th refinement level. If* $\vartheta^{**} < 1$ *and*

$$\begin{cases} \chi_7 = \sum\limits_{\nu=0}^{m-2} \left[\frac{(\nu+1)(\nu+2)}{2} \{(\nu+1)\rho_{m-\nu-2} - (2\nu+3)\varrho_{m-\nu-2} + (\nu+2)\sigma_{m-\nu-2}\} \right] = 0, \\ \chi_8 = \sum\limits_{\nu=0}^{m-2} \left[(\nu+1)(\nu+2) \left\{ \frac{(\nu-1)}{2}\rho_{m-\nu-2} + \frac{(\nu+4)}{2}\varrho_{m-\nu-2} - (\nu+3)\sigma_{m-\nu-2} \right\} \right] = 0, \\ \chi_9 = \sum\limits_{\nu=0}^{m-2} \left[(\nu+1)(\nu+2) \left\{ -\nu\rho_{m-\nu-2} + \frac{(\nu-1)}{2}\varrho_{m-\nu-2} + \frac{(\nu+4)}{2}\sigma_{m-\nu-2} \right\} \right] = 0, \end{cases} \tag{3.39}$$

then the deviation between two consecutive points at $(k+1)$*th level is*

$$\max_i \left\|d_{i+1}^{[3](k+1)} - d_i^{[3](k+1)}\right\| \leq (\vartheta^{**})^{k+1} \max_i \left\|d_{i+1}^{[3]0} - d_i^{[3]0}\right\|, \tag{3.40}$$

where

$$\vartheta^{**} = \max\{\vartheta_7, \vartheta_8, \vartheta_9\}, \tag{3.41}$$

$$\begin{cases} \vartheta_7 = 3^3 \sum\limits_{\mu=0}^{m-2} \left| \sum\limits_{\nu=0}^{\mu} \left\{ \nu(\nu+1)(\nu+2)\varrho_{\mu-\nu} - \frac{\nu(\nu+1)(\nu+3)}{2}\rho_{\mu-\nu} - \frac{(\nu-1)(\nu+1)(\nu+2)}{2}\sigma_{\mu-\nu} \right\} \right|, \\ \vartheta_8 = 3^3 \sum\limits_{\mu=0}^{m-3} \left| \sum\limits_{\nu=0}^{\mu} \left\{ \frac{-(\nu-1)(\nu+1)(\nu+2)}{2}\rho_{\mu-\nu} - \frac{(\nu+1)(\nu+2)(\nu+4)}{2}\varrho_{\mu-\nu} + (\nu+1)(\nu+2)(\nu+3)\sigma_{\mu-\nu} \right\} \right|, \\ \vartheta_9 = 3^3 \sum\limits_{\mu=0}^{m-3} \left| \sum\limits_{\nu=0}^{\mu} \left\{ \nu(\nu+1)(\nu+2)\rho_{\mu-\nu} - \frac{(\nu-1)(\nu+1)(\nu+2)}{2}\varrho_{\mu-\nu} - \frac{(\nu+1)(\nu+2)(\nu+4)}{2}\sigma_{\mu-\nu} \right\} \right|. \end{cases} \tag{3.42}$$

Theorem 3.8. *If* $d^{[3]0} = \left\{\left(i/3^0, d_i^{[3]0}\right)\right\}$ *is the initial polygon and* $d^{[3](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[3](k+1)}\right)\right\}$ *is the polygon obtained by third order DD TRS at* $(k+1)$*th refinement level. If* $\vartheta^{***} < 1$ *and*

$$\begin{cases} \chi_{10} = -\frac{1}{3^3} + \sum\limits_{\nu=0}^{m-2} \left[\frac{(\nu+1)(\nu+2)}{2} \{\sigma_{m-2-\nu} + \nu\varrho_{m-2-\nu} - \nu\rho_{m-2-\nu}\} \right] = 0, \\ \chi_{11} = -\frac{1}{3^3} + \sum\limits_{\nu=0}^{m-2} \left[\frac{(\nu+1)(\nu+2)}{2} \{\rho_{m-\nu-2} - (\nu+3)\varrho_{m-\nu-2} + (\nu+3)\sigma_{m-\nu-2}\} \right] = 0, \\ \chi_{12} = -\frac{1}{3^3} + \sum\limits_{\nu=0}^{m-2} \left[\frac{(\nu+1)(\nu+2)}{2} \{\varrho_{m-\nu-2} - (\nu+3)\sigma_{m-\nu-2} + \nu\rho_{m-\nu-2}\} \right] = 0, \end{cases} \tag{3.43}$$

then the deviation between two successive polygons at kth and $(k+1)$th levels is

$$\left\|d^{[3](k+1)} - d^{[3]k}\right\|_\infty \leq \vartheta^{***} \max_i \left\|d_{i+1}^{[3]k} - d_i^{[3]k}\right\|, \tag{3.44}$$

where

$$\vartheta^{***} = \max\{\vartheta_{10}, \vartheta_{11}, \vartheta_{12}\}, \tag{3.45}$$

$$\begin{cases} \vartheta_{10} = 3^3 \sum_{\mu=0}^{m-3} \left| \frac{1}{3^3} + \sum_{\nu=0}^{\mu} \left\{ \frac{(\nu+1)(\nu+2)}{2} (\nu\rho_{\mu-\nu} - \nu\varrho_{\mu-\nu} - \sigma_{\mu-\nu}) \right\} \right|, \\ \vartheta_{11} = 3^3 \left|\frac{2}{3^4}\right| + 3^3 \sum_{\mu=1}^{m-2} \left| \frac{1}{3^3} + \sum_{\nu=1}^{\mu} \left\{ \frac{(\nu+1)}{2} \{-\nu\rho_{\mu-\nu} + (\nu+2)\varrho_{\mu-\nu} - (\nu+2)\sigma_{\mu-\nu}\} \right\} \right|, \\ \vartheta_{12} = 3^3 \left|\frac{1}{3^4}\right| + 3^3 \sum_{\mu=1}^{m-2} \left| \left\{ \frac{1}{3^3} + \sum_{\nu=1}^{\mu} \frac{\nu(\nu+1)}{2} \left\{ (\nu+2)\sigma_{\mu-\nu} - \varrho_{\mu-\nu} - (\nu-1)\rho_{\mu-\nu} \right\} \right\} \right|. \end{cases} \tag{3.46}$$

4. The analysis of the ternary refinement schemes

Now we present the analysis of the ternary refinement schemes. A function is called C^m-continuous if its mth order derivative is continuous. The common class of continuous function is $C = C^0$. It is natural to think of a C^m function as being a little bit rough, but the graph of a C^3 function looks smooth. So we discuss the analysis of the schemes up to C^3-continuity. The results can be extended similarly. Let $\{\pi^m[0,n],\ m = 0,1,2,3\}$ denotes the set of C^m-continuous functions on the closed and bounded interval $[0,n]$.

Theorem 4.1. *If $f^0 = \{\left(i/3^0, f_i^0\right)\}$ is the initial polygon and $f^{k+1} = \{\left(i/3^{k+1}, f_i^{k+1}\right)\}$ is the polygon obtained by TRS at $(k+1)$th refinement level. If $\varpi,\ \varpi^* < 1$ then $\lim_{k\to\infty} f^k = f \in \pi[0,n] = \pi^0[0,n]$.*

Proof. The (3.7), gives

$$\left\|f^{k+1} - f^k\right\|_\infty \leq \varpi^* \max_i \left\|f_{i+1}^k - f_i^k\right\|,$$

and (3.1), gives

$$\left\|f^{k+1} - f^k\right\|_\infty \leq \varpi^* (\varpi)^k \max_i \left\|f_{i+1}^0 - f_i^0\right\|,$$

Since $\varpi,\ \varpi^* < 1$ therefore $\{f^k\}_{k=0}^\infty$ is a Cauchy sequence on closed and bounded interval $[0,n]$. Hence it is convergent. That is

$$\lim_{k\to\infty} f^k = f \in \pi[0,n] = \pi^0[0,n].$$

Hence the proof. □

Lemma 4.2. *If $d^{[1]0} = \left\{\left(i/3^0, d_i^{[1]0}\right)\right\}$ is the initial polygon and $d^{[1](k+1)} = \left\{\left(i/3^{k+1}, d_i^{[1](k+1)}\right)\right\}$ is the polygon obtained by first order DD TRS at $(k+1)$th refinement level. If $\varpi^{**},\ \varpi^{***} < 1$ and $c_1, c_2, c_3, c_4, c_5, c_6 = 0$, then $\lim_{k\to\infty} d^{[1]k} = d^{[1]} \in \pi[0,n] = \pi^0[0,n]$.*

Proof. By (3.23), we have

$$\left\|d^{[1](k+1)} - d^{[1]k}\right\|_\infty \le \varpi^{***} \max_i \left\|d^{[1]k}_{i+1} - d^{[1]k}_i\right\|,$$

Using (3.16), we have

$$\left\|d^{[1](k+1)} - d^{[1]k}\right\|_\infty \le \varpi^{***}(\varpi^{**})^k \max_i \left\|d^{[1]0}{}_{i+1} - d^{[1]0}_i\right\|,$$

Since $\varpi^{**},\ \varpi^{**} < 1$, therefore $\{d^{[1]k}\}_{k=0}^{\infty}$ defines a Cauchy sequence on $[0, n]$ and

$$\lim_{k\to\infty} d^{[1]k} = d^{[1]} \in \pi[0, n] = \pi^0[0, n].$$

This completes the proof. □

Theorem 4.3. *Let $f \in \pi^0[0, n]$ be the limiting curve produced by the ternary refinement scheme. If $\varpi^{**},\ \varpi^{***} < 1$ and $c_1, c_2, c_3, c_4, c_5, c_6 = 0$ then $f \in \pi^1[0, n]$.*

Proof. Lemma 4.2 leads to $\lim_{k\to\infty} d^{[1]k} = d^{[1]} \in \pi^0[0, n]$. Lemma 2.4 gives $d^{[1]} = f'$. Therefore, we have the result, $f \in \pi^1[0, n]$. □

Lemma 4.4. *If $d^{[2]0} = \left\{\left(i/3^0, d^{[2]0}_i\right)\right\}$ is the initial polygon and $d^{[2](k+1)} = \left\{\left(i/3^{k+1}, d^{[2](k+1)}_i\right)\right\}$ is the polygon obtained by second order DD TRS at $(k + 1)$th refinement level. If $\vartheta,\ \vartheta^* < 1$ and $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6 = 0$ then $\lim_{k\to\infty} d^{[2]k} = d^{[2]} \in \pi^0[0, n]$.*

Proof. By (3.36), we have

$$\left\|d^{[2](k+1)} - d^{[2]k}\right\|_\infty \le \vartheta^* \max_i \left\|d^{[2]k}_{i+1} - d^{[2]k}_i\right\|,$$

Using (3.32), we have

$$\left\|d^{[2](k+1)} - d^{[2]k}\right\|_\infty \le \vartheta^*(\vartheta)^k \max_i \left\|d^{[2]0}_{i+1} - d^{[2]0}_i\right\|,$$

Since $\vartheta,\ \vartheta^* < 1$ therefore $\{d^{[2]k}\}_{k=0}^{\infty}$ is a Cauchy convergent sequence. So

$$\lim_{k\to\infty} d^{[2]k} = d^{[2]} \in \pi^0[0, n].$$

Hence the proof. □

Theorem 4.5. *Let $f \in \pi^0[0, n]$ be the limiting curve produced by the ternary refinement scheme. If $\vartheta,\ \vartheta^* < 1$ and $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 = 0$ then $f \in \pi^2[0, n]$.*

Proof. Lemma 4.4 gives $\lim_{k\to\infty} d^{[2]k} = d^{[2]} \in \pi^0[0, n]$. Lemma 2.5 leads to $d^{[2]} = f''$. This implies $f \in \pi^2[0, n]$. □

Lemma 4.6. *If $d^{[3]0} = \left\{\left(i/3^0, d^{[3]0}_i\right)\right\}$ is the initial polygon and $d^{[3](k+1)} = \left\{\left(i/3^{k+1}, d^{[3](k+1)}_i\right)\right\}$ is the polygon obtained by third order DD TRS at $(k + 1)$th refinement level. If $\vartheta^{**},\ \vartheta^{***} < 1$ and $\varphi_7, \varphi_8, \varphi_9, \varphi_{10}, \varphi_{11}, \varphi_{12} = 0$ then $\lim_{k\to\infty} d^{[3]k)} = d^{[3]} \in \pi^0[0, n]$.*

Proof. By (3.44), we have

$$\left\|d^{[3](k+1)} - d^{[3]k}\right\|_\infty \leq \vartheta^{***} \max_i \left\|d_{i+1}^{[3]k} - d_i^{[3]k}\right\|,$$

Using (3.40), we have

$$\left\|d^{[3](k+1)} - d^{[3]k}\right\|_\infty \leq \vartheta^{***}(\vartheta^{**})^k \max_i \left\|d_{i+1}^{[3]0} - d_i^{[3]0}\right\|,$$

Since $\vartheta^{**},\ \vartheta^{***} < 1$ therefore $\{d^{[3]k}\}_{k=0}^{\infty}$ is a Cauchy convergent sequence. Thus

$$\lim_{k\to\infty} d^{[3]k} = d^{[3]} \in \pi^0[0,n].$$

Hence the proof. □

Theorem 4.7. *Let $f \in \pi^0[0,n]$ be the limiting curve produced by the ternary refinement scheme. If $\vartheta,\ \vartheta^* < 1$ and $\varphi_7, \varphi_8, \varphi_9, \varphi_{10}, \varphi_{11}, \varphi_{12} = 0$ then $f \in \pi^3[0,n]$.*

Proof. Lemma 4.6, implies that $\lim_{k\to\infty} d^{[3]k)} = d^{[3]} \in \pi^0[0,n]$ while Lemma 2.6 leads to $d^{[3]} = f'''$. This implies $f \in \pi^3[0,n]$. □

Application and authenticity of the results

We discuss the analysis of the ternary refinement schemes introduced by [8–10, 16, 17]. We conclude that the results obtained by our methods are always equivalent to the one returned by the Laurent polynomial method. We add the Figure 2 for the interest of general readers by the direction of anonymous reviewer of this paper. The Figure 2(d) depicts the smoothness of the limiting curve produced by the ternary scheme. The Figure 2(a), (b) and (c) represent the initial, first and second refinement level of the ternary scheme.

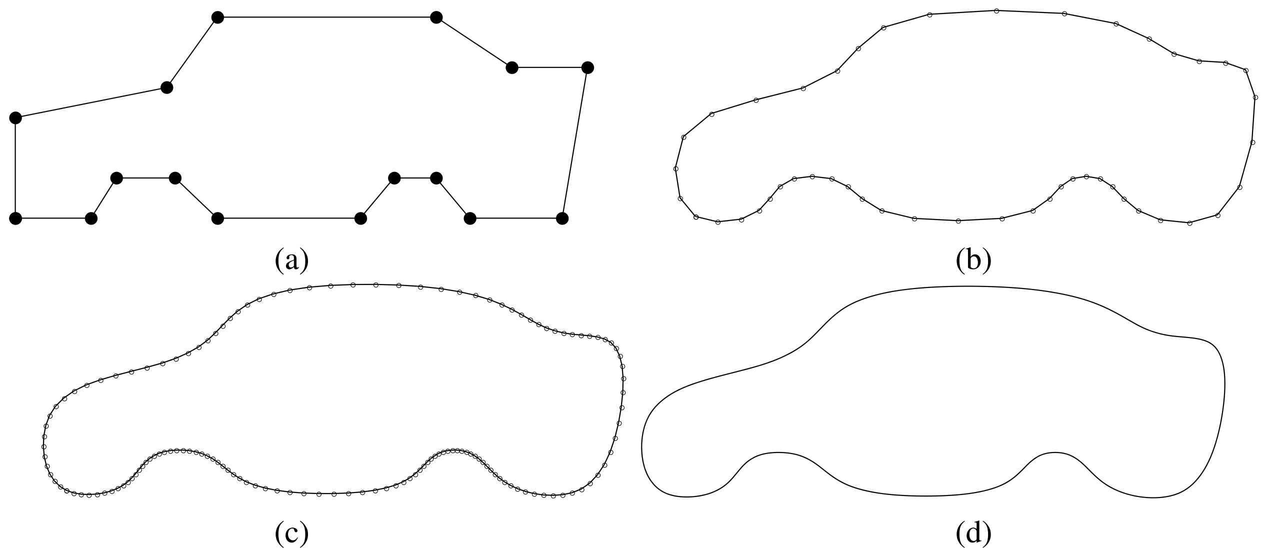

Figure 2. Applications of ternary refinement scheme [17]. Here, the value of μ is $-\frac{1}{54}$.

Example 4.1. If the curve is produced by the following 4-point ternary refinement scheme [17]

$$\begin{cases} f_{3i}^{k+1} = \rho_0 f_{i-1}^k + \rho_1 f_i^k + \rho_2 f_{i+1}^k + \rho_3 f_{i+2}^k, \\ f_{3i+1}^{k+1} = \varrho_0 f_{i-1}^k + \varrho_1 f_i^k + \varrho_2 f_{i+2}^k + \varrho_3 f_{i+3}^k, \\ f_{3i+2}^{k+1} = \sigma_0 f_{i-1}^k + \sigma_1 f_i^k + \sigma_2 f_{i+2}^k + \sigma_3 f_{i+3}^k, \end{cases}$$

where

$$\begin{cases} \rho_0 = \left(\frac{5}{81} + \frac{5}{3}\mu\right), & \rho_1 = \left(\frac{19}{27} - 3\mu\right), & \rho_2 = \left(\frac{13}{54} + \mu\right), & \rho_3 = \left(\frac{-1}{162} + \frac{1}{3}\mu\right), \\ \varrho_0 = \mu, & \varrho_1 = (\frac{1}{2} - \mu), & \varrho_2 = (\frac{1}{2} - \mu), & \varrho_3 = \mu, \\ \sigma_0 = (\frac{-1}{162} + \frac{1}{3}\mu), & \sigma_1 = (\frac{13}{54} + \mu), & \sigma_2 = (\frac{19}{27} - 3\mu), & \sigma_3 = (\frac{5}{81} + \frac{5}{3}\mu), \end{cases}$$

then by Laurent polynomial method the scheme produces C^3-continuous curve for $\mu \in (-\frac{1}{27}, \frac{2}{27})$. By Theorems 4.1, 4.3, 4.5 and 4.7, we also get the same results.

Example 4.2. If the curve is produced by the following 4-point ternary refinement scheme [9]

$$\begin{cases} f_{3i}^{k+1} = \rho_1 f_i^k, \\ f_{3i+1}^{k+1} = \varrho_0 f_{i-1}^k + \varrho_1 f_i^k + \varrho_2 f_{i+2}^k + \varrho_3 f_{i+3}^k, \\ f_{3i+2}^{k+1} = \sigma_0 f_{i-1}^k + \sigma_1 f_i^k + \sigma_2 f_{i+2}^k + \sigma_3 f_{i+3}^k, \end{cases}$$

where

$$\begin{cases} \rho_0 = 0, & \rho_1 = 1, & \rho_2 = 0, & \rho_3 = 0, \\ \varrho_0 = \frac{-1}{18} - \frac{1}{6}\mu, & \varrho_1 = \frac{13}{18} + \frac{1}{2}\mu, & \varrho_2 = \frac{7}{18} - \frac{1}{2}\mu, & \varrho_3 = \frac{-1}{18} + \frac{1}{6}\mu, \\ \sigma_0 = \frac{-1}{18} + \frac{1}{6}\mu, & \sigma_1 = \frac{7}{18} - \frac{1}{2}\mu, & \sigma_2 = \frac{13}{18} + \frac{1}{2}\mu, & \sigma_3 = \frac{-1}{18} - \frac{1}{6}\mu, \end{cases}$$

then by Laurent polynomial method the scheme produces C^2-continuous curve over the interval $\mu \in (\frac{1}{15}, \frac{1}{9})$. By Theorems 4.1, 4.3, and 4.5, we also get the same results.

Example 4.3. If the curve is produced by the following 3-point ternary refinement scheme [8]

$$\begin{cases} f_{3i}^{k+1} = \rho_0 f_{i-1}^k + \rho_1 f_i^k + \rho_2 f_{i+1}^k, \\ f_{3i+1}^{k+1} = \varrho_1 f_i^k, \\ f_{3i+2}^{k+1} = \sigma_0 f_{i-1}^k + \sigma_1 f_i^k + \sigma_2 f_{i+1}^k, \end{cases}$$

where

$$\begin{cases} \rho_0 = \left(\rho + \frac{1}{3}\right), & \rho_1 = \left(\frac{2}{3} - 2\rho\right), & \rho_2 = \rho, \\ \varrho_0 = 0, & \varrho_1 = 1, & \varrho_2 = 0, \\ \sigma_0 = \rho, & \sigma_1 = \left(\frac{2}{3} - 2\rho\right), & \sigma_2 = \left(\rho + \frac{1}{3}\right), \end{cases}$$

then by both methods the scheme produces C^1-continuous curve for $w \in \left(-\frac{1}{9}, 0\right)$.

Example 4.4. If the curve is produced by the following 6-point ternary refinement scheme [10]

$$\begin{cases} f_{3i}^{k+1} = \rho_2 f_i^k, \\ f_{3i+1}^{k+1} = \varrho_0 f_{i-2}^k + \varrho_1 f_{i-1}^k + \varrho_2 f_i^k + \varrho_3 f_{i+1}^k + \varrho_4 f_{i+2}^k + \varrho_5 f_{i+3}^k, \\ f_{3i+2}^{k+1} = \sigma_0 f_{i-2}^k + \sigma_1 f_{i-1}^k + \sigma_2 f_i^k + \sigma_3 f_{i+1}^k + \sigma_4 f_{i+2}^k + \sigma_5 f_{i+3}^k, \end{cases} \tag{4.1}$$

where

$$\begin{cases} \rho_0 = 0, & \rho_1 = 0, & \rho_2 = 1, \\ \rho_3 = 0, & \rho_4 = 0, & \rho_5 = 0, \\ \varrho_0 = \left(-\frac{11}{81} + 13\omega\right), & \varrho_1 = \left(\frac{13}{27} - 51\omega\right), & \varrho_2 = \left(-\frac{2}{27} + 74\omega\right), \\ \varrho_3 = \left(\frac{74}{81} - 46\omega\right), & \varrho_4 = \left(-\frac{5}{27} + 9\omega\right), & \varrho_5 = \omega, \\ \sigma_0 = \omega, & \sigma_1 = \left(-\frac{5}{27} + 9\omega\right), & \sigma_2 = \left(\frac{74}{81} - 46\omega\right), \\ \sigma_3 = \left(-\frac{2}{27} + 74\omega\right), & \sigma_4 = \left(\frac{13}{27} - 51\omega\right), & \sigma_5 = \left(-\frac{11}{81} + 13\omega\right), \end{cases}$$

then by both methods the scheme produces C^2-continuous curve over the interval $w \in \left(\frac{14}{1215}, \frac{23}{1944}\right)$.

Example 4.5. If the curve is produced by the following 3-point ternary refinement scheme [16]

$$\begin{cases} f_{3i}^{k+1} = \rho_0 f_{i-1}^k + \rho_1 f_i^k + \rho_2 f_{i+1}^k, \\ f_{3i+1}^{k+1} = \varrho_0 f_{i-1}^k + \varrho_1 f_i^k + \varrho_2 f_{i+2}^k, \\ f_{3i+2}^{k+1} = \sigma_0 f_{i-1}^k + \sigma_1 f_i^k + \sigma_2 f_{i+2}^k, \end{cases} \tag{4.2}$$

where

$$\begin{cases} \rho_0 = \left(\frac{25}{72} + \mu\right), \rho_1 = \left(\frac{23}{36} - 2\mu\right), \rho_2 = \left(\frac{1}{72} + \mu\right), \\ \varrho_0 = \left(\frac{1}{8} + \mu\right), \ \varrho_1 = \left(\frac{3}{4} - 2\mu\right), \ \varrho_2 = \left(\frac{1}{8} + \mu\right), \\ \sigma_0 = \left(\frac{1}{72} + \mu\right), \sigma_1 = \left(\frac{23}{36} - 2\mu\right), \ \sigma_2 = \left(\frac{25}{72} + \mu\right), \end{cases}$$

then by both methods the scheme produces C^2-continuous curve for $\mu \in \left(-\frac{1}{72}, \frac{7}{72}\right)$.

5. Summary of the work

Here we present the brief summary of the work done so for in this paper.

- If $\max\{\varpi_1, \varpi_2, \varpi_3\} < 1$ where $\varpi_1, \varpi_2, \varpi_3$ are defined in (3.3) then TRS will generate C^0 continuous curve.
- If c_1, c_2, c_3, c_4, c_5, c_6 defined in (3.15) and (3.22) are equal to zero and $\max\left\{\varpi_1^*, \varpi_2^*, \varpi_3^*\right\} < 1$, where $\varpi_1^*, \varpi_2^*, \varpi_3^*$ are defined in (3.18) then TRS will generate C^1 continuous curve.
- If χ_1, χ_2 , χ_3 , χ_4, χ_5, χ_6 defined in (3.31) and (3.35) are equal to zero and $\max\{\vartheta_1, \vartheta_2, \vartheta_3\} < 1$, where $\vartheta_1, \vartheta_2, \vartheta_3$ are defined in (3.34) then TRS will generate C^2 continuous curve.
- If χ_7, χ_8 , χ_9 $\chi_{10}, \chi_{11}, \chi_{12}$ defined in (3.39) and (3.43) are equal to zero and $\max\{\vartheta_7, \vartheta_8, \vartheta_9\} < 1$, where $\vartheta_7, \vartheta_8, \vartheta_9$ are defined in (3.42) then TRS will generate C^3 continuous curve.

6. Comparison and advantages

There are two major techniques to analyze the refinement schemes. These are called Laurent polynomial and divided difference (DD) techniques.

- Our technique is the generalization of the DD technique for ternary refinement schemes.

- We have presented the explicit form of the general inequalities for the analysis of the ternary refinement schemes. These inequalities contain simple algebraic expressions. In these inequalities the polynomial factorization, division and summation are not involved. Simple arithmetic operations such as subtraction and multiplication are involved to evaluate the inequalities.
- While in Laurent polynomial technique, the polynomial factorization, division and summation are involved to evaluate the inequalities. In this technique, the explicit form of the general inequalities are also not available.
- So it is obvious that the computational complexity of our techniques is less than the complexity of Laurent polynomial technique.

7. Conclusions

In this paper, we have introduced an alternative technique to analyze a class of ternary refinement schemes. A comparative study of the proposed technique with other existing technique has been presented to prove the effectiveness of the proposed technique. It has been observed that the alternative technique has less computational cost comparative to the existing Laurent polynomial technique.

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Contributions of the authors

Conceptualization, Ghulam Mustafa and Dumitru Baleanu; Formal analysis, Dumitru Baleanu and Yu-Ming Chu; Methodology, Ghulam Mustafa and Syeda Tehmina Ejaz ; Supervision, Ghulam Mustafa; Writing original draft, Syeda Tehmina Ejaz and Ghulam Mustafa; Writing, review and editing, Syeda Tehmina Ejaz and Yu-Ming Chu.

Conflicts of interest

The authors declare no conflict of interest.

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