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The Lie symmetry analysis and exact Jacobi elliptic solutions for the Kawahara–KdV type equations

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ABSTRACT

In this article, we aim to employ two analytical methods including, the Lie symmetry method and the Jacobi elliptical solutions finder method to acquire exact solitary wave solutions in various forms of (1+1)-dimensional Kawahara–KdV type equation and modified Kawahara–KdV type equation. These models are famous models that arise in the modeling of many complex physical phenomena. At the outset, we have generated geometric vector fields and infinitesimal generators of Kawahara–KdV type equations. The (1+1)-dimensional Kawahara–KdV type equations reduced into ordinary differential equations (ODEs) using Lie symmetry reductions. Furthermore, numerous exact solitary wave solutions are obtained utilizing the Jacobi elliptical solutions finder method with the help of symbolic computation with Maple. The obtained results are new in the formulation, and more useful to explain complex physical phenomena. The results reveal that these mathematical approaches are straightforward, effective, and powerful methods that can be adopted for solving other nonlinear evolution equations.

Introduction

During the last decades, abundant powerful nonlinear models have been utilized to describe various real-world problems in different areas such as optical fiber, plasma physics, chemical physics, acoustics, solid-state physics, and fluid dynamics. Due to this importance, determining exact solutions to such equations has a high priority. For this reason, seeking solutions to such type of equation is a laborious task. The exact solutions to such equations can be calculated in very limited cases. In recent years, many significant developments have been done in finding explicit exact solutions of nonlinear partial differential equations (NLPDEs), and various techniques have been proposed [1–16]. Among these techniques, the Lie group of transformation method is an effective, reliable, and very impressive method to obtain the exact solutions of NLPDEs. Lie group of transformation method of symmetries of differential equations is an encouraging source for numerous generalizations attempting to find the methods for obtaining explicit exact solutions. The Lie group method provides a benchmark method [17,18] for obtaining the Lie symmetries of a nonlinear complex system. Most of all, the Lie group of infinitesimal transformations method makes it possible to reduce the dimension of the equation by one after applying once. Thus, the Lie symmetry method is a standard, effective, and highly powerful among group theoretical methods and has a wide range of equations [19–24], which is solved with the help of this technique. Solitons have been presented in the investigation of nonlinear complex physical phenomena. A large number of researchers have extensively studied the dynamical behaviors of the solitons for various nonlinear evolution equations by using different techniques [25–27]. The Kawahara–KdV equation [28],

$$\Delta := u_t + uu_x + u_{xxx} - ku_{xxxxx} = 0, \tag{1}$$

and the modified Kawahara–KdV equation [29]

$$\Delta := u_t + au^2u_x + bu_{xxx} - ku_{xxxxx} = 0. \tag{2}$$

In this model, a, b and k are real parameters corresponding to the theory of gravity-capillary waves on shallow-water waves with surface tension and magneto-acoustic waves in plasma. These equations are empirical models while studying plasma waves, capillarity-gravity water waves on shallow water and other dispersive physical phenomena when the cubic KdV-type equation is weak. The formulation (1) was first considered by Kawahara in 1972, as a framework for

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describing the propagation of different waves of solitons in complex media [28]. Some exact solutions of these two equations, which contain two dispersed terms and a different degree of nonlinearity, were also considered earlier. In particular, the simplest solitary and periodic waves of the Kawakhara equation were found thirty years ago in the work [30]. In recent literature, many researchers have applied various analytical and numerical techniques to deal with the Kawahara–KdV type equations [28,29,31–38]. Liu et al. [39] obtained some exact solutions for the Kawahara–KdV type equations through similarity reductions using the optimal system method. Demina et al. [40] studied and discussed the traveling wave solutions of the Kawahara and the modified Kawahara equations. They have also attained some families of meromorphic solutions including traveling wave, rational, and simply periodic solutions to the autonomous nonlinear ODEs. Further, Kudryashov et al. [41] investigated new exact solutions for the generalized Brethenton equation.

The main objective of the present work is to seek new explicit exact solitary wave solutions of the Kawahara–KdV type equations via the Lie symmetry method and Jacobi elliptical solutions finder method. First of all, by utilizing the Lie symmetry method, the (1+1)-dimensional Kawahara–KdV type equations are reduced into several nonlinear ODEs. Thereafter, we apply the Jacobi elliptical solutions finder method to the reduced nonlinear ODES with the help of symbolic computation via symbolic packages. Thus, abundant exact analytic solutions are obtained in various forms of solitons, namely the interaction between single soliton, lump-soliton, lump-type soliton, trigonometric and hyperbolic solitons, and solitary waves. The obtained exact solitary wave solutions involve many rational form solutions, thereby exhibiting rich physical structures and including the existing solutions in the previous results. Some of the obtained solutions are entirely new and completely different from the earlier established findings. These exact solitary solutions have many significant applications in plasma physics, fiber optics, dynamics of solitons, mathematical physics, fluid dynamics, and various areas of applied sciences. Further, we illustrate the dynamical behavior of solitons graphically as well as physically using 3D-graphics and contour plots. In recent years, extensive research works on the solitons or solitary waves have been growing increasingly that led to favorable results. Consequently, theoretically researches on solitons or solitary waves are helpful to better predicting feasible dynamics for nonlinear evolution equations.

This research consists of the following sections: Section “Introduction” presents a brief sketch of the historical background of our study. Section “Lie symmetry analysis” includes a brief introduction of Lie symmetry analysis for the Kawahara–KdV type equations. In Section “The Lie symmetry reductions and exact analytical solutions”, infinitesimals and the Lie symmetry are studied under the Lie group of transformation method. Section “A soliton wave solution finder method” explained the Jacobi elliptical solutions finder method. The methods have been implemented in Sections “The method implementation on solving Eq. (14)” and “The method implementation on solving Eq. (40)” for solving two considered equations, respectively.

Lie symmetry analysis

Let us consider a system of one-parameter (ε) Lie group transformation as

$$\bar{x} = x + \varepsilon\xi^x(x,t,u) + O(\varepsilon^2),$$
$$\bar{t} = t + \varepsilon\xi^t(x,t,u) + O(\varepsilon^2),$$
$$\bar{u} = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

where ξ^x, ξ^t, η are infinitesimals for the variable x, t and u, respectively. The vector field $\mathbf{V}$ associated with the one-parameter transformation for the Kawahara–KdV equations can be written as

$$\mathbf{V} = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \quad (3)$$

To obtain a Lie point symmetry of the Kawahara–KdV type equations (1) and (2). The vector field (3) is used and $\mathbf{V}$ must satisfy the invariant surface conditions

$$Pr^{(5)}\mathbf{V}(\Delta) = 0, \text{ whenever } \Delta = 0$$

where $Pr^{(5)}$ represent the fifth prolongation of the vector field $\mathbf{V}$, which are expressed as

$$Pr^{(5)}\mathbf{V} = \mathbf{V} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}.$$

Applying $Pr^{(5)}$ to the Kawahara–KdV type equations (1) and (2), we must have the following Lie invariant surface conditions

$$\eta^t + u\eta^x + \eta u_x + \eta^{xxx} - k\eta^{xxxxx} = 0, \quad (4)$$

and

$$\eta^t + au^2\eta^x + a\eta^2 u_x + b\eta^{xxx} - k\eta^{xxxxx} = 0, \quad (5)$$

where $\eta^t, \eta^x, \eta^{xxx}$ and η^{xxxxx} can be defined, respectively [42,43]

$$\eta^t = D_t\eta - u_x D_t\xi^x - u_t D_t\xi^t,$$
$$\eta^x = D_x\eta - u_x D_x\xi^x - u_t D_x\xi^t,$$
$$\eta^{xxx} = D_x\eta^{xx} - u_{xxx} D_x\xi^x - u_{xxt} D_x\xi^t,$$
$$\eta^{xxxxx} = D_x\eta^{xxxx} - u_{xxxxx} D_x\xi^x - u_{xxxxt} D_x\xi^t,$$

in which D_t and D_x denote the total derivatives. In this illustration, they can be defined such as

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots.$$

Putting the above these values of $\eta^t, \eta^x, \eta^{xxx}$ and η^{xxxxx} into Eqs. (4) and (5) and equating to zero the coefficient of various monomials, a system of over-determining equations are obtained as

$$(\xi^t)_t = 0,\ (\xi^t)_u = 0,\ (\xi^t)_x = 0,\ (\xi^x)_u = 0,\ (\xi^x)_x = 0,\ (\xi^x)_{tt} = 0,\ \eta = (\xi^x)_t, \quad (6)$$

and

$$(\xi^t)_t = 0,\ (\xi^t)_u = 0,\ (\xi^t)_x = 0,\ (\xi^x)_u = 0,\ (\xi^x)_t = 0,\ (\xi^x)_x = 0,\ \eta = 0, \quad (7)$$

respectively.

Lie symmetry analysis for Kawahara–KdV equation

To simplify system (6), we obtain the following infinitesimals for the Kawahara–Korteweg–de Vries equation (1)

$$\xi^x = a_2 t + a_3,\ \xi^t = a_1,\ \eta = a_2. \quad (8)$$

where a_1, a_2, and a_3 are arbitrary constants. The associated vector field $\mathbf{X}$ of the Kawahara–KdV equation (1) can be written in the form of the vectors X_1, X_2, and X_3.

$$\mathbf{X} = a_1 X_1 + a_2 X_2 + a_3 X_3 \quad (9)$$

where the vectors X_1, X_2 and X_3 are defined as

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$
$$X_3 = \frac{\partial}{\partial x}.$$

Then Lie algebra and commutative relation of these vectors are calculated via the Lie bracket's $[X_i, X_j] = X_i X_j - X_j X_i$ (see Table 1).

The Lie series to compute the adjoint relation is represented as

$$Ad(exp(\varepsilon)X_i)X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i,[X_i,X_j]] - \cdots.$$

Table 1
Commutator table for Kawahara–KdV equation.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	X_3	0
X_2	$-X_3$	0	0
X_3	0	0	0

Table 2
Adjoint table for Kawahara–KdV equation.

$Ad(exp(\varepsilon)X_i)X_j$	X_1	X_2	X_3
X_1	X_1	$X_2 - \varepsilon X_3$	X_3
X_2	$X_2 + \varepsilon X_3$	X_2	X_3
X_3	X_1	X_2	X_3

The adjoint representation of $X_i's$ $(1 \leq i \leq 3)$ is demonstrated in Table 2.

Thus, the adjoint Table 2 describe the calculation of the commutative relation of these vector fields for the Kawahara–KdV system. Taking the adjoint Table 2 into account, Eq. (1) has following types of cases

$$a_1X_1 + a_3X_3,\ X_1,\ X_2.$$

Lie symmetry analysis for the modified Kawahara–KdV equation

On simplify the system of Eqs. (7), we obtain the following set of infinitesimals for the modified Kawahara–KdV equation (2)

$$\xi^x = C_2,\ \xi^t = C_1,\ \eta = 0. \tag{10}$$

The vector field **V** of the modified Kawahara–KdV equation (2) can be generated with the help of the vectors V_1 and V_2 corresponding to the constants C_1 and C_2, respectively,

$$\mathbf{V} = C_1V_1 + C_2V_2 \tag{11}$$

where the vectors V_1 and V_2 are defined as

$$V_1 = \frac{\partial}{\partial t},$$
$$V_2 = \frac{\partial}{\partial x}.$$

The Lie symmetry reductions and exact analytical solutions

In this section, we obtain several exact solitary wave solutions of the Kawahara–KdV type equations (1) and (2). Now, we deal with the Lie symmetry reductions and explicit exact solutions to the Kawahara–KdV type equations.

Lie Symmetry reductions and exact solutions for the Kawahara–KdV equation

For Lie symmetry reductions, we consider the following three cases: $a_1X_1 + a_3X_3$, X_1 and X_2.

For vector field $a_1X_1 + a_3X_3 = a_1\frac{\partial}{\partial t} + a_3\frac{\partial}{\partial x}$

The associated Lagrange equation is

$$\frac{dx}{a_3} = \frac{dt}{a_1} = \frac{du}{0}. \tag{12}$$

Solving the first two factors of (12), we get the similarity variable of the form $X = x - A_2t$, where $A_2 = \frac{a_3}{a_1}$, provided $a_1 \neq 0$.

Then, again solving the last two factors of (12), thus one obtains

$$u(x,t) = \mathscr{U}(X), \tag{13}$$

where $\mathscr{U}(X)$ is similarity function with similarity variable $X = x - A_2t$.

Substituting (13) into (1), then we get the reduction equation

$$(\mathscr{U}(X) - A_2)\mathscr{U}'(X) - k\mathscr{U}^{(5)}(X) + \mathscr{U}^{(3)}(X) = 0.$$

On integration, we get

$$\frac{\mathscr{U}(X)^2}{2} - A_2\mathscr{U}(X) - k\mathscr{U}^{(4)}(X) + \mathscr{U}^{(2)}(X) = 0. \tag{14}$$

where $A_2 = \frac{a_3}{a_1}$ is constant, provided $a_1 \neq 0$. Applying the balancing principle to the terms $\mathscr{U}^2$ and $\mathscr{U}^{(4)}$ in Eq. (14) which yield $2\aleph = \aleph + 4$ implies $\aleph = 4$.

Let the solution of (14) can be furnished by assuming *sech* function, use the following

$$\mathscr{U}(X) = sech^4(AX + B), \tag{15}$$

where A and B are arbitrary constants.

Now, equating coefficients of same powers of *sech* function to zero yields following system of equations:

$$2\left(256A^4k - 16A^2 + A_2\right) = 0,$$
$$40A^2\left(52A^2k - 1\right) = 0,$$
$$1 - 1680A^4k = 0. \tag{16}$$

Solving the system of Eqs. (16), we get the following two solution sets

$$A = \frac{1}{2}\sqrt{\frac{13}{105}},\ A_2 = \frac{12}{35},\ k = \frac{105}{169},$$

and

$$A = -\frac{1}{2}\sqrt{\frac{13}{105}},\ A_2 = \frac{12}{35},\ k = \frac{105}{169}.$$

Putting the above these values into (15), then one obtains

$$\mathscr{U}(X) = sech^4\left(B + \frac{1}{2}\sqrt{\frac{13}{105}}X\right),\quad \mathscr{U}(X) = sech^4\left(B - \frac{1}{2}\sqrt{\frac{13}{105}}X\right). \tag{17}$$

By back substitution from (17) into (13), the exact solitary wave solutions for the Kawahara–KdV equation are as followed

$$u(x,t) = sech^4\left(B + \frac{1}{2}\sqrt{\frac{13}{105}}\left(x - \frac{12}{35}t\right)\right), \tag{18}$$

$$u(x,t) = sech^4\left(B - \frac{1}{2}\sqrt{\frac{13}{105}}\left(x - \frac{12}{35}t\right)\right). \tag{19}$$

Again, let the solution of (14) can be taken as

$$\mathscr{U}(X) = Asech^4(X) + Bsech^2(X), \tag{20}$$

where A and B are arbitrary constants.

Proceeding in the similar way, equating coefficients of same powers of *sech* function to zero yields following system of equations:

$$2B\left(A_2 + 16k - 4\right) = 0,$$
$$A(32 - 512k) - 2AA_2 + B(B + 240k - 12) = 0,$$
$$2(A(B + 1040k - 20) - 120Bk) = 0,$$
$$A(A - 1680k) = 0. \tag{21}$$

Solving system of Eqs. (21), we obtain the following sets

$$A = \frac{1}{13}\left(-651 - 63i\sqrt{31}\right),\ B = \frac{364560 + 1148\left(-651 - 63i\sqrt{31}\right)}{13\left(-651 - 63i\sqrt{31}\right)},$$
$$A_2 = \frac{1}{105}\left(420 + \frac{1}{13}\left(651 + 63i\sqrt{31}\right)\right),$$
$$k = \frac{-651 - 63i\sqrt{31}}{21840}\ \text{ and }\ A = \frac{420}{13},\ B = 0,\ A_2 = \frac{144}{13},\ k = \frac{1}{52}.$$

Putting the above these values into (20), thus, the corresponding solutions of (14) are given by

$$\mathscr{U}(X) = \frac{1}{13}\left(-651 - 63i\sqrt{31}\right)sech^4(X)$$

$$+\frac{\left(364560+1148\left(-651-63i\sqrt{31}\right)\right)sech^2(X)}{13\left(-651-63i\sqrt{31}\right)},$$

$$\mathcal{U}(X)=\frac{420}{13}sech^4(X). \tag{22}$$

By back substitution from (22) into (13), the exact solitary wave solutions for Kawahara–KdV equation are follows as

$$u(x,t)=\frac{1}{13}\left(-651-63i\sqrt{31}\right)sech^4(x-A_2t)$$
$$+\frac{\left(364560+1148\left(-651-63i\sqrt{31}\right)\right)sech^2(x-A_2t)}{13\left(-651-63i\sqrt{31}\right)}, \tag{23}$$

$$u(x,t)=\frac{420}{13}sech^4\left(\frac{144t}{13}-x\right), \tag{24}$$

respectively, where $A_2=\frac{1}{105}\left(420+\frac{1}{13}\left(651+63i\sqrt{31}\right)\right)$ (see Figs. 1–4).

For vector field $X_1=\frac{\partial}{\partial t}$

The associated Lagrange equation is

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0},$$

which immediately yields

$$u(x,t)=\mathcal{U}(X), \tag{25}$$

where $\mathcal{U}(X)$ is a similarity function with similarity variable $X=x$. The Lie symmetry reduction is easily obtained as

$$\mathcal{U}(X)\mathcal{U}'(X)+\mathcal{U}^{(3)}(X)-k\,\mathcal{U}^{(5)}(X)=0. \tag{26}$$

To simplify (26), we assume that

$$\mathcal{U}(X)=\frac{B_4}{\cot^4(X)}+\frac{B_3}{\cot^3(X)}+\frac{B_2}{\cot^2(X)}+\frac{B_1}{\cot(X)}+N_4\cot^4(X)$$
$$+N_3\cot^3(X)+N_2\cot^2(X)$$
$$+N_1\cot(X)+N_0, \tag{27}$$

be the solution for Eq. (26). On comparing the coefficients of like powers of *cot* function to zero yields following system of equations:

$$4B_4^2-6720B_4k=0,\quad 7B_3B_4-2520B_3k=0,$$
$$-720B_2k-19200B_4k+3B_3^2+4B_4^2+6B_2B_4+120B_4=0,$$
$$120B_1k+6600B_3k-5B_4B_1-5B_2B_3-60B_3-7B_3B_4=0,$$
$$1680B_2k+19264B_4k-4B_4N_0-2B_2^2-6B_4B_2$$
$$-24B_2-3B_3^2-4B_1B_3-248B_4=0,\quad -240B_1k-5808B_3k$$
$$+3B_3N_0+3B_4N_1+3B_2B_1+5B_4B_1+6B_1$$
$$+5B_2B_3+114B_3=0,\quad -1232B_2k-7744B_4k+2B_2N_0+4B_4N_0$$
$$+2B_3N_1+2B_4N_2+B_1^2+4B_3B_1+2B_2^2$$
$$+40B_2+152B_4=0,\quad -136B_1k-1848B_3k+B_1N_0+3B_3N_0$$
$$+B_2N_1+3B_4N_1+B_3N_2+B_4N_3+3B_2B_1$$
$$+8B_1+60B_3=0,\quad -272B_2k-960B_4k+2B_2N_0+2B_3N_1$$
$$+2B_4N_2+B_1^2+16B_2+24B_4=0,$$
$$16B_1k+120B_3k-B_1N_0+B_1N_2-B_2N_1-B_3N_2+B_2N_3$$
$$-B_4N_3+B_3N_4-2B_1-6B_3-16kN_1-120kN_3$$
$$N_0N_1+2N_1+6N_3=0,\quad -2B_1N_3-2B_2N_4+272kN_2$$
$$+960kN_4-N_1^2-2N_0N_2-16N_2-24N_4=0,$$
$$B_1N_2+B_2N_3+3B_1N_4+B_3N_4-136kN_1-1848kN_3+N_0N_1$$
$$+3N_2N_1+8N_1+3N_0N_3+60N_3=0,$$
$$2B_1N_3+2B_2N_4-1232kN_2-7744kN_4+N_1^2+4N_3N_1+2N_2^2$$
$$+2N_0N_2+40N_2+4N_0N_4+152N_4=0,$$
$$3B_1N_4-240kN_1-5808kN_3+3N_2N_1+5N_4N_1+6N_1+3N_0N_3$$
$$+5N_2N_3+114N_3=0,$$
$$1680kN_2+19264kN_4-2N_2^2-6N_4N_2-24N_2-3N_3^2-4N_1N_3$$
$$-4N_0N_4-248N_4=0,$$
$$120kN_1+6600kN_3-5N_4N_1-5N_2N_3-60N_3-7N_3N_4=0,$$
$$2520kN_3-7N_3N_4=0, 6720kN_4-4N_4^2=0,$$
$$720kN_2+19200kN_4-3N_3^2-4N_4^2-6N_2N_4-120N_4=0. \tag{28}$$

Solving system of algebraic equations (28), we obtain the following sets of solutions

Solution Set 1.

$$N_0=\frac{12}{65}\left(-2+9i\sqrt{31}\right),\ N_1=0,\ N_2=0,\ N_3=0,\ N_4=0,\ B_1=0,$$
$$B_2=\frac{84}{13}\left(7+i\sqrt{31}\right),\ B_3=0,$$
$$B_4=\frac{651}{13}+\frac{63i\sqrt{31}}{13},\ k=\frac{31+3i\sqrt{31}}{1040}.$$

Using the above values into (27) and (25), one gets

$$u(x,t)=\left(\frac{651}{13}+\frac{63i\sqrt{31}}{13}\right)\tan^4(x)+\frac{84}{13}\left(7+i\sqrt{31}\right)\tan^2(x)$$
$$+\frac{12}{65}\left(-2+9i\sqrt{31}\right). \tag{29}$$

Solution Set 2.

$$N_0=\frac{12}{65}\left(-2-9i\sqrt{31}\right),\ N_1=0,\ N_2=\frac{84}{13}\left(7-i\sqrt{31}\right),$$
$$N_3=0,\ N_4=\frac{651}{13}-\frac{63i\sqrt{31}}{13},$$
$$B_1=0,\ B_2=0,\ B_3=0,\ B_4=0,\ k=\frac{31-3i\sqrt{31}}{1040}.$$

Using the above values into (27) and (25) yields

$$u(x,t)=\left(\frac{651}{13}-\frac{63i\sqrt{31}}{13}\right)\cot^4(x)+\frac{84}{13}\left(7-i\sqrt{31}\right)\cot^2(x)$$
$$+\frac{12}{65}\left(-2-9i\sqrt{31}\right). \tag{30}$$

Solution Set 3.

$$N_0=\frac{3}{130}\left(-729+43i\sqrt{31}\right),\ N_1=0,\ N_2=\frac{21}{13}\left(-3+i\sqrt{31}\right),$$
$$N_3=0,\ N_4=\frac{651}{52}+\frac{63i\sqrt{31}}{52},$$
$$B_1=0,\ B_2=\frac{21}{13}\left(-3+i\sqrt{31}\right),\ B_3=0,\ B_4=\frac{651}{52}+\frac{63i\sqrt{31}}{52},$$
$$k=\frac{31+3i\sqrt{31}}{4160}.$$

Taking the above values along with (27) and (25), we have

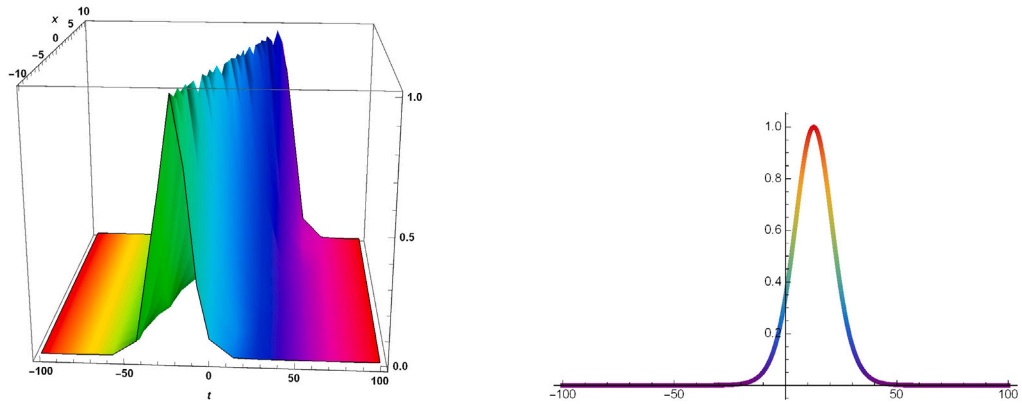
$$u(x,t)=\left(\frac{651}{52}+\frac{63i\sqrt{31}}{52}\right)\tan^4(x)+\frac{21}{13}\left(-3+i\sqrt{31}\right)\tan^2(x)$$
$$+\left(\frac{651}{52}+\frac{63i\sqrt{31}}{52}\right)\cot^4(x)$$
$$+\frac{21}{13}\left(-3+i\sqrt{31}\right)\cot^2(x)+\frac{3}{130}\left(-729+43i\sqrt{31}\right). \tag{31}$$

Solution Set 4.

$$N_0=-\frac{276}{13},\ N_1=-B_1,\ N_2=\frac{1}{13}\left(-13B_2-840\right),\ N_3=-B_3,$$
$$N_4=\frac{1}{13}\left(-13B_4-420\right),\ k=-\frac{1}{52}.$$

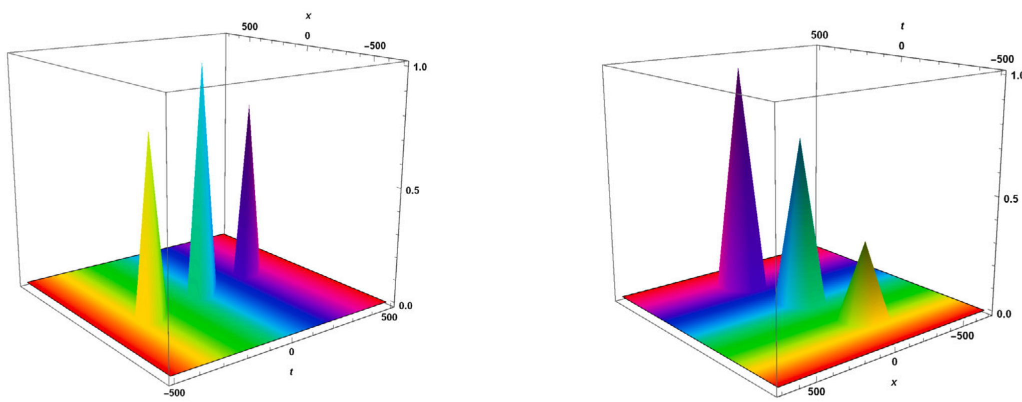
Taking the above values along with (27) and (25), we have

$$u(x,t)=B_4\cot^4(x)+\frac{1}{13}\left(-13B_4-420\right)\cot^4(x)+B_2\cot^2(x)$$
$$+\frac{1}{13}\left(-13B_2-840\right)\cot^2(x)-\frac{276}{13}. \tag{32}$$



(a) (b)

Fig. 1. Sketch of Eq. (18) for $B = 1.5$ and $-10 \le x \le 10$, $-100 \le t \le 100$; (a) 3D shape of single soliton solution; (b) 2D shape of single soliton solution.



(a) (b)

Fig. 2. Sketch of Eq. (19) (a) 3D shape of triply-soliton solution for $B = 0$ and $-700 \le x \le 700$, $-500 \le t \le 500$; (b) 3D shape of sharped triply-soliton solution for $B = 0.4$ and $-700 \le x \le 700$, $-500 \le t \le 500$.

For vector field $X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$

The associated Lagrange equation is

$$\frac{dx}{t} = \frac{dt}{0} = \frac{du}{1},$$

which immediately yields

$$u(x,t) = \frac{x}{T} + \mathscr{U}(T), \tag{33}$$

where $\mathscr{U}(T)$ is similarity function with variable with $T = t$. The Lie symmetry reduction is easily obtained from (1),

$$\frac{\mathscr{U}(T)}{T} + \mathscr{U}'(T) = 0. \tag{34}$$

Solving Eq. (34), we obtain

$$\mathscr{U}(T) = \frac{\alpha}{T} \tag{35}$$

where α is arbitrary constant. By back substitution from (35) into (33), we obtain

$$u(x,t) = \frac{(x+\alpha)}{t}. \tag{36}$$

The Lie symmetry reductions and exact solutions for a modified Kawahara–KdV equation

For vector field $C_1V_1 + C_2V_2 = C_1\frac{\partial}{\partial t} + C_2\frac{\partial}{\partial x}$

To obtain the exact solutions of the modified Kawahara–KdV equation (2). Using, Eq. (10) then, the corresponding Lagrange system is

$$\frac{dx}{\xi^x} = \frac{dt}{\xi^t} = \frac{du}{\eta}. \tag{37}$$

From (10) and (37), we obtain

$$\frac{dx}{C_2} = \frac{dt}{C_1} = \frac{du}{0}.$$

The similarity solution is of the form

$$u(x,t) = \mathscr{U}(X), \tag{38}$$

with the similarity variables $X = x - Ct$, where $C = \frac{C_2}{C_1}$. Using the value of u from (38) into Eq. (2), we obtain the reduction equation as

$$\mathscr{U}'(X)\left(a\,\mathscr{U}^2(X) - C\right) + b\left(\mathscr{U}^{(3)}(X) - k\,\mathscr{U}^{(5)}(X)\right) = 0. \tag{39}$$

Integrating (39) with respect to X, we get

$$-C\mathscr{U}(X) + \frac{a}{3}\mathscr{U}^3(X) + b(\mathscr{U}^{(2)}(X) - k\mathscr{U}^{(4)}(X)) = 0. \tag{40}$$

Applying the balancing principle to the terms $\mathscr{U}^2$ and $\mathscr{U}^{(4)}$ in Eq. (40) which yield $2\aleph = \aleph + 4$ implies $\aleph = 4$. Now, seeking the solution by *tan* function in Eq. (40), use the following

$$\mathscr{U}(X) = M_0 + M_1\tan(X) + M_2\tan^2(X) + \frac{M_3}{\tan(X)} + \frac{M_4}{\tan^2(X)}. \tag{41}$$

Now equating coefficients of same powers of *tan* function to zero yields following system of equations:

$$1440kM_4 - 4aM_4^3 = 0, \quad 240kM_3 - 10aM_3M_4^2 = 0,$$
$$4aM_2^3 - 1440kM_2 = 0, \quad 10aM_1M_2^2 - 240kM_1 = 0,$$
$$4aM_4^3 + 8aM_0M_4^2 + 8aM_3^2M_4 + 48bM_4 - 3360kM_4 = 0,$$
$$2aM_3^3 + 10aM_4^2M_3 + 12aM_0M_4M_3 + 6aM_1M_4^2 + 12bM_3 - 480kM_3 = 0,$$
$$4aM_4M_0^2 + 4aM_3^2M_0 + 8aM_4^2M_0 + 4aM_2M_4^2 + 8aM_3^2M_4$$

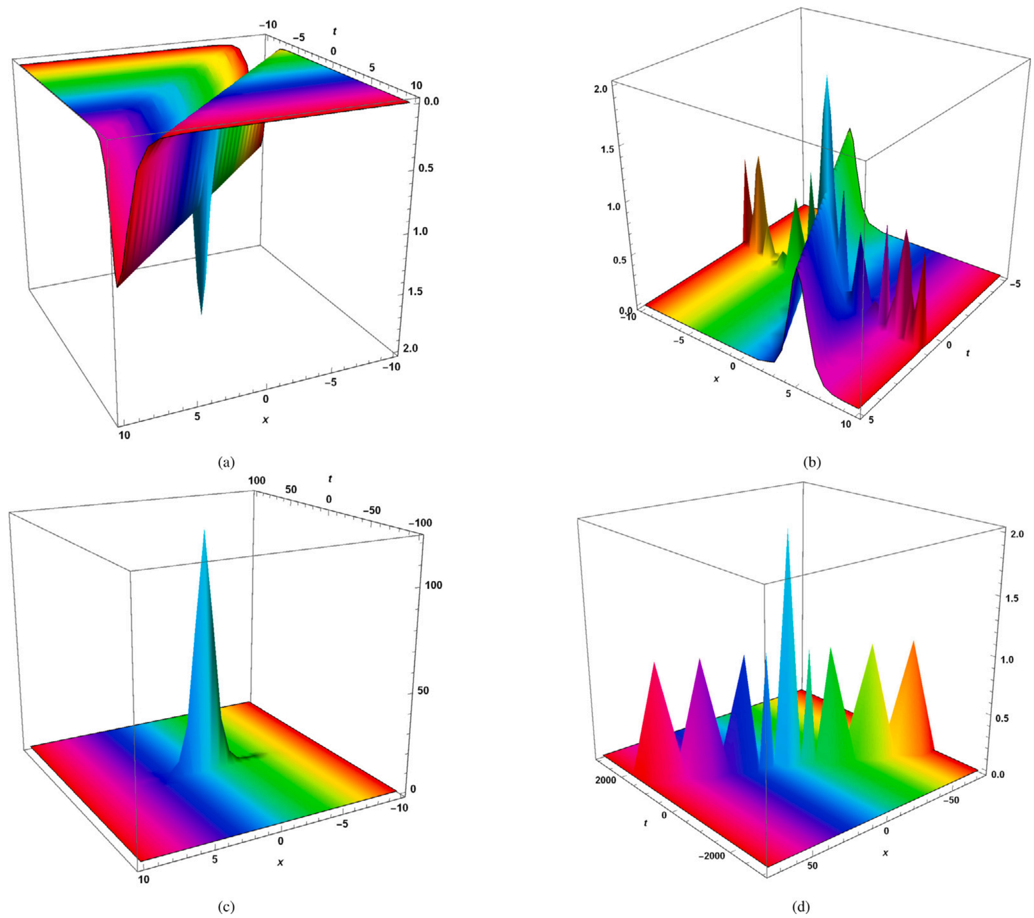


Fig. 3. Sketch of Eq. (23), (a) 3D shape of interaction between single soliton and lump-soliton solution for the values $A_2 = \frac{144}{13}$, $B_1 = 1$, $B_2 = 1$, $-10 \le x \le 10$, $-10 \le t \le 10$; (b) interaction between multi soliton and kink wave profile for $-10 \le x \le 10$, $-5 \le t \le 5$; (c) 3D shape of single soliton solution for the values $A_2 = 20$, $B_1 = 1$, $B_2 = 1$, $-10 \le x \le 10$, $-100 \le t \le 100$; (d) 3D shape of multi-soliton solution for the values $A_2 = \frac{44}{13}$, $B_1 = 1$, $B_2 = 1$, $-75 \le x \le 75$, $-3000 \le t \le 3000$.

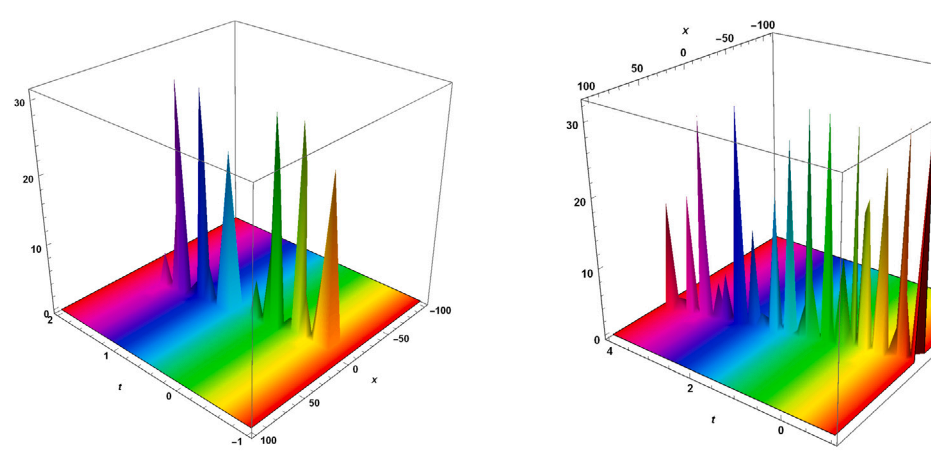


Fig. 4. Sketch of Eq. (24) (a) 3D shape of multi-soliton solution for $-100 \le x \le 100$, $-1 \le t \le 2$; (b) 3D shape of oscillating multi-soliton solution for $-109 \le x \le 102$, $-1 \le t \le 4$.

$$+8aM_1M_3M_4 + 80bM_4 - 4cM_4 - 2464kM_4 = 0,$$
$$2aM_3^3 + 2aM_1M_3^2 + 2aM_0^2M_3 + 12aM_0M_4M_3 + 4aM_2M_4M_3$$
$$+6aM_1M_4^2 + 4aM_0M_1M_4 + 16bM_3 - 2cM_3$$
$$-272kM_3 = 0, \quad 4aM_4M_0^2 + 4aM_3^2M_0 + 4aM_2M_4^2 + 8aM_1M_3M_4$$
$$+32bM_4 - 4cM_4 - 544kM_4 = 0,$$
$$2aM_1M_0^2 - 2aM_3M_0^2 + 4aM_2M_3M_0 - 4aM_1M_4M_0 - 2aM_1M_3^2$$
$$+2aM_1^2M_3 + 4aM_1M_2M_4 - 4aM_2M_3M_4$$
$$+4bM_1 - 4bM_3 - 2cM_1 + 2cM_3 - 32kM_1 + 32kM_3 = 0,$$
$$4aM_2M_0^2 + 4aM_1^2M_0 + 8aM_1M_2M_3 + 4aM_2^2M_4 + 32bM_2 - 4cM_2$$
$$-544kM_2 = 0,$$
$$2aM_1^3 + 2aM_3M_1^2 + 2aM_0^2M_1 + 12aM_0M_2M_1 + 4aM_2M_4M_1$$
$$+6aM_2^2M_3 + 4aM_0M_2M_3 + 16bM_1 - 2cM_1$$
$$-272kM_1 = 0, \quad 4aM_2M_0^2 + 4aM_1^2M_0 + 8aM_2^2M_0 + 8aM_1^2M_2$$
$$+8aM_1M_2M_3 + 4aM_2^2M_4 + 80bM_2$$

$$-4cM_2-2464kM_2=0,\quad 2aM_1^3+10aM_2^2M_1+12aM_0M_2M_1+6aM_2^2M_3+12bM_1-480kM_1=0,$$
$$4aM_2^3+8aM_0M_2^2+8aM_1^2M_2+48bM_2-3360kM_2=0. \tag{42}$$

Solving the system of Eqs. (42), we have

Solution Set 1.

$$M_0=\frac{\sqrt{\frac{30aC-6\sqrt{5}\sqrt{a^2(5C^2-48b^2)}}{a^2}}\left(\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}+5a(C-8b)\right)}{120ab},$$
$$M_1=0,$$
$$M_2=-\sqrt{\frac{3}{2}}\sqrt{\frac{5C}{a}-\frac{\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}}{a^2}},\quad M_3=0,\quad M_4=0,$$
$$k=\frac{1}{240}\left(5C-\frac{\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}}{a}\right)$$

Substituting above values into Eq. (41), we get the corresponding solution of Eq. (40) read as

$$\mathscr{U}(X)=\frac{\sqrt{30aC-6\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}}}{120a^2b}\left(\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}+5a(C-8b)-60ab\tan^2(X)\right).$$

Hence, one gets

$$u(x,t)=\frac{\sqrt{30aC-6\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}}}{120a^2b}\times\left(\sqrt{5}\sqrt{a^2\left(5C^2-48b^2\right)}-60ab\tan^2(-x+Ct)+5a(C-8b)\right). \tag{43}$$

Solution Set 2.

$$M_0=\frac{\sqrt{30aC-6\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}}{480a^2b}\times\left(\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}+5a(C-8b)\right),\quad M_1=0,$$
$$M_2=-\frac{1}{4}\sqrt{\frac{3}{2}}\sqrt{\frac{5aC-\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}{a^2}},$$
$$M_3=0,\quad M_4=-\frac{1}{4}\sqrt{\frac{3}{2}}\sqrt{\frac{5C}{a}-\frac{\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}{a^2}},$$
$$k=\frac{5aC-\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}{3840a}$$

Substituting above values into Eq. (41), we get the corresponding solution of Eq. (40) read as

$$\mathscr{U}(X)=\frac{\sqrt{30aC-6\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}}{480a^2b}\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}+5a(16b+C)-240ab\csc^2(2X).$$

Hence, one gets

$$u(x,t)=\frac{\sqrt{30aC-6\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}}}{480a^2b}\sqrt{5}\sqrt{a^2\left(5C^2-768b^2\right)}+5a(16b+C)-240ab\csc^2(2(x-Ct)). \tag{44}$$

For vector field $V_1=\frac{\partial}{\partial t}$

The associated Lagrange equation is

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0},$$

which immediately yields

$$u(x,t)=\mathscr{U}(X), \tag{45}$$

where $\mathscr{U}(X)$ is similarity function with variable with $X=x$. The Lie symmetry reduction is easily obtained as

$$a\,\mathscr{U}(X)^2\mathscr{U}'(X)+b\,\mathscr{U}^{(3)}(X)-k\,\mathscr{U}^{(5)}(X)=0. \tag{46}$$

To simplify Eq. (46), we assume that

$$\mathscr{U}(X)=N_0+N_1\tan(X)+N_2\tan^2(X)+\frac{N_3}{\tan(X)}+\frac{N_4}{\tan^2(X)}, \tag{47}$$

be the solution for Eq. (46). Equating coefficients of same powers of *tan* function to zero yields following system of equations:

$$720kN_4-2aN_4^3=0,\quad 120kN_3-5aN_3N_4^2=0,\quad 2aN_2^3-720kN_2=0,$$
$$5aN_1N_2^2-120kN_1=0,$$
$$2aN_4^3+4aN_0N_4^2+4aN_3^2N_4+24bN_4-1680kN_4=0,\quad 2aN_2^3+4aN_0N_2^2+4aN_1^2N_2+24bN_2-1680kN_2=0,$$
$$aN_3^3+5aN_4^2N_3+6aN_0N_4N_3+3aN_1N_4^2+6bN_3-240kN_3=0,$$
$$aN_3^3+5aN_4^2N_3+6aN_0N_4N_3+3aN_1N_4^2+6bN_3-240kN_3=0,\quad -2aN_4N_0^2-2aN_3^2N_0-4aN_4^2N_0-2aN_2N_4^2-4aN_3^2N_4-4aN_1N_3N_4-40bN_4+1232kN_4=0,$$
$$aN_3^3+aN_1N_3^2+aN_0^2N_3+6aN_0N_4N_3+2aN_2N_4N_3+3aN_1N_4^2+2aN_0N_1N_4+8bN_3-136kN_3=0,$$
$$-2aN_4N_0^2-2aN_3^2N_0-2aN_2N_4^2-4aN_1N_3N_4-16bN_4+272kN_4=0,$$
$$aN_1N_0^2-aN_3N_0^2+2aN_2N_3N_0-2aN_1N_4N_0-aN_1N_3^2+aN_1^2N_3+2aN_1N_2N_4-2aN_2N_3N_4+2bN_1-2bN_3-16kN_1+16kN_3=0,$$
$$2aN_2N_0^2+2aN_1^2N_0+4aN_1N_2N_3+2aN_2^2N_4+16bN_2-272kN_2=0,$$
$$aN_1^3+aN_3N_1^2+aN_0^2N_1+6aN_0N_2N_1+2aN_2N_4N_1+3aN_2^2N_3+2aN_0N_2N_3+8bN_1-136kN_1=0,$$
$$2aN_2N_0^2+2aN_1^2N_0+4aN_2^2N_0+4aN_1^2N_2+4aN_1N_2N_3+2aN_2^2N_4+40bN_2-1232kN_2=0,$$
$$aN_1^3+5aN_2^2N_1+6aN_0N_2N_1+3aN_2^2N_3+6bN_1-240kN_1=0. \tag{48}$$

Solving system of Eqs. (48), we obtain the following sets of constants:

Solution Set 1.

$$N_0=-\frac{i\left(3(-5)^{3/4}\sqrt{2}\sqrt[4]{3}-10\sqrt[4]{-5}\sqrt{2}3^{3/4}\right)\sqrt{b}}{15\sqrt{a}},\qquad N_1=0,$$
$$N_2=\frac{i\sqrt[4]{-5}\sqrt{2}3^{3/4}\sqrt{b}}{\sqrt{a}},\qquad N_3=0,\quad N_4=0,$$
$$k=-\frac{ib}{4\sqrt{15}}.$$

Using above values into both (47) and (45), one obtains

$$u(x,t)=\frac{i\sqrt[4]{-5}\sqrt{2}3^{3/4}\sqrt{b}\tan^2(x)}{\sqrt{a}}-\frac{i\left(3(-5)^{3/4}\sqrt{2}\sqrt[4]{3}-10\sqrt[4]{-5}\sqrt{2}3^{3/4}\right)\sqrt{b}}{15\sqrt{a}}. \tag{49}$$

Solution Set 2.

$$N_0=-\sqrt{\frac{b}{a}}\sqrt{-8-\frac{14i}{\sqrt{15}}},\quad N_1=0,\quad N_2=\frac{3^{3/4}\sqrt[4]{-5}\sqrt{b}}{\sqrt{2}\sqrt{a}},\quad N_3=0,$$

$$N_4=\frac{\sqrt[4]{-5}3^{3/4}\sqrt{b}}{\sqrt{2}\sqrt{a}},\quad k=\frac{ib}{16\sqrt{15}}.$$

Using above values into both (47) and (45), one obtains

$$u(x,t)=\frac{\sqrt[4]{-5}3^{3/4}\sqrt{b}\tan^2(x)}{\sqrt{2}\sqrt{a}}+\frac{\sqrt[4]{-5}3^{3/4}\sqrt{b}\cot^2(x)}{\sqrt{2}\sqrt{a}}-\frac{\sqrt{b}}{\sqrt{a}}\sqrt{-8-\frac{14i}{\sqrt{15}}}. \tag{50}$$

Solution Set 3.

$$N_0=-\frac{\sqrt{b}}{\sqrt{a}}\sqrt{-8+\frac{14i}{\sqrt{15}}},\quad N_1=0,\quad N_2=-\frac{i\sqrt[4]{-5}3^{3/4}\sqrt{b}}{\sqrt{2}\sqrt{a}},\quad N_3=0,$$

$$N_4=-\frac{i\sqrt[4]{-5}3^{3/4}\sqrt{b}}{\sqrt{2}\sqrt{a}},\quad k=-\frac{ib}{16\sqrt{15}}.$$

Using above values into both (47) and (45), one obtains

$$u(x,t)=-\frac{i\sqrt[4]{-5}3^{3/4}\sqrt{b}\tan^2(x)}{\sqrt{2}\sqrt{a}}-\frac{i\sqrt[4]{-5}3^{3/4}\sqrt{b}\cot^2(x)}{\sqrt{2}\sqrt{a}}-\frac{\sqrt{b}}{\sqrt{a}}\sqrt{-8+\frac{14i}{\sqrt{15}}}. \tag{51}$$

Solution Set 4.

$$N_0=\frac{2}{15}\left(5N_4+2i\sqrt{15}N_4\right),\quad N_1=0,\quad N_2=N_4,$$

$$N_3=0,\quad b=-\frac{2iaN_4^2}{3\sqrt{15}},\quad k=\frac{aN_4^2}{360}.$$

Using above values into both (47) and (45), one obtains

$$u(x,t)=N_4\tan^2(x)+N_4\cot^2(x)+\frac{2}{15}\left(5N_4+2i\sqrt{15}N_4\right). \tag{52}$$

A soliton wave solution finder method

Very recently, a new efficient technique has been developed by Ghanbari et al. to solve the resonance nonlinear Schrödinger equation [44]. Other successful applications of the technique in solving different types of PDEs have also been reported in [45,46]. One of the prominent features of this method is determining the solutions are given in terms of Jacobi elliptic functions. The required steps in this method can be summarized as follows.

1. Here, we are going to solve an equation with the following structure:

$$\mathscr{PDE}(\delta,\delta_x,\delta_t,\delta_{xx},\ldots)=0. \tag{53}$$

2. Taking $\delta=\delta(X)$ and $X=v_1x-v_2t$ into account in Eq. (53) yields

$$\mathscr{ODE}(\delta,\delta',\delta'',\ldots)=0, \tag{54}$$

   where v_1 and v_2 are two unknown values.

3. The following structure is suggested to construct the solution to Eq. (54) :

$$\delta(X)=\frac{\lambda_0+\mu\theta_{k=1}^{2\aleph}\lambda_k\Psi(X)^k}{\gamma_0+\mu\theta_{k=1}^{2\aleph}\gamma_k\Psi(X)^k}, \tag{55}$$

   where λ_0,γ_0 and $\lambda_k,\gamma_k(1\le k\le 2\aleph)$ are chosen so that (55) satisfies Eq. (54), and $\aleph$ is also obtained from balance principles.

Table 3
Jacobi elliptic solutions of Eq. (58).

Item	l_0	l_2	l_4	$\Psi(X)$
1	1	$-(1+\theta^2)$	θ^2	$sn(X,\theta)$ or $cd(X,\theta)$
2	$1-\theta^2$	$2m^2-1$	$-\theta^2$	$cn(X,\theta)$
3	θ^2-1	$2-\theta^2$	-1	$dn(X,\theta)$
4	θ^2	$-(m^2+1)$	1	$ns(X,\theta)$ or $dc(X,\theta)$
5	$-\theta^2$	$2m^2-1$	$1-\theta^2$	$nc(X,\theta)$
6	-1	$2-\theta^2$	$-(1-m^2)$	$nd(X,\theta)$
7	1	$2-\theta^2$	$1-\theta^2$	$sc(X,\theta)$
8	1	$2\theta^2-1$	$-\theta^2(1-\theta^2)$	$sd(X,\theta)$
9	$1-\theta^2$	$2-\theta^2$	1	$cs(X,\theta)$
10	$-\theta^2(1-m^2)$	$2m^2-1$	1	$ds(X,\theta)$
11	$\frac{1-\theta^2}{4}$	$\frac{1+\theta^2}{2}$	$\frac{1-\theta^2}{4}$	$nc(X,\theta)\pm sc(X,\theta)$ or $\frac{cn(X,\theta)}{1\pm sn(X,\theta)}$
12	$\frac{-(1-\theta^2)^2}{4}$	$\frac{\theta^2+1}{2}$	$-\frac{1}{4}$	$\theta cn(X,\theta)\pm dn(X,\theta)$
13	$\frac{1}{4}$	$\frac{1-2\theta^2}{2}$	$\frac{1}{4}$	$\frac{sn(X,\theta)}{1\pm cn(X,\theta)}$
14	$\frac{1}{4}$	$\frac{1+\theta^2}{2}$	$\frac{(1-\theta^2)^2}{4}$	$\frac{sn(X,\theta)}{cn(X,\theta)\pm dn(X,\theta)}$

Table 4
Jacobi elliptic functions and their limits.

Function	$\theta\to 0$	$\theta\to 1$
$sn(X)=sn(X,\theta)$	$\sin(X)$	$\tanh(X)$
$cn(X)=cn(X,\theta)$	$\cos(X)$	$sech(X)$
$dn(X)=dn(X,\theta)$	1	$sech(X)$
$ns(X)=ns(X,\theta)$	$\csc(X)$	$\coth(X)$
$cs(X)=cs(X,\theta)$	$\cot(X)$	$csch(X)$
$ds(X)=ds(X,\theta)$	$\csc(X)$	$csch(X)$
$sc(X)=sc(X,\theta)$	$\tan(X)$	$\sinh(X)$
$sd(X)=sd(X,\theta)$	$\sin(X)$	$\sinh(X)$
$nc(X)=nc(X,\theta)$	$\sec(X)$	$\cosh(X)$
$cd(X)=cd(X,\theta)$	$\cos(X)$	1
$nd(X)=nd(X,\theta)$	1	$\cosh(X)$

4. Principles of balance can be utilized in Eq. (55) to determine the amount of $\aleph$. Moreover, $\Psi(X)$ is considered as a solution to the following equation:

$$\begin{aligned}\Psi(X)'^2&=s_0+s_2\Psi(X)^2+s_4\Psi(X)^4+s_6\Psi(X)^6,\\ \Psi(X)''&=s_2\Psi(X)+2s_4\Psi(X)^3+3s_6\Psi(X)^5.\end{aligned} \tag{56}$$

5. The system introduced in relation (56) admits the following solution

$$\Psi(X)=\frac{\Delta(X)}{\sqrt{p\Delta(X)^2+q}}, \tag{57}$$

   where $p\Delta(X)^2+q>0$, and $\Delta(X)$ satisfies the following equation

$$(\Delta'(X))^2=l_0+l_2\Delta(X)^2+l_4\Delta(X)^4, \tag{58}$$

   and $l_j(j=0,2,4)$ are disposal parameters. We must also have:

$$\begin{aligned}p&=\frac{s_4(l_2-s_2)}{(l_2-s_2)^2+3l_0l_4-2l_2(l_2-s_2)},\\ q&=\frac{3s_4l_0}{(l_2-s_2)^2+3l_0l_4-2l_2(l_2-s_2)}.\end{aligned} \tag{59}$$

   The necessary condition for the parameters are

$$s_2^4(l_2-s_2)[9l_0l_4-(l_2-s_2)(2l_2+s_2)]+3s_6[3l_0l_4-(l_2^2-s_2^2)]^2=0. \tag{60}$$

6. Substituting both (58) and (57) into Eq. (55) yields the wave solutions of Eq. (53) (see Table 4).

The method implementation on solving Eq. (14)

Our primary assumption in the framework of Eq. (55) and from the balance principles we get $\aleph=2$. So, the following general form of the solution to the Eq. (14) is suggested as

$$\mathscr{U}(X)=\frac{\lambda_0+\lambda_1\Psi(X)+\lambda_2\Psi^2(X)+\lambda_3\Psi^3(X)+\lambda_4\Psi^4(X)}{\gamma_0+\gamma_1\Psi(X)+\gamma_3\Psi^3(X)+\gamma_1\Psi(X)+\gamma_4\Psi^4(X)}. \tag{61}$$

Again λ_0, γ_0 and $\lambda_k, \gamma_k (1 \le k \le 4)$ should be determined so that (61) satisfies Eq. (14). Determination of these unknown coefficients are obtained using symbolic computational software, such as Mathematica, and following the outlined steps in the method. As a result, the following set of solutions is determined

$$A_2 = -\frac{2\begin{pmatrix} 38038603273222400k^6 s_2^6 + 1132881530653440k^5 s_2^5 \\ +12945493144896k^4 s_2^4 + 4711438323104k^3 s_2^3 \\ +462881064024k^2 s_2^2 + 17754393360k s_2 + 247843850 \end{pmatrix}^{1/2}}{5318460k^2 s_2 + 146853k},$$

$\lambda_0 = \frac{2\gamma_0\left(\mu_1 - 2\mu_2^{1/2} - 25670\right)}{5318460k^2 s_2 + 146853k}, \lambda_1 = 0, \lambda_2 = \frac{272\gamma_0 s_4 (248ks_2 - 5)}{63}, \lambda_3 = 0,$
$\lambda_4 = 1632\gamma_0 k s_4{}^2,$

$\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0, \gamma_4 = 0, s_0 = \frac{166884160k^3 s_2{}^3 + 6559368k^2 s_2{}^2 - 12840ks_2 - 3775}{571536k^2 s_4 (1340ks_2 + 37)},$
$s_6 = 0,$
$s_2, s_4, \gamma_0 = arbitrary.$

where two notations μ_1 and μ_2 as

$$\mu_1 = \begin{pmatrix} 507689120k^3 s_2^3 \\ -16673424k^2 s_2^2 \\ -1473816ks_2 \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} 38038603273222400k^6 s_2^6 + 1132881530653440k^5 s_2^5 \\ +12945493144896k^4 s_2^4 + 4711438323104k^3 s_2^3 + \\ 462881064024k^2 s_2^2 + 17754393360ks_2 + 247843850 \end{pmatrix}$$

are used throughout the article.

- Using item 1 in Table 3 yields

$$\mathscr{U}(X) = \lambda_0 + \frac{67456 (sn(X,\theta))^2 \left(m^4 - m^2 - s_2{}^2 + 1\right)\left(ks_2 - \frac{5}{248}\right)}{-189 + \left(63m^2 + 63s_2 + 63\right)(sn(X,\theta))^2} + \frac{1632k (sn(X,\theta))^4 \left(m^4 - m^2 - s_2{}^2 + 1\right)^2}{\left(-3 + \left(m^2 + s_2 + 1\right)(sn(X,\theta))^2\right)^2},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 - 2\right)\left(\theta^2 + s_2 + 1\right) s_4{}^2 = 0.$$

Consequently, we find that Eq. (14) admits the following solution

$$u(x,t) = \lambda_0 + \frac{67456 \left(sn\left(x - A_2 t, \theta\right)\right)^2 \left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)\left(ks_2 - \frac{5}{248}\right)}{-189 + \left(63\theta^2 + 63s_2 + 63\right)\left(sn\left(x - A_2 t,\theta\right)\right)^2} + \frac{1632k \left(sn\left(x - A_2 t,\theta\right)\right)^4 \left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)^2}{\left(-3 + \left(\theta^2 + s_2 + 1\right)\left(sn\left(x - A_2 t,\theta\right)\right)^2\right)^2}. \quad (62)$$

Also,

$$\mathscr{U}(X) = \lambda_0 + \frac{272\left(248ks_2 - 5\right)\left((sn(\xi,\theta))^2 - 1\right)\left(-m^4 + m^2 + s_2{}^2 - 1\right)}{\left(63m^2 + 63s_2 + 63\right)(cn(\xi,\theta))^2 - 189(dn(\xi,\theta))^2} + \frac{1632k\left((sn(\xi,\theta))^2 - 1\right)^2\left(-m^4 + m^2 + s_2{}^2 - 1\right)^2}{\left(\left(m^2 + s_2 + 1\right)(cn(\xi,\theta))^2 - 3(dn(\xi,\theta))^2\right)^2}.$$

Therefore, it can easily be seen that Eq. (14) admits the following solution

$$u(x,t) = \lambda_0 + \frac{272\left(248ks_2 - 5\right)\left(\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 1\right)\left(-\theta^4 + \theta^2 + s_2{}^2 - 1\right)}{\left(63m^2 + 63s_2 + 63\right)\left(cn\left(x - A_2 t,\theta\right)\right)^2 - 189\left(dn\left(x - A_2 t,\theta\right)\right)^2} + \frac{1632k\left(\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 1\right)^2\left(-\theta^4 + m^2 + s_2{}^2 - 1\right)^2}{\left(\left(\theta^2 + s_2 + 1\right)\left(cn\left(x - A_2 t,\theta\right)\right)^2 - 3\left(dn\left(x - A_2 t,\theta\right)\right)^2\right)^2}. \quad (63)$$

- Using item 2 in Table 3 yields

$$\mathscr{U}(X) = \lambda_0 + \frac{272\left(248ks_2 - 5\right)\left((sn(\xi,\theta))^2 - 1\right)\left(-m^4 + m^2 + s_2{}^2 - 1\right)}{\left(-126m^2 + 63s_2 + 63\right)(cn(\xi,\theta))^2 + 189m^2 - 189} + \frac{1632k\left((sn(\xi,\theta))^2 - 1\right)^2\left(-m^4 + m^2 + s_2{}^2 - 1\right)^2}{\left(\left(-2m^2 + s_2 + 1\right)(cn(\xi,\theta))^2 + 3m^2 - 3\right)^2},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 - 2\right)\left(\theta^2 + s_2 + 1\right) s_4{}^2 = 0.$$

Thus, Eq. (14) possesses the following solution

$$u(x,t) = \lambda_0 + \frac{272\left(248ks_2 - 5\right)\left(\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 1\right)\left(-\theta^4 + \theta^2 + s_2{}^2 - 1\right)}{\left(-126m^2 + 63s_2 + 63\right)\left(cn\left(x - A_2 t,\theta\right)\right)^2 + 189\theta^2 - 189} + \frac{1632k\left(\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 1\right)^2\left(-m^4 + \theta^2 + s_2{}^2 - 1\right)^2}{\left(\left(-2m^2 + s_2 + 1\right)\left(cn\left(x - A_2 t,\theta\right)\right)^2 + 3\theta^2 - 3\right)^2}. \quad (64)$$

- Using item 6 in Table 3 yields

$$\mathscr{U}(X) = \lambda_0 + \frac{\left(67456ks_2 - 1360\right)\left(m^4 - m^2 - s_2{}^2 + 1\right)}{189(dn(\xi,\theta))^2 + 63m^2 + 63s_2 - 126} + \frac{1632k\left(m^4 - m^2 - s_2{}^2 + 1\right)^2}{\left(3(dn(\xi,\theta))^2 + m^2 + s_2 - 2\right)^2},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 - 2\right)\left(\theta^2 + s_2 + 1\right) s_4{}^2 = 0.$$

Consequently, it is found that Eq. (14) admits the following solution

$$u(x,t) = \lambda_0 + \frac{\left(67456ks_2 - 1360\right)\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{189\left(dn\left(x - A_2 t,\theta\right)\right)^2 + 63\theta^2 + 63s_2 - 126} + \frac{1632k\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)^2}{\left(3\left(dn\left(x - A_2 ti,\theta\right)\right)^2 + \theta^2 + s_2 - 2\right)^2}. \quad (65)$$

- Using item 8 in Table 3 yields

$$\mathscr{U}(X) = \lambda_0 - \frac{67456\left(-m^4 + m^2 + s_2{}^2 - 1\right)(sn(\xi,\theta))^2}{\left(-126m^2 + 63s_2 + 63\right)(sn(\xi,\theta))^2 - 189(dn(\xi,\theta))^2\left(ks_2 - \frac{5}{248}\right)} + \frac{1632k(sn(\xi,\theta))^4\left(-m^4 + m^2 + s_2{}^2 - 1\right)^2}{\left(\left(-2m^2 + s_2 + 1\right)(sn(\xi,\theta))^2 - 3(dn(\xi,\theta))^2\right)^2},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 - 2\right)\left(\theta^2 + s_2 + 1\right) s_4{}^2 = 0.$$

Hence, it can easily be seen that Eq. (14) admits the following solution

$$u(x,t) = \lambda_0 - \frac{67456\left(-\theta^4 + \theta^2 + s_2{}^2 - 1\right)\left(sn\left(x - A_2 t,\theta\right)\right)^2}{\left(-126\theta^2 + 63s_2 + 63\right)\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 189\left(dn\left(x - A_2 t,\theta\right)\right)^2\left(ks_2 - \frac{5}{248}\right)} + \frac{1632k\left(sn\left(x - A_2 t,\theta\right)\right)^4\left(-\theta^4 + \theta^2 + s_2{}^2 - 1\right)^2}{\left(\left(-2\theta^2 + s_2 + 1\right)\left(sn\left(x - A_2 t,\theta\right)\right)^2 - 3\left(dn\left(x - A_2 t,\theta\right)\right)^2\right)^2}. \quad (66)$$

- Using item 9 in Table 3, one gets

$$\mathscr{U}(X) = \lambda_0 + \frac{272\left(248ks_2 - 5\right)\left((sn(\xi,\theta))^2 - 1\right)\left(-m^4 + m^2 + s_2{}^2 - 1\right)}{\left(189m^2 - 189\right)(sn(\xi,\theta))^2 + 63(cn(\xi,\theta))^2\left(m^2 + s_2 - 2\right)} + \frac{1632k\left((sn(\xi,\theta))^2 - 1\right)^2\left(-m^4 + m^2 + s_2{}^2 - 1\right)^2}{\left(\left(3m^2 - 3\right)(sn(\xi,\theta))^2 + (cn(\xi,\theta))^2\left(m^2 + s_2 - 2\right)\right)^2}$$

whenever we have

$$\left(2\theta^2-s_2-1\right)\left(\theta^2+s_2-2\right)\left(\theta^2+s_2+1\right)s_4{}^2=0.$$

Thus, it can easily be verified that Eq. (14) admits the following solution

$$u(x,t)=\lambda_0+\frac{272\left(248ks_2-5\right)\left(\left(sn\left(x-A_2t,\theta\right)\right)^2-1\right)\left(-\theta^4+\theta^2+s_2{}^2-1\right)}{\left(189m^2-189\right)\left(sn\left(x-A_2t,\theta\right)\right)^2+63\left(cn\left(x-A_2t,\theta\right)\right)^2\left(\theta^2+s_2-2\right)}+\frac{1632k\left(\left(sn\left(x-A_2t,\theta\right)\right)^2-1\right)^2\left(-\theta^4+\theta^2+s_2{}^2-1\right)^2}{\left(\left(3\theta^2-3\right)\left(sn\left(x-A_2t,\theta\right)\right)^2+\left(cn\left(x-A_2t,\theta\right)\right)^2\left(\theta^2+s_2-2\right)\right)^2}. \quad (67)$$

Using item 10 in Table 3 yields

$$\mathscr{U}(X)=\lambda_0+\frac{272\left(248ks_2-5\right)\left((sn(\xi,\theta))^2m^2-1\right)\left(-m^4+m^2+s_2{}^2-1\right)}{\left(-189m^4+189m^2\right)(sn(\xi,\theta))^2+63(dn(\xi,\theta))^2\left(-2m^2+s_2+1\right)}+\frac{1632k\left((sn(\xi,\theta))^2m^2-1\right)^2\left(-m^4+m^2+s_2{}^2-1\right)^2}{\left((dn(\xi,\theta))^2\left(-2m^2+s_2+1\right)+\left(-3m^4+3m^2\right)(sn(\xi,\theta))^2\right)^2}$$

whenever we have

$$\left(2\theta^2-s_2-1\right)\left(\theta^2+s_2-2\right)\left(\theta^2+s_2+1\right)s_4{}^2=0.$$

Thus, the following solution is obtained for Eq. (14) giving by

$$u(x,t)=\lambda_0+\frac{272\left(248ks_2-5\right)\left(\left(sn\left(x-A_2t,\theta\right)\right)^2\theta^2-1\right)\left(-\theta^4+\theta^2+s_2{}^2-1\right)}{\left(-189\theta^4+189\theta^2\right)\left(sn\left(x-A_2t,\theta\right)\right)^2+63\left(dn\left(x-A_2t,\theta\right)\right)^2\left(-2\theta^2+s_2+1\right)} \quad (68)$$

$$+\frac{1632k\left(\left(sn\left(x-A_2t,\theta\right)\right)^2\theta^2-1\right)^2\left(-\theta^4+\theta^2+s_2{}^2-1\right)^2}{\left(\left(dn\left(x-A_2t,\theta\right)\right)^2\left(-2\theta^2+s_2+1\right)+\left(-3m^4+3\theta^2\right)\left(sn\left(x-A_2t,\theta\right)\right)^2\right)^2}.$$

- Using item 13 in Table 3 yields

$$\mathscr{U}(X)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-16m^4+16m^2+16s_2{}^2-1\right)(sn(\xi,\theta))^2}{\left(252m^2+252s_2-126\right)(sn(\xi,\theta))^2-189(1+cn(\xi,\theta))^2}+\frac{1632\left(-m^4+m^2-1/16+s_2{}^2\right)^2(sn(\xi,\theta))^4k}{\left(\left(m^2-1/2+s_2\right)(sn(\xi,m))^2-3/4(1+cn(\xi,\theta))^2\right)^2},$$

whenever we have

$$s_4{}^2\left(2\theta^2+2s_2-1\right)\left(32m^4+16\theta^2s_2-32\theta^2-16s_2{}^2-8s_2-1\right)=0.$$

As a consequence, it can easily be seen that Eq. (14) admits the following solution

$$u(x,t)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-16\theta^4+16\theta^2+16s_2{}^2-1\right)\left(sn\left(x-A_2t,\theta\right)\right)^2}{\left(252\theta^2+252s_2-126\right)\left(sn\left(x-A_2t,\theta\right)\right)^2-189\left(1+cn\left(x-A_2t,\theta\right)\right)^2}+\frac{1632\left(-\theta^4+\theta^2-1/16+s_2{}^2\right)^2\left(sn\left(x-A_2t,\theta\right)\right)^4k}{\left(\left(\theta^2-1/2+s_2\right)\left(sn\left(x-A_2t,\theta\right)\right)^2-3/4\left(1+cn\left(x-A_2t,\theta\right)\right)^2\right)^2}. \quad (69)$$

Also,

$$\mathscr{U}(X)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-16m^4+16m^2+16s_2{}^2-1\right)(sn(\xi,\theta))^2}{\left(252m^2+252s_2-126\right)(sn(\xi,\theta))^2-189(-1+cn(\xi,\theta))^2}+\frac{1632\left(-m^4+m^2-1/16+s_2{}^2\right)^2(sn(\xi,\theta))^4k}{\left(\left(m^2-1/2+s_2\right)(sn(\xi,m))^2-3/4(-1+cn(\xi,\theta))^2\right)^2}.$$

As a consequence, it is found that Eq. (14) admits the following solution

$$u(x,t)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-16\theta^4+16\theta^2+16s_2{}^2-1\right)\left(sn\left(x-A_2t,\theta\right)\right)^2}{\left(252\theta^2+252s_2-126\right)\left(sn\left(x-A_2t,\theta\right)\right)^2-189\left(-1+cn\left(x-A_2t,\theta\right)\right)^2}+\frac{1632\left(-\theta^4+\theta^2-1/16+s_2{}^2\right)^2\left(sn\left(x-A_2t,\theta\right)\right)^4k}{\left(\left(\theta^2-1/2+s_2\right)\left(sn\left(x-A_2t,\theta\right)\right)^2-3/4\left(-1+cn\left(x-A_2t,\theta\right)\right)^2\right)^2}. \quad (70)$$

- Using item 14 in Table 3 yields

$$\mathscr{U}(X)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-m^4-14m^2+16s_2{}^2-1\right)(cn(\xi,\theta))^2}{\left(-126m^2+252s_2-126\right)(cn(\xi,\theta))^2-189(cn(\xi,\theta)+dn(\xi,\theta))^2}+\frac{51k\left(m^4+14m^2-16s_2{}^2+1\right)^2(cn(\xi,\theta))^4}{2\left(\left(m^2-2s_2+1\right)(cn(\xi,\theta))^2+3/2(cn(\xi,\theta)+dn(\xi,\theta))^2\right)^2},$$

whenever we have

$$s_4{}^2\left(\theta^2-2s_2+1\right)\left(\theta^2+6m+4s_2+1\right)\left(\theta^2-6\theta+4s_2+1\right)=0.$$

Therefore, it is found that Eq. (14) admits the following solution

$$u(x,t)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-\theta^4-14\theta^2+16s_2{}^2-1\right)\left(cn\left(x-A_2t,\theta\right)\right)^2}{\left(-126\theta^2+252s_2-126\right)\left(cn\left(x-A_2t,\theta\right)\right)^2-189\left(cn\left(x-A_2t,\theta\right)+dn\left(x-A_2t,\theta\right)\right)^2}+\frac{51k\left(m^4+14\theta^2-16s_2{}^2+1\right)^2\left(cn\left(x-A_2t,\theta\right)\right)^4}{2\left(\left(\theta^2-2s_2+1\right)\left(cn\left(x-A_2t,\theta\right)\right)^2+3/2\left(cn\left(x-A_2t,\theta\right)+dn\left(x-A_2t,\theta\right)\right)^2\right)^2}.$$

Also, it reads

$$\mathscr{U}(X)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-m^4-14m^2+16s_2{}^2-1\right)(sn(\xi,\theta))^2}{\left(-126m^2+252s_2-126\right)(sn(\xi,\theta))^2-189(cn(\xi,\theta)-dn(\xi,\theta))^2}+\frac{51k\left(m^4+14m^2-16s_2{}^2+1\right)^2(sn(\xi,\theta))^4}{2\left(\left(m^2-2s_2+1\right)(sn(\xi,\theta))^2+3/2(cn(\xi,\theta)-dn(\xi,\theta))^2\right)^2}.$$

Hence, Eq. (14) admits the following solution (see Figs. 5–8)

$$u(x,t)=\lambda_0-\frac{68\left(248ks_2-5\right)\left(-\theta^4-14\theta^2+16s_2{}^2-1\right)\left(sn\left(x-A_2t,\theta\right)\right)^2}{\left(-126\theta^2+252s_2-126\right)\left(sn\left(x-A_2t,\theta\right)\right)^2-189\left(cn\left(x-A_2t,\theta\right)-dn\left(x-A_2t,\theta\right)\right)^2}+\frac{51k\left(m^4+14\theta^2-16s_2{}^2+1\right)^2\left(sn\left(x-A_2t,\theta\right)\right)^4}{2\left(\left(\theta^2-2s_2+1\right)\left(sn\left(x-A_2t,\theta\right)\right)^2+3/2\left(cn\left(x-A_2t,\theta\right)-dn\left(x-A_2t,\theta\right)\right)^2\right)^2}.$$

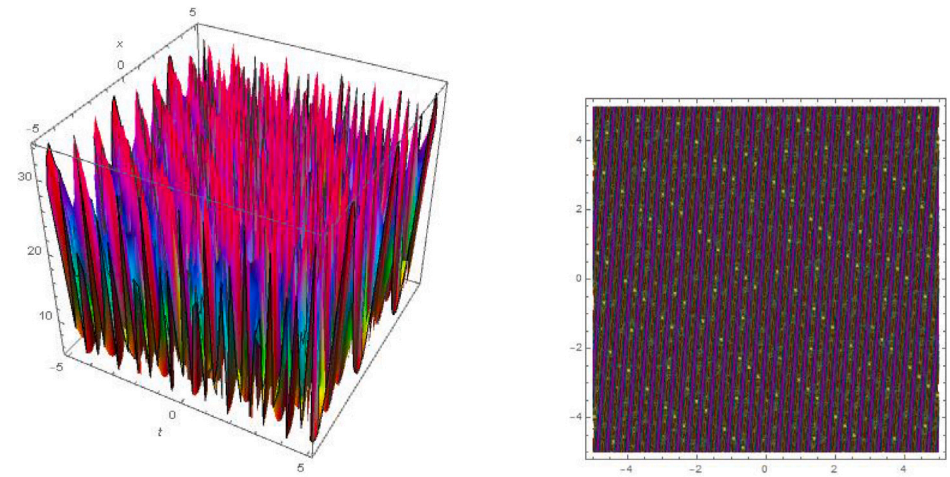
The method implementation on solving Eq. (40)

Our main assumption in the framework of Eq. (55) is to consider the following general form of the solution the equation to the Eq. (40) as

$$\mathscr{U}(X)=\frac{\lambda_0+\lambda_1\Psi(X)+\lambda_2\Psi^2(X)+\lambda_3\Psi^3(X)+\lambda_4\Psi^4(X)}{\gamma_0+\gamma_1\Psi(X)+\gamma_3\Psi^3(X)+\gamma_1\Psi(X)+\gamma_4\Psi^4(X)}.$$

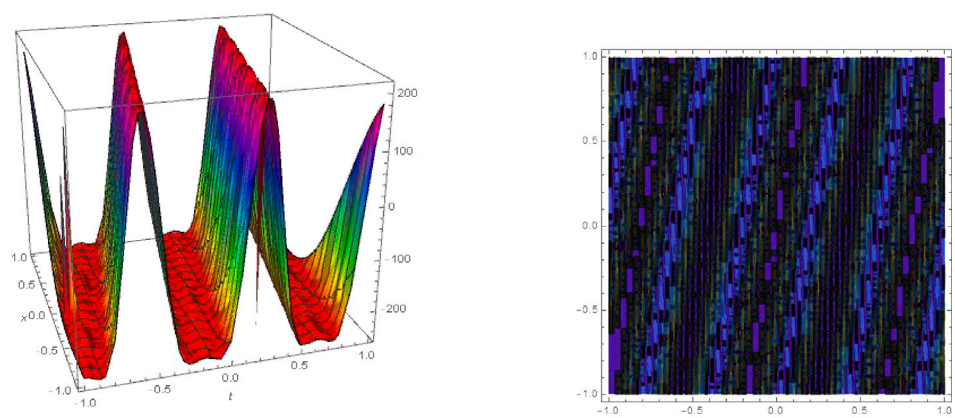
The following solutions are derived from the above methodology stated in Section "A soliton wave solution finder method". First, we obtain

$$C=\frac{800bk^3s_2{}^3+10bks_2+b}{100k^2s_2+5k},\ \lambda_0=\frac{\sqrt{10b}\gamma_0(20ks_2-1)}{10\sqrt{ak}},\ \lambda_1=0,\ \lambda_2=\frac{6\sqrt{10bk}s_4\gamma_0}{\sqrt{a}},$$
$$\lambda_3=0,\ \lambda_4=0,$$
$$\gamma_0=\gamma_0,\gamma_1=0,\gamma_2=0,\gamma_3=0,\gamma_4=0,\ s_0=\frac{3200k^3s_2{}^3+240k^2s_2{}^2-1}{720k^2s_4(20ks_2+1)},$$
$$s_2=s_2,\ h_4=s_4,\ s_6=0.$$



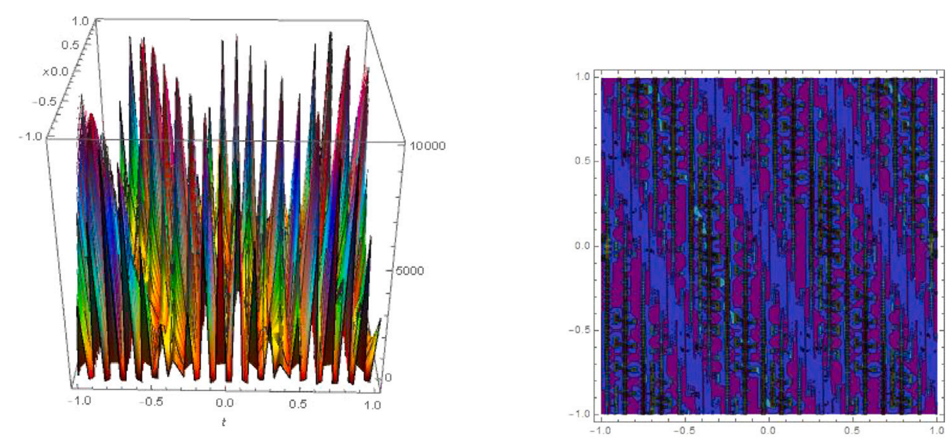
(a) (b)

Fig. 5. Sketch of Eq. (62) for the values $s_2 = -1.25, m = 0.5, k = 0.1$ (a) 3D shape of soliton-cnoidal wave profile; (b) Corresponding contour plot.



(a) (b)

Fig. 6. Sketch of Eq. (64) for the values $s_2 = 1.36, m = 0.8, k = 0.9$ (a) 3D shape of solitary wave profile; (b) Corresponding contour plot.



(a) (b)

Fig. 7. Sketch of Eq. (65) for the values $s_2 = 1.1, m = 0.95, k = 0.25$ (a) 3D shape of multi-solitons profile; (b) Corresponding contour plot.

- Using item 1 in Table 3 yields

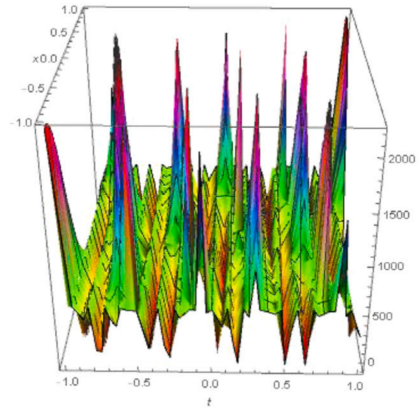
$$\mathscr{U}(X) = \frac{b\left(20ks_2 - 1\right)}{\sqrt{10abk}} + \frac{6\sqrt{10bk}\,(sn(X,\theta))^2\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(-3 + \left(\theta^2 + s_2 + 1\right)(sn(X,\theta))^2\right)},$$

whenever we have

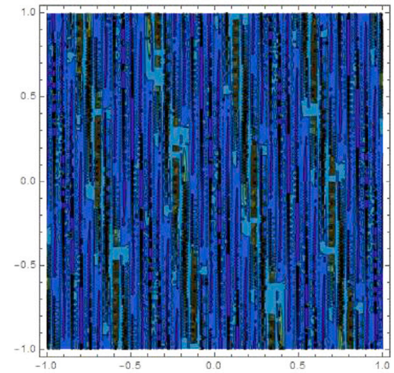
$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)s_4{}^2\left(\theta^2 + s_2 - 2\right) = 0.$$

Consequently, we find a solution for that Eq. (40) as follows

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\,(sn(X,\theta))^2\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(-3 + \left(\theta^2 + s_2 + 1\right)(sn(X,\theta))^2\right)}. \tag{71}$$



(a)



(b)

Fig. 8. Sketch of Eq. (71) for the values $s_2 = -1.8, m = 0.9, k = 4$ (a) 3D shape of elastic multi-solitons profile; (b) Corresponding contour plot.

Also,

$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left((sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 + 1\right)(cn(X,\theta))^2 - 3\,(dn(X,\theta))^2\right)}.$$

Consequently, we obtain that Eq. (40) has the solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left((sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 + 1\right)(cn(X,\theta))^2 - 3\,(dn(X,\theta))^2\right)}, \tag{72}$$

where

$$X = x - \frac{b\left(800k^3 s_2{}^3 + 10ks_2 + 1\right)}{5k\left(20ks_2 + 1\right)}t.$$

- Using item 2 in Table 3 yields

$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left((sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(2\,(cn(X,\theta))^2\,\theta^2 - (cn(X,\theta))^2\,s_2 - (cn(X,\theta))^2 - 3\theta^2 + 3\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right)s_4{}^2 = 0.$$

So, one finds that Eq. (40) admits the following solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left((sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(2\,(cn(X,\theta))^2\,\theta^2 - (cn(X,\theta))^2\,s_2 - (cn(X,\theta))^2 - 3\theta^2 + 3\right)}, \tag{73}$$

- Using item 3 in Table 3 yields

$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 - 2\right)(dn(X,\theta))^2 - 3\theta^2 + 3\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right)s_4{}^2 = 0.$$

Therefore, we find a solution for that Eq. (40) as follows

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 - 2\right)(dn(X,\theta))^2 - 3\theta^2 + 3\right)}. \tag{74}$$

- Using item 4 in Table 3 yields

$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(3\theta^2\,(sn(X,\theta))^2 - \theta^2 - s_2 - 1\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right)s_4{}^2 = 0.$$

Hence, we obtain that Eq. (40) admits the following solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(3\theta^2\,(sn(X,\theta))^2 - \theta^2 - s_2 - 1\right)}. \tag{75}$$

Also,

$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{2\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left((cn(X,\theta))^2\,\theta^2 - 1/3\left(\theta^2 + s_2 + 1\right)(dn(X,\theta))^2\right)}.$$

Hence, we find a solution for that Eq. (40) as follows

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{2\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left((cn(X,\theta))^2\,\theta^2 - 1/3\left(\theta^2 + s_2 + 1\right)(dn(X,\theta))^2\right)}. \tag{76}$$

- Using item 5 in Table 3 yields

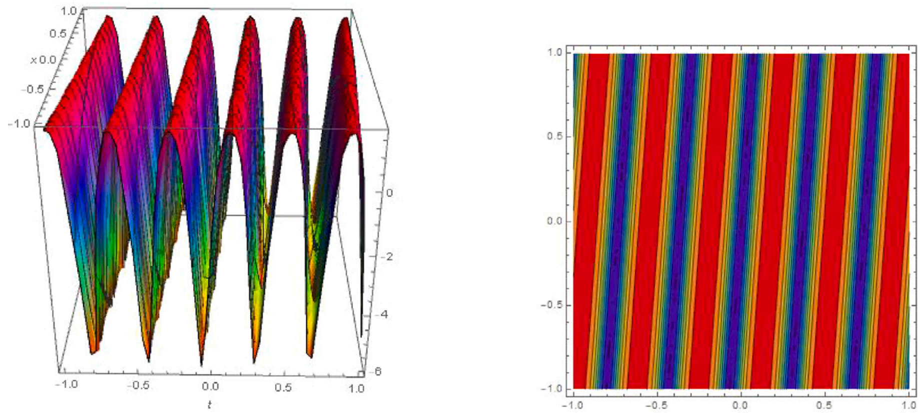
$$\mathscr{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(3\,(cn(X,\theta))^2\,\theta^2 - 2\theta^2 + s_2 + 1\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right)s_4{}^2 = 0.$$

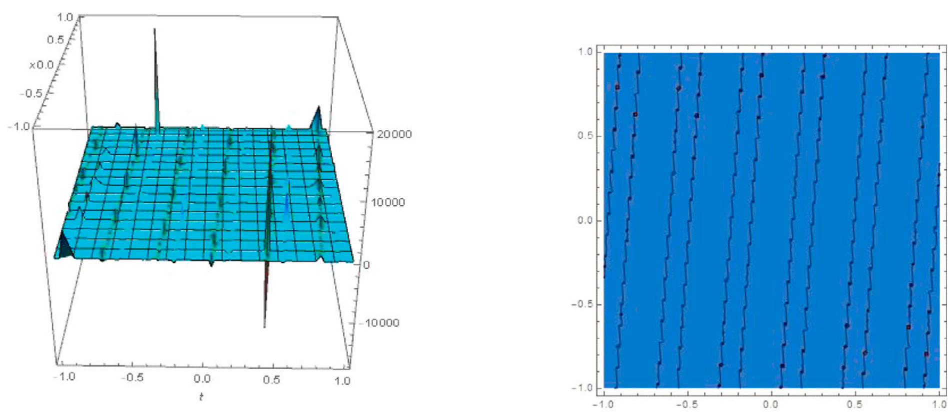
So, one obtains that Eq. (40) has the solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - s_2{}^2 + 1\right)}{\sqrt{a}\left(3\,(cn(X,\theta))^2\,\theta^2 - 2\theta^2 + s_2 + 1\right)}. \tag{77}$$



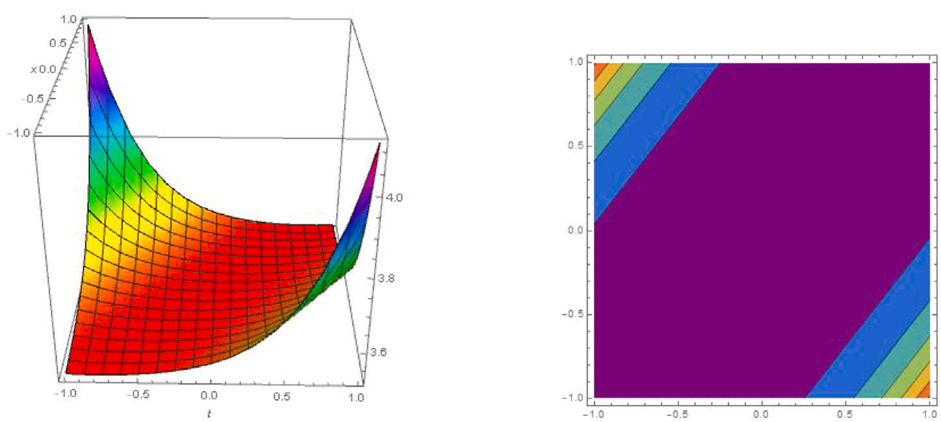
(a) (b)

Fig. 9. Sketch of Eq. (79) for the values $s_2 = -1.81, m = 0.9, k = 0.5, a = 2, b = 1$ (a) 3D shape of solitary wave profile; (b) Corresponding contour plot.



(a) (b)

Fig. 10. Sketch of Eq. (81) for the values $s_2 = -1.81, m = 0.9, k = 0.5, a = 2, b = 1$ (a) 3D shape of multi-soliton behavior of traveling wave profile; (b) Corresponding contour plot.



(a) (b)

Fig. 11. Sketch of Eq. (82) for the values $s_2 = 0.905, m = 0.9, k = 2, a = 0.5, b = 0.1$ (a) 3D shape of elastic behavior of single soliton; (b) Corresponding contour plot.

- Using item 6 in Table 3 yields

$$\mathcal{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(3\left(dn(X,\theta)\right)^2 + \theta^2 + s_2 - 2\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right){s_4}^2 = 0.$$

Therefore, we find the following solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(3\left(dn(X,\theta)\right)^2 + \theta^2 + s_2 - 2\right)}. \quad (78)$$

- Using item 7 in Table 3 yields

$$\mathcal{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\left(sn(X,\theta)\right)^2\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 - 2\right)\left(sn(X,\theta)\right)^2 - 3\left(cn(X,\theta)\right)^2\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right) {s_4}^2 = 0.$$

Thus, it is found that Eq. (40) has the solution (see Fig. 9)

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{6\sqrt{10bk}\,(sn(X,\theta))^2\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 + s_2 - 2\right)(sn(X,\theta))^2 - 3\,(cn(X,\theta))^2\right)}. \tag{79}$$

- Using item 8 in Table 3 yields

$$\mathcal{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{3\sqrt{10bk}\,(sn(X,\theta))^2\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 - 1/2s_2 - 1/2\right)(sn(X,\theta))^2 + 3/2\,(dn(X,\theta))^2\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right) {s_4}^2 = 0.$$

Therefore, one can easily check that Eq. (40) admits the following solution

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{3\sqrt{10bk}\,(sn(X,\theta))^2\left(\theta^4 - \theta^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(\theta^2 - 1/2s_2 - 1/2\right)(sn(X,\theta))^2 + 3/2\,(dn(X,\theta))^2\right)}. \tag{80}$$

- Using item 10 in Table 3 yields

$$\mathcal{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{2\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(2/3\theta^2 - 1/3s_2 - 1/3\right)(dn(X,\theta))^2 + \left(\theta^4 - \theta^2\right)(sn(X,\theta))^2\right)},$$

whenever we have

$$\left(2\theta^2 - s_2 - 1\right)\left(\theta^2 + s_2 + 1\right)\left(\theta^2 + s_2 - 2\right) {s_4}^2 = 0.$$

Consequently, we obtain that Eq. (40) has the solution (see Fig. 10)

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} + \frac{2\sqrt{10bk}\left(\theta^2\,(sn(X,\theta))^2 - 1\right)\left(\theta^4 - m^2 - {s_2}^2 + 1\right)}{\sqrt{a}\left(\left(2/3\theta^2 - 1/3s_2 - 1/3\right)(dn(X,\theta))^2 + \left(\theta^4 - \theta^2\right)(sn(X,\theta))^2\right)}. \tag{81}$$

- Using item 14 in Table 3 yields

$$\mathcal{U}(X) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{3\sqrt{10bk}\left(\theta^4 + 14\theta^2 - 16{s_2}^2 + 1\right)(sn(X,\theta))^2}{4\sqrt{a}\left(\left(\theta^2 - 2s_2 + 1\right)(sn(X,\theta))^2 + 3/2\,(cn(X,\theta) + dn(X,\theta))^2\right)},$$

whenever we have

$${s_4}^2\left(\theta^2 - 2s_2 + 1\right)\left(\theta^2 + 6m + 4s_2 + 1\right)\left(\theta^2 - 6\theta + 4s_2 + 1\right) = 0.$$

So, one can easily check that Eq. (40) admits the following solution (see Fig. 11)

$$u(x,t) = \frac{\sqrt{b}\left(20ks_2 - 1\right)}{\sqrt{10ak}} - \frac{3\sqrt{10bk}\left(\theta^4 + 14\theta^2 - 16{s_2}^2 + 1\right)(sn(X,\theta))^2}{4\sqrt{a}\left(\left(\theta^2 - 2s_2 + 1\right)(sn(X,\theta))^2 + 3/2\,(cn(X,\theta) + dn(X,\theta))^2\right)}. \tag{82}$$

In all the solutions retrieved in this section, we have assumed that

$$X = x - \frac{b\left(800k^3{s_2}^3 + 10ks_2 + 1\right)}{5k\left(20ks_2 + 1\right)}t.$$

It should also be emphasized that by the aid of Table 2, and considering the limit cases of 0 and 1 for θ, several new trigonometric-type solutions to the equations are determined.

Conclusions

In this paper, novel applications of the Lie symmetry analysis along with a Jacobi elliptic finder method were employed to integrate the Kawahara–KdV type equations. Firstly, the Lie symmetry analysis is successfully applied to obtain new exact analytical solutions of the Kawahara–KdV equations. This method determines the solutions of the model in terms of Jacobi elliptic functions. that for some of its particular cases, they reduce to some known trigonometric functions. In the structure of these solutions, there is an index that for some of the specific limit cases produces some known trigonometric functions. Besides, numerical simulations corresponding to some of the obtained solutions are presented. The single soliton, lump-soliton, lump-type soliton, trigonometric and hyperbolic solitons, and solitary waves reported in this paper are entirely new and valid, which have not been presented in previous articles for this equation. These exact solutions reflect the dynamics of different wave structures of solitons which can be used to test exactness, comparison and study of numerical results in the field. This is one of the significant benefits of our results in this article. Also, the two methods used in this contribution can be adopted to solve other problems in the field of mathematics and physics. Further research seems to be required about the design of new efficient analytical methods to solve partial differential equations. Taking advantage of the superior capabilities of such techniques, the exact solutions are determined for more real-world problems in science and engineering. And this could be one of the best motivations for researchers to focus more on this outstanding research field.

CRediT authorship contribution statement

Behzad Ghanbari: Conceptualization, Software. **Sachin Kumar:** Conceptualization, Methodology. **Monika Niwas:** Validation. **Dumitru Baleanu:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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All authors have read and agreed to the published version of the manuscript.

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