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Research Article

Weighted Estimates for Commutator of Rough *p*-Adic Fractional Hardy Operator on Weighted *p*-Adic Herz–Morrey Spaces

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The current article investigates the boundedness criteria for the commutator of rough p-adic fractional Hardy operator on weighted p-adic Lebesgue and Herz-type spaces with the symbol function from weighted p-adic bounded mean oscillations and weighted p-adic Lipschitz spaces.

1. Introduction

For a fixed prime p, it is always possible to write a nonzero rational number x in the form $x = p^{\gamma}(m/n)$, where p is not divisible by $m, n \in \mathbb{Z}$ and γ is an integer. The p-adic norm is defined as $|x|_p = \{p^{-\gamma} \cup \{0\}: \gamma \in \mathbb{Z}\}$. The p-adic norm $|\cdot|_p$ fulfills all the properties of a real norm along with a stronger inequality:

$$|x + y|_p \leq \max\left\{|x|_p, |y|_p\right\}. \tag{1}$$

The completion of the field of rational number with respect to $|\cdot|_p$ leads to the field of p-adic numbers $\mathbb{Q}_p$. In [1], it can be seen that any $x \in \mathbb{Q}_p \backslash \{0\}$ can be represented in the formal power series form as

$$x = p^{\gamma} \sum_{j=0}^{\infty} \beta_j p^j, \tag{2}$$

where $\beta_j, \gamma \in \mathbb{Z}, \beta_j \in (\mathbb{Z}/(p\mathbb{Z}_p)), \beta_0 \neq 0$. The convergence of series (2) is followed from $|p^{\gamma}\beta_k p^j|_p = p^{-\gamma-j}$.

The n-dimensional vector space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of tuples $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p, i = 1, 2, \ldots, n$, with the following norm:

$$|\mathbf{x}|_p = \max_{1\leq i\leq n} |x_i|_p. \tag{3}$$

The ball $B_{\gamma}(\mathbf{a})$ and the corresponding sphere $S_{\gamma}(\mathbf{a})$ with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius p^{γ} in non-Archimedean geometry are given by

$$\begin{aligned} B_{\gamma}(\mathbf{a}) &= \left\{\mathbf{x} \in \mathbb{Q}_p^n: |\mathbf{x} - \mathbf{a}|_p \leq p^{\gamma}\right\}, \\ S_{\gamma}(\mathbf{a}) &= \left\{\mathbf{x} \in \mathbb{Q}_p^n: |\mathbf{x} - \mathbf{a}|_p = p^{\gamma}\right\}. \end{aligned} \tag{4}$$

When $\mathbf{a} = 0$, we write $B_{\gamma}(0) = B_{\gamma}, S_{\gamma}(0) = S_{\gamma}$.

Since the space $\mathbb{Q}_p^n$ is locally compact commutative group under addition, it cements the fact from the standard analysis that there exists a translation invariant Haar measure $d\mathbf{x}$. Also, the measure is normalized by

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1, \tag{5}$$

where $|E|_H$ represents the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. Furthermore, one can easily show that $|B_{\gamma}(\mathbf{a})|_H = p^{n\gamma}$, $|S_{\gamma}(\mathbf{a})|_H = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

The last several decades have seen a growing interest in the p-adic models appearing in various branches of science. The p-adic analysis has cemented its role in the field of mathematical physics (see, for example, [2–4]). Many researchers have also paid relentless attention to harmonic analysis in the p-adic fields [5–11]. The present paper can be considered as an extension of investigation of Hardy-type operators started in [6, 7, 12–16].

The one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\mathrm{d}y, \quad x > 0, \tag{6}$$

was introduced by Hardy in [17] for measurable functions $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which satisfies the inequality

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty, \tag{7}$$

where the constant $q/(q-1)$ is sharp. In [18], Faris proposed an extension of an operator H on higher dimensional space $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{y}|\leq|\mathbf{x}|} f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{8}$$

where $|\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{(1/2)}$ for $\mathbf{x} = (x_1, \ldots, x_n)$. In addition, Christ and Grafakos [23] obtained the exact value of the norm of an operator H defined by (8). Over the years, Hardy operator has gained a significant amount of attention due to its boundedness properties [19–22]. For complete understanding of Hardy-type operators, we refer the interested readers to study [12, 23–29] and the references therein.

In what follows, the n-dimensional p-adic fractional Hardy operator

$$H_\alpha^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{9}$$

was defined and studied for $f \in L_1^{\mathrm{loc}}(\mathbb{Q}_p^n)$ and $0\leq\alpha < n$ in [15]. When $\alpha = 0$, the operator H_α^p transfers to the p-adic Hardy operator (see [30] for more details). Fu et al. in [30] acquired the optimal bounds of p-adic Hardy operator on $L^q(\mathbb{Q}_p^n)$. On the central Morrey spaces, the p-adic Hardy-type operators and their commutators are discussed in [16]. In this link, see also [6, 7, 14, 27].

From now on, we turn our attention towards the rough kernel version of an operator which recently received a substantial attention in analysis (see for instance [11, 31–37]). The roughness of Hardy operator was first time studied by Fu et al. in [12]. Motivated from the results of rough Hardy-type operators in Euclidean space, we define a special kind of rough fractional Hardy operator and its commutator in the p-adic field.

Let $f: \mathbb{Q}_p^n \longrightarrow \mathbb{R}$, $b: \mathbb{Q}_p^n \longrightarrow \mathbb{R}$ and $\Omega: S_0 \longrightarrow \mathbb{R}$ be measurable functions and let $0 < \alpha < n$. Then, for $\mathbf{x} \in \mathbb{Q}_p^n\setminus\{0\}$, we define a rough p-adic fractional Hardy operator $H_{\Omega,\alpha}^p$ and its commutator $H_{\Omega,\alpha}^{p,b}$ as

$$H_{\Omega,\alpha}^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{10}$$

$$H_{\Omega,\alpha}^{p,b} f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} (b(\mathbf{x}) - b(\mathbf{y}))\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{11}$$

whenever

$$\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty, \tag{12}$$

$$\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \left|b(\mathbf{y})\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty. \tag{13}$$

Remark 1. Obviously

$$\left\{|\mathbf{y}|_p: \mathbf{y} \in \mathbb{Q}_p^n\right\} = \{p^\gamma: \gamma \in \mathbb{Z}\} \cup \{0\}, \tag{14}$$

holds for every integer $n \geq 1$ and prime $p \geq 2$. Since the inclusion

$$\{0\} \cup \{p^\gamma: \gamma \in \mathbb{Z}\} \subseteq \mathbb{Q}_p, \tag{15}$$

holds and $\mathbb{Q}_p^n$ is a linear space over field $\mathbb{Q}_p$, the product $|\mathbf{y}|_p\mathbf{y}$ is correctly defined. Moreover, if a nonzero $\mathbf{y} \in \mathbb{Q}_p^n$ has a form $\mathbf{y} = (y_1, \ldots, y_n)$ and

$$y_i = p^{\gamma_i}\left(\beta_{0,i} + \beta_{1,i}p + \beta_{2,i}p^2 + \cdots\right), \quad i = 1, \ldots, n, \tag{16}$$

(see (2)), then there is $i_0 \in \{1, \ldots, n\}$ such that

$$\left|y_{i_0}\right|_p = p^{-\gamma_{i_0}} \geq p^{-\gamma_i} = \left|y_i\right|_p, \tag{17}$$

whenever $y_i \neq 0$. Using (3), we obtain $|\mathbf{y}|_p = p^{-\gamma_{i_0}}$. Now from (16) and (17), it follows that

$$\left||\mathbf{y}|_p\mathbf{y}\right|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} \left|p^{\gamma_i-\gamma_{i_0}}\right|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} p^{\gamma_{i_0}-\gamma_i} = p^{\gamma_{i_0}-\gamma_{i_0}} = 1. \tag{18}$$

Thus, for every nonzero $\mathbf{y} \in \mathbb{Q}_p^n$, the vector $|\mathbf{y}|_p\mathbf{y}$ belongs to the sphere

$$S_0(0) = \left\{\mathbf{y} \in \mathbb{Q}_p^n: |\mathbf{y}|_p = 1\right\}. \tag{19}$$

From (12), it directly follows that $H_{\Omega,\alpha}^p \in \mathbb{R}$ for every nonzero $\mathbf{x} \in \mathbb{Q}_p^n$, and using (12) and (13), we have

$$\begin{aligned}\left|H_{\Omega,\alpha}^{p,b} f(\mathbf{x})\right| &\leq \frac{|b(\mathbf{x})|}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}\left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\right|\mathrm{d}\mathbf{y} \\ &\quad + \frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}\left|b(\mathbf{y})\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})\right|\mathrm{d}\mathbf{y} < \infty,\end{aligned} \tag{20}$$

for every $\mathbf{x} \in \mathbb{Q}_p^n\setminus\{0\}$. Consequently, the operators $H_{\Omega,\alpha}^p$ and $H_{\Omega,\alpha}^{p,b}$ are correctly defined.

The aim of the present paper is to study the weighted central mean oscillations (CMO) and weighted p-adic Lipschitz estimates of $H_{\Omega,\alpha}^{p,b}$ on weighted p-adic function spaces like weighted p-adic Lebesgue spaces, weighted p-adic Herz spaces and p-adic Herz–Morrey spaces. Throughout this article, the letter C represents a constant whose value may differ at all of its occurrence. Before turning to our key results, let us define and denote the relevant p-adic function spaces.

2. Notations and Definitions

Suppose $w(\mathbf{x})$ is a weight function on $\mathbb{Q}_p^n$, which is non-negative and locally integrable function on $\mathbb{Q}_p^n$. The weighted measure of E is denoted and defined as $w(E) = \int_E w(\mathbf{x})d\mathbf{x}$. Let $L^q(w, \mathbb{Q}_p^n)$, $(0 < q < \infty)$ be the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^q(w,\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{(1/q)} < \infty. \tag{21}$$

Definition 1. Suppose $1 \le q < \infty$ and w is a weight function. The p-adic space $\mathrm{CMO}^q(w, \mathbb{Q}_p^n)$ is defined as follows:

$$\|f\|_{\mathrm{CMO}^q(w,\mathbb{Q}_p^n)} = \sup_{\gamma\in Z}\left(\frac{1}{w(B_\gamma)}\int_{B_\gamma}\left|f(\mathbf{x}) - f_{B_\gamma}\right|^q w(\mathbf{x})^{1-q}d\mathbf{x}\right)^{(1/q)}, \tag{22}$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|}\int_{B_\gamma} f(\mathbf{x})d\mathbf{x}. \tag{23}$$

Definition 2 (see [5]). Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and w_1 and w_2 are weight functions. Then, the weighted p-adic Herz space $K_q^{\alpha,p}(w_1, w_2)$ is defined by

$$K_q^{\alpha,p}(w_1, w_2) = \left\{f \in L_{\mathrm{loc}}^q\left(w_2, \mathbb{Q}_p^n\backslash\{0\}\right): \|f\|_{K_q^{\alpha,p}(w_1,w_2)} < \infty\right\}, \tag{24}$$

where

$$\|f\|_{K_q^{\alpha,p}(w_1,w_2)} = \left(\sum_{k=-\infty}^{\infty} w_1(B_k)^{((\alpha p)/n)}\|f\chi_k\|_{L^q(w_2,\mathbb{Q}_p^n)}^p\right)^{(1/p)} \tag{25}$$

and χ_k is the characteristic function of the sphere $S_k = B_k \setminus B_{k-1}$.

Remark 2. Obviously $K_q^{0,q}(w_1, w_2) = L^q(w_2, \mathbb{Q}_p^n)$.

Definition 3 (see [5]). Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, w_1 and w_2 be weight functions and λ be a non-negative real number. Then, the weighted p-adic Herz–Morrey space $MK_{p,q}^{\alpha,\lambda}(w_1, w_2)$ is defined as follows:

$$MK_{p,q}^{\alpha,\lambda}(w_1, w_2) = \left\{f \in L_{\mathrm{loc}}^q\left(w_2, \mathbb{Q}_p^n\backslash\{0\}\right): \|f\|_{MK_{p,q}^{\alpha,\lambda}(w_1,w_2)} < \infty\right\}, \tag{26}$$

where

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}(w_1,w_2)} = \sup_{k_0\in\mathbb{Z}} w_1\left(B_{k_0}\right)^{(-\lambda/n)}\left(\sum_{k=-\infty}^{k_0} w_1(B_k)^{((\alpha p)/n)}\|f\chi_k\|_{L^q(w_2,\mathbb{Q}_p^n)}^p\right)^{(1/p)}. \tag{27}$$

Remark 3. It is evident that $MK_{p,q}^{\alpha,0}(w_1, w_2) = K_q^{\alpha,p}(w_1, w_2)$.

Now, we define the weighted p-adic Lipschitz space.

Definition 4. Suppose $1 \le q < \infty$, $0 < \gamma < 1$ and w is a weight function. The p-adic space $\mathrm{Lip}_\gamma(w, \mathbb{Q}_p^n)$ is defined as

$$\|f\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)} = \sup_{B\subset\mathbb{Q}_p^n}\frac{1}{w(B)^{(\gamma/n)}}\left(\frac{1}{w(B)}\int_B\left|f(\mathbf{x}) - f_B\right|^q w(\mathbf{x})^{1-q}d\mathbf{x}\right)^{(1/q)}, \tag{28}$$

where

$$f_B = \frac{1}{|B|}\int_B f(\mathbf{x})d\mathbf{x}. \tag{29}$$

Muckenhoupt introduced the theory of A_q weights on $\mathbb{R}^n$ in [38]. Let us define the A_q weights in the p-adic field.

Definition 5. A weight function $w \in A_q(1 \le q < \infty)$, if there exists a constant C free from choice of $B \subset \mathbb{Q}_p^n$ such that

$$\left(\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x}\right)\left(\frac{1}{|B|}\int_B w(\mathbf{x})^{-(1/(q-1))}d\mathbf{x}\right)^{(1/q)} \le C. \tag{30}$$

For the case $q = 1, w \in A_1$, we have

$$\frac{1}{|B|}\int_B w(\mathbf{x})d\mathbf{x} \le C\operatorname{ess}\inf_{\mathbf{x}\in B} w(\mathbf{x}), \tag{31}$$

for every $B \subset \mathbb{Q}_p^n$.

Remark 4. A weight function $w \in A_\infty$ if it undergoes the stipulation of $A_q(1 \le q < \infty)$ weights.

3. Weighted CMO Estimates of $H_{\Omega,\alpha}^{p,b}$ on Weighted p-Adic Herz-Type Spaces

The present section discusses the boundedness of $H_{\Omega,\alpha}^{p,b}$ on weighted p-adic Lebesgue spaces as well as on the weighted p-adic Herz-type spaces. We begin the section with some useful lemmas to prove our main results.

Lemma 1 (see [39]). *Suppose $w \in A_1$; then, there exists constants C_1, C_2 and $0<\mu<1$ such that*

$$C_1\frac{|A|}{|B|}\le\frac{w(A)}{w(B)}\le C_2\left(\frac{|A|}{|B|}\right)^{\mu}, \tag{32}$$

for measurable subset A of a ball B.

Remark 5. If $w\in A_1$, then it follows from Lemma 1 that there exists a constant C and $\mu(0<\mu<1)$ such that $(w(B_k)/w(B_i))\le Cp^{(k-i)n}$ as $i<k$ and $(w(B_k)/w(B_i))\le Cp^{(k-i)n\mu}$ as $i\ge k$.

Lemma 2. *Suppose $w\in A_1$ and $b\in CMO^q(w,\mathbb{Q}_p^n)$; then, there is a constant C such that for i, $k\in\mathbb{Z}$,*

$$\left|b_{B_i}-b_{B_k}\right|\le C(i-k)\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}\frac{w(B_k)}{|B_k|}. \tag{33}$$

Proof. Firstly, we consider

$$\begin{aligned}\left|b_{2B_\gamma}-b_{B_\gamma}\right| &\le\frac{1}{\left|B_\gamma\right|}\int_{B_\gamma}\left|b(x)-b_{2B_\gamma}\right|\mathrm{d}x\\ &\le\frac{1}{\left|B_\gamma\right|}\int_{2B_\gamma}\left|b(x)-b_{2B_\gamma}\right|\mathrm{d}x\\ &\le\frac{C}{\left|B_\gamma\right|}\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}w\left(2B_\gamma\right).\end{aligned} \tag{34}$$

We assume without loss of generality that $i>k$; then, using Lemma 1, we are down to

$$\begin{aligned}\left|b_{B_i}-b_{B_k}\right| &\le\left|b_{B_i}-b_{B_{i-1}}\right|+\cdots+\left|b_{B_{k+1}}-b_{B_k}\right|\\ &\le\frac{1}{\left|B_{i-1}\right|}\int_{B_i}\left|b(x)-b_{B_i}\right|\mathrm{d}x+\cdots+\frac{1}{\left|B_k\right|}\int_{B_{k+1}}\left|b(x)-b_{B_{k+1}}\right|\mathrm{d}x\\ &\le C\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}\left(\frac{w(B_i)}{\left|B_{i-1}\right|}+\cdots+\frac{w(B_{k+1})}{\left|B_k\right|}\right)\\ &\le C(i-k)\|b\|_{\mathrm{CMO}^q\left(w,\mathbb{Q}_p^n\right)}\frac{w(B_k)}{\left|B_k\right|}.\end{aligned} \tag{35}$$

Lemma 3. *Suppose $w\in A_1$; then for $1<q<\infty$,*

$$\int_B w(x)^{1-q'}\mathrm{d}x\le C|B|^{q'}w(B)^{1-q'}, \tag{36}$$

where $(1/q)+(1/q')=1$.

Proof. Since $A_1\subset A_q\,(q>1)$, w satisfies the A_q conditions

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B (w(x))^{-(1/(q-1))}\mathrm{d}x\right)^{q-1}\mathrm{d}x\le C, \tag{37}$$

for every $B\subset\mathbb{Q}_p^n$.

From here, we easily get

$$\int_B w(x)^{1-q'}\mathrm{d}x\le C|B|^{q'}w(B)^{1-q'}. \tag{38}$$

Theorem 1. *Let $1\le p,q<\infty$, $w\in A_1$, $(\alpha/n)+1=(1/s')$; then*

$$\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}\le C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{L^q\left(w,\mathbb{Q}_p^n\right)}, \tag{39}$$

holds for all $b\in CMO^{p\max\{q,q'\}}(w,\mathbb{Q}_p^n)$, $\Omega\in L^s(S_0(0))$, $1<s<\infty$ and $f\in L_{loc}(\mathbb{Q}_p^n)$.

Now we state the results about the boundedness of commutator of rough p-adic fractional Hardy operator on weighted p-adic Herz-type spaces.

Theorem 2. *Let $0<p_1\le p_2<\infty$, $1\le p,q<\infty$ and let $w\in A_1$, $(\alpha/n)+1=1/s'$.*

If $\beta<(n\mu/q')$, then the inequality

$$\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{\dot K_q^{\beta,p_2}(w,w^{1-q})}\le C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{\dot K_q^{\beta,p_1}(w,w)} \tag{40}$$

holds for all $b\in\mathrm{CMO}^{p\max\{q,q'\}}(w,\mathbb{Q}_p^n)$, $\Omega\in L^s(S_0(0))$, $1<s<\infty$ and $f\in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Remark 6. If $\beta=0$, $p_1=p_2=q$, then Theorem 1 becomes a special case of Theorem 2.

Theorem 3. *Let $0<p_1\le p_2<\infty$, $1\le p,q<\infty$ and let $w\in A_1$, $(\alpha/n)+1=(1/s')$ and $\lambda>0$. If $\beta<(n\mu/q')+\lambda$, then*

$$\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{M\dot K_{p_2,q}^{\beta,\lambda}(w,w^{1-q})}\le C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot K_{p_1,q}^{\beta,\lambda}(w,w)}, \tag{41}$$

holds for all $b\in CMO^{p\max\{q,q'\}}(w,\mathbb{Q}_p^n)$, $\Omega\in L^s(S_0(0))$, $1<s<\infty$ and $f\in L_{loc}(\mathbb{Q}_p^n)$.

Proof. of Theorem 2. By definition, we firstly have

$$
\begin{aligned}
&\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^q_{L^q\left(w^{1-q},\mathbb{Q}^n_p\right)} \\
&= \int_{S_k}|\mathbf{x}|_p^{-q(n-\alpha)}\left|\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})(b(\mathbf{x})-b(\mathbf{y}))d\mathbf{y}\right|^q w(\mathbf{x})^{1-q}d\mathbf{x} \\
&\le Cp^{-kq(n-\alpha)}\int_{S_k}\left(\int_{|\mathbf{y}|_p\le p^k}\left|\Omega\left(|\mathbf{y}|_p\mathbf{y}\right)f(\mathbf{y})(b(\mathbf{x})-b(\mathbf{y}))\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\
&\le Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{x})-b_{B_k}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\
&\quad + Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_k}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\
&= I + II.
\end{aligned} \tag{42}
$$

For $j,k\in\mathbb{Z}$ with $j\le k$, we get

$$
\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s d\mathbf{y} = \int_{|\mathbf{z}|_p=1}|\Omega(\mathbf{z})|^s p^{jn}d\mathbf{z}\le Cp^{kn}. \tag{43}
$$

Also, since $w\in A_1\subset A_q$, by the application of Hölder's inequality $(((1/q)+(1/q'))=1)$ together with Lemma 3, we have

$$
\begin{aligned}
\int_{S_j}f(\mathbf{y})d\mathbf{y} &\le \left(\int_{S_j}|f(\mathbf{y})|^q w(\mathbf{y})d\mathbf{y}\right)^{(1/q)}\left(\int_{S_j}w(\mathbf{y})^{(-q'/q)}d\mathbf{y}\right)^{(1/q')} \\
&\le C\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|w\left(B_j\right)^{(-1/q)}.
\end{aligned} \tag{44}
$$

To estimate I, we make use of Hölder's inequality, Remark 5, and $(\alpha/n)+1=(1/s')$ along with (43) and (44) to have

$$
\begin{aligned}
I &\le Cp^{-kq(n-\alpha)}\int_{B_k}\left|b(\mathbf{x})-b_{B_k}\right|^q \\
&\quad\times\left\{\sum_{j=-\infty}^{k}\left(\int_{S_j}|f(\mathbf{y})|^{s'}d\mathbf{y}\right)^{1/s'}\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s d\mathbf{y}\right)^{1/s}\right\}^q w(\mathbf{x})^{1-q}d\mathbf{x} \\
&\le Cp^{kq(n-\alpha)}\|b\|^q_{CMO^q\left(w,\mathbb{Q}^n_p\right)}w\left(B_k\right)\left\{\sum_{j=-\infty}^{k}p^{kn/s}\int_{S_j}|f(\mathbf{y})|d\mathbf{y}\right\}^q \\
&\le Cp^{-kqn((1-\alpha)/(n-1)/s)}\|b\|^q_{CMO^q\left(w,\mathbb{Q}^n_p\right)}w\left(B_k\right)\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|w\left(B_j\right)^{-1/q}\right\}^q \\
&\le Cp^{-knq}\|b\|^q_{CMO^q\left(w,\mathbb{Q}^n_p\right)}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\left|B_j\right|\left(\frac{w\left(B_k\right)}{w\left(B_j\right)}\right)^{1/q}\right\}^q \\
&\le C\|b\|^q_{CMO^q\left(w,\mathbb{Q}^n_p\right)}\left(p^{(k-j)n/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}^n_p\right)}\right)^q.
\end{aligned} \tag{45}
$$

Now, we turn our attention towards estimating II.

$$\begin{aligned} II &\le Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_j}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\ &\quad + Cp^{-kq(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b_{B_k}-b_{B_j}\right)\right|d\mathbf{y}\right)^q w(\mathbf{x})^{1-q}d\mathbf{x} \\ &= II_1 + II_2. \end{aligned} \tag{46}$$

In order to evaluate II_1, we need the following preparation. Apply Hölder's inequality at the outset to deduce

$$\begin{aligned} &\int_{S_j}\left|f(\mathbf{y})\left(b(\mathbf{y})-b_{B_j}\right)\right|d\mathbf{y} \\ &\le \left(\int_{S_j}|f(\mathbf{y})|^q w(\mathbf{y})d\mathbf{y}\right)^{(1/q)}\left(\int_{S_j}\left|b(\mathbf{y})-b_{B_j}\right|^{q'} w(\mathbf{y})^{(-q'/q)}d\mathbf{y}\right)^{(1/q')} \\ &\le w\left(B_j\right)^{(-1/q')}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\|b\|_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}. \end{aligned} \tag{47}$$

We imply Hölder's inequality, inequality (47), Lemma 3, and Remark 5 to estimate II_1.

$$\begin{aligned} II_1 &\le Cp^{-kq(n-\alpha)}\int_{S_k}\left\{\sum_{j=-\infty}^{k}\left(\int_{S_j}\left|f(\mathbf{y})b(\mathbf{y})-b_{B_j}\right|^{s'}d\mathbf{y}\right)^{1/s'}\right. \\ &\quad\left.\times\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s d\mathbf{y}\right)^{(1/s)}\right\}^q w(\mathbf{x})^{1-q}d\mathbf{x} \\ &\le Cp^{-kqn((1-\alpha)/(n-1)/s)}\int_{S_k}w(\mathbf{x})^{1-q}d\mathbf{x}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})b(\mathbf{y})-b_{B_j}\right|d\mathbf{y}\right)^q \\ &\le Cp^{-kqn((1-\alpha)/(n-1)/s)}\left|B_k\right|^q w\left(B_k\right)^{1-q}\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{1/q'}\right)^q \\ &\le C\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-(1/q)}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q \\ &\le C\|b\|^q_{\mathrm{CMO}^{q'}\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}p^{(j-k)n\mu/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q. \end{aligned} \tag{48}$$

In a similar fashion, we can estimate II_2. Using Hölder's inequality, Lemmas 2 and 3, Remark 5, and inequality (44), we get

$$\begin{aligned}
II_2 &\leq Cp^{-kq(n-\alpha)} \\
&\quad\times \int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)(j-k)\|b\|_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)}\frac{w\left(B_j\right)}{\left|B_j\right|}\right|\mathbf{dy}\right)^q w(\mathbf{x})^{1-q}\mathbf{dx} \\
&\leq Cp^{-kqn((1-\alpha)/(n-1)/s)}\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)}\left|B_k\right|^q w\left(B_k\right)^{1-q} \\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)\frac{w\left(B_j\right)}{\left|B_j\right|}\left(\int_{S_j}|f(\mathbf{y})|^{s'}\mathbf{dy}\right)^{1/s'}\left(\int_{S_j}\left|\Omega\left(p^j\mathbf{y}\right)\right|^s\mathbf{dy}\right)^{1/s}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)}w\left(B_k\right)^{1-q} \\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)\frac{w\left(B_j\right)}{\left|B_j\right|}\int_{S_j}|f(\mathbf{y})|\mathbf{dy}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)} \\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-(1/q)}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q \\
&\leq C\|b\|^q_{\mathrm{CMO}^p\left(w,\mathbb{Q}_p^n\right)} \\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n\mu/q'}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^q.
\end{aligned} \tag{49}$$

From (45), (48), and (49) together with Jensen inequality, we have

$$\begin{aligned}
\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{\dot{K}_q^{\beta,p_2}(w,w^{1-q})} &= \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_2)/n)} \left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}^{p_2}\right)^{1/p_2} \\
&\leq \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}^{p_1}\right)^{1/p_1} \\
&\leq C\|b\|_{CMO^q\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} p^{((j-k)n)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&\quad + C\|b\|_{CMO^{q'}\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} p^{((j-k)n\mu)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&\quad + C\|b\|_{CMO^p\left(w,\mathbb{Q}_p^n\right)} \left(\sum_{k=-\infty}^{\infty} w(B_k)^{((\beta p_1)/n)} \left(\sum_{j=-\infty}^{k} (k-j) p^{((j-k)n\mu)/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}\right)^{1/p_1} \\
&= S.
\end{aligned} \tag{50}$$

Consequently,

$$\begin{aligned}
S^{p_1} &\leq C\|b\|_{CMO^{p\max(q,q')}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} w(B_k)^{(\beta p_1)/n} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n\mu/q'} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1} \\
&\leq C\|b\|_{CMO^{p\max(q,q')}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n\mu/q'-\beta} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{p_1}.
\end{aligned} \tag{51}$$

From here on in the proof we consider couple of cases, $0 < p_1 \leq 1$ and $p_1 > 1$. □

Case 1. When $0 < p_1 \leq 1$, noticing that $\beta < (n\mu/q')$, we proceed as follows.

$$\begin{aligned}
S^{p_1} &\leq C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k} (k-j)^{p_1} w\left(B_j\right)^{(\beta p_1)/n} p^{(j-k)((n\mu/q')-\beta)p_1} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}^{p_1} = C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \\
&\quad \times \sum_{k=-\infty}^{\infty} w\left(B_j\right)^{((\beta p_1)/n)} \left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}^{p_1} \sum_{k=j}^{\infty} (k-j)^{p_1} p^{(j-k)((n\mu/q')-\beta)p_1} \\
&= C\|b\|_{CMO^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}^{p_1} \|f\|_{\dot{K}_q^{\beta,p_1}(w,w)}^{p_1}.
\end{aligned} \tag{52}$$

Case 2. When $p_1 > 1$, applying Hölder's inequality with $\beta < (n\mu/q')$, we get

$$\begin{aligned} S^{p_1} &\leq C\|b\|^{p_1}_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} w\left(B_j\right)^{\left(\left(\beta p_1\right)/n\right)}\left\|f\chi_j\right\|^{p_1}_{L^q\left(0w,\mathbb{Q}_p^n\right)} p^{\left((j-k)\left(\left(n\mu/q'\right)-\beta\right)p_1\right)/2} \\ &\quad\times\left(\sum_{j=-\infty}^{k}(k-j)^{p_1'}p^{(j-k)\left(\left(n\mu/q'\right)-\beta\right)p_1'/2}\right)^{p_1/p_1'} \\ &= C\|b\|^{p_1}_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)} \sum_{k=-\infty}^{\infty} w\left(B_j\right)^{\left(\left(\beta p_1\right)/n\right)}\left\|f\chi_j\right\|^{p_1}_{L^q\left(w,\mathbb{Q}_p^n\right)} \sum_{k=j}^{\infty} p^{(j-k)\left(\left(n\mu/q'\right)-\beta\right)p_1/2} \\ &= C\|b\|^{p_1}_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|^{p_1}_{\dot{K}^{\beta,p_1}_{q_1}(w,w)}. \end{aligned} \tag{53}$$

Therefore, the proof of theorem is completed.

Proof of Theorem 3. From Theorem 2, we have

$$\left\|\left(H^{p,b}_{\Omega,\alpha}\right)f\chi_k\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)} \leq C\|b\|^{p_1}_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\sum_{j=-\infty}^{k}(k-j)p^{(j-k)\left(n\mu/q'\right)}\left\|f\chi_j\right\|_{L^q\left(w,\mathbb{Q}_p^n\right)}. \tag{54}$$

By definition of weighted p-adic Herz–Morrey space and Jensen inequality together with $\beta < (n\mu/q') + \lambda$, $\lambda > 0$ and $1 < p_1 < \infty$, it follows that

$$\begin{aligned} \left\|H^{p,b}_{\Omega,\alpha}f\right\|_{M\dot{K}^{\beta,\lambda}_{p_2,q}\left(w,w^{1-q}\right)} &= \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{\left(\left(\beta p_2\right)/n\right)}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{p_2}_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}\right)^{\left(1/p_2\right)} \\ &\leq \sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{\left(\left(\beta p_1\right)/n\right)}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{p_1}_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)}\right)^{1/p_1} \\ &\leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n} \\ &\quad\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{\left(\left(\lambda p_1\right)/n\right)}\left(\sum_{j=-\infty}^{k}(k-j)p^{\left((j-k)n\mu/q'\right)}\left(\frac{w\left(B_k\right)}{w\left(B_j\right)}\right)^{\left(\left(\lambda p_1\right)/n\right)}\right.\right. \\ &\quad\left.\left.\times w\left(B_j\right)^{-\lambda/n}\left(\sum_{l=-\infty}^{j} w\left(B_l\right)^{\beta p_1/n}\left\|f\chi_j\right\|^{p_1}_{L^q\left(w,\mathbb{Q}_p^n\right)}\right)^{1/p_1}\right)^{p_1}\right)^{1/p_1} \\ &\leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n} \\ &\quad\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{\left(\left(\lambda p_1\right)/n\right)}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)\left(\left((n\mu)/q'\right)-\beta+\lambda\right)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}\right)^{p_1}\right)^{1/p_1} \\ &\leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\sup_{k_0\in\mathbb{Z}} w\left(B_{k_0}\right)^{-\lambda/n}\times\left(\sum_{k=-\infty}^{k_0} w\left(B_k\right)^{\left(\left(\lambda p_1\right)/n\right)}\right)^{1/p_1}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)} \\ &\leq C\|b\|_{\mathrm{CMO}^{p\max\{q,q'\}}\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}. \end{aligned} \tag{55}$$

4. Weighted Lipschitz Estimates for the Commutator of Rough p-Adic Fractional Hardy Operator on Herz–Morrey Spaces

In this section, we obtain the weighted p-adic Lipschitz estimates for the commutator of rough p-adic fractional Hardy operator on p-adic Lebesgue spaces and p-adic Herz-type spaces. We begin the section with a useful lemma which can be proved in the similar lines as Lemma 2.

Lemma 5. *Suppose $w \in A_1$ and $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$; then, there is a constant C such that for i, $k \in \mathbb{Z}$,*

$$\left|b_{B_i} - b_{B_k}\right| \le C(i-k)\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)} w(B_i)^{\gamma/n}\frac{w(B_k)}{|B_k|}. \tag{56}$$

Theorem 4. *Let $1\le p, q<\infty$, $(1/q_1) - (1/q_2) = (\gamma/n)$, $w \in A_1$, $(\alpha/n) + 1 = (1/s')$; then,*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{L^q\left(w^{1-q},\mathbb{Q}_p^n\right)} \le C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{L^q\left(w,\mathbb{Q}_p^n\right)}, \tag{57}$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Now we state the results about the boundedness of commutator of rough p-adic fractional Hardy operator on weighted p-adic Herz-type spaces.

Theorem 5. *Let $0 < p_1 \le p_2 < \infty$, $1\le q_1, q_2 < \infty$, $(1/q_1)-(1/q_2) = (\gamma/n)$ and let $w \in A_1$, $(\alpha/n) + 1 = (1/s')$. If $\beta < (n\mu/q_1')$, then the inequality*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{\dot{K}_{q_2}^{\beta,p_2}\left(w,w^{1-q_2}\right)} \le C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{\dot{K}_{q_1}^{\beta,p_1}(w,w)}, \tag{58}$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Remark 7. If $\beta = 0$, $p_1 = q_1 = p$ and $p_2 = q_2 = q$, then Theorem 4 can easily be obtained from Theorem 5.

Theorem 6. *Let $0 < p_1 \le p_2 < \infty$, $1\le q_1, q_2 < \infty$, $(1/q_1)-(1/q_2) = (\gamma/n)$ and let $w \in A_1$, $(\alpha/n) + 1 = (1/s')$, and $\lambda > 0$. If $\beta < (n\mu/q_1') + \lambda$, then*

$$\left\|H_{\Omega,\alpha}^{p,b} f\right\|_{M\dot{K}_{p_2,q_2}^{\beta,\lambda}\left(w,w^{1-q_2}\right)} \le C\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\|f\|_{M\dot{K}_{p_1,q_1}^{\beta,\lambda}(w,w)}, \tag{59}$$

holds for all $b \in Lip_\gamma(w, \mathbb{Q}_p^n)$, $\Omega \in L^s(S_0(0))$, $1 < s < \infty$, and $f \in L_{loc}(\mathbb{Q}_p^n)$.

Proof of Theorem 5. Following the same pattern of Theorem 2, we have

$$\begin{aligned}
&\left\|\left(H_{\Omega,\alpha}^{p,b} f\right)\chi_k\right\|_{L^{q_2}\left(w^{1-q_2},\mathbb{Q}_p^n\right)}^{q_2} \\
&\le C p^{-kq_2(n-\alpha)} \int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}|f(\mathbf{y})|\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{x}) - b_{B_k}\right)|d\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x} \\
&\quad + C p^{-kq_2(n-\alpha)} \int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}|f(\mathbf{y})|\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y}) - b_{B_k}\right)|d\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}d\mathbf{x} \\
&= J + JJ.
\end{aligned} \tag{60}$$

To estimate J, we make use of Hölder's inequality, Remark 5, $(\alpha/n) + (1) = (1/s')$, $(\gamma/n) = (1/q_1) - (1/q_2)$, and $w \in A_1 \subset A_{q_1}$ along with (43) and (44) to have

$$\begin{aligned}
J &\le C p^{-kq_2 n((1-\alpha)/((n-1)/s))}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2} w(B_k)^{((1+\gamma q_2)/n)}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\left|B_j\right| w\left(B_j\right)^{(-1/q_1)}\right\}^{q_2} \\
&\le C p^{-knq_2}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2}\left\{\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\left|B_j\right|\left(\frac{w(B_k)}{w\left(B_j\right)}\right)^{1/q_1}\right\}^{q_2} \\
&\le C\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}^{q_2}\left(p^{(k-j)n/q_1'}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}.
\end{aligned} \tag{61}$$

For the estimation of JJ, we need to decompose it as

$$\begin{aligned} JJ &\le Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}\mathrm{d}\mathbf{x}\\ &\quad + Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega\left(p^j\mathbf{y}\right)\left(b_{B_k}-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\right)^{q_2} w(\mathbf{x})^{1-q_2}\mathrm{d}\mathbf{x}\\ &= JJ_1 + JJ_2. \end{aligned} \tag{62}$$

We need the following preparation to estimate JJ_1. Apply Hölder's inequality to get

$$\begin{aligned} &\int_{S_j}\left|f(\mathbf{y})\left(b(\mathbf{y})-b_{B_j}\right)\right|\mathrm{d}\mathbf{y}\\ &\le\left(\int_{S_j}|f(\mathbf{y})|^{q_1}w(\mathbf{y})\mathrm{d}\mathbf{y}\right)^{1/q_1}\left(\int_{S_j}\left|b(\mathbf{y})-b_{B_j}\right|^{q_1{}'}w(\mathbf{y})^{(-q_1'/q_1)}\mathrm{d}\mathbf{y}\right)^{(1/q_1')}\\ &\le w\left(B_j\right)^{(-1/q_1')+(\gamma/n)}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\|b\|_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}. \end{aligned} \tag{63}$$

We imply Hölder's inequality, inequality (63), Lemma 3, and Remark 5 to estimate JJ_1.

$$\begin{aligned} JJ_1 &\le Cp^{-kq_2n((1-\alpha)/(n-1)/s)}\int_{S_k}w(\mathbf{x})^{1-q_2}d\mathbf{x}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{(1/q_1'+\gamma/n)}\|b\|_{Lip_\gamma\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}\\ &\le Cp^{-kq_2n((1-\alpha)/(n-1)/s)}\left|B_k\right|^{q_2}w\left(B_k\right)^{1-q_2}\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}w\left(B_j\right)^{(1/q_1'+\gamma/n)}\right)^{q_2}\\ &\le C\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}\left(\frac{w\left(B_j\right)}{w\left(B_k\right)}\right)^{1-1/q_2}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}\\ &\le C\|b\|^{q_2}_{\mathrm{Lip}_\gamma\left(w,\mathbb{Q}_p^n\right)}\left(\sum_{j=-\infty}^{k}p^{(j-k)n\mu/q_2'}\left\|f\chi_j\right\|_{L^{q_1}\left(w,\mathbb{Q}_p^n\right)}\right)^{q_2}. \end{aligned} \tag{64}$$

Now we turn towards JJ_2. Using once again Hölder's inequality, Lemmas 5 and 3, Remark 5, and inequality (44), we get

$$\begin{aligned} JJ_2 &\le Cp^{-kq_2n((1-\alpha)/(n-1)/s)}\|b\|^{q_2}_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}|B_k|^{q_2}w(B_k)^{1-q_2} \\ &\quad\times\left(\sum_{j=-\infty}^{k}(k-j)w(B_k)^{\gamma/n}\frac{w(B_j)}{|B_j|}|B_j|w(B_j)^{-1/q_1}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2} \\ &= C\|b\|^{q_2}_{Lip_\gamma(w,\mathbb{Q}_p^n)} \\ &\quad\times\left(\sum_{j=-\infty}^{k}(k-j)\left(\frac{w(B_j)}{w(B_k)}\right)^{1-1/q_1}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2} \\ &\le C\|b\|^{q_2}_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)} \\ &\quad\times\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n\mu/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{q_2}. \end{aligned} \tag{65}$$

Rest of the proof is similar to the proof of Theorem 2. Thus, we come to an end of proof.

Proof of Theorem 6. Let $\beta < n\mu/q_1' + \lambda$. By the definition of weighted p-adic Herz–Morrey spaces along with inequalities (61), (64), and (65), we are down to

$$\begin{aligned} \left\|H^{p,b}_{\Omega,\alpha}f\right\|_{M\dot{K}^{\beta,\lambda}_{p_2,q_2}(w,w^{1-q_2})} &\le C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty}w(B_k)^{(\beta p_2)/n}\left(\sum_{j=-\infty}^{k}p^{(j-k)n/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2} \\ &\quad+C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty}w(B_k)^{((\beta p_2)/n)}\left(\sum_{j=-\infty}^{k}p^{(j-k)n\mu/q_2'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2} \\ &\quad+C\|b\|_{Lip_\gamma(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}w(B_{k_0})^{-\lambda/n}\left(\sum_{k=-\infty}^{\infty}w(B_k)^{(\beta p_2)/n}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n\mu/q_1'}\|f\chi_j\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{p_2}\right)^{1/p_2} \\ &= L_1+L_2+L_3. \end{aligned} \tag{66}$$

Next by applying the similar arguments as in Theorem 3, we get

$$\begin{aligned} L_1 &\le C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta<\frac{n}{q_1'}+\lambda, \\ L_2 &\le C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta<\frac{n\mu}{q_2'}+\lambda, \\ L_3 &\le C\|b\|_{\mathrm{Lip}_\gamma(w,\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\beta,\lambda}_{p_1,q}(w,w)}, \quad \beta<\frac{n\mu}{q_1'}+\lambda. \end{aligned} \tag{67}$$

Therefore, we conclude the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors made equal contributions and read and supported the last original copy.

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