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Time-fractional nonlinear Swift-Hohenberg equation: Analysis and numerical simulation

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Abstract In this paper, a new scheme based on the exponential fitting technique is presented for solving the nonlinear time-fractional Swift-Hohenberg equation, where the first and second-order derivatives are replaced by Caputo fractional derivative. The exponential fitting technique depends on a parameter that led to getting high-order accuracy. The convergence and unconditional stability will be discussed using the Fourier method and the analysis is built on uniform and nonuniform time steps, to solve initial singularity by using the graded meshes. Applicability and theoretical results will be demonstrated and enhanced with numerical examples.

1. Introduction

The fractional nonlinear cubic-quintic standard and modified Swift-Hohenberg (S-H) equation are given by:

$$\beta D_t^{\gamma} u + D_t^{\alpha} u + D_x^4 u + \epsilon u_{xx} + (1-\sigma)u + f(x,t,u(x,t)) = 0, \quad < \alpha \le 1, 1 < \gamma \le 2, \tag{1.1}$$

according to the initial conditions:

$$u(x,0) = \varphi(x), u_t(x,0) = \bar{\varphi}(x), 0 \le x \le L, \tag{1.2}$$

and end conditions:

$$u(0,t) = u(L,t) = D_x^2 u(0,t) = D_x^2 u(L,t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

where $\varphi(x)$ is a smooth continuous function and ϵ, σ are real positive constants. The nonlinear term $f(x,t,u)$ satisfies the Lipschitz condition $|f(x,t,u) - f(x,t,w)| \le L_f |u - w|$, $0 < L_f < 1$, where L_f is called a Lipschitz constant [1]. If $\alpha = 1$, and $\beta = 0$, Eq. (1.1) turns into the standard S-H equation of cubic type with $f(x,t,u) = u^3$ and cubic-quintic type with $f(x,t,u) = cu^5 - bu^3$ (see [2,3]). If $\alpha = 1, \gamma = 2$ and $\beta = 1$, Eq. (1.1) turns into the modified S-H equation of cubic type with $f(x,t,u) = u^3$ and cubic-quintic type with $f(x,t,u) = cu^5 - bu^3$ (see [4,5]). If the nonlinear source term $f(x,t,u) = u^3$, and $\beta = 0$, Eq. (1.1) becomes a time-fractional

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standard cubic S-H equation, while $\beta = 1$ turns it into the time-fractional modified cubic type [6]. But if the nonlinear source term $f(x, t, u) = cu^5 - bu^3$, $\beta = 0$ turns Eq. (1.1) into the time-fractional standard cubic-quintic S-H equation, while $\beta = 1$ turns it into the time-fractional modified cubic-quintic type. The operator D_t^{ϑ} represents the Caputo fractional derivative such that [7,8]:

$$D_t^{\vartheta} u(x,t) = \frac{1}{\Gamma(n-\vartheta)} \int_0^t (t-\zeta)^{n-\vartheta-1} D_{\zeta}^{n} u(x,\zeta) d\zeta, n-1 < \vartheta \le n \tag{1.4}$$

In recent years, fractional partial differential equations have been used in modeling several systems and processes in engineering and science [9], such as fractal porous media [10], predator–prey dynamical system [11], fractional differential equations [12], shallow water equations [13], heat conduction problem [14], laser drilling process [15], Brownian motion [16], vibrations [17], reaction–diffusion [18,19], optical solitons [20], and the reference therein. In 1977, Swift and Hohenberg derived an order parameter equation for the temperature and fluid velocity dynamics of the convection. The mathematical model for the Rayleigh–Benard convection involves the Navier–Stokes equations coupled with the transport equation for temperature [21]. Nowadays, there is a great interest in the S-H model as it describes many phenomena such as localized stationary structures in modern physics and microstructures in solidification [6,22]. Almost all the work concerning the S-H equation is depending on numerical methods, little work has been done analytically to find exact solutions of the stationary case of (1.1) in the explicit form [5]. In [6,23–26], the homotopy analysis, fractional variational iteration, and differential transform methods are proposed for solving the nonlinear time-fractional S-H equation.

In this paper, we will deal with the time-fractional standard or modified Swift-Hohenberg equation nonlinear source term. We target to construct a new numerical method depending on the exponentially fitted (EF) method. This technique based on detecting the best choice of the μ-parameter needed to cut down the error and raise the order of accuracy.

The rest of the article is structured as. In Section 2, the temporal-discretization of the time-fractional derivative operator is presented. In Section 3, the steps needed for constructing EF methods are presented. Section 4 will focus on deducing the numerical scheme, while the local truncation error and the choice of the free-parameter are discussed in Section 5. The stability and convergence analysis of our method are introduced in Section 6, respectively. In Section 7, the numerical solution of the time-fractional standard or modified Swift-Hohenberg equation (with the cubic or the cubic-quintic types) in different cases are illustrated. In Section 8, we present a conclusion.

2. Temporal discretization

We introduce a compact approximated form of the derivatives $D_t^{\gamma} u$and$D_t^{\alpha} u$. We consider the Riemann-Liouville (R–L) operator which is given by:

$${}^{RL}D_t^{\vartheta} u(x,t) = \frac{1}{\Gamma(n-\vartheta)} \frac{\partial^n}{\partial t^n} \int_0^t (t-\zeta)^{n-\vartheta-1} u(x,\zeta) d\zeta, \quad n-1 < \vartheta \le n, \tag{2.1}$$

where $\Gamma(.)$ denotes Gamma function.

Let $\mathscr{H}^{n+\alpha}(\mathbb{R}) = \{u/u\epsilon L^1(\mathbb{R}); \int_{-\infty}^{\infty} (1+|\kappa|)^{n+\alpha} |\hat{u}| d\kappa < \infty\}$, where $\hat{u} = \int_{-\infty}^{\infty} e^{i\kappa x} u(x) dx$ is the Fourier transform of $u(x)$, see [27].

Also, the shifted Grünwald-Letnikov (SGL) operator is:

$$D_{t,s}^{\alpha} u(x,t) = \lim_{\tau \to 0} \frac{1}{\tau^{\alpha}} \sum_{k=0}^{[\frac{t}{\tau}]} g_k^{\alpha} u(x, t-(k-s)\tau),$$

such that $\tau = \frac{T}{n}$ is the temporal discretization step. If $s = 0$, we get the Grünwald-Letnikov formula [28]. So, the R–L fractional derivative is equivalent to the shifted Grünwald-Letnikov (SGL) fractional derivative as follows [28–31] and [41]:

$$D_{t,s}^{\alpha} u(x,t_j) = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{j} g_k^{\alpha} u(x, t_{j-(k-s)}) + O(\tau) \tag{2.2}$$

where $t_j = j\tau, j = 0, 1, \cdots, n$ and the coefficientsg_k^{ϑ} can be computed by:

$$g_0^{\vartheta} = 1, g_k^{\vartheta} = \left(1 - \frac{\vartheta+1}{k}\right) g_{k-1}^{\vartheta}, k = 1, 2, 3, \cdots$$

Lemma 2.1.. *The coefficients of the SGL formula (2.2) for $0 < \vartheta \le 1$ satisfy the following, see [30]:*

$$g_0^{\vartheta} = 1, g_1^{\vartheta} = -\vartheta, g_2^{\vartheta} \le g_3^{\vartheta} \le g_4^{\vartheta} \le \cdots \le 0, \quad \sum_{k=0}^{\infty} g_k^{\vartheta} = 0, \sum_{k=0}^{n} g_k^{\vartheta} \ge 0, n \ge 1. \tag{2.3}$$

For $1 < \vartheta \le 2$ satisfy the following, see [31]:

$$g_0^{\vartheta} = 1, g_1^{\vartheta} = -\vartheta, 1 \ge g_2^{\vartheta} \ge g_3^{\vartheta} \ge g_4^{\vartheta} \ge \cdots \ge 0, \quad \sum_{k=0}^{\infty} g_k^{\vartheta} = 0, \sum_{k=1}^{n} g_k^{\vartheta} < 0, n \ge 1. \tag{2.4}$$

Now, the Caputo and R–L fractional derivatives are related by the following, see [29]:

$$D_t^{\vartheta} u(x,t_j) = {}^{RL}D_t^{\vartheta} u(x,t_j) - \sum_{k=0}^{r-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t_j^{k-\vartheta}}{\Gamma(\mathrm{k}-\vartheta+1)}.$$

Hence, we can easily get the time-fractional derivative approximation for αandγ in the following two forms:

$$D_t^{\alpha} u(x,t_j) = \sum_{k=0}^{j+1} \frac{g_k^{\alpha}}{\tau^{\alpha}} u(x, t_{j-k+1}) - \frac{u(x,0)}{t_j^{\alpha}\Gamma(1-\alpha)} + O(\tau). \tag{2.5}$$

$$D_t^{\gamma} u(x,t_j) = \sum_{k=0}^{j+1} \frac{g_k^{\gamma}}{\tau^{\gamma}} u(x, t_{j-k+1}) - \frac{t_j^{-\gamma} u(x,0)}{\Gamma(1-\gamma)} - \frac{t_j^{1-\gamma} u_t(x,0)}{\Gamma(2-\gamma)} + O(\tau). \tag{2.6}$$

Combining both the time-fractional derivative operators in Eqs. (2.5) and (2.6) into one operator, we have

$$D_t^{\vartheta} = D_t^{\alpha} + \beta D_t^{\gamma}, \tag{2.7}$$

which can be easily converted to the following form

$$D_t^{\vartheta}u(x,t_j) = \sum_{k=0}^{j+1} w_k^{\vartheta}u(x,t_{j-k+1}) - \psi_{\vartheta}\varphi(x) - \psi_{\gamma}\bar{\varphi}(x) + O(\tau), \tag{2.8}$$

where $w_k^{\vartheta} = \frac{g_k^{\alpha}}{\tau^{\alpha}} + \beta\frac{g_k^{\gamma}}{\tau^{\gamma}}, \psi_{\vartheta} = \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)} + \beta\frac{t_j^{-\gamma}}{\Gamma(1-\gamma)}$ and $\psi_{\gamma} = \beta\frac{t_j^{1-\gamma}}{\Gamma(2-\gamma)}$.

For the seek of higher-order accuracy, we introduce the following lemma:

Lemma 2.2.. *Let $u \in \mathscr{H}^{1+\vartheta}(\mathbb{R})$ and u belongs to $L^1(\mathbb{R})$, then we get the following:*

For $0 < \alpha \le 1$, see [30]:

$$D_t^{\alpha}u(x,t_j) = \tau^{-\alpha}\sum_{k=0}^{\infty}\lambda_k^{\alpha}u(x,t_{j-k+1}) - \frac{u(x,0)}{t^{\alpha}\Gamma(1-\alpha)} + O(\tau^2), \tag{2.9}$$

where

$$\lambda_0^{\alpha} = 1 + \frac{\alpha}{2}, \lambda_1^{\alpha} = -\frac{1}{2}(\alpha+3)\alpha,$$
$$\lambda_2^{\alpha} = \frac{1}{4}(\alpha^2 + 3\alpha - 2),$$
$$\lambda_k^{\alpha} = \left[\left(1+\frac{\alpha}{2}\right)\left(1-\frac{\alpha+1}{k}\right) - \frac{\alpha}{2}\right]g_{k-1}^{\alpha},\ k \ge 3.$$

And for $1 < \gamma \le 2$, see [19]:

$$D_t^{\gamma}u(x,t_j) = \sum_{k=0}^{j+1}\lambda_k^{\gamma}u(x,t_{j-k+1}) - \frac{t_j^{-\gamma}u(x,0)}{\Gamma(1-\gamma)} - \frac{t_j^{1-\gamma}u_t(x,0)}{\Gamma(2-\gamma)} + O(\tau^2) \tag{2.10}$$

where

$$\lambda_0^{\gamma} = \frac{\gamma}{2}, \lambda_1^{\gamma} = -\frac{1}{2}(2-\gamma-\gamma^2),$$
$$\lambda_2^{\gamma} = \frac{\gamma}{4}(-4+\gamma+\gamma^2),$$
$$\lambda_k^{\gamma} = \left[\frac{\gamma}{2}\left(1-\frac{\gamma+1}{k}\right) + \frac{2-\gamma}{2}\right]g_{k-1}^{\gamma},\ k \ge 3.$$

Then the time-fractional derivative operators in Eqs. (2.9) and (2.10) can be written as:

$$D_t^{\vartheta}u(x,t_j) = \sum_{k=0}^{j+1}\Theta_k^{\vartheta}u(x,t_{j-k+1}) - \psi_{\vartheta}\varphi(x) - \psi_{\gamma}\bar{\varphi}(x) + O(\tau^2), \tag{2.11}$$

where $\Theta_k^{\vartheta} = \frac{\lambda_k^{\alpha}}{\tau^{\alpha}} + \beta\frac{\lambda_k^{\gamma}}{\tau^{\gamma}}, \psi_{\vartheta} = \frac{t_j^{-\alpha}}{\Gamma(1-\alpha)} + \beta\frac{t_j^{-\gamma}}{\Gamma(1-\gamma)}$ and $\psi_{\gamma} = \beta\frac{t_j^{1-\gamma}}{\Gamma(2-\gamma)}$.

3. Exponentially fitted method

The technique presented in [32–39] will be adjusted for solving the time-fractional nonlinear standard and modified Swift-Hohenberg Eqs. (1.1)–(1.3). There are other parametric methods for solving differential equations, but the most defect of these methods is the difficulty of determining the value of the parameter, for more details, we may refer to [7,8,34,37,40–42], and the references therein.

Our purpose is to adjust the approach [32], to implement new exponentially-fitted methods for approximating (1.1–1.3). Also, obtain the best choice of the μ -parameter to develop the accurate EF method. Furthermore, to ensure a high-order accuracy for Eq. (1.1), an exponentially fitted scheme has been constructed for both the second and the fourth-order spatial derivatives.

3.1. Constructing an EF scheme for the fourth-order derivative

We start with the following fourth-order differential equation (4ODE):

$$u^{(4)} = f(x,u) \tag{3.1}$$

Following the chart discussed in [20] and [24].

Let u_i be the approximate value of $u(x_i)$, with $x_i = ih, i = 0,1,\cdots,\mathrm{m}, h = \frac{L}{m}$, then we have the following scheme:

$$u_{i-2} + a_1u_{i-1} + a_0u_i + a_1u_{i+1} + u_{i+2}$$
$$= h^4(b_2M_{i-2} + b_1M_{i-1} + b_0M_i + b_1M_{i+1} + b_2M_{i+2}), \tag{3.2}$$

along with the begin and end formulas as follows

$$(a_1+2)u_0 + (a_0-1)u_1 + a_1u_2 + u_3$$
$$= d_1h^2W_0 + h^4(d_2M_0 + d_3M_1 + d_4M_2 + d_5M_3), \tag{3.3}$$

$$u_{m-3} + a_1u_{m-2} + (a_0-1)u_{m-1} + (a_1+2)u_m$$
$$= d_1h^2W_m$$
$$+ h^4(d_5M_{m-3} + d_4M_{m-2} + d_3M_{m-1} + d_2M_m), \tag{3.4}$$

where $M_i = u^{(4)}(x_i), W_i = u^{(2)}(x_i)$.

Apply the following steps:

Step i: with $\boldsymbol{a} := [a_0, a_1, b_0, b_1, b_2]$ we define the operator $\int[h,a]$ as

$$\int[h,\boldsymbol{a}]u(x) := u(x-2h) + a_1u(x-h) + a_0u(x) + a_1u(x+h)$$
$$+ u(x+2h) - h^4\big(b_2u^{(4)}(x-2h) + b_1u^{(4)}(x-h)$$
$$+ b_0u^{(4)}(x) + b_1u^{(4)}(x+h) + b_2u^{(4)}(x+2h)\big).$$

Step ii: we determine the maximum value of N such that the algebraic system

$$L_m^*(a) = 0|m = 0,1,2,...,N-1\},\text{with}$$
$$L_m^*(a) = h^{-m}\int[h,a]x^m|_{x=0}. \tag{3.5}$$

can be solved. Due to symmetry, we get that $L_{2k+1}^* = 0$ for any integer value of$k$. Hence, we have

$$L_0^*(a) = 2 + 2a_1 + a_0,$$

$$L_2^*(a) = 8 + 2a_1,$$

$$L_4^*(a) = 32 + 2a_1 - 48b_2 - 48b_1 - 24b_0,$$

$$L_6^*(a) = 128 + 2a_1 - 2880b_2 - 720b_1,$$

$$L_8^*(a) = 512 + 2a_1 - 53760b_2 - 3360b_1,$$

$$L_{10}^*(a) = 2048 + 2a_1 - 645120b_2 - 10080b_1,$$

such that $N = 10$ and the solution of the corresponding system is $(a_0,a_1,b_0,b_1,b_2) = \left(6,-4,\frac{79}{120},\frac{31}{180},\frac{-1}{720}\right)$. But if we assumed $b_2 = 0$ then we have $N = 8$ and $(a_0,a_1,b_0,b_1) = \left(6,-4,\frac{2}{3},\frac{1}{6}\right)$

and we will need both of them to get a higher-order as possible when dealing with Eq. (1.1).

Step iii: to construct EF methods, let

$$E_0^*(\pm z, a) := \exp(\mp\mu x)\int [\exp(\pm\mu x)], \tag{3.6}$$

where $z := \mu h$ and we build

$$G^+(Z,a) := \frac{\left(E_0^*(z,a) + E_0^*(-z,a)\right)}{2} \text{ and}$$
$$G^-(Z,a) := \frac{E_0^*(z,a) - E_0^*(-z,a)}{2z}, \text{ where } Z = z^2.$$

Also, the derivatives $G^{\pm(m)}(Z,a)$ w.r. t. Z easily computed. Due to the symmetry, $G^-(Z,a) \equiv 0$ and $G^+(Z,a) = 2\eta_{-1}(4Z) + 2a_1\eta_{-1}(Z) + a_0 - Z^2(2b_2\eta_{-1}(4Z) + b_0 + 2b_1\eta_{-1}(Z))$, where η_{-1} and η_0 are defined as:

$$\eta_{-1}(Z) = \begin{cases} \cos\left(|Z|^{\frac{1}{2}}\right), ifZ < 0, \\ \cosh\left(Z^{\frac{1}{2}}\right), ifZ \geq 0, \end{cases} \quad \eta_0(Z) = \begin{cases} \sin\frac{|Z|^{\frac{1}{2}}}{|Z|^{\frac{1}{2}}}, ifZ < 0, \\ 1, ifZ = 0, \\ \sinh\frac{Z^{\frac{1}{2}}}{Z^{\frac{1}{2}}}, ifZ > 0. \end{cases} \tag{3.7}$$

These functions η_{-1} and η_0 have the following properties:

$$\eta_n(Z) := \frac{1}{Z}[\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)],$$
$$n = 1, 2, 3, \cdots \tag{3.8}$$

$$\eta_n'(Z) = \frac{1}{2}\eta_{n+1}(Z), n = 1, 2, 3, \cdots \tag{3.9}$$

Step iv: We consider the following reference set of N functions:

$$\{1, x, x^2, \cdots, x^K\} \cup \{\exp(\pm\mu x), xexp(\pm\mu x), x^2exp(\pm\mu x), \cdots, x^P\exp(\pm\mu x)\}, \text{where } K + 2P = N - 3.$$

Step v: solve the algebraic system

$$\begin{cases} L_k^*(a) = 0, 0 \leq k \leq K, \\ G^{\pm(p)}(Z,a) = 0, 0 \leq p \leq P, \end{cases} \tag{3.10}$$

Step vi: write down the error expressions.

The following Lemma lists the above results.

Lemma 3.1. *The EF methods* (3.1–3.4 *and* 3.10) *for the case* $N = 8$ *give the following methods:*

Case (i)$(K,P) = (7,-1)$:

$$(a_0, a_1, b_0, b_1) = \left(6, -4, \frac{2}{3}, \frac{1}{6}\right) \text{ and } lte_i = \frac{-h^8}{720}D_x^8 u_i.$$

Case (ii) $(K,P) = (5,0)$

$(a_0, a_1) = (6, -4)$,

$$b_0 = \frac{2}{3} + \frac{z}{360} - \frac{13z^2}{30240} + \frac{19z^3}{1814400} - \frac{23z^4}{59875200} + \frac{7571z^5}{653837184000} + \mathcal{O}(h^{12}),$$

$$b_1 = \frac{1}{6} - \frac{z}{720} + \frac{13z^2}{60480} - \frac{19z^3}{3628800} + \frac{23z^4}{119750400} - \frac{7571z^5}{1307674368000} + \mathcal{O}(h^{12}),$$

and

$$lte_i = \frac{-h^8}{720}\left(D_x^8 u_i - \mu^2 D_x^6 u_i\right).$$

Case (iii) $(K,P) = (3,1)$

$(a_0, a_1) = (6, -4)$,

$$b_0 = \frac{2}{3} + \frac{z}{180} - \frac{z^2}{7560} - \frac{19z^3}{64800} + \frac{223z^4}{29937600} - \frac{348091z^5}{217945728000} + \mathcal{O}(h^{12}),$$

$$b_1 = \frac{1}{6} - \frac{z}{360} + \frac{23z^2}{30240} - \frac{31z^3}{453600} + \frac{841z^4}{119750400} - \frac{309839z^5}{435891456000} + \mathcal{O}(h^{12}),$$

And

$$lte_i = \frac{-h^8}{720}\left(\mu^4 D_x^4 u_i - 2\mu^2 D_x^6 u_i + D_x^8 u_i\right)$$

Case (iv) $(K,P) = (1,2)$

$$a_0 = 6 - \frac{z^3}{360} + \frac{13z^4}{10080} - \frac{59z^5}{604800} + \mathcal{O}(h^{12}),$$
$$a_1 = -4 + \frac{z^3}{720} - \frac{13z^4}{20160} + \frac{59z^5}{1209600} + \mathcal{O}(h^{12}),$$

$$b_0 = \frac{2}{3} + \frac{z}{120} + \frac{z^2}{1120} - \frac{2221z^3}{1814400} + \frac{2069z^4}{39916800} - \frac{458743z^5}{24216192000} + \mathcal{O}(h^{12}),$$

$$b_1 = \frac{1}{6} - \frac{z}{240} + \frac{11z^2}{6720} - \frac{899z^3}{3628800} + \frac{629z^4}{15966720} - \frac{920611z^5}{145297152000} + \mathcal{O}(h^{12}),$$

And

$$lte_i = \frac{-h^8}{720}\left(D_x^8 u_i - 3\mu^2 D_x^6 u_i + 3\mu^4 D_x^4 u_i - \mu^6 D_x^2 u_i\right)$$

Case (v) $(K,P) = (-1,3)$

$$a_0 = 6 - \frac{z^3}{90} + \frac{z^4}{189} + \frac{z^5}{16200} + \mathcal{O}(h^{12}),$$
$$a_1 = -4 + \frac{z^3}{180} - \frac{101z^4}{30240} + \frac{181z^5}{453600} + \mathcal{O}(h^{12}),$$

$$b_0 = \frac{2}{3} + \frac{z}{90} + \frac{z^2}{378} - \frac{13z^3}{4200} + \frac{z^4}{5544} - \frac{1864201z^5}{20432412000} + \mathcal{O}(h^{12}),$$

$$b_1 = \frac{1}{6} - \frac{z}{180} + \frac{43z^2}{15120} - \frac{13z^3}{21600} + \frac{49z^4}{380160} - \frac{9210037z^5}{326918592000} + \mathcal{O}(h^{12}),$$

and

$$lte_i = \frac{-h^8}{720}\left(\mu^8 u_i - 4\mu^6 D_x^2 u_i + 6\mu^4 D_x^4 u_i - 4\mu^2 D_x^6 u_i + D_x^8 u_i\right)$$

Following the same procedure of Lemma 3.1, the coefficients (a_0, a_1, b_0, b_1) for $N = 10$ can be obtained. In the same way, the coefficients of the begin and end formulas can be obtained for the same reference set. Consider the case $M = 10$ to be more general, where we can get six cases $P = -1$ to 4 but it seems that all these cases will not have any significant effect on the order of convergence of the scheme. So, we will use only the case $P=-1$ for begin and end formulas for ease of calculations where the coefficients and its local truncation error will be as follows:

$$(d_1,d_2,d_3,d_4,d_5)=\left(-1,\frac{7}{90},\frac{49}{72},\frac{7}{45},\frac{1}{360}\right)\text{and}$$

$$lte_i=\frac{-241h^8}{60480}D_x^8u_i.$$

3.2. Constructing an EF scheme for the second-order derivative

Consider the 2ODE

$$u^{(2)}=f(x,u), \tag{3.11}$$

By the same manner as in Section 3.1, see [20] and [25], then the central finite difference scheme will take the form:

$$c_1u_{i-1}+c_0u_i+c_1u_{i+1}=h^2W_i, \tag{3.12}$$

where $W_i=u^{(2)}(x_i)$. Three cases for the coefficients of (3.12) are given in lemma 3.2.

Lemma 3.2.. *The EF technique* (3.11–3.12 *and* 3.10) *for the case* $N=4$ *gives:*

Case (i)$(K,P)=(3,-1):(c_0,c_1)=(-2,1)andlte_i=\frac{h^4}{12}D_x^4u_i$.

Case (ii) $(K,P)=(1,0):(c_0,c_1)=\left(\frac{-Z}{\eta_{-1}(Z)-1},\frac{Z}{2\eta_{-1}(Z)-2}\right)$,And

$$lte_i=\frac{h^4}{12}\left(D_x^4u_i-v^2D_x^2u_i\right)\ \text{where}\ \rho^{(0)}\left(v^2\right)=D_x^4u_i-v^2D_x^2u_i.$$

Case (iii) $(K,P)=(-1,1):(c_0,c_1)=\left(Z-\frac{2\eta_{-1}(Z)}{\eta_0(Z)},\frac{1}{\eta_0(Z)}\right)$,

And

$$lte_i=\frac{h^4}{12}\left(v^4u_i-2v^2D_x^2u_i+D_x^4u_i\right),\text{where}$$

$$\rho^{(1)}\left(v^2\right)=v^4u_i-2v^2D_x^2u_i+D_x^4u_i.$$

To get the order of accuracy as high as possible, we should use the case $P=0$ or $P=1$ as an approximation for the second derivative as we can annihilate the leading term of their local truncation error by using the most suited value of its free-parameter v. For the fourth-order derivative, we can increase the order of accuracy by increasing the value of N but because of second-order existence i.e. $\epsilon\neq0$, there are bounds that N will lose its significance out of them. If $\epsilon=0$ this means that only the fourth-order derivative exists, so we can raise the accuracy by increasing the value of N.

4. Numerical approximation

4.1. Uniform time-stepping scheme

Now, we consider the EF relations (3.2–3.4 and 3.12) for solving Eqs. (1.1)–(1.3) at the grid point(x_i,t_j). Let u_i^j and $u(x_i,t_j)$ the approximate and the exact solutions respectively. Then our scheme is given by:

$$\left(u_{i-2}^j+a_1u_{i-1}^j+a_0u_i^j+a_1u_{i+1}^j+u_{i+2}^j\right)=h^4\left(b_2M_{i-2}^j+b_1M_{i-1}^j+b_0M_i^j+b_1M_{i+1}^j+b_2M_{i+2}^j\right) \tag{4.1}$$

Using the Eq. (4.1) at j and $j+1$ time levels, we have:

$$\begin{aligned}&\left(u_{i-2}^j+a_1u_{i-1}^j+a_0u_i^j+a_1u_{i+1}^j+u_{i+2}^j\right)\\&\quad+\left(u_{i-2}^{j+1}+a_1u_{i-1}^{j+1}+a_0u_i^{j+1}+a_1u_{i+1}^{j+1}+u_{i+2}^{j+1}\right)\\&=h^4\left[b_2\left(\mathrm{M}_{i-2}^j+\mathrm{M}_{i-2}^{j+1}\right)+b_1\left(\mathrm{M}_{i-1}^j+\mathrm{M}_{i-1}^{j+1}\right)+b_0\left(\mathrm{M}_i^j+\mathrm{M}_i^{j+1}\right)\right.\\&\quad\left.+b_1\left(\mathrm{M}_{i+1}^j+\mathrm{M}_{i+1}^{j+1}\right)+b_2\left(\mathrm{M}_{i+2}^j+\mathrm{M}_{i+2}^{j+1}\right)\right].\end{aligned} \tag{4.2}$$

Apply the weighted θ-scheme to Eq. (1.1), we have:

$$\begin{aligned}D_t^{\vartheta}u(x_i,t_j)&=-(1-\theta)\left(\mathrm{M}_i^j+\epsilon\mathrm{W}_i^j+(1-\sigma)\mathrm{u}_i^j+f_i^j\right)\\&\quad-\theta\left(\left(\mathrm{M}_i^{j+1}+\epsilon\mathrm{W}_i^{j+1}+(1-\sigma)\mathrm{u}_i^{j+1}+f_i^{j+1}\right)\right),\end{aligned} \tag{4.3}$$

where

$$f_i^j=f\left(x_i,t_j,u_i^j\right).$$

Now, combining the two Eqs. (4.2) and (4.3) with Eq. (2.8) and $\theta=1/2$, we get the following scheme for solving Eqs. (1.1)–(1.3)

$$\begin{aligned}&\left(u_{i-2}^j+a_1u_{i-1}^j+a_0u_i^j+a_1u_{i+1}^j+u_{i+2}^j\right)\\&\quad+\left(u_{i-2}^{j+1}+a_1u_{i-1}^{j+1}+a_0u_i^{j+1}+a_1u_{i+1}^{j+1}+u_{i+2}^{j+1}\right)\\&=-h^4\left[b_2\left(\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-3}^j+c_0\mathrm{u}_{i-2}^j+c_1\mathrm{u}_{i-1}^j\right)\right.\right.\\&\quad+\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-3}^{j+1}+c_0\mathrm{u}_{i-2}^{j+1}+c_1\mathrm{u}_{i-1}^{j+1}\right)+(1-\sigma)\left(\mathrm{u}_{i-2}^j+\mathrm{u}_{i-2}^{j+1}\right)\\&\quad\left.+\mathrm{f}_{i-2}^j+\mathrm{f}_{i-2}^{j+1}+2\sum_{k=0}^{j+1}w_k^{\vartheta}\mathrm{u}_{i-2}^{j-k+1}-2\psi_{\vartheta}\varphi_{i-2}-2\psi_{\gamma}\bar{\varphi}_{i-2}\right)\\&\quad+b_1\left(\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-2}^j+c_0\mathrm{u}_{i-1}^j+c_1\mathrm{u}_i^j\right)\right.\\&\quad+\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-2}^{j+1}+c_0\mathrm{u}_{i-1}^{j+1}+c_1\mathrm{u}_i^{j+1}\right)+(1-\sigma)\left(\mathrm{u}_{i-1}^j+\mathrm{u}_{i-1}^{j+1}\right)\\&\quad\left.+\mathrm{f}_{i-1}^j+\mathrm{f}_{i-1}^{j+1}+2\sum_{k=0}^{j+1}w_k^{\vartheta}\mathrm{u}_{i-1}^{j-k+1}-2\psi_{\vartheta}\varphi_{i-1}-2\psi_{\gamma}\bar{\varphi}_{i-1}\right)\\&\quad+b_0\left(\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-1}^j+c_0\mathrm{u}_i^j+c_1\mathrm{u}_{i+1}^j\right)\right.\\&\quad+\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i-1}^{j+1}+c_0\mathrm{u}_i^{j+1}+c_1\mathrm{u}_{i+1}^{j+1}\right)+(1-\sigma)\left(\mathrm{u}_i^j+\mathrm{u}_i^{j+1}\right)\\&\quad\left.+\mathrm{f}_i^j+\mathrm{f}_i^{j+1}+2\sum_{k=0}^{j+1}w_k^{\vartheta}\mathrm{u}_i^{j-k+1}-2\psi_{\vartheta}\varphi_i-2\psi_{\gamma}\bar{\varphi}_i\right)\\&\quad+b_1\left(\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_i^j+c_0\mathrm{u}_{i+1}^j+c_1\mathrm{u}_{i+2}^j\right)\right.\\&\quad+\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_i^{j+1}+c_0\mathrm{u}_{i+1}^{j+1}+c_1\mathrm{u}_{i+2}^{j+1}\right)+(1-\sigma)\left(\mathrm{u}_{i+1}^j+\mathrm{u}_{i+1}^{j+1}\right)\\&\quad\left.+\mathrm{f}_{i+1}^j+\mathrm{f}_{i+1}^{j+1}+2\sum_{k=0}^{j+1}w_k^{\vartheta}\mathrm{u}_{i+1}^{j-k+1}-2\psi_{\vartheta}\varphi_{i+1}-2\psi_{\gamma}\bar{\varphi}_{i+1}\right)\\&\quad+b_2\left(\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i+1}^j+c_0\mathrm{u}_{i+2}^j+c_1\mathrm{u}_{i+3}^j\right)\right.\\&\quad+\frac{\epsilon}{h^2}\left(c_1\mathrm{u}_{i+1}^{j+1}+c_0\mathrm{u}_{i+2}^{j+1}+c_1\mathrm{u}_{i+3}^{j+1}\right)+(1-\sigma)\left(\mathrm{u}_{i+2}^j+\mathrm{u}_{i+2}^{j+1}\right)\\&\quad\left.\left.+\mathrm{f}_{i+2}^j+\mathrm{f}_{i+2}^{j+1}+2\sum_{k=0}^{j+1}w_k^{\vartheta}\mathrm{u}_{i+2}^{j-k+1}-2\psi_{\vartheta}\varphi_{i+2}-2\psi_{\gamma}\bar{\varphi}_{i+2}\right)\right],\end{aligned}$$

$$\text{where}\varphi_i=\varphi(x_i),\bar{\varphi}_i=\bar{\varphi}(x_i). \tag{4.4}$$

As we mentioned before, for the ease of calculation we will use the case $N = 8$ to approximate the fourth-order derivative with the existence of the second derivative i.e. $\epsilon \neq 0$. While for $\epsilon = 0$ we will use the $N = 10$ case as an approximation for the fourth-order derivative.

Let $U^j = \left(u_i^j\right)$ and $\boldsymbol{u}^j = \left(u\left(x_i, t_j\right)\right)$ and the local truncation errors are $\boldsymbol{T}^j = \left(T_i^j\right)$ where the nonlinear source term $\boldsymbol{f}^j = f\left(x_i, t_j, u\left(x_i, t_j\right)\right)$. Then our scheme can be written in the following matrix form:

$$\begin{aligned}&\left(A - \epsilon h^2 BC + h^4(1-\sigma)B + 2w_1^{\vartheta} h^4 B\right)\boldsymbol{u}^{j+1} + h^4 B\boldsymbol{f}^{j+1}\\ &\quad = -\left(A - \epsilon h^2 BC + h^4(1-\sigma)B\right)\boldsymbol{u}^j - h^4 B\boldsymbol{f}^j\\ &\qquad - 2h^4 B\sum_{k=2}^{j+1} w_k^{\vartheta}\boldsymbol{u}^{j-k+1} - 2h^4 B\psi_{\vartheta}\varphi + \boldsymbol{T}^j, j = 0,1,2,\cdots,n,\end{aligned} \tag{4.5}$$

where A is a pentadiagonal matrix $Penta(a_0, a_1, 1)$ of order $m-1$ with first row elements $(a_0 - 1, a_1, 1)$ and last row elements are the reverse of the first row. While B is also a pentadiagonal matrix$Penta(b_0, b_1, b_2)$ with first row elements (d_3, d_4, d_5) and last row elements are the reverse of the first row of order $m-1$ and C is a tridiagonal matrix $Tri(c, c_1)$ of order $m-1$, such that the tridiagonal and pentadiagonal matrices are defined by

$$Tri(a,b) = \begin{cases} a, i = j \\ b, |i-j| = 1 \\ 0, otherwise. \end{cases} \text{ and } Penta(a,b,c) = \begin{cases} a, i = j \\ b, |i-j| = 1 \\ c, |i-j| = 2 \\ 0, otherwise. \end{cases}$$

4.2. Nonuniform time-stepping scheme

In case of a singular behavior occurs at the initial time $t = 0$ in the solution of the given problem, then nonuniform mesh would represent a suitable technique to deal with such a problem. Many papers such as [43,44], and [45] discussed fractional subdiffusion with nonsmooth data and developed a second-order accurate scheme based on a nonuniform time mesh.

If the solution of Eq. (1.1) shows a singular behavior at the initial time $t = 0$, then we will need to adapt the scheme (4.4) to compensate for the initial singularity. Let $0 = t_0 < t_1 < t_2 <$ Â · Â · Â· $< t_n = T$ be a nonuniform time-stepping, with $\tau_j = t_j - t_{j-1}, j = 1,2,...,n$, and define the graded mesh such that $t_j = T(j/n)^r$, $r > 0$, that turns into a uniform mesh at $r = 1$. The graded mesh is described by $t_{j-1} \le t_j, 2 \le j \le n$, concentrating more grid points near the initial time to deal with the lack of smoothness. Now, using L1-scheme, the R–L operator (with the aid of the relation between Caputo and R–L fractional derivatives) can be discretized for $\alpha \in (0,1)$ as follows:

$$\begin{aligned}D_t^{\alpha} u_i^j = &\sum_{k=0}^{j-1} \frac{u_i^{k+1} - u_i^k}{\tau_{k+1}\Gamma(1-\alpha)}\left[\left(t_j - t_k\right)^{1-\alpha} - \left(t_j - t_{k+1}\right)^{1-\alpha}\right]\\ &+ \frac{u_i^0}{t_j^{\alpha}\Gamma(1-\alpha)},\end{aligned}$$

which can be written as:

$$D_t^{\alpha} u_i^j = \sum_{k=0}^{j-1} \omega_{j,k}\left(u_i^{k+1} - u_i^k\right) + \omega_i^0, \tag{4.6}$$

where $\omega_{j,k} = \frac{1}{\tau_{k+1}\Gamma(1-\alpha)}\left[\left(t_j - t_k\right)^{1-\alpha} - \left(t_j - t_{k+1}\right)^{1-\alpha}\right]$and $\omega_i^0 = \frac{u_i^0}{t_j^{\alpha}\Gamma(1-\alpha)}$.

And for $\gamma \in (1,2)$ as in [29] as follows:

$$\begin{aligned}D_t^{\gamma} u_i^j = &\sum_{k=0}^{j-1} \frac{u_i^{k+2} - 2u_i^{k+1} + u_i^k}{\tau_{k+1}^2\Gamma(3-\gamma)}\left[\left(t_j - t_k\right)^{2-\gamma} - \left(t_j - t_{k+1}\right)^{2-\gamma}\right]\\ &+ \frac{t_j^{-\gamma}\varphi_i}{\Gamma(1-\gamma)} + \frac{t_j^{1-\gamma}\bar{\varphi}_i}{\Gamma(2-\gamma)},\end{aligned}$$

which can be written as:

$$D_t^{\gamma} u_i^j = \sum_{k=0}^{j-1} \bar{\omega}_{j,k}\left(u_i^{k+2} - 2u_i^{k+1} + u_i^k\right) + \omega_i^0 + \bar{\omega}_i^0, \tag{4.7}$$

where$\bar{\omega}_{j,k} = \frac{1}{\tau_{k+1}^2\Gamma(3-\gamma)}\left[\left(t_j - t_k\right)^{2-\gamma} - \left(t_j - t_{k+1}\right)^{2-\gamma}\right], \omega_i^0 = \frac{u_i^0}{t_j^{\gamma}\Gamma(1-\gamma)}$ and $\bar{\omega}_i^0 = \frac{t_j^{1-\gamma}\bar{\varphi}_i}{\Gamma(2-\gamma)}$.

Then, from Eqs. (4.3), (4.6) and (4.7) into Eq. (4.2), we get that:

$$\begin{aligned}{u^{(4)}}_i^j + {u^{(4)}}_i^{j+1} = &\frac{-\epsilon}{h^2}\left(c_1 u_{i-1}^j + c_0 u_i^j + c_1 u_{i+1}^j\right)\\ &- \frac{\epsilon}{h^2}\left(c_1 u_{i-1}^{j+1} + c_0 u_i^{j+1} + c_1 u_{i+1}^{j+1}\right)\\ &- (1-\sigma)\left(u_i^j + u_i^{j+1}\right) - f_i^j - f_i^{j+1}\\ &- 2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_i^{k+1} - u_i^k\right) - 2\omega_i^0 - 2\beta\\ &\times\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_i^{k+2} - 2u_i^{k+1} + u_i^k\right) + \omega_i^0 + \bar{\omega}_i^0\right),\\ &\text{where}\varphi_i = \varphi(x_i), \bar{\varphi}_i = \bar{\varphi}(x_i).\end{aligned} \tag{4.8}$$

Using Eq. (4.8) in Eq. (4.1), we can get a new relation valid for the singular near $t = 0$ and the smooth as follows:

$$
\begin{aligned}
&\left(u_{i-2}^{j}+a_1u_{i-1}^{j}+a_0u_i^{j}+a_1u_{i+1}^{j}+u_{i+2}^{j}\right)\\
&\quad+\left(u_{i-2}^{j+1}+a_1u_{i-1}^{j+1}+a_0u_i^{j+1}+a_1u_{i+1}^{j+1}+u_{i+2}^{j+1}\right)\\
&=-h^4\Bigg[b_2\Bigg(\frac{\epsilon}{h^2}\left(c_1u_{i-3}^{j}+c_0u_{i-2}^{j}+c_1u_{i-1}^{j}\right)\\
&\quad+\frac{\epsilon}{h^2}\left(c_1u_{i-3}^{j+1}+c_0u_{i-2}^{j+1}+c_1u_{i-1}^{j+1}\right)+(1-\sigma)\left(u_{i-2}^{j}+u_{i-2}^{j+1}\right)\\
&\quad+f_{i-2}^{j}+f_{i-2}^{j+1}-2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_{i-2}^{k+1}-u_{i-2}^{k}\right)-2\omega_{i-2}^{0}\\
&\quad-2\beta\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_{i-2}^{k+2}-2u_{i-2}^{k+1}+u_{i-2}^{k}\right)+\omega_{i-2}^{0}+\bar{\omega}_{i-2}^{0}\right)\Bigg)\\
&\quad+b_1\Bigg(\frac{\epsilon}{h^2}\left(c_1u_{i-2}^{j}+c_0u_{i-1}^{j}+c_1u_i^{j}\right)\\
&\quad+\frac{\epsilon}{h^2}\left(c_1u_{i-2}^{j+1}+c_0u_{i-1}^{j+1}+c_1u_i^{j+1}\right)+(1-\sigma)\left(u_{i-1}^{j}+u_{i-1}^{j+1}\right)\\
&\quad+f_{i-1}^{j}+f_{i-1}^{j+1}-2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_{i-1}^{k+1}-u_{i-1}^{k}\right)-2\omega_{i-1}^{0}\\
&\quad-2\beta\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_{i-1}^{k+2}-2u_{i-1}^{k+1}+u_{i-1}^{k}\right)+\omega_{i-1}^{0}+\bar{\omega}_{i-1}^{0}\right)\Bigg)\\
&\quad+b_0\Bigg(\frac{\epsilon}{h^2}\left(c_1u_{i-1}^{j}+c_0u_i^{j}+c_1u_{i+1}^{j}\right)\\
&\quad+\frac{\epsilon}{h^2}\left(c_1u_{i-1}^{j+1}+c_0u_i^{j+1}+c_1u_{i+1}^{j+1}\right)+(1-\sigma)\left(u_i^{j}+u_i^{j+1}\right)\\
&\quad+f_i^{j}+f_i^{j+1}-2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_i^{k+1}-u_i^{k}\right)-2\omega_i^{0}\\
&\quad-2\beta\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_i^{k+2}-2u_i^{k+1}+u_i^{k}\right)+\omega_i^{0}+\bar{\omega}_i^{0}\right)\Bigg)\\
&\quad+b_1\Bigg(\frac{\epsilon}{h^2}\left(c_1u_i^{j}+c_0u_{i+1}^{j}+c_1u_{i+2}^{j}\right)\\
&\quad+\frac{\epsilon}{h^2}\left(c_1u_i^{j+1}+c_0u_{i+1}^{j+1}+c_1u_{i+2}^{j+1}\right)+(1-\sigma)\left(u_{i+1}^{j}+u_{i+1}^{j+1}\right)\\
&\quad+f_{i+1}^{j}+f_{i+1}^{j+1}-2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_{i+1}^{k+1}-u_{i+1}^{k}\right)-2\omega_{i+1}^{0}\\
&\quad-2\beta\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_{i+1}^{k+2}-2u_{i+1}^{k+1}+u_{i+1}^{k}\right)+\omega_{i+1}^{0}+\bar{\omega}_{i+1}^{0}\right)\Bigg)\\
&\quad+b_2\Bigg(\frac{\epsilon}{h^2}\left(c_1u_{i+1}^{j}+c_0u_{i+2}^{j}+c_1u_{i+3}^{j}\right)\\
&\quad+\frac{\epsilon}{h^2}\left(c_1u_{i+1}^{j+1}+c_0u_{i+2}^{j+1}+c_1u_{i+3}^{j+1}\right)+(1-\sigma)\left(u_{i+2}^{j}+u_{i+2}^{j+1}\right)\\
&\quad+f_{i+2}^{j}+f_{i+2}^{j+1}-2\sum_{k=0}^{j-1}\omega_{j,k}\left(u_{i+2}^{k+1}-u_{i+2}^{k}\right)-2\omega_{i+2}^{0}\\
&\quad-2\beta\left(\sum_{k=0}^{j-1}\bar{\omega}_{j,k}\left(u_{i+2}^{k+2}-2u_{i+2}^{k+1}+u_{i+2}^{k}\right)+\omega_{i+2}^{0}+\bar{\omega}_{i+2}^{0}\right)\Bigg)\Bigg]
\end{aligned}
$$

where $\varphi_i=\varphi(x_i), \bar{\varphi}_i=\bar{\varphi}(x_i)$ (4.9)

5. The evaluation of the μ -parameter

We will deduce the local truncation error (LTE) of the suggested methods and introduce how to select the best-suited values for the free-parameters μ and hence the appropriate scheme to be used.

Theorem 5.1.. *The LTE T_i^j of the difference relation (4.5) at $N=8$ is given by:*

$$
\begin{aligned}
T_i^j&=(4+2a_0+4a_1)\left(2u_i^j+\tau D_tu_i^j\right)\\
&\quad+(4+a_1)\left(2h^2D_x^2u_i^j+h^2\tau D_x^2D_tu_i^j\right)\\
&\quad+\frac{1}{12}(16+a_1-12b_0-24b_1)\left(2h^4D_x^4u_i^j+h^4\tau D_x^4D_tu_i^j\right)\\
&\quad+\frac{1}{360}(64+a_1-360b_1)\left(2h^6D_x^6u_i^j+h^6\tau D_x^6D_tu_i^j\right)\\
&\quad+\frac{1}{20160}(256+a_1-1680b_1)\left(2h^8D_x^8u_i^j+h^8\tau D_x^8D_tu_i^j\right)\\
&\quad+\mathcal{O}\left(\epsilon h^6+h^{10}+\tau^2\right).
\end{aligned}
\tag{5.1}
$$

Also, methods of higher orders can be obtained as:

$$
T_i^j=\begin{cases}
\frac{-h^8}{360}D_x^8u_i^j+O\left(\epsilon h^6+h^{10}+h^8\tau+\tau^2\right),(K,P)=(7,-1),\mu=0,\\
\frac{-h^8}{360}\rho^{(0)}(\mu^2)+O\left(\epsilon h^6+h^{10}+h^8\tau+\tau^2\right),(K,P)=(5,0),\\
\frac{-h^8}{360}\rho^{(1)}(\mu^2)+O\left(\epsilon h^6+h^{10}+h^8\tau+\tau^2\right),(\mathrm{K},\mathrm{P})=(3,1),\\
\frac{-h^8}{360}\rho^{(2)}(\mu^2)+O\left(\epsilon h^6+h^{10}+h^8\tau+\tau^2\right),(\mathrm{K},\mathrm{P})=(1,2),\\
\frac{-h^8}{360}\rho^{(3)}(\mu^2)+O\left(\epsilon h^6+h^{10}+h^8\tau+\tau^2\right),(\mathrm{K},\mathrm{P})=(-1,3),\\
\rho^{(0)}(\mu^2)=\left(D_x^8u_i^j-\mu^2D_x^6u_i^j\right),\rho^{(1)}(\mu^2)=\left(\mu^4D_x^4u_i-2\mu^2D_x^6u_i+D_x^8u_i\right),\\
\rho^{(2)}(\mu^2)=\left(D_x^8u_i-3\mu^2D_x^6u_i+3\mu^4D_x^4u_i-\mu^6D_x^2u_i\right),\\
\rho^{(3)}(\mu^2)=\left(\mu^8u_i-4\mu^6D_x^2u_i+6\mu^4D_x^4u_i-4\mu^2D_x^6u_i+D_x^8u_i\right).
\end{cases}
\tag{5.2}
$$

where the local truncation error T_i^j of the difference relation (4.5) at $N=10$ can be computed by the same procedure.

Proof. *Following the same proceeding in* [39].

Now, we aim to choose high order EF method for the problem (1.1–1.3) and the value of the μ-parameter.. This can be done by solving the equation of each EF method $\rho^{(p)}(\mu^2), p=0,1,2,3$ as defined above. Minimizing the values of $\rho^{(p)}(\mu^2)$, lead to decreasing the LTE. If μ is constant, then the solution is in the fitting space, otherwise, we select the value of P for which μ^2 is minimal. The coefficients of the equation $\rho^{(p)}(\mu^2)$ can be computed numerically. We may refer to the algorithm presented in [26,27].

6. Stability and convergence analysis

Here, the stability and convergence analysis of (4.5) will be analyzed using the Fourier method. Let $U^j, j = 0, 1, \cdots, n$, be an approximate solution to Eq. (4.5), we get:

$$\left(A - \epsilon h^2 BC + h^4(1-\sigma)B + 2w_1^\vartheta h^4 B\right)U^{j+1} + h^4 BF^{j+1} = -\left(A - \epsilon h^2 BC + h^4(1-\sigma)B\right)U^j - h^4 BF^j - 2h^4 B \times \sum_{k=2}^{j+1} w_k^\vartheta U^{j-k+1} - 2h^4 B\psi_\vartheta U^0, = 0, 1, 2, \cdots, n \quad (6.1)$$

Let $\bar{u}_i^j$ be the numerical solution and $\bar{f}_i^j = f\left(x_i, t_j, \bar{u}_i^j\right)$, then we have the round-off error at the point $\left(x_i, t_j\right)$ is:

$$\varepsilon_i^j = u_i^j - \bar{u}_i^j \text{and} \eta_i^j = f_i^j - \bar{f}_i^j, i = 0, 1, \cdots, m; j = 0, 1, \cdots, n, \quad (6.2)$$

such that

$$\varepsilon_0^j = \varepsilon_m^j = 0, \varepsilon_i^0 = 0; i = 0, 1, \cdots, m; j = 0, 1, \cdots, n. \quad (6.3)$$

Now, we define $\varepsilon^j(x_i) = \varepsilon_i^j, \eta^j(x_i) = \eta_i^j$, where $x \in [x_i, x_i + h], i = 0, 1, \cdots, m$, and

$$\|\varepsilon^j(x)\|_2 \equiv \left(\sum_{i=1}^{n-1} h\left|\varepsilon_i^j\right|^2\right)^{\frac{1}{2}} \text{and} \|\eta^j(x)\|_2 \equiv \left(\sum_{i=1}^{n-1} h\left|\eta_i^j\right|^2\right)^{\frac{1}{2}}$$

Then, the round-off of Eq. (6.2) is

$$\varepsilon_i^j = \xi_j e^{i\omega\sqrt{-1}h}, \eta_i^j = \lambda_j e^{i\omega\sqrt{-1}h}, \omega = \frac{2\pi k}{m}, k = 0, \pm 1, \pm 2. \quad (6.4)$$

We also define

$$\bar{\varepsilon}_i^j = u\left(x_i, t_j\right) - u_i^j \text{and} \bar{\eta}_i^j = f\left(x_i, t_j, u\left(x_i, t_j\right)\right) - f_i^j; \quad i = 0, 1, \cdots, m; j = 0, 1, \cdots, n, \quad (6.5)$$

where

$$\bar{\varepsilon}_0^j = \bar{\varepsilon}_m^j = \bar{\varepsilon}_i^0 = 0; i = 0, 1, \cdots, m; j = 0, 1, \cdots, n \quad (6.6)$$

Now, we define $\bar{\varepsilon}^j(x_i) = \bar{\varepsilon}_i^j$, $\bar{\eta}^j(x_i) = \bar{\eta}_i^j$ and $\boldsymbol{T}^j(x_i) = T_i^j$, where$x \in [x_i, x_i + h], i = 0, 1, \cdots, m$.

$$\|\bar{\varepsilon}^j(x)\|_2 \equiv \left(\sum_{i=1}^{n-1} h\left|\bar{\varepsilon}_i^j\right|^2\right)^{\frac{1}{2}}, \|\bar{\eta}^j(x)\|_2 \equiv \left(\sum_{i=1}^{n-1} h\left|\bar{\eta}_i^j\right|^2\right)^{\frac{1}{2}} \text{and} \|\boldsymbol{T}^j\|_2 \equiv \left(\sum_{i=1}^{n-1} h\left|T_i^j\right|^2\right)^{\frac{1}{2}}. \quad (6.7)$$

$$\bar{\varepsilon}_i^j = \bar{\xi}_j e^{i\omega\sqrt{-1}h}, \bar{\eta}_i^j = \bar{\lambda}_j e^{i\omega\sqrt{-1}h} \text{and} T_i^j = \rho_j e^{i\omega\sqrt{-1}h}, \quad (6.8)$$

and using Eq. (5.2), we obtain

$$\left|T_i^j\right| = k_1\left(\epsilon h^6 + h^{10} + h^8\tau + \tau^2\right), k_1 > 0, 0 \le i \le m, 0 \le j \le n \quad (6.9)$$

6.1. Stability

Using Eqs. (4.4) and (6.1)–(6.3), we get:

$$\begin{aligned}
&\left(\varepsilon_{i-2}^j + a_1\varepsilon_{i-1}^j + a_0\varepsilon_i^j + a_1\varepsilon_{i+1}^j + \varepsilon_{i+2}^j\right) + \left(\varepsilon_{i-2}^{j+1} + a_1\varepsilon_{i-1}^{j+1} + a_0\varepsilon_i^{j+1} + a_1\varepsilon_{i+1}^{j+1} + \varepsilon_{i+2}^{j+1}\right) \\
&= -h^4\Bigg[b_2\Bigg(\frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-3}^j + c_0\varepsilon_{i-2}^j + c_1\varepsilon_{i-1}^j\right) + \frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-3}^{j+1} + c_0\varepsilon_{i-2}^{j+1} + c_1\varepsilon_{i-1}^{j+1}\right) + (1-\sigma)\left(\varepsilon_{i-2}^j + \varepsilon_{i-2}^{j+1}\right) + \eta_{i-2}^j \\
&\quad + \eta_{i-2}^{j+1} + 2\sum_{k=0}^{j+1} w_k^\vartheta \varepsilon_{i-2}^{j-k+1}\Bigg) \\
&\quad + b_1\Bigg(\frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-2}^j + c_0\varepsilon_{i-1}^j + c_1\varepsilon_i^j\right) + \frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-2}^{j+1} + c_0\varepsilon_{i-1}^{j+1} + c_1\varepsilon_i^{j+1}\right) \\
&\quad + (1-\sigma)\left(\varepsilon_{i-1}^j + \varepsilon_{i-1}^{j+1}\right) + \eta_{i-1}^j + \eta_{i-1}^{j+1} + 2\sum_{k=0}^{j+1} w_k^\vartheta \varepsilon_{i-1}^{j-k+1}\Bigg) \\
&\quad + b_0\Bigg(\frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-1}^j + c_0\varepsilon_i^j + c_1\varepsilon_{i+1}^j\right) + \frac{\epsilon}{h^2}\left(c_1\varepsilon_{i-1}^{j+1} + c_0\varepsilon_i^{j+1} + c_1\varepsilon_{i+1}^{j+1}\right) \\
&\quad + (1-\sigma)\left(\varepsilon_i^j + \varepsilon_i^{j+1}\right) + \eta_i^j + \eta_i^{j+1} + 2\sum_{k=0}^{j+1} w_k^\vartheta \varepsilon_i^{j-k+1}\Bigg) \\
&\quad + b_1\Bigg(\frac{\epsilon}{h^2}\left(c_1\varepsilon_i^j + c_0\varepsilon_{i+1}^j + c_1\varepsilon_{i+2}^j\right) + \frac{\epsilon}{h^2}\left(c_1\varepsilon_i^{j+1} + c_0\varepsilon_{i+1}^{j+1} + c_1\varepsilon_{i+2}^{j+1}\right) \\
&\quad + (1-\sigma)\left(\varepsilon_{i+1}^j + \varepsilon_{i+1}^{j+1}\right) + \eta_{i+1}^j + \eta_{i+1}^{j+1} + 2\sum_{k=0}^{j+1} w_k^\vartheta \varepsilon_{i+1}^{j-k+1}\Bigg) \\
&\quad + b_2\Bigg(\frac{\epsilon}{h^2}\left(c_1\varepsilon_{i+1}^j + c_0\varepsilon_{i+2}^j + c_1\varepsilon_{i+3}^j\right) + \frac{\epsilon}{h^2}\left(c_1\varepsilon_{i+1}^{j+1} + c_0\varepsilon_{i+2}^{j+1} + c_1\varepsilon_{i+3}^{j+1}\right) \\
&\quad + (1-\sigma)\left(\varepsilon_{i+2}^j + \varepsilon_{i+2}^{j+1}\right) + \eta_{i+2}^j + \eta_{i+2}^{j+1} + 2\sum_{k=0}^{j+1} w_k^\vartheta \varepsilon_{i+2}^{j-k+1}\Bigg)\Bigg] \quad (6.1.1)
\end{aligned}$$

Now, substituting Eq. (6.4) into Eq. (6.1.1) we get the following

$$\begin{aligned}
&\left(e^{-2\sqrt{-1}\omega h} + a_1 e^{-\omega\sqrt{-1}h} + a_0 + a_1 e^{-\omega\sqrt{-1}h} + e^{-2\omega\sqrt{-1}h}\right)\left(\xi_{j+1} + \xi_j\right) \\
&= -\epsilon h^2\Big[b_2\left(c_1 e^{-3\omega\sqrt{-1}h} + c_0 e^{-2\omega\sqrt{-1}h} + c_1 e^{-\omega h}\right) \\
&\quad + b_1\left(c_1 e^{-2\omega\sqrt{-1}h} + c_0 e^{-\omega\sqrt{-1}h} + c_1\right) \\
&\quad + b_0\left(c_1 e^{-\omega\sqrt{-1}h} + c_0 + c_1 e^{\omega\sqrt{-1}h}\right) \\
&\quad + b_1\left(c_1 + c_0 e^{\omega\sqrt{-1}h} + c_1 e^{2\omega\sqrt{-1}h}\right) \\
&\quad + b_2\left(c_1 e^{\omega\sqrt{-1}h} + c_0 e^{2\omega\sqrt{-1}h} + c_1 e^{3\omega\sqrt{-1}h}\right)\Big]\left(\xi_{j+1} + \xi_j\right) \\
&\quad - h^4\left(b_2 e^{-2\omega\sqrt{-1}h} + b_1 e^{-\omega\sqrt{-1}h} + b_0 + b_1 e^{-\omega\sqrt{-1}h} + b_2 e^{-2\omega\sqrt{-1}h}\right) \\
&\quad \times \left[(1-\sigma)\left(\xi_{j+1} + \xi_j\right) + \left(\lambda_{j+1} + \lambda_j\right) + 2\sum_{k=0}^{j+1} w_k^\vartheta \xi_{j-k+1}\right] \quad (6.1.2)
\end{aligned}$$

After simplification, we obtain

$$(\theta + \psi[1 + 2\tau^{-\alpha} + 2\beta\tau^{-\gamma}])\xi_{j+1} = -(\theta + \psi)\xi_j - \psi \times \left(\left(\lambda_{j+1} + \lambda_j\right) + 2\tau^{-\alpha}\sum_{k=1}^{j+1} g_k^\alpha \xi_{j-k+1} + +2\beta\tau^{-\gamma}\sum_{k=1}^{j+1} g_k^\gamma \xi_{j-k+1}\right) \quad (6.1.3)$$

such that,

$$\theta = 2\epsilon h^2 b_2 c_1 \cos(3\omega h) + 2\left[1 + \epsilon h^2(b_2 c_0 + b_1 c_1) + h^4 b_2(1-\sigma)\right]\cos(2\omega h) + 2\left[a_1 + \epsilon h^2(b_2 c_1 + b_1 c_0 + b_0 c_1) + h^4 b_1(1-\sigma)\right]\cos(\omega h) + \left[a_0 + \epsilon h^2(2b_1 c_1 + b_0 c_0) + h^4 b_0(1-\sigma)\right]\cos(\omega h) + h^4(2b_2\cos(2\omega h) + 2b_1\cos(\omega h) + b_0), \text{and} \psi = h^4(2b_2\cos(2\omega h) + 2b_1\cos(\omega h) + b_0).$$

Using the condition, $|\gamma_i^j| \le L_f|\varepsilon_i^j|$ into Eq. (6.1.3) gives the following:

$$|\theta + \psi[2\tau^{-\alpha} + 2\beta\tau^{-\gamma}]||\xi_{j+1}| \le |\theta||\xi_j| + |\psi|(L_f|\xi_{j+1}| + L_f|\xi_j|) + 2\tau^{-\alpha}|\psi|\left|\sum_{k=1}^{j+1} g_k^\alpha\right||\xi_{j-k+1}| + 2\beta\tau^{-\gamma}|\psi|\left|\sum_{k=1}^{j+1} g_k^\gamma\right||\xi_{j-k+1}|. \tag{6.1.4}$$

It is clear that $2\tau^{-\alpha}|\psi| < |\theta + 2\psi[\tau^{-\alpha} + \beta\tau^{-\gamma}]|$ and $2\tau^{-\gamma}\beta|\psi| < |\theta + 2\psi[\tau^{-\alpha} + \beta\tau^{-\gamma}]|$, then using Lemma 2.1, we have

$$|\xi_{j+1}| \le K_1|\xi_j| + (1+\gamma)|\xi_0|, \tag{6.1.5}$$

where $K_1 = \frac{|\theta| + |\psi|L_f}{|\theta + 2\psi[\tau^{-\alpha} + \beta\tau^{-\gamma}]| - |\psi|L_f} < 1$.

Repeating $j+1$ times, in Eq. (6.1.5), we get the following

$$|\xi_{j+1}| \le \boldsymbol{K}|\xi_0|$$

where $= \frac{(1+\gamma)}{1-K_1} > 0$, and free of h and τ. We outline these results in theorem 3.

Theorem 3. *The numerical scheme of Eq. (1.1) defined by Eq. (4.5) satisfies* $\|\varepsilon^j\|_2 \le \boldsymbol{K}\|\varepsilon^0\|_2$.

6.2. Convergence

Follow the same procedure in Section 6.1, our numerical scheme defined by Eq. (4.5) is convergent, and easily, we can prove the following theorem.

Theorem 4. *The EF method given by Eq. (4.5) is convergent with* $\mathcal{O}(\epsilon h^6 + h^{10} + h^8\tau + \tau^2)$ *for the cases* $P = 0, 1, 2, 3$.

7. Numerical results and discussion

In this section, we will prove the accuracy and the effectiveness of the exponential fitting methods by testing them in solving time-fractional nonlinear cubic-quintic standard and modified Swift-Hohenberg equation at different cases. Since these types of equations have no exact solutions yet, we will study them and the effect of changing the significant parameters which they depend on such as length, the time-fractional derivative order, and the parameters β and σ.

7.1. Test problem (1)

Consider the initial condition and nonlinear term as follows

$$\varphi(x) = 0.1\sin\left(\frac{\pi x}{L}\right), f(x,t,u) = u^3$$

In this example, we will study both cases of the time-fractional nonlinear cubic Swift-Hohenberg equation which is the standard i.e. $\beta = 0$ and the modified i.e. $\beta = 1$ using the different cases of exponential fitting technique. Also, we will illustrate the effect of changing the parameter σ, length, the time-fractional derivative, and the type of the equation whether it is standard or modified on the nature and behavior of the solution graphically. In Fig. 1, we show the behavior of the solution at $\alpha = 1, T = 2$ using the EF method $P = 0$ such that Figs. 1.a–1.b represent the solution at $\sigma = 0.3$ with different values of the length $L = 3, 8$. It is obvious that the solution increases at $L = 3$ but decreases at $L = 8$. Figs. 1.c-1.d represent the solution at $\sigma = 0.9$ such that the solution increases in both cases $L = 3, 8$ but it increases for $L = 3$ faster than $L = 8$.

Fig. 2 shows the behavior of the solution at $\alpha = 0.8, T = 2$ using the EF method $P = 1$ such that Figs. 2.a–2.b represent the solution at $\sigma = 0.3$ with different values of the length $L = 3, 8$. It is obvious that the solution increases at $L = 3$ but decreases at $L = 8$. Figs. 2.c–2.d represent the solution at $\sigma = 0.9$ such that the solution increases in both cases $L = 3, 8$ but it increases for $L = 3$ faster than $L = 8$. We can compare Figs. 1 and 2 to see the difference between the solution nature for integer and fractional derivative orders of standard cubic Swift-Hohenberg equation.

Now, we study the solution behavior for the modified cubic type for integer and fractional order derivatives as shown in Fig. 3 and Fig. 4, respectively. Fig. 3 shows the solution at $\alpha = 1, \gamma = 2$, and $T = 2$ using the EF method $P = 2$ such that Figs. 3.a-3.b represent the solution at $\sigma = 0.3$ with different values of the length $L = 3, 8$. The solution increases for $L = 3$ but for $L = 8$ increases and starts to decrease at the end. Figs. 3.c-3.d represent the solution at $\sigma = 0.9$ such that the solution increases in both cases $L = 3, 8$ where the solution increases in both cases but it increases for $L = 3$ faster than $L = 8$. Fig. 4 shows the behavior of the solution at $\alpha = 0.8, \gamma = 1.8$, and $T = 2$ using the EF method $P = 3$ such that Figs. 4.a-4.b represent the solution at $\sigma = 0.3$ and Figs. 4.c-4.d represent the solution at $\sigma = 0.9$. The solution behavior for Figs. 3 and 4 are the same but the solution for integer-order derivative increases faster than the fractional one.

7.2. Test problem (2)

Consider the problem (1.1–1.3) with the initial condition and nonlinear term as follows

$$\varphi(x) = 0.1\sin\left(\frac{\pi x}{L}\right), f(x,t,u) = cu^5 - bu^3$$

In this example, we will follow the same procedure of the previous problem to study both cases of the time-fractional nonlinear cubic-quintic Swift-Hohenberg equation which is the standard i.e. $\beta = 0$ and the modified i.e. $\beta = 1$ with $c = \frac{3}{2}, b = 2$ using the different cases of exponential fitting technique. Also, we will illustrate the effect of changing the

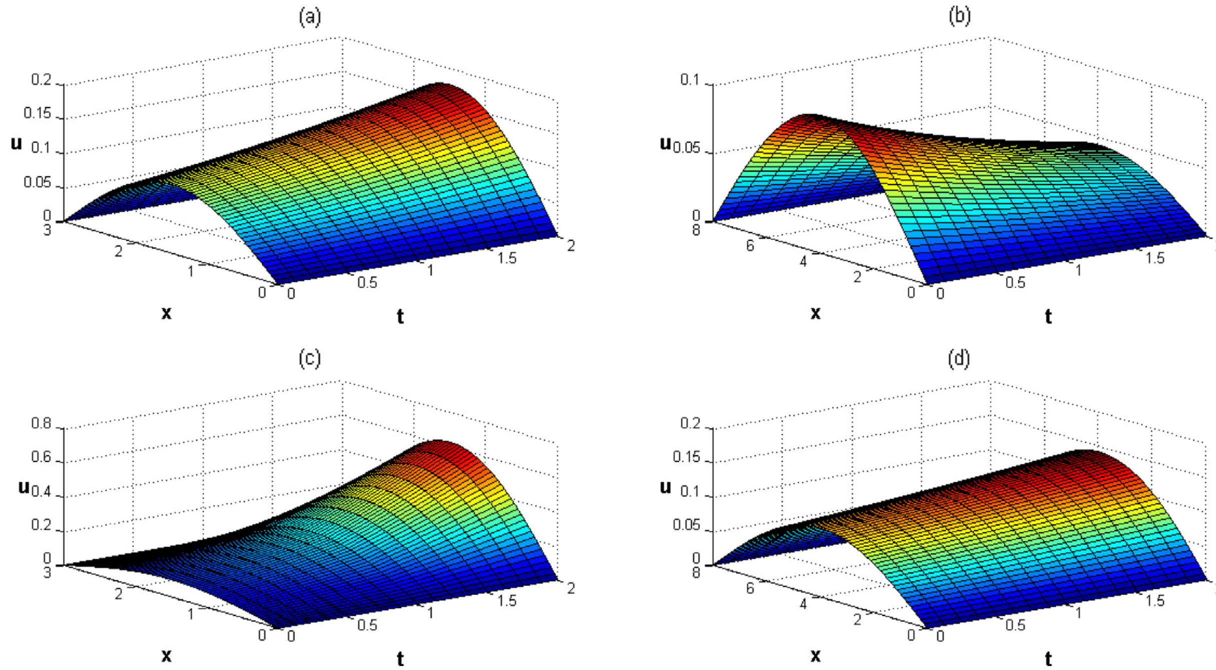

Fig. 1 The numerical solution behavior for the standard cubic Swift-Hohenberg equation at $\alpha = 1$ for problem 1.

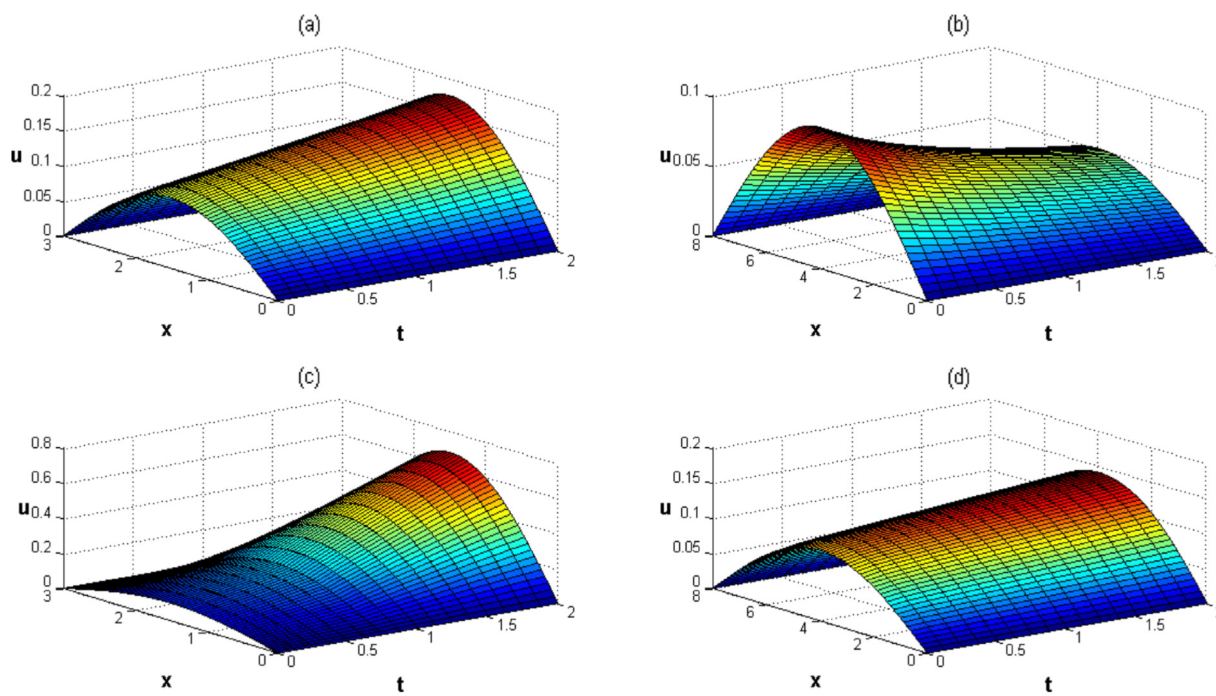

Fig. 2 The numerical solution behavior for the time-fractional standard cubic Swift-Hohenberg equation at $\alpha = 0.8$ for problem 1.

parameter σ, length, the time-fractional derivative, and the type of the equation whether it is standard or modified on the nature and behavior of the solution graphically by the same sequence of the previous example. In Fig. 5 the EF method $P = 0$ has been used to show the behavior of the solution at $\alpha = 1, T = 2$ such that Figs. 5.a-5.b represent the solution at $\sigma = 0.3$ where the solution increases at $L = 3$ but decreases for $L = 8$, but at $\sigma = 0.9$ illustrated in Figs. 5.c-5.d the solution increases for both cases but for $L = 3$ faster than $L = 8$. the EF method $P = 1$ has been used in Fig. 6 to show the behavior of the solution at $\alpha = 0.8, T = 2$ such that Figs. 6.a-6.b represent the solution at $\sigma = 0.3$ with different values of the length $L = 3, 8$. For $L = 3$ the solution decreases then increases but for $L = 8$ it decreases smoothly. Figs. 6.c-6.d represent the solution at $\sigma = 0.9$ where the solution increases for both cases but for $L = 3$ faster than $L = 8$.

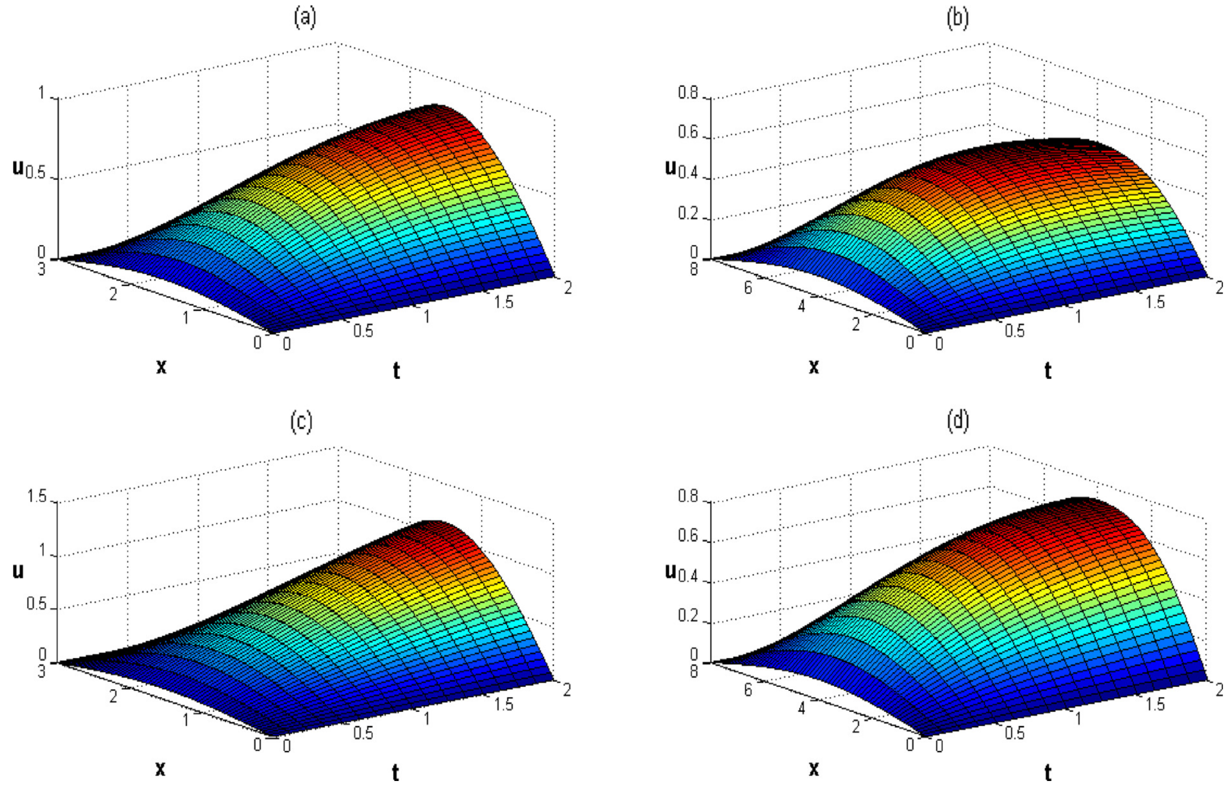

Fig. 3 The numerical solution behavior for the modified cubic Swift-Hohenberg equation at $\alpha = 1, \gamma = 2$ for problem 1.

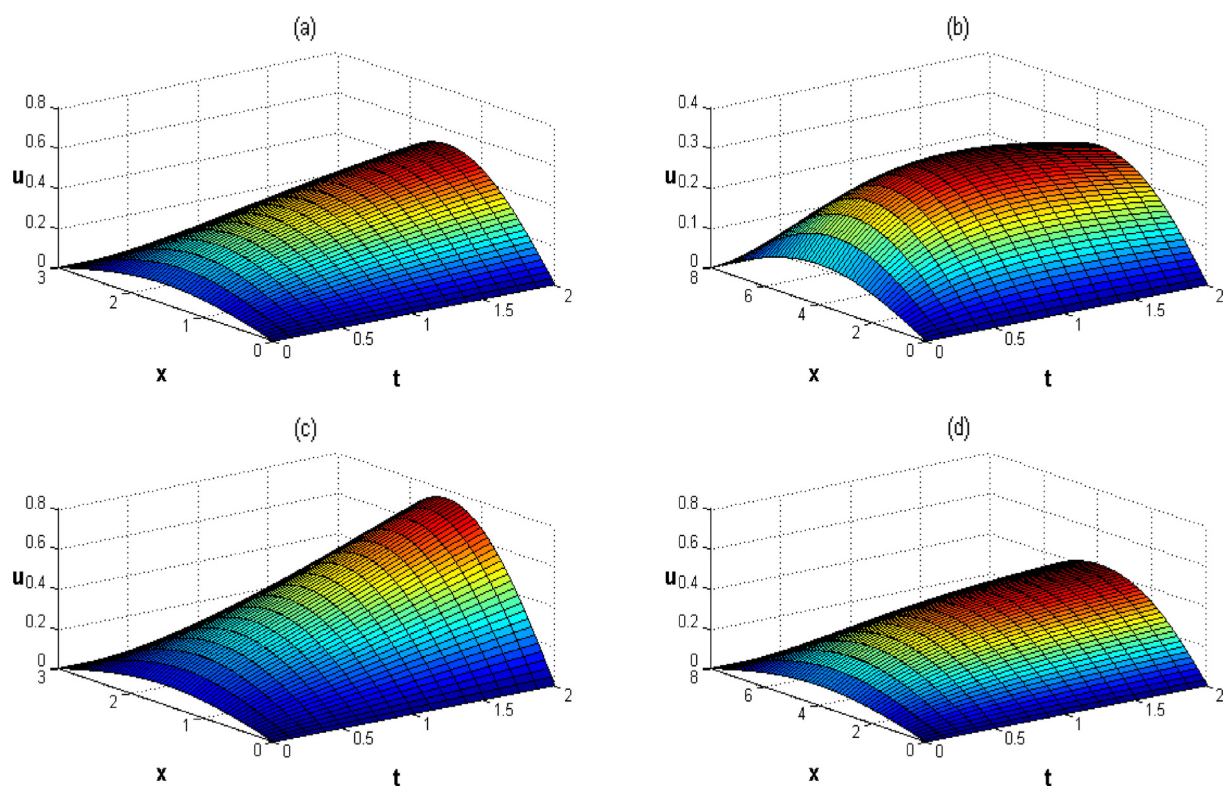

Fig. 4 The numerical solution behavior for the time-fractional modified cubic Swift-Hohenberg equation at $\alpha = 0.8, \gamma = 1.8$ for problem 1.

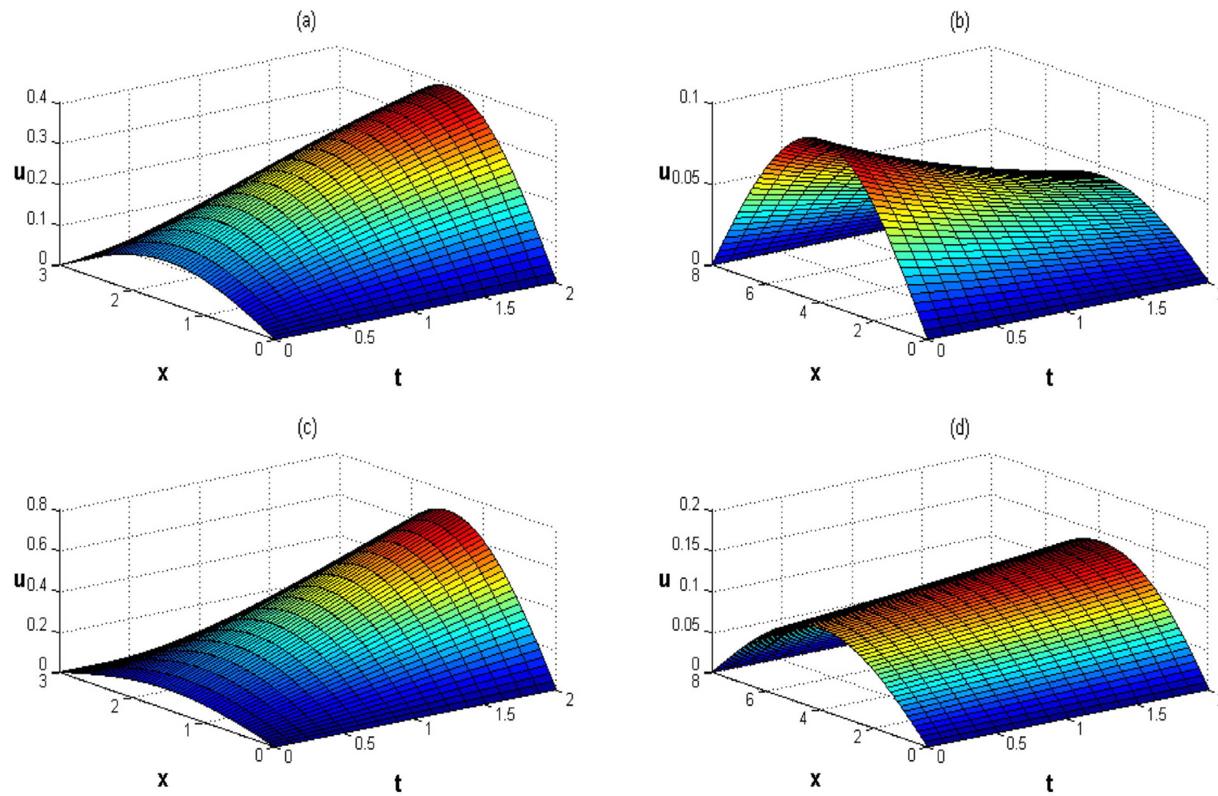

Fig. 5 The numerical solution behavior for the standard cubic-quintic Swift-Hohenberg equation at $\alpha = 1$ for problem 2.

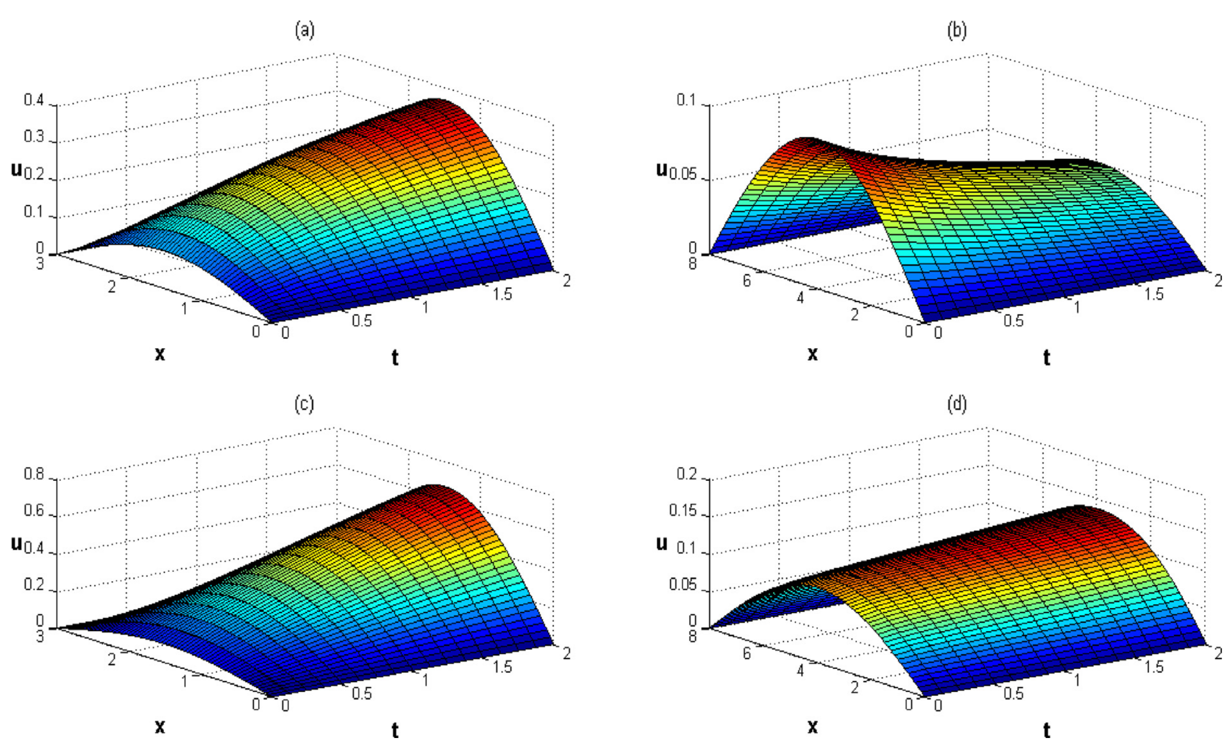

Fig. 6 The numerical solution behavior for the time-fractional standard cubic-quintic Swift-Hohenberg equation at $\alpha = 0.8$ for problem 2.

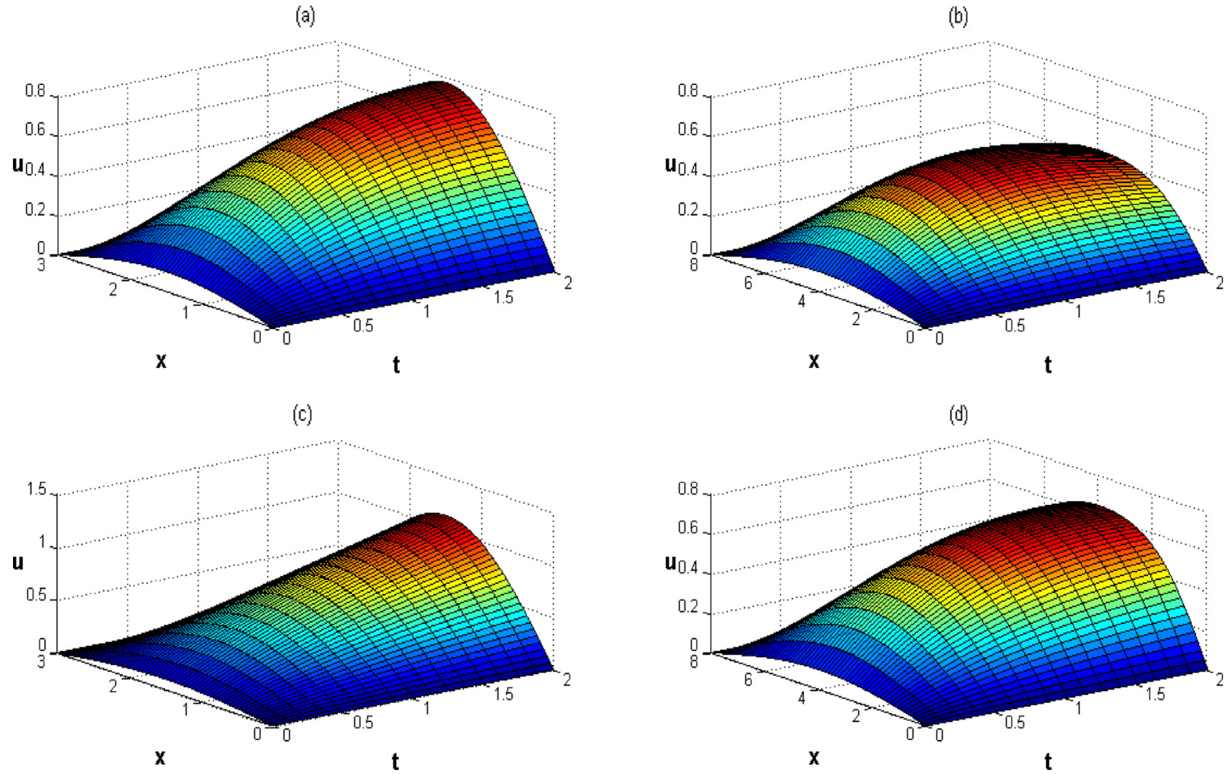

Fig. 7 The numerical solution behavior for the modified cubic-quintic Swift-Hohenberg equation at $\alpha = 1, \gamma = 2$ for problem 2.

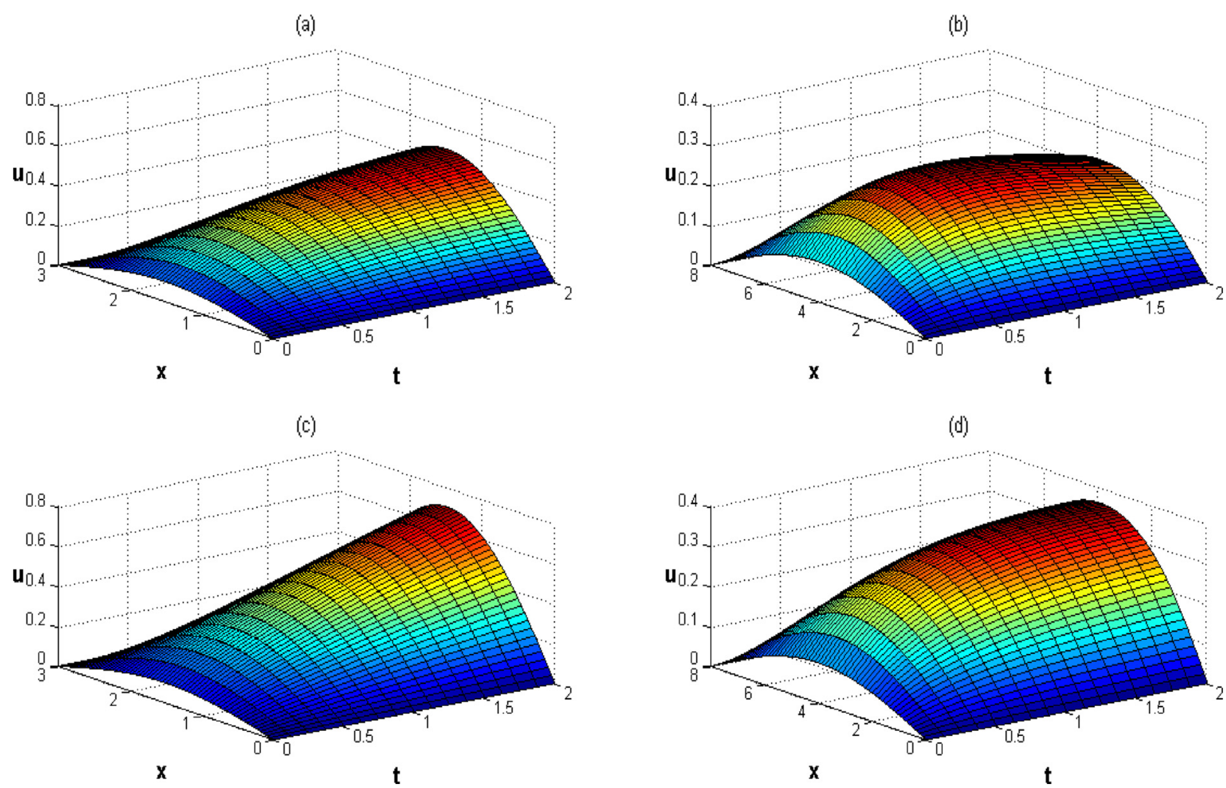

Fig. 8 The numerical solution behavior for the time-fractional modified cubic-quintic Swift-Hohenberg equation at $\alpha = 0.8, \gamma = 1.8$ for problem 2.

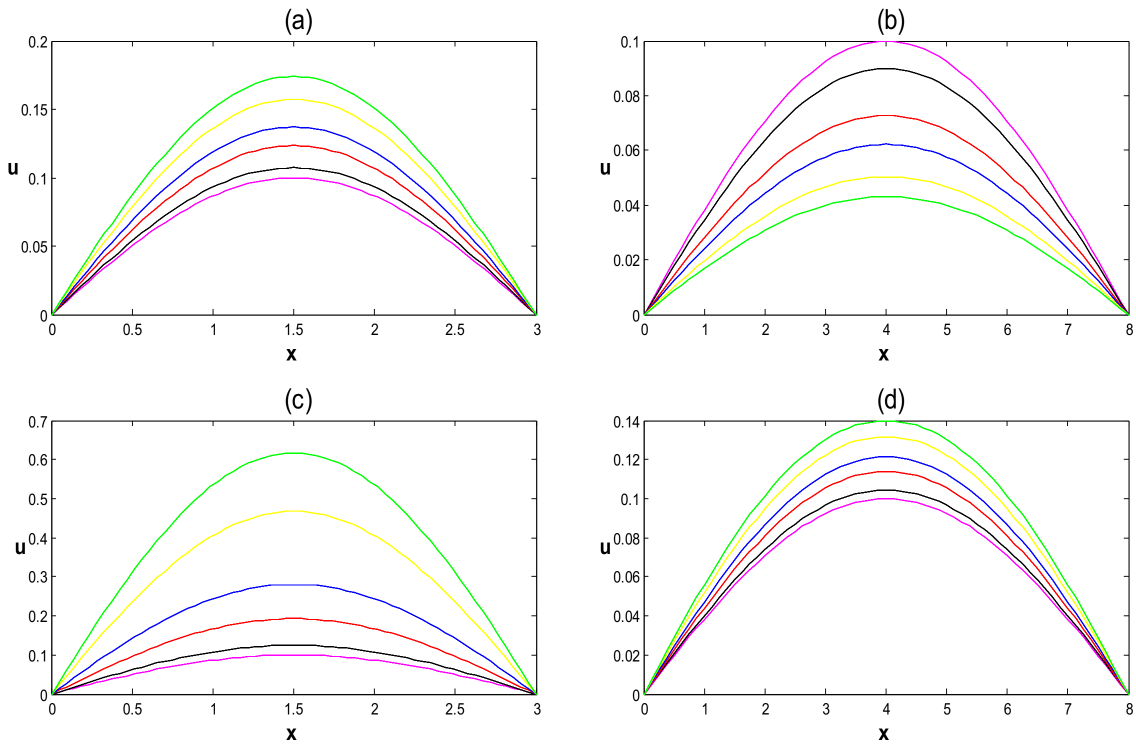

Fig. 9 The numerical solution behavior for the standard cubic Swift-Hohenberg equation at $\alpha = 1$ for problem 1.

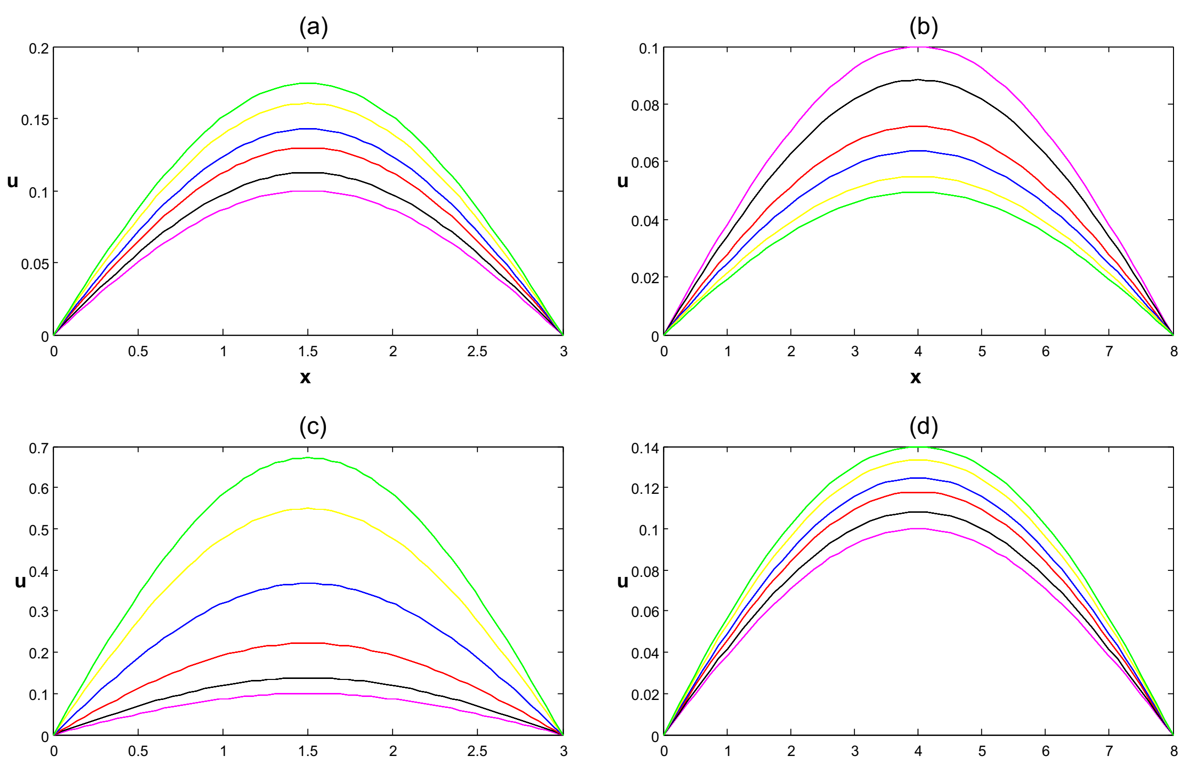

Fig. 10 The numerical solution behavior for the time-fractional standard cubic Swift-Hohenberg equation at $\alpha = 0.8$ for problem 1.

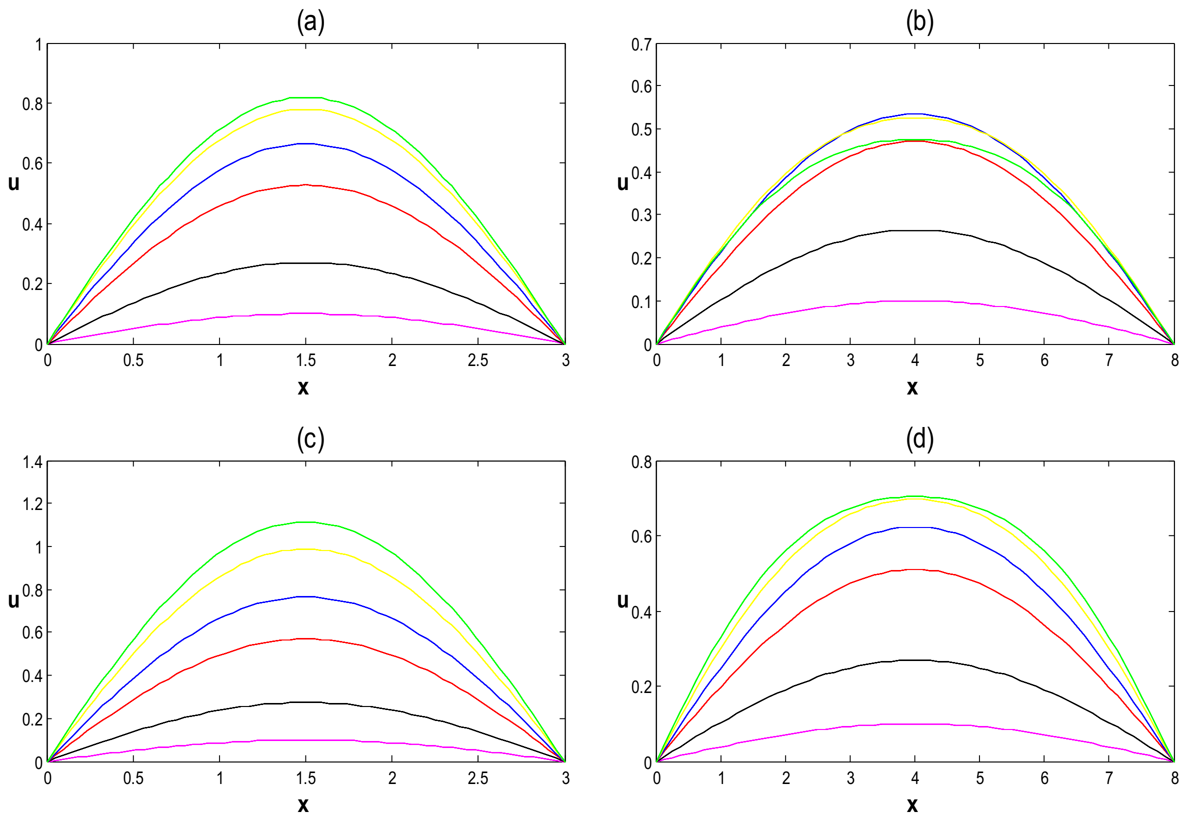

Fig. 11 The numerical solution behavior for the modified cubic Swift-Hohenberg equation at $\alpha = 1, \gamma = 2$ for problem 1.

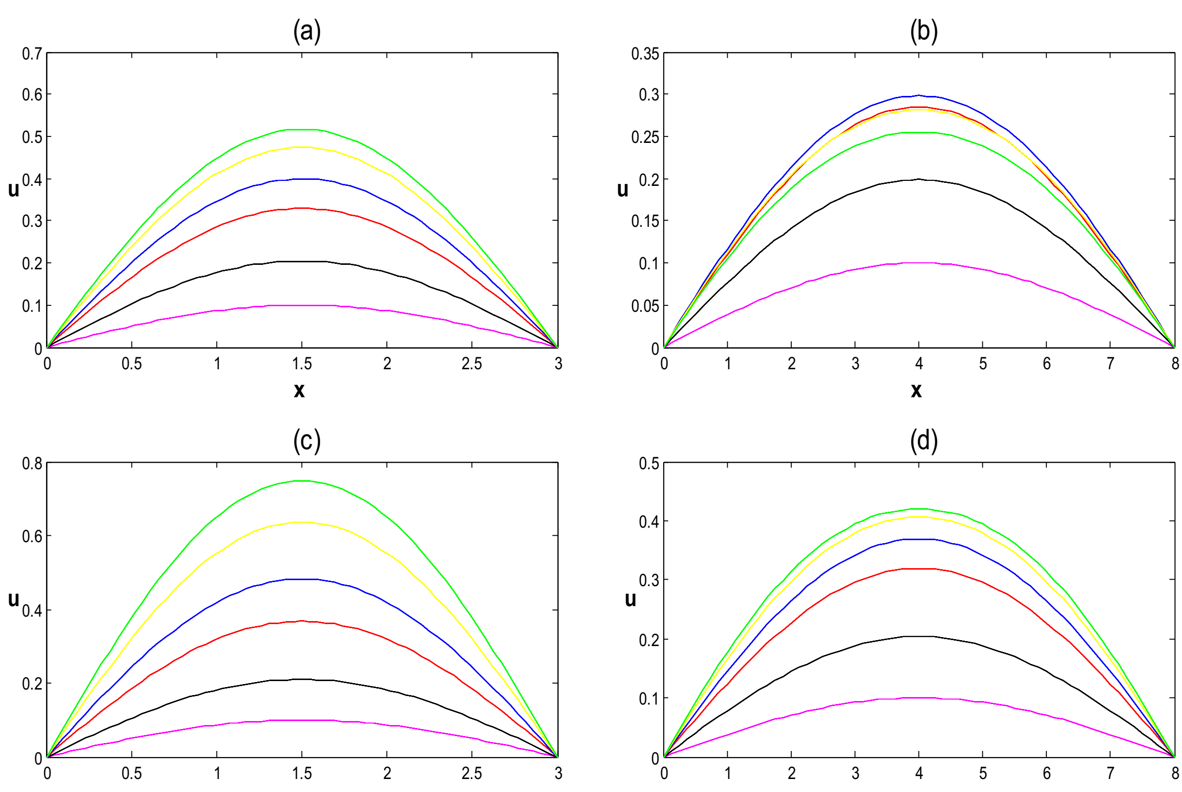

Fig. 12 The numerical solution behavior for the time-fractional modified cubic Swift-Hohenberg equation at $\alpha = 0.8, \gamma = 1.8$ for problem 1.

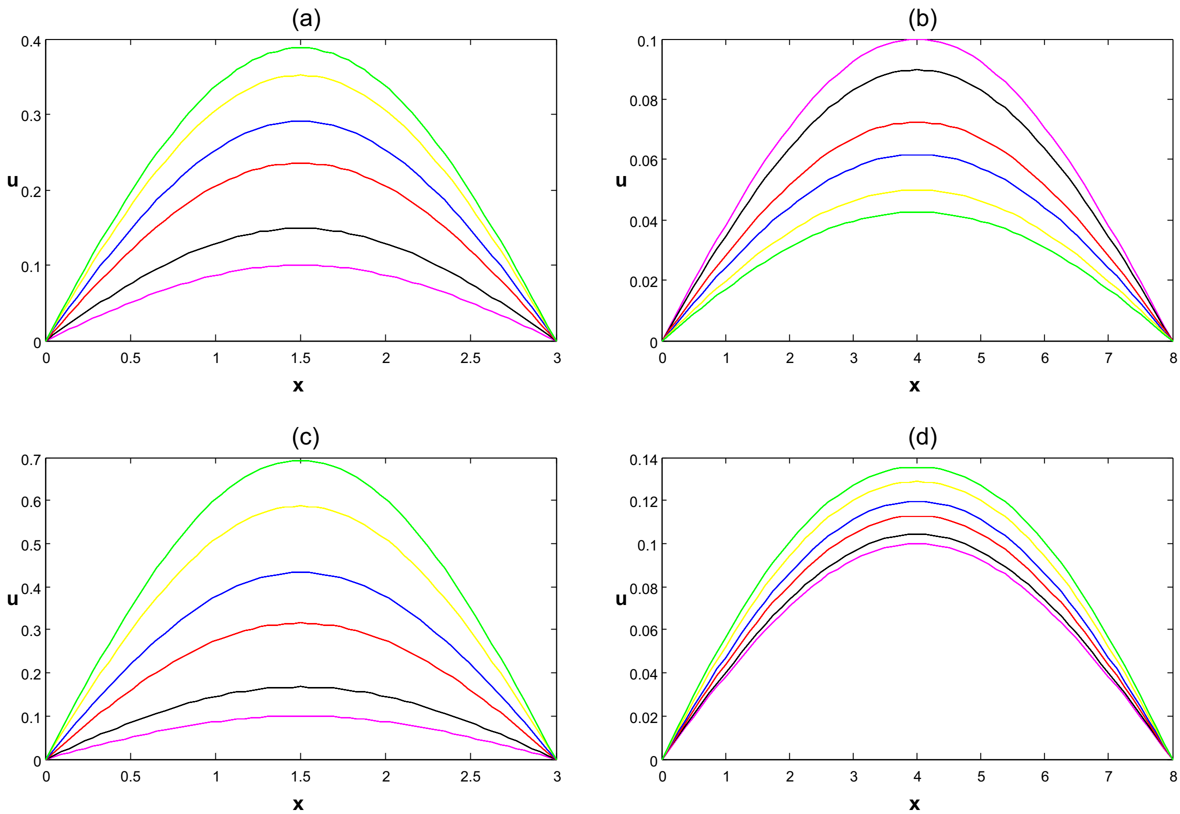

Fig. 13 The numerical solution behavior for the standard cubic-quintic Swift-Hohenberg equation at $\alpha = 1$ for problem 2.

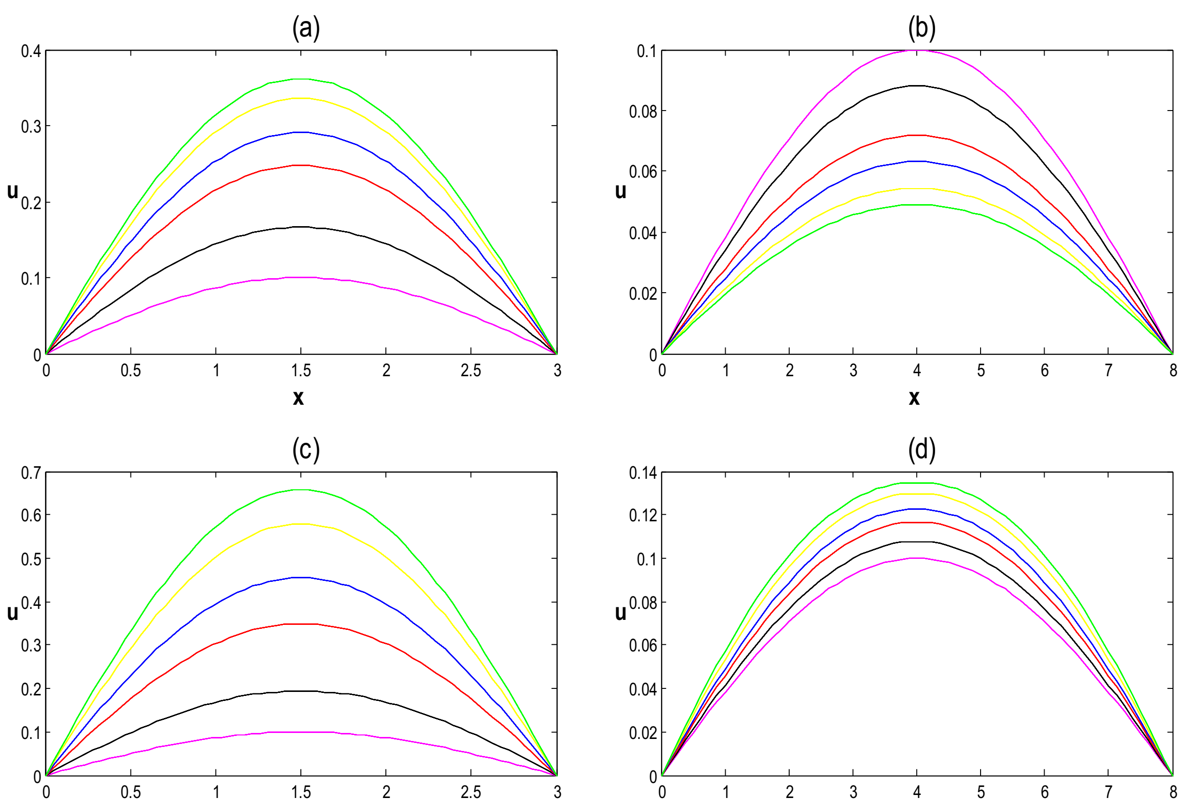

Fig. 14 The numerical solution behavior for the time-fractional standard cubic-quintic Swift-Hohenberg equation at $\alpha = 0.8$ for problem 2.

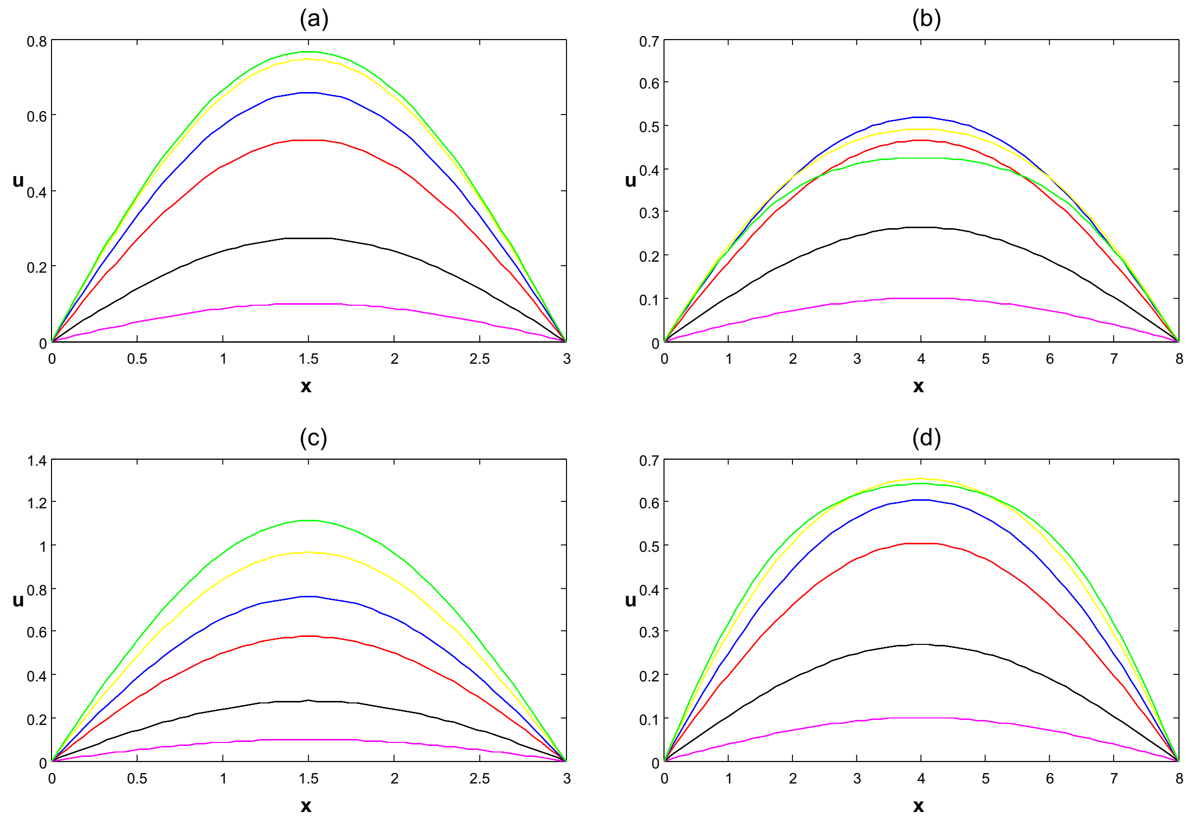

Fig. 15 The numerical solution behavior for the modified cubic-quintic Swift-Hohenberg equation at $\alpha = 1, \gamma = 2$ for problem 2.

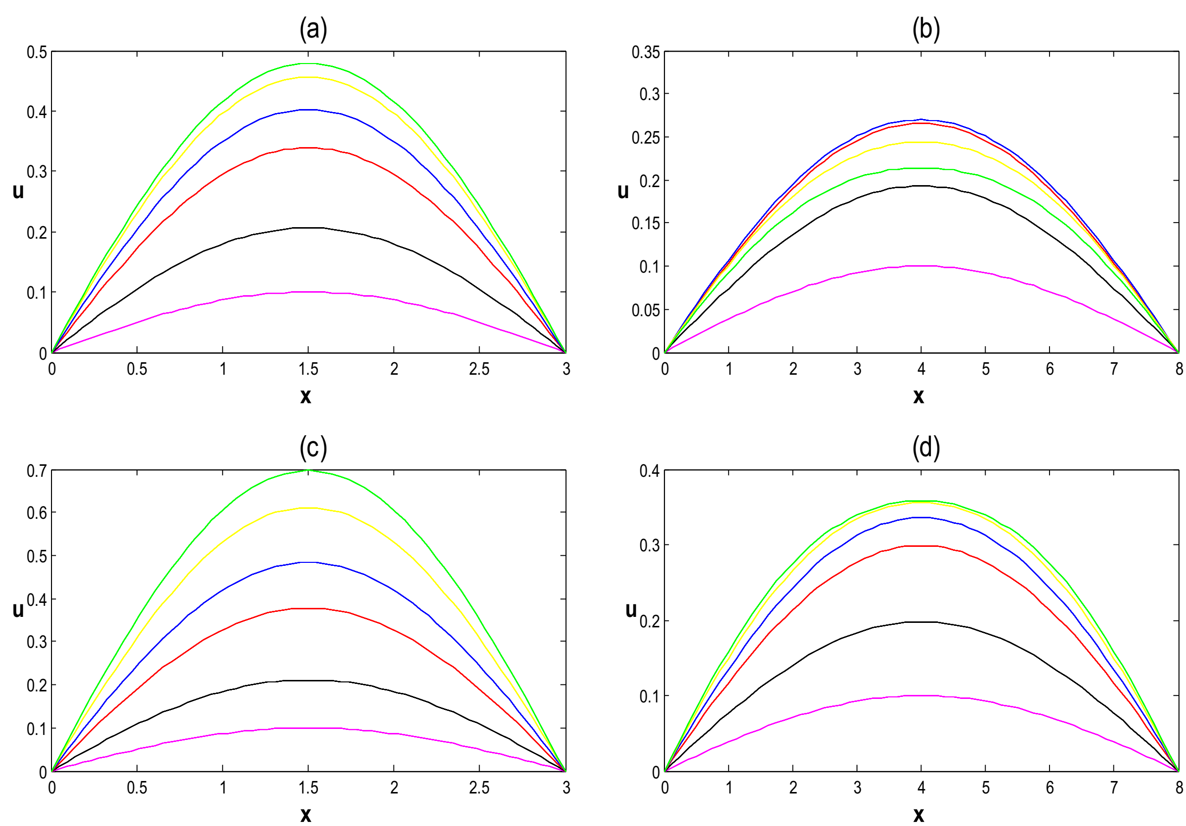

Fig. 16 The numerical solution behavior for the time-fractional modified cubic-quintic Swift-Hohenberg equation at $\alpha = 0.8, \gamma = 1.8$ for problem 2.

Now, we study the solution behavior for the modified cubic-quintic type for integer and fractional order derivatives as shown in Fig. 7 and Fig. 8, respectively. Fig. 7 shows the solution at $\alpha = 1, \gamma = 2$ and $T = 2$ using the EF method $P = 2$ such that Figs. 7.a-7.b represent the solution at $\sigma = 0.3$ and Figs. 7.c-7.d represent the solution at $\sigma = 0.9$ where the solution increases for $L = 3$ but increases and start to decrease at the end for $L = 8$ at$\alpha = 1$. Fig. 8 used the EF method $P = 2$ to show the behavior of the solution at $\alpha = 0.8, \gamma = 1.8$ and $T = 2$such that Figs. 8.a-8.b represent the solution at $\sigma = 0.3$ and Figs. 8.c-8.d represent the solution at $\sigma = 0.9$. The solution behavior for both integer and fractional order derivatives are the same except that the solution increases faster for integer order.

For more illustration of the solution behavior of all the previous cases, we will introduce all the previous figures in 2-D representation at different time levels $t = 0$ (magenta),0.4 (black),0.8 (red),1.2 (blue), 1.8 (yellow), 2 (green). Figs. 9–16 represents the 2-D representation of Figs. 1–8, respectively.

7.3. Test problem (3)

Consider the following problem and the conditions given by (1.3)

$$D_t^\alpha u + D_x^4 u + \pi^4 u + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(\pi x) = 0, \varphi(x) = 0, \text{ and } \alpha = \frac{1}{2}.$$

The case $(P = -1)$ is used for approximating the second- derivative to prove the effectiveness and efficiency of our technique. The absolute errors are shown in Table 1 with $h = \frac{1}{12}, \tau = h^4$ as in [39], and [40]. Also, Fig. 17 shows the graphs of the solutions, while Fig. 18 shows the graph for the absolute error at $h = \frac{1}{12}$. The mean absolute error MAE and the spatial order of convergence $OC(h)$ at different values of the spatial step size h is presented in Table 2.

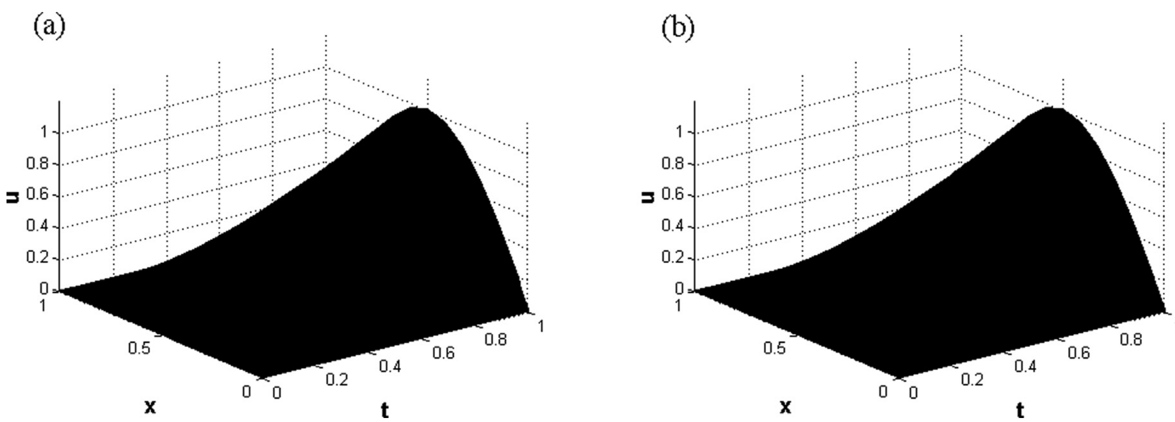

Fig. 17 (a) The exact solution, and (b) The approximate solution, $\alpha = 0.5$ for problem 3.

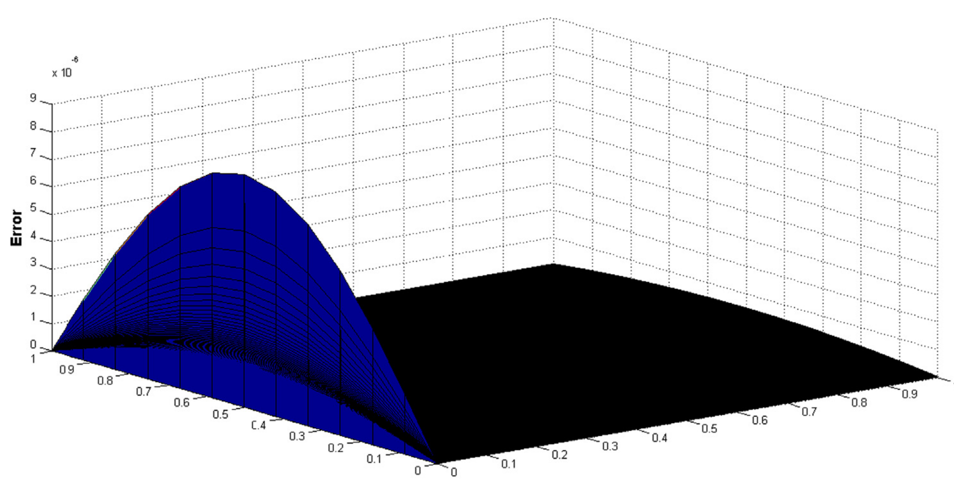

Fig. 18 The absolute error between exact and approximate solutions at $h = \frac{1}{12}$ for problem 3.

Table 1 Absolute errors for problem 3 by the method$(P=-1)$ at $h=\frac{1}{12}, \tau=h^4$and.$\alpha=\frac{1}{2}$.

x/t	0.2	0.4	0.6	0.8	1
0.2	9.7147e-09	3.3248e-08	6.9690e-08	1.0404e-07	1.3742e-07
0.4	4.3813e-08	8.8638e-08	1.9926e-07	3.0290e-07	4.0328e-07
0.6	5.2826e-08	9.7880e-08	2.2338e-07	3.4081e-07	4.5449e-07
0.8	3.3722e-08	7.6475e-08	1.6885e-07	2.5552e-07	3.3954e-07
1	9.7147e-09	3.3248e-08	6.9690e-08	1.0404e-07	1.3742e-07

Table 2 The mean absolute error and the spatial order of convergence at different values of h.

h	Method$(P=-1)$	
	MAE	$OC(h)$
1/6	1.4957e-05	
1/12	1.3848e-07	6.7550
1/24	1.2938e-09	6.7419

8. Conclusion

In this paper, we have constructed a new numerical method for solving the cubic-quintic standard and modified time-fractional nonlinear Swift-Hohenberg equation. The proposed method is based on the exponential fitting technique and the shifted Grünwald-Letnikov fractional derivative with uniform time- stepping. Also, for the lack of smoothness near the initial time, the proposed scheme is adapted with L1-scheme along with the graded mesh in time to deal with the given problem. Our approach depends on a free-parameter which is used to raise the order of accuracy of the suggested methods. Convergence and unconditional stability of our schemes are proposed. Test problems are presented to confirm the accuracy and effectiveness of our schemes. Also, a graphical study of the nature and behavior of the pre-mentioned problem in several cases is illustrated.

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