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Optical solitons for Lakshmanan–Porsezian–Daniel equation with Kerr law non-linearity using improved $\tan\left(\frac{\psi(\eta)}{2}\right)$-expansion technique

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ABSTRACT

The Lakshmanan–Porsezian–Daniel (LPD) equation, with spatio-temporal dispersion as well as group velocity dispersion, is investigated to retrieve new optical solitons. Abundant bright, dark, singular, kink and periodic optical solitons solutions of the LPD equation are constructed for Kerr law of non-linearity by improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method. Some of the obtained solutions are graphically illustrated using 3D-surface plots and the corresponding 2D-contour plots. The novelty of the constructed solutions is established by comparison of the obtained results with the results available in the literature for LPD equation which shows the effectiveness of the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method.

Introduction

Nonlinear evolution equations are of great interest due to the fact that many of the physical systems are non-linear in nature. The theory of evolution equations is a major area of research in mathematics, physics, engineering, biology and other fields of science. Lakshmanan–Porsezian–Daniel (LPD) equation is a well-known partial differential equation. This evolution equation governs the dynamical behavior of optical solitons just like the non-linear Schrödinger equation [1,2]. The LPD model describes the transmission of solitons through a variety of waveguides. It has been established that the LPD model is among the infinite hierarchies of the non-linear Schrödinger equation [3].

The LPD equation has been studied by many mathematical methods such as method of undetermined coefficients [4], semi-inverse variational principle [5], extended trial equation method [6] and many others. In the present study, the dynamics of solitons is studied with spatio-temporal dispersion (STD) as well as group velocity dispersion (GVD). The well posedness of this equation due to this STD was identified in 2012 [7].

During the last few years, various effective methods have been introduced to find the solutions of non-linear evolution equations (NLEEs) such as; extended tanh function method [8], modified exp-function method [9], variational iteration method [10], Hirota's direct method [11], first integral method [12], new extended generalized Kudryashov method [13] and many more.

In this paper, abundant optical soliton solutions of the Lakshmanan–Porsezian–Daniel equation are constructed using the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method. The improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method is a recently developed direct technique. It has been successfully used in many recent studies [14–16]. It is a valuable method due to the variety of solutions it retrieves.

The improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method was first proposed by Manafian and Lakestani. Manafian and Lakestani called it an improved expansion method as it is a modified form of $\frac{G'}{G}$-expansion method and generally provides a wider range of wave solutions than the generalized and improved $\frac{G'}{G}$-expansion methods [17].

Many authors have solved various models using the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method and abundant interesting solutions have been extracted in the literature. Manafian et al. [18] used the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$-expansion method to solve $(2+1)$-dimensional Zoomeron equation, the Duffing equation and the SRLW equation. Biswas–Milovic equation, Schrödinger type non-linear equations, Kundu–Eckhaus equation and Tzitzéica type non-linear equations have also been solved by the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$-expansion method [19–22].

The paper is organized as follows: The governing model is described in Section "Governing model". The improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method is briefly explained in Section "Description of method". The exact solutions of the considered LPD equation are constructed in Section "Exact solutions of LPD equation". Graphical illustrations for some

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of the obtained solutions are presented in Section "Graphical illustration". Section "Comparison with other methods" provides a comparison with other methods. The results are discussed in Section "Results and discussion". Section "Conclusion" contains the concluding remarks.

Governing model

The dimensionless form of Lakshmanan–Porsezian–Daniel equation model with GVD, non-linearity and STD is considered [4], as

$$ip_t + lp_{xx} + mp_{xt} + nF(|p|^2)p = \tau p_{xxxx} + \epsilon(p_x)^2 p^* + \rho|p_x|^2 p + \nu|p|^2 p_{xx} + \pi p^2 p^*_{xx} + \eta|p|^4 p. \quad (1)$$

Here, the dependent variable $p(x,t)$ is representing the complex valued function, whereas the independent variables x and t are donating space and time respectively. The term ip_t is representing the temporal evolution of pulse, l is GVD and m is STD. The functional F is a real-valued function and it is essential to have its smoothness $F(|p|^2)p : C \to C$. The function $F(|p|^2)p$ is j-times continuously differentiable when the complex plane C is treated as 2D linear space R^2, thus

$$F(|p|^2)p \in \bigcup_{i,k=1}^{\infty} C^j((-k,k)\times(-i,i) : R^2).$$

The constants τ and η represent fourth-order dispersion and two-photon absorption respectively. The parameters ϵ, ρ, ν and π are indicating the non-linear forms of dispersion.

Description of method

Step 1 Suppose a non-linear evolution equation for $v(x,t)$ in the form

$$\Gamma(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots) = 0. \quad (2)$$

The above PDE can be converted into an ODE by using the transformation $\eta = x - \lambda t$, as

$$\Gamma(V, V', -\lambda V', V'', \lambda^2 V'' \ldots) = 0, \quad (3)$$

where η is a wave variable and λ is to be found.

Step 2 The traveling wave solution of Eq. (3) is considered, as

$$U(\eta) = Q(\Phi) = \sum_{j=0}^{k} A_j \left[s + \tan\left(\frac{\psi(\eta)}{2}\right)\right]^j + \sum_{j=1}^{k} B_j \left[s + \tan\left(\frac{\psi(\eta)}{2}\right)\right]^{-j}, \quad (4)$$

where $A_j (0 \le j \le k)$ and $B_j (1 \le j \le k)$ are constants to be determined. Also, $A_k \ne 0, B_k \ne 0$ and $\psi = \psi(\eta)$ satisfies the ordinary differential equation

$$\psi'(\eta) = a_0 \sin(\psi(\eta)) + a_1 \cos(\psi(\eta)) + a_2. \quad (5)$$

Following are the special solutions for Eq. (5).

Family 1 For $a_0^2 + a_1^2 - a_2^2 < 0$ and $a_1 - a_2 \ne 0$,

$$\psi(\eta) = 2\arctan\left[\frac{a_0}{a_1 - a_2} - \frac{\sqrt{a_2^2 - a_1^2 - a_0^2}}{a_1 - a_2}\tan\left(\frac{\sqrt{a_2^2 - a_1^2 - a_0^2}}{2}(\eta + C)\right)\right].$$

Family 2 For $a_0^2 + a_1^2 - a_2^2 > 0$ and $a_1 - a_2 \ne 0$,

$$\psi(\eta) = 2\arctan\left[\frac{a_0}{a_1 - a_2} - \frac{\sqrt{a_1^2 + a_0^2 - a_2^2}}{a_1 - a_2}\tanh\left(\frac{\sqrt{a_1^2 + a_0^2 - a_2^2}}{2}(\eta + C)\right)\right].$$

Family 3 For $a_0^2 + a_1^2 - a_2^2 > 0$, $a_1 \ne 0$ and $a_2 = 0$,

$$\psi(\eta) = 2\arctan\left[\frac{a_0}{a_1} - \frac{\sqrt{a_1^2 + a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2 + a_0^2}}{2}(\eta + C)\right)\right].$$

Family 4 For $a_0^2 + a_1^2 - a_2^2 < 0$, $a_2 \ne 0$ and $a_1 = 0$,

$$\psi(\eta) = 2\arctan\left[-\frac{a_0}{a_2} + \frac{\sqrt{a_2^2 - a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2 - a_0^2}}{2}(\eta + C)\right)\right].$$

Family 5 For $a_0^2 + a_1^2 - a_2^2 > 0$, $a_1 - a_2 \ne 0$ and $a_0 = 0$,

$$\psi(\eta) = 2\arctan\left[\sqrt{\frac{a_1 + a_2}{a_1 - a_2}}\tanh\left(\frac{\sqrt{a_1^2 - a_2^2}}{2}(\eta + C)\right)\right].$$

Family 6 For $a_0 = 0$ and $a_2 = 0$,

$$\psi(\eta) = \arctan\left[\frac{e^{2a_1(\eta+C)} - 1}{e^{2a_1(\eta+C)} + 1}, \frac{2e^{a_1(\eta+C)}}{e^{2a_1(\eta+C)} + 1}\right].$$

Family 7 For $a_1 = 0$ and $a_2 = 0$,

$$\psi(\eta) = \arctan\left[\frac{2e^{a_0(\eta+C)}}{e^{2a_0(\eta+C)} + 1}, \frac{e^{2a_0(\eta+C)} - 1}{e^{2a_0(\eta+C)} + 1}\right].$$

Family 8 For $a_0^2 + a_1^2 = a_2^2$,

$$\psi(\eta) = -2\arctan\left[\frac{(a_1 + a_2)(a_0(\eta + C) + 2)}{a_0^2(\eta + C)}\right].$$

Family 9 For $a_0 = a_1 = a_2 = ia_0$,

$$\psi(\eta) = 2\arctan\left[e^{ia_0(\eta+C)} - 1\right].$$

Family 10 For $a_0 = a_2 = ia_0$ and $a_1 = -ia_0$,

$$\psi(\eta) = -2\arctan\left[\frac{e^{ia_0(\eta+C)}}{-1 + e^{ia_0(\eta+C)}}\right].$$

Family 11 For $a_2 = a_0$,

$$\psi(\eta) = -2\arctan\left[\frac{(a_0 + a_1)e^{a_1(\eta+C)} - 1}{(a_0 - a_1)e^{a_1(\eta+C)} - 1}\right].$$

Family 12 For $a_0 = a_2$,

$$\psi(\eta) = 2\arctan\left[\frac{(a_1 + a_2)e^{a_1(\eta+C)} + 1}{(a_1 - a_2)e^{a_1(\eta+C)} - 1}\right].$$

Family 13 For $a_2 = -a_0$,

$$\psi(\eta) = 2\arctan\left[\frac{e^{a_1(\eta+C)} + a_1 - a_0}{e^{a_1(\eta+C)} - a_1 - a_0}\right].$$

Family 14 For $a_1 = -a_2$,

$$\psi(\eta) = -2\arctan\left[\frac{a_0 e^{a_0(\eta+C)}}{a_2 e^{a_0(\eta+C)}}\right].$$

Family 15 For $a_1 = 0$, $a_0 = a_2$,

$$\psi(\eta) = -2\arctan\left[\frac{a_2(\eta + C) + 2}{a_2(\eta + C)}\right].$$

Family 16 For $a_0 = 0$ and $a_1 = a_2$,

$$\psi(\eta) = 2\arctan\left[a_2(\eta + C)\right].$$

Family 17 For $a_0 = 0$, $a_1 = -a_2$,

$$\psi(\eta) = -2\arctan\left[\frac{1}{a_2(\eta + C)}\right].$$

Family 18 For $a_0 = 0$ and $a_1 = 0$,

$$\psi(\eta) = a_2\eta + C,$$

where values of a_0, a_1, a_2 and A_j, B_j ($j = 1, 2, \ldots, k$) are to be determined. By considering highest order derivatives and highest non-linear terms occurring in Eq. (5) the positive integer j can be determined using homogeneous balance principle. If j is not an integer, then a suitable transformation is used.

Step 3 Eq. (4) is substituted into Eq. (3) using the value of j obtained in Step 2. The coefficients of $\tan\left(\frac{\psi(\eta)}{2}\right)^j$, $\cot\left(\frac{\psi(\eta)}{2}\right)^j$ are collected and then each coefficient is set equal to zero. With the help of Maple, the set of algebraic equations for A_0, A_j, B_j ($j = 1, 2, \ldots, k$), a_0, a_1, a_2 and s can be obtained.

Step 4 The obtained set of over determined equations in Step 3 is solved and A_0, A_1, B_1, $\ldots$, A_k, B_k, λ and s are substituted in Eq. (4).

Exact solutions of LPD equation

The following wave transformation can be used to study Eq. (1).

$$p(x,t) = V(\eta)e^{i\phi(x,t)}, \qquad p^*(x,t) = V(\eta)e^{-i\phi(x,t)}, \tag{6}$$

where $V(\eta)$ is representing the structure of the pulse and $\eta = x - \mu t$. Also,

$$\phi(x,t) = -\lambda x + \sigma t + \theta.$$

The phase component of the soliton is $\phi(x,t)$, λ is representing the frequency of the soliton, μ is the velocity of the soliton, while phase constant here is θ and σ is the wave number. Substituting Eq. (6) into Eq. (1) the resulting expression can then be divided into a couplet containing real and imaginary parts.

Real part is stated, as

$$\begin{aligned}\tau V''' - (6\lambda^2\tau - m\mu + l)V'' - (m\lambda\sigma - \lambda^4\tau - l\lambda^2 - \sigma)V \\ - \lambda^2(\pi - \rho + \nu + \epsilon)V^3 + oV^5 \\ -nF(V^2)V + (\epsilon + \rho)V(V'^2) + (\pi + \nu)V^2V'' = 0.\end{aligned} \tag{7}$$

while the imaginary part is stated, as

$$(m\lambda\mu - \mu + m\sigma - 2l\lambda - 4\lambda^3\tau)V' + 2\lambda(\nu + \epsilon - \pi)V^2V' + 4\lambda\tau V''' = 0. \tag{8}$$

Taking into account, the linearly independent functions of Eq. (7), (8) and setting the coefficients equal to zero, the following results can be obtained.

$$\epsilon + \rho = 0, \tag{9}$$

$$\pi + \zeta = 0, \tag{10}$$

$$\tau = 0, \tag{11}$$

$$\zeta + \epsilon - \pi = 0, \tag{12}$$

and accordingly the speed of soliton falls out to be

$$\mu = \frac{m\sigma - 2l\lambda}{1 - m\lambda},$$

whenever $1 \neq m\lambda$.

Implementing the constraints from (9)–(12) in Eq. (7), the resulting expression is obtained, as

$$(l - m\mu)V'' + (m\lambda\sigma - l\lambda^2 - \sigma)V - 4\zeta\lambda^2V^3 - \eta V^5 + nF(V^2)V = 0. \tag{13}$$

Kerr law

Kerr law of non-linearity arises from the fact that due to external electric field in an optical fiber non-linear responses are faced by the light wave.

In this case, $F(r) = r$. Hence, Eq. (1) reduces to

$$ip_t + lp_{xx} + mp_{xt} + n|p|^2p = -2\zeta(p_x)^2p^* + 2\zeta|p_x|^2p + \zeta|p|^2p_{xx} - \zeta p^2p^*_{xx} + \eta|p|^4p. \tag{14}$$

Eq. (13) yields

$$(l - m\mu)V'' + (m\lambda\sigma - l\lambda^2 - \sigma)V + (n - 4\zeta\lambda^2)V^3 - \eta V^5 = 0. \tag{15}$$

Implementing the transformation $V = U^{\frac{1}{2}}$, Eq. (15) becomes

$$(l - m\mu)(-(U')^2 + 2UU'') + 4(m\lambda\sigma - l\lambda^2 - \sigma)U^2 + 4(n - 4\zeta\lambda^2)U^3 - 4\eta U^4 = 0. \tag{16}$$

$N = 1$ is obtained using homogeneous balance principle by comparing UU'' or $(U')^2$ with U^4 in Eq. (16). Thus the trail solution is

$$U(\eta) = A_0 + A_1\left[\tan\left(\frac{\psi(\eta)}{2}\right)\right] + B_1\left[\tan\left(\frac{\psi(\eta)}{2}\right)\right]^{-1}. \tag{17}$$

Collecting all terms with same order of $\tan(\frac{\psi(\eta)}{2})$, a set of algebraic equations can be obtained in $\tan(\frac{\psi(\eta)}{2})$ by setting each polynomial equal to zero. Equations obtained for A_0, A_1, B_1, l, m, n, λ, σ, ζ and μ are as follows:

$$\begin{aligned}\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^0 =\; & 3m\mu B_1^2a_1^2 + 6m\mu B_1^2a_1a_2 \\ & + 3m\mu B_1^2a_2^2 + 16\eta B_1^4 - 3lB_1^2a_1^2 \\ & - 6lB_1^2a_1a_2 - 3lB_1^2a_2^2,\end{aligned}$$

$$\begin{aligned}\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^1 =\; & 64\zeta\lambda^2B_1^3 + 4m\mu A_0B_1a_1^2 \\ & + 8m\mu A_0B_1a_1a_2 + 4m\mu A_0B_1a_2^2 \\ & + 8m\mu B_1^2a_0a_1 + 8m\mu B_1^2a_0a_2 + 64\eta A_0B_1^3 \\ & - 4lA_0B_1a_1^2 - 8lA_0B_1a_1a_2 \\ & - 4lA_0B_1a_2^2 - 8lB_1^2a_0a_1 - 8lB_1^2a_0a_2 - 16nB_1^3,\end{aligned}$$

$$\begin{aligned}\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^2 =\; & 192\zeta\lambda^2A_0B_1^2 + 12m\mu A_0B_1a_0a_1 \\ & + 12m\mu A_0B_1a_0a_2 + 6m\mu A_1B_1a_1^2 \\ & + 12m\mu A_1B_1a_1a_2 + 6m\mu A_1B_1a_2^2 \\ & + 4m\mu B_1^2a_0^2 - 2m\mu B_1^2a_1^2 \\ & + 2m\mu B_1^2a_2^2 + 96\eta A_0^2B_1^2 + 64\eta A_1B_1^3 \\ & + 16l\lambda^2B_1^2 - 12lA_0B_1a_0a_1 \\ & - 12lA_0B_1a_0a_2 - 6lA_1B_1a_1^2 - 12lA_1B_1a_1a_2 \\ & - 6lA_1B_1a_2^2 - 4lB_1^2a_0^2 \\ & + 2lB_1^2a_1^2 - 2lB_1^2a_2^2 - 16\lambda m\sigma B_1^2 \\ & - 48nA_0B_1^2 + 16\sigma B_1^2,\end{aligned}$$

$$\begin{aligned}\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^3 =\; & 192\zeta\lambda^2A_0^2B_1 + 192\zeta\lambda^2A_1B_1^2 \\ & + 8m\mu A_0B_1a_0^2 - 4m\mu A_0B_1a_1^2 \\ & + 4m\mu A_0B_1a_2^2 + 24m\mu A_1B_1a_0a_1 \\ & + 24m\mu A_1B_1a_0a_2 + 64\eta A_0^3B_1 \\ & + 192\eta A_0A_1B_1^2 + 32l\lambda^2A_0B_1 - 8lA_0B_1a_0^2 \\ & + 4lA_0B_1a_1^2 - 4lA_0B_1a_2^2 \\ & - 24lA_1B_1a_0a_1 - 24lA_1B_1a_0a_2 - 32\lambda m\sigma A_0B_1 \\ & - 48nA_0^2B_1 - 48nA_1B_1^2 \\ & + 32\sigma A_0B_1,\end{aligned}$$

$$\begin{aligned}\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^4 =\; & 64\zeta\lambda^2A_0^3 + 384\zeta\lambda^2A_0A_1B_1 + 4m\mu A_0A_1a_0a_1 \\ & + 4m\mu A_0A_1a_0a_2\end{aligned}$$

$$
\begin{aligned}
&-4\,m\mu\,A_0B_1a_0a_1+4\,m\mu\,A_0B_1a_0a_2-m\mu\,{A_1}^2{a_1}^2\\
&-2\,m\mu\,{A_1}^2a_1a_2-m\mu\,{A_1}^2{a_2}^2\\
&+24\,m\mu\,A_1B_1{a_0}^2-12\,m\mu\,A_1B_1{a_1}^2\\
&+12\,m\mu\,A_1B_1{a_2}^2-m\mu\,{B_1}^2{a_1}^2\\
&+2\,m\mu\,{B_1}^2a_1a_2-m\mu\,{B_1}^2{a_2}^2+16\,\eta\,{A_0}^4\\
&+192\,\eta\,{A_0}^2A_1B_1+96\,\eta\,{A_1}^2{B_1}^2\\
&+16\,l\lambda^2{A_0}^2+32\,l\lambda^2A_1B_1-4\,lA_0A_1a_0a_1\\
&-4\,lA_0A_1a_0a_2+4\,lA_0B_1a_0a_1\\
&-4\,lA_0B_1a_0a_2+l{A_1}^2{a_1}^2+2\,l{A_1}^2a_1a_2\\
&+l{A_1}^2{a_2}^2-24\,lA_1B_1{a_0}^2+12\,lA_1B_1{a_1}^2\\
&-12\,lA_1B_1{a_2}^2+l{B_1}^2{a_1}^2-2\,l{B_1}^2a_1a_2\\
&+l{B_1}^2{a_2}^2-16\,\lambda\,m\sigma\,{A_0}^2-32\,\lambda\,m\sigma\,A_1B_1\\
&-16\,n{A_0}^3-96\,nA_0A_1B_1+16\,\sigma\,{A_0}^2+32\,\sigma\,A_1B_1,
\end{aligned}
$$

$$
\begin{aligned}
\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^5 &= 192\,\zeta\,\lambda^2{A_0}^2A_1+192\,\zeta\,\lambda^2{A_1}^2B_1+8\,m\mu\,A_0A_1{a_0}^2\\
&-4\,m\mu\,A_0A_1{a_1}^2+4\,m\mu\,A_0A_1{a_2}^2\\
&-24\,m\mu\,A_1B_1a_0a_1+24\,m\mu\,A_1B_1a_0a_2\\
&+64\,\eta\,{A_0}^3A_1+192\,\eta\,A_0{A_1}^2B_1+32\,l\lambda^2A_0A_1\\
&-8\,lA_0A_1{a_0}^2+4\,lA_0A_1{a_1}^2-4\,lA_0A_1{a_2}^2\\
&+24\,lA_1B_1a_0a_1-24\,lA_1B_1a_0a_2\\
&-32\,\lambda\,m\sigma\,A_0A_1-48\,n{A_0}^2A_1\\
&-48\,n{A_1}^2B_1+32\,\sigma\,A_0A_1,
\end{aligned}
$$

$$
\begin{aligned}
\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^6 &= 192\,\zeta\,\lambda^2A_0{A_1}^2-12\,m\mu\,A_0A_1a_0a_1\\
&+12\,m\mu\,A_0A_1a_0a_2+4\,m\mu\,{A_1}^2{a_0}^2\\
&-2\,m\mu\,{A_1}^2{a_1}^2+2\,m\mu\,{A_1}^2{a_2}^2\\
&+6\,m\mu\,A_1B_1{a_1}^2-12\,m\mu\,A_1B_1a_1a_2\\
&+6\,m\mu\,A_1B_1{a_2}^2+96\,\eta\,{A_0}^2{A_1}^2\\
&+64\,\eta\,{A_1}^3B_1+16\,l\lambda^2{A_1}^2+12\,lA_0A_1a_0a_1\\
&-12\,lA_0A_1a_0a_2-4\,l{A_1}^2{a_0}^2+2\,l{A_1}^2{a_1}^2\\
&-2\,l{A_1}^2{a_2}^2-6\,lA_1B_1{a_1}^2\\
&+12\,lA_1B_1a_1a_2-6\,lA_1B_1{a_2}^2-16\,\lambda\,m\sigma\,{A_1}^2\\
&-48\,nA_0{A_1}^2+16\,\sigma\,{A_1}^2,
\end{aligned}
$$

$$
\begin{aligned}
\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^7 &= 64\,\zeta\,\lambda^2{A_1}^3+4\,m\mu\,A_0A_1{a_1}^2-8\,m\mu\,A_0A_1a_1a_2\\
&+4\,m\mu\,A_0A_1{a_2}^2-8\,m\mu\,{A_1}^2a_0a_1\\
&+8\,m\mu\,{A_1}^2a_0a_2+64\,\eta\,A_0{A_1}^3-4\,lA_0A_1{a_1}^2\\
&+8\,lA_0A_1a_1a_2-4\,lA_0A_1{a_2}^2\\
&+8\,l{A_1}^2a_0a_1-8\,l{A_1}^2a_0a_2-16\,n{A_1}^3,
\end{aligned}
$$

$$
\begin{aligned}
\left(\tan\left(\frac{\psi(\eta)}{2}\right)\right)^8 &= 3\,m\mu\,{A_1}^2{a_1}^2-6\,m\mu\,{A_1}^2a_1a_2+3\,m\mu\,{A_1}^2{a_2}^2\\
&+16\,\eta\,{A_1}^4-3\,l{A_1}^2{a_1}^2+6\,l{A_1}^2a_1a_2\\
&-3\,l{A_1}^2{a_2}^2.
\end{aligned}
$$

Following solutions of system of algebraic equations are obtained by using Maple software.

Set I $\zeta=\zeta,\quad l=\frac{16}{3}\frac{\eta{B_1}^2}{{a_1}^2+2\,a_1a_2+{a_2}^2},\quad \lambda=\lambda,\quad m=0,\quad \mu=\mu,\quad n=\frac{8\,B_1\eta\sqrt{{a_0}^2+{a_1}^2-{a_2}^2}+12\,\zeta\,\lambda^2(a_1+a_2)}{3\,a_1+3\,a_2}$,

$\sigma=\frac{-4}{3}\frac{\eta\left(4\,\lambda^2-{a_0}^2-{a_1}^2+{a_2}^2\right){B_1}^2}{{a_1}^2+2\,a_1a_2+{a_2}^2}$, $A_0=\frac{\left(a_0+\sqrt{{a_0}^2+{a_1}^2-{a_2}^2}\right)B_1}{a_1+a_2}$, $A_1=0$, $B_1=B_1$,

$$V(\eta)=A_0+B_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]^{-1}, \tag{18}$$

where a_0, a_1, a_2 are arbitrary constants, $p(x,t)=V(\eta)e^{i\phi(x,t)}$.

Using Eq. (18) and **Families 1, 2, 3** and **4** respectively, yields

$$p_1(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}-\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{19}$$

$$p_2(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}+\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{20}$$

$$p_3(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\frac{a_0}{a_1}+\frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{21}$$

$$p_4(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{22}$$

where $\eta=x-\mu t$.

Using Eq. (18) and **Families 5, 6, 8, 9, 13, 14, 15, 16** and **18** respectively, yields

$$p_5(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\sqrt{\frac{a_1+a_2}{a_1-a_2}}\tanh\left(\frac{\sqrt{a_1^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{23}$$

$$p_6(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\tan\frac{1}{2}\left(\arctan\left[\frac{e^{2a_1(\eta+C)}-1}{e^{2a_1(\eta+C)}+1},\frac{2e^{a_1(\eta+C)}}{e^{2a_1(\eta+C)}+1}\right]\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{24}$$

$$p_7(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{25}$$

$$p_8(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(e^{\sigma a_0(\eta+C)}-1\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{26}$$

$$p_9(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{27}$$

$$p_{10}(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{28}$$

$$p_{11}(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(-\frac{a_2(\eta+C)+2}{a_2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{29}$$

$$p_{12}(\eta)=\left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right)+B_1\left(\tan\left(\frac{1}{2}\arctan\left[\eta a_2+C\right]\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))). \tag{30}$$

Set II
$\zeta=\zeta, l=\frac{1}{3}\frac{\left(4\,\eta\,{B_1}^2+3\,\lambda\,m\sigma-3\,\sigma\right){a_1}^2+6\,\sigma\,a_2(\lambda\,m-1)a_1+\left(-4\,\eta\,{B_1}^2+3\,\lambda\,m\sigma-3\,\sigma\right){a_2}^2+4\,\eta\,{B_1}^2{a_0}^2}{(a_1+a_2)^2\lambda^2}$,

$\lambda = \lambda,\ \sigma = \sigma,\ A_0 = \frac{\left(a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)B_1}{a_1+a_2},\ A_1 = 0,\ B_1 = B_1,\ m = m,\ \mu = \frac{1}{3}\frac{(4\eta B_1{}^2+3\lambda m\sigma-3\sigma)a_1{}^2+6\sigma a_2(\lambda m-1)a_1+(-4\eta B_1{}^2+3\lambda m\sigma-3\sigma)a_2{}^2-16\eta\lambda^2B_1{}^2+4\eta B_1{}^2a_0{}^2}{\lambda^2m(a_1+a_2)^2},$

$n = \frac{8B_1\eta\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}+12\zeta\lambda^2(a_1+a_2)}{3a_1+3a_2},$

$$V(\eta) = A_0 + B_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]^{-1}, \tag{31}$$

where a_0, a_1, a_2 are arbitrary constants.

Using Eq. (31) and **Families 1, 2, 3, 4, 5, 6, 8, 9** and **13, 15, 16, 18** respectively, yields

$$p_{13}(\eta) = \left[\frac{\left(a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)B_1}{a_1+a_2} + B_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2} - \tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{32}$$

$$p_{14}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2} + \tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{33}$$

$$p_{15}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(\frac{a_0}{a_1} + \frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{34}$$

$$p_{16}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(-\frac{a_0}{a_2} + \frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{35}$$

$$p_{17}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(\sqrt{\frac{a_1+a_2}{a_1-a_2}}\tanh\left(\frac{\sqrt{a_1^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{36}$$

$$p_{18}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(\tan\frac{1}{2}\left(\arctan\left[\frac{e^{2a_1(\eta+C)}-1}{e^{2a_1(\eta+C)}+1}, \frac{2e^{a_1(\eta+C)}}{e^{2a_1(\eta+C)}+1}\right]\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{37}$$

$$p_{19}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{38}$$

$$p_{20}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left[e^{\sigma a_0(\eta+C)}-1\right]^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{39}$$

$$p_{21}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left[\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right]^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{40}$$

$$p_{22}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(-\frac{a_2(\eta+C)+2}{a_2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{41}$$

$$p_{23}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left[\tan\left(\frac{1}{2}\arctan\left[\eta a_2+C\right]\right)\right]^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{42}$$

$$p_{24}(\eta) = \left[\left(\frac{(a_0+\sqrt{a_0^2+a_1^2-a_2^2})B_1}{a_1+a_2}\right) + B_1\left(\tan\left(\frac{1}{2}\arctan\left[\eta a_2+C\right]\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))). \tag{43}$$

Set III $\zeta = \zeta,\ l = \frac{16}{3}\frac{\eta A_1{}^2}{(a_1-a_2)^2},\ \lambda = \lambda,\ m = 0,\ \mu = \mu,\ n = \frac{8\eta A_1\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}+12\zeta\lambda^2(a_1-a_2)}{3a_1-3a_2},$

$\sigma = \frac{-4}{3}\frac{\eta(4\lambda^2-a_0{}^2-a_1{}^2+a_2{}^2)A_1{}^2}{(a_1-a_2)^2},\ A_0 = \frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2},\ A_1 = A_1,\ B_1 = 0,$

$$V(\eta) = A_0 + A_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right], \tag{44}$$

where a_0, a_1, a_2 are arbitrary constants.

Using Eq. (44) and **Families 1, 2, 3, 4, 5, 6, 8** and **10, 13, 14, 17, 18** respectively, yields

$$p_{25}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2} - \tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{45}$$

$$p_{26}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(\frac{a_0}{a_1-a_2}\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2} + \tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{46}$$

$$p_{27}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(\frac{a_0}{a_1} + \frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{47}$$

$$p_{28}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(-\frac{a_0}{a_2} + \frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{48}$$

$$p_{29}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(\sqrt{\frac{a_1+a_2}{a_1-a_2}}\tanh\left(\frac{\sqrt{a_1^2-a_2^2}}{2}(\eta+C)\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{49}$$

$$p_{30}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(\tan\frac{1}{2}\left(\arctan\left[\frac{e^{2a_1(\eta+C)}-1}{e^{2a_1(\eta+C)}+1}, \frac{2e^{a_1(\eta+C)}}{e^{2a_1(\eta+C)}+1}\right]\right)\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{50}$$

$$p_{31}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{51}$$

$$p_{32}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left[-\frac{e^{\sigma a_0(\eta+C)}}{-1+e^{\sigma a_0(\eta+C)}}\right]\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{52}$$

$$p_{33}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left[\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right]\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{53}$$

$$p_{34}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left[-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right]\right]^{\frac{1}{2}}$$

$$(\exp(i(\lambda x+\sigma t+\theta))), \quad (54)$$

$$p_{35}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left[-\frac{1}{c(\eta+C)}\right]\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (55)$$

$$p_{36}(\eta) = \left[\frac{\left(-a_0+\sqrt{a_0{}^2+a_1{}^2-a_2{}^2}\right)A_1}{a_1-a_2} + A_1\left[\tan\left(\frac{1}{2}\arctan[\eta\, a_2+C]\right)\right]\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))). \quad (56)$$

Set IV $\zeta=\zeta,\ \ l=\frac{1}{3}\frac{\left(\eta\, A_0{}^2+3\,\lambda\, m\sigma-3\,\sigma\right)a_0{}^2+\eta\, A_0{}^2\left(a_1{}^2-a_2{}^2\right)}{\lambda^2 a_0{}^2},\ \ \lambda=\lambda,\ \ m=m,\ \ \mu=\frac{1}{3}\frac{-4\left(\lambda^2-\frac{1}{4}a_0{}^2-\frac{1}{4}a_1{}^2+\frac{1}{4}a_2{}^2\right)\eta\, A_0{}^2+3\,\sigma\, a_0{}^2(\lambda\, m-1)}{m\lambda^2 a_0{}^2},\ \ n=4\,\zeta\,\lambda^2+\frac{4}{3}\,\eta\, A_0,$
$\sigma=\sigma,\ \ A_0=A_0,\ \ A_1=\frac{-1}{2}\frac{A_0(a_1-a_2)}{a_0},\ \ B_1=\frac{1}{2}\frac{A_0(a_1+a_2)}{a_0},$

$$V(\eta)=A_0+A_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]+B_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]^{-1}, \quad (57)$$

where a_0, a_1, a_2 are arbitrary constants.

Using Eq. (57) and **Families 1, 2, 3, 4, 8, 10, 13** and **14, 15, 16** respectively, yields

$$p_{37}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (58)$$

$$p_{38}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\right.\right. \quad (59)$$

$$\left.\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\right. \quad (60)$$

$$\left.\left.\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (61)$$

$$p_{39}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{a_0}{a_1}+\frac{\sqrt{a_1^2+a_0^2}}{a_1}\right.\right. \quad (62)$$

$$\left.\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)+ \frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{a_0}{a_1}\frac{\sqrt{a_1^2+a_0^2}}{a_1}\right. \quad (63)$$

$$\left.\left.\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (64)$$

$$p_{40}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}} \times(\exp(i(\lambda x+\sigma t+\theta))), \quad (65)$$

$$p_{41}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}} \times(\exp(i(\lambda x+\sigma t+\theta))), \quad (66)$$

$$p_{42}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{e^{\sigma\, a_0(\eta+C)}}{-1+e^{\sigma\, a_0(\eta+C)}}\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{e^{\sigma\, a_0(\eta+C)}}{-1+e^{\sigma\, a_0(\eta+C)}}\right)^{-1}\right]^{\frac{1}{2}} \times(\exp(i(\lambda x+\sigma t+\theta))), \quad (67)$$

$$p_{43}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (68)$$

$$p_{44}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right)^{-1}\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (69)$$

$$p_{45}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{a_2\,(\eta+C)+2}{a_2\,(\eta+C)}\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\left(-\frac{a_2\,(\eta+C)+2}{a_2\,(\eta+C)}\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (70)$$

$$p_{46}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\tan\left(\frac{1}{2}\arctan\left[\eta\, a_2+C\right]\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\tan\left(\frac{1}{2}\arctan\left[\eta\, a_2+C\right]\right)\right)^{-1}\right]^{\frac{1}{2}} \times(\exp(i(\lambda x+\sigma t+\theta))). \quad (71)$$

Set V $\zeta=\zeta,\ \ l=\frac{4}{3}\frac{\eta A_0^2}{a_0^2},\ \ \lambda=\lambda,\ \ m=0,\ \ \mu=\mu,\ \ n=4\,\zeta\,\lambda^2+\frac{4}{3}\,\eta\, A_0,$
$\sigma=\frac{-1}{3}\frac{\left(4\,\lambda^2-a_0{}^2-a_1{}^2+a_2{}^2\right)A_0{}^2\eta}{a_0{}^2},\ A_0=A_0,\ A_1=\frac{-1}{2}\frac{A_0(a_1-a_2)}{a_0},\ B_1=\frac{1}{2}\frac{A_0(a_1+a_2)}{a_0},$

$$V(\eta)=A_0+A_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]+B_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]^{-1}, \quad (72)$$

where a_0, a_1, a_2 are arbitrary constants.

Using Eq. (72) and **Families 1,2,3,4,8,9,10,13** and **14,15,18** respectively, yields

$$p_{47}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (73)$$

$$p_{48}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\times\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right) +\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\times\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (74)$$

$$p_{49}(\eta) = \left[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\right.$$

$$\times\left(\frac{a_0}{a_1}+\frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)$$
$$+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{a_0}{a_1}\frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}$$
$$(\exp(i(\lambda x+\sigma t+\theta))), \tag{75}$$

$$p_{50}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\times\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{76}$$

$$p_{51}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)^{-1}\Bigg]^{\frac{1}{2}}\times(\exp(i(\lambda x+\sigma t+\theta))), \tag{77}$$

$$p_{52}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}A_1\left(e^{\sigma a_0(\eta+C)}-1\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}A_1\left(e^{\sigma a_0(\eta+C)}-1\right)^{-1}\Bigg]^{\frac{1}{2}}\times(\exp(i(\lambda x+\sigma t+\theta))), \tag{78}$$

$$p_{53}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{e^{\sigma a_0(\eta+C)}}{-1+e^{\sigma a_0(\eta+C)}}\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{e^{\sigma a_0(\eta+C)}}{-1+e^{\sigma a_0(\eta+C)}}\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{79}$$

$$p_{54}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{80}$$

$$p_{55}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{81}$$

$$p_{56}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(-\frac{a_2\,(\eta+C)+2}{a_2\,(\eta+C)}\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\left(-\frac{a_2\,(\eta+C)+2}{a_2\,(\eta+C)}\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{82}$$

$$p_{57}(\eta) = \Bigg[A_0+\frac{-1}{2}\frac{A_0\left(a_1-a_2\right)}{a_0}\left(\tan\left(\frac{1}{2}\arctan[\eta\,a_2+C]\right)\right)+\frac{1}{2}\frac{A_0\left(a_1+a_2\right)}{a_0}\left(\tan\left(\frac{1}{2}\arctan[\eta\,a_2+C]\right)\right)^{-1}\Bigg]^{\frac{1}{2}}\times(\exp(i(\lambda x+\sigma t+\theta))). \tag{83}$$

Set VI $\zeta=\zeta,\quad l=\frac{16}{3}\,\frac{\eta\,{A_1}^2}{{a_1}^2-2\,a_1a_2+{a_2}^2},\quad \lambda=\lambda,\quad m=0,\quad \mu=\mu,$
$n=\frac{16\,\eta\,\sqrt{-{a_1}^2+{a_2}^2}A_1+8\,a_0A_1\eta+12\,\zeta\,\lambda^2(a_1-a_2)}{3\,a_1-3\,a_2},$
$\sigma=\frac{-4}{3}\,\frac{\left(4\,\lambda^2-6\,\sqrt{-{a_1}^2+{a_2}^2}a_0-{a_0}^2+5\,{a_1}^2-5\,{a_2}^2\right){A_1}^2\eta}{(a_1-a_2)^2},$
$A_0=\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1,\quad A_1=A_1,\quad B_1=-\frac{(a_1+a_2)A_1}{a_1-a_2},$

$$V(\eta)=A_0+A_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]+B_1\left[\tan\left(\frac{\Psi(\eta)}{2}\right)\right]^{-1}, \tag{84}$$

where a_0, a_1, a_2 are arbitrary constants.

Using Eq. (84) and **Families 1, 2, 3, 4, 6, 8, 10** and **13, 14, 15, 17** respectively, yields

$$p_{58}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\left(\frac{a_0}{a_1-a_2}-\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{a_1-a_2}\tan\left(\frac{\sqrt{a_2^2-a_1^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{85}$$

$$p_{59}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\times\left(\frac{a_0}{a_1-a_2}+\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{a_1-a_2}\tanh\left(\frac{\sqrt{a_1^2+a_0^2-a_2^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{86}$$

$$p_{60}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(\frac{a_0}{a_1}+\frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\left(\frac{a_0}{a_1}\frac{\sqrt{a_1^2+a_0^2}}{a_1}\tanh\left(\frac{\sqrt{a_1^2+a_0^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{87}$$

$$p_{61}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\left(-\frac{a_0}{a_2}+\frac{\sqrt{a_2^2-a_0^2}}{a_2}\tan\left(\frac{\sqrt{a_2^2-a_0^2}}{2}(\eta+C)\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{88}$$

$$p_{62}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(\tan\frac{1}{2}\left(\arctan\left[\frac{e^{2\,a_1(\eta+C)}-1}{e^{2\,a_1(\eta+C)}+1},\frac{2e^{a_1(\eta+C)}}{e^{2\,a_1(\eta+C)}+1}\right]\right)\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\times\left(\tan\frac{1}{2}\left(\arctan\left[\frac{e^{2\,a_1(\eta+C)}-1}{e^{2\,a_1(\eta+C)}+1},\frac{2e^{a_1(\eta+C)}}{e^{2\,a_1(\eta+C)}+1}\right]\right)\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{89}$$

$$p_{63}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\left(-\frac{(a_1+a_2)(a_0(\eta+C)+2)}{a_0^2(\eta+C)}\right)^{-1}\Bigg]^{\frac{1}{2}}\times(\exp(i(\lambda x+\sigma t+\theta))), \tag{90}$$

$$p_{64}(\eta) = \Bigg[\sqrt{-\frac{4\,a_1+4\,a_2}{a_1-a_2}}A_1+A_1\left(-\frac{e^{\sigma a_0(\eta+C)}}{-1+e^{\sigma a_0(\eta+C)}}\right)-\frac{\left(a_1+a_2\right)A_1}{a_1-a_2}\left(-\frac{e^{\sigma a_0(\eta+C)}}{-1+e^{\sigma a_0(\eta+C)}}\right)^{-1}\Bigg]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \tag{91}$$

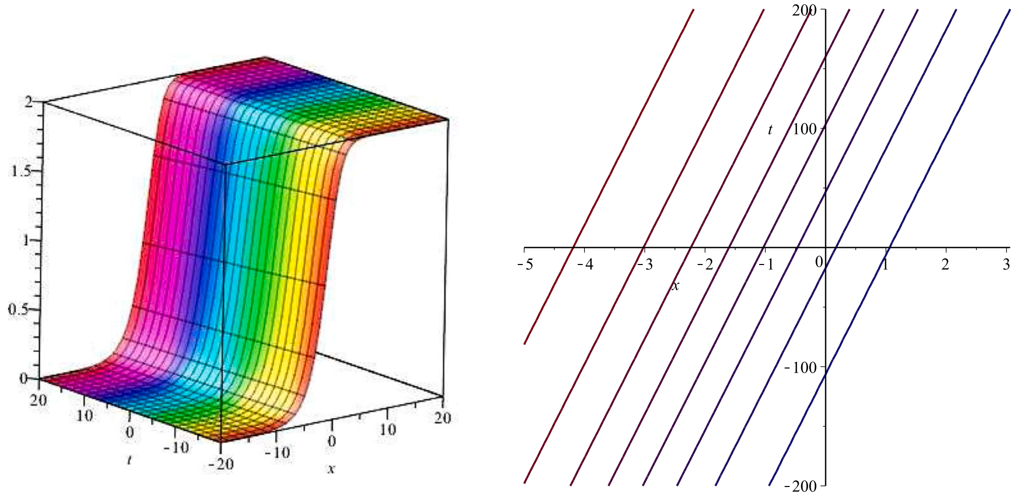

Fig. 1. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = -1.5, a_1 = -1, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 0.01, \lambda = 1, A_0 = -1, A_1 = 0, B_1 = 1, n = 1.467, \sigma = -3.2, l = 4.267, C = 1, \theta = 0.2$.

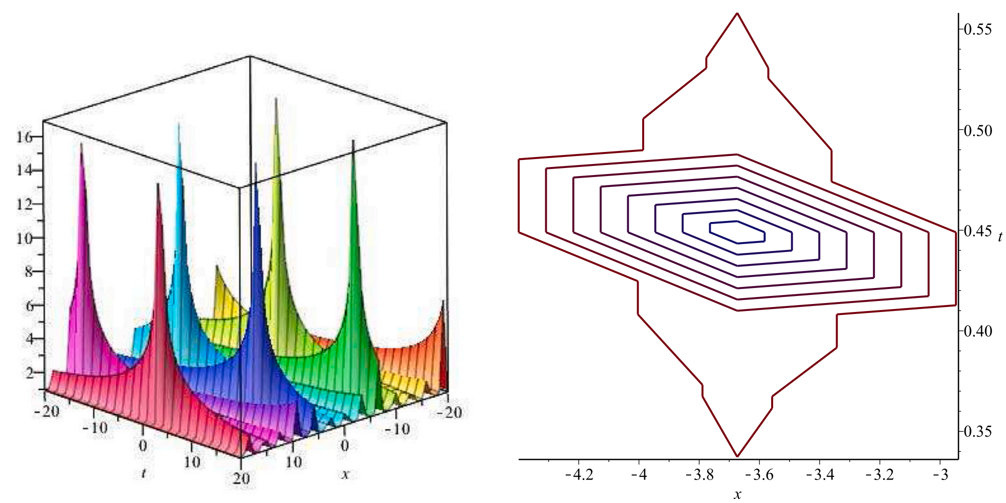

Fig. 2. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 0, a_2 = 2, \eta = 0.2, \zeta = 0.1, \mu = 0.01, \lambda = 1, A_0 = \frac{1}{2} + (\frac{1}{2} I)\sqrt{3}, A_1 = 0, B_1 = 1, n = 0.267I\sqrt{3} + 0.4, \sigma = -0.467, l = 0.267, C = 1, \theta = 0.2$.

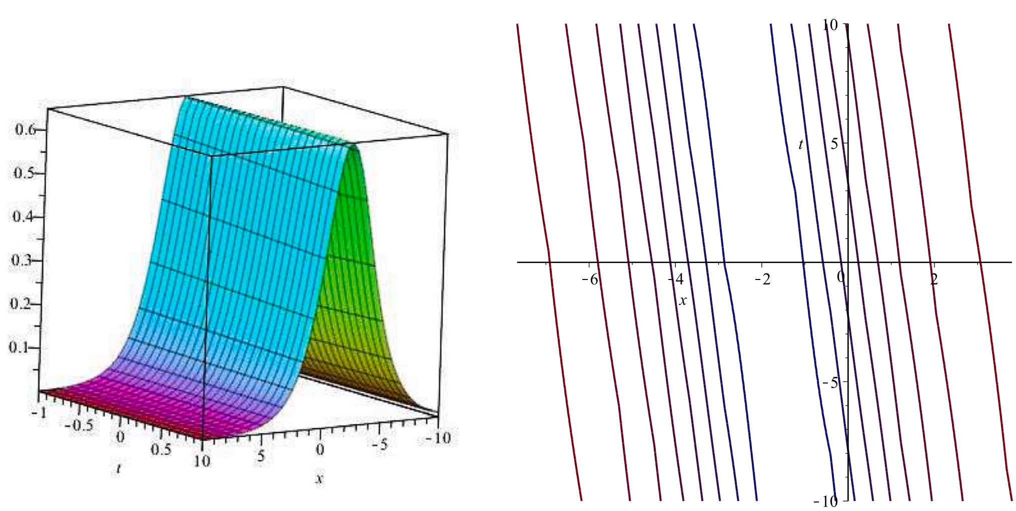

Fig. 3. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1.5, a_1 = 0.5, a_2 = 1, \eta = 0.2, \zeta = 0.1, \mu = -0.074, \lambda = 1, A_0 = 1, A_1 = 0.16, B_1 = 0.5, n = 0.67, \sigma = 0.2901, l = 0.443, C = 1, \theta = 0.2$.

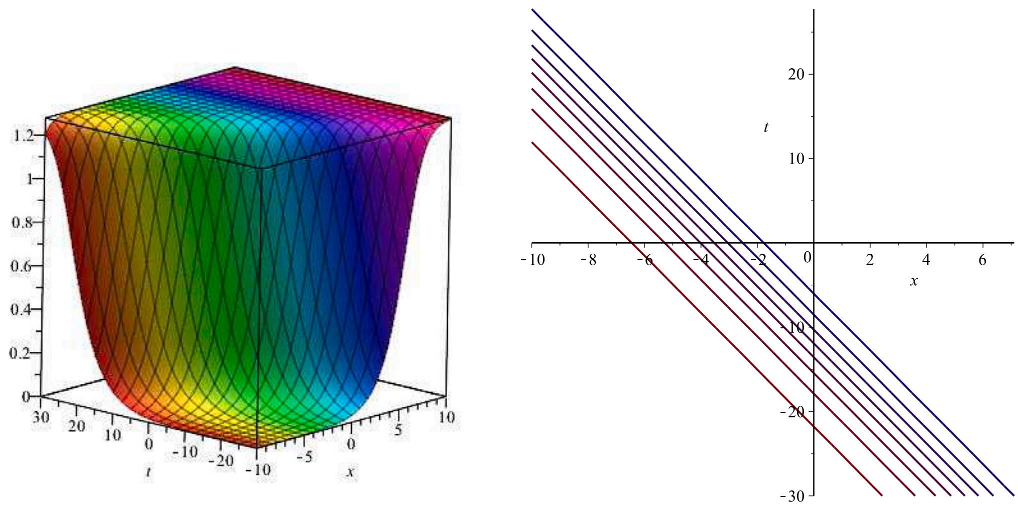

Fig. 4. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1.5, a_1 = 0.5, a_2 = 1, \eta = 0.2, \zeta = 0.1, \mu = -0.296, \lambda = 1, A_0 = 1.8164, A_1 = 0, B_1 = 1, n = 0.8354, \sigma = 1, l = 0.1778, C = 1, \theta = 0.2, m = 1, k = 1$.

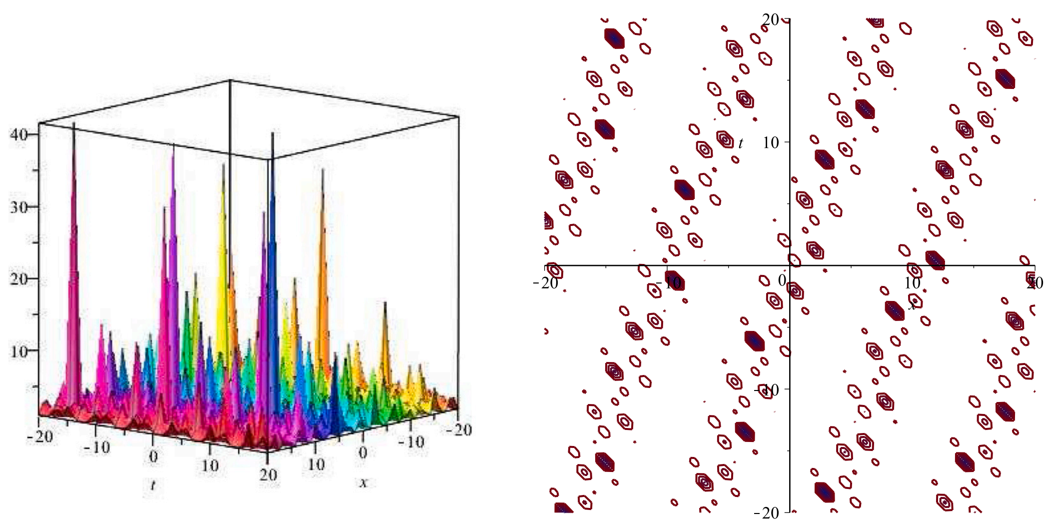

Fig. 5. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 0, a_2 = 2, \eta = 0.2, \zeta = 0.1, \mu = -0.4667, \lambda = 1, A_0 = 1, A_1 = 1, B_1 = 1, n = 0.67, \sigma = 1.933, l = -0.2, C = 1, \theta = 0.2, m = 1, k = 1$.

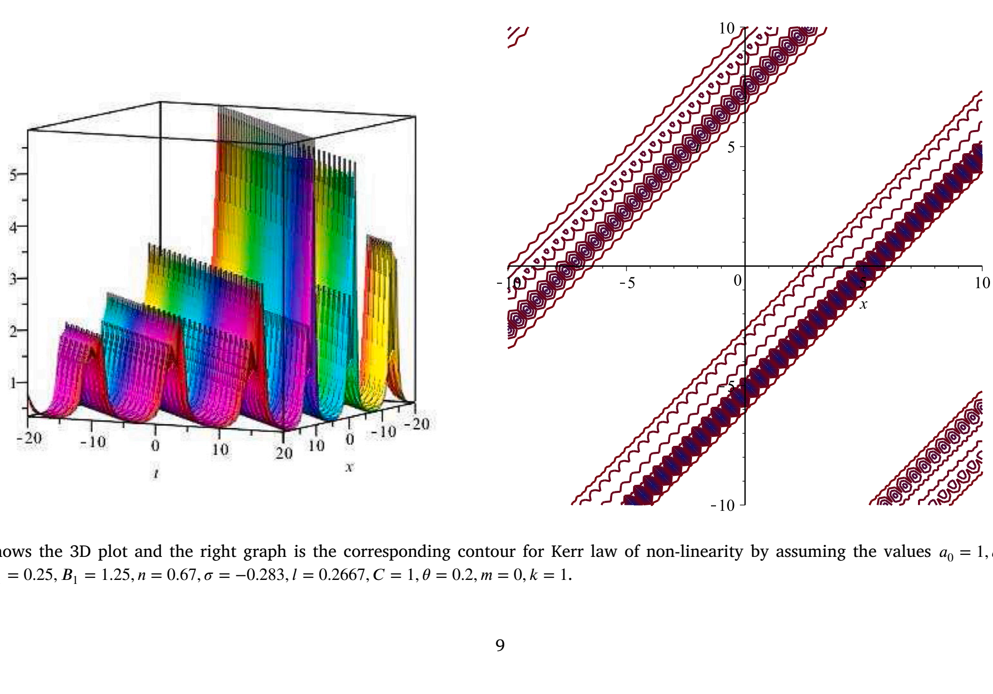

Fig. 6. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 1, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 1, \lambda = 1, A_0 = 1, A_1 = 0.25, B_1 = 1.25, n = 0.67, \sigma = -0.283, l = 0.2667, C = 1, \theta = 0.2, m = 0, k = 1$.

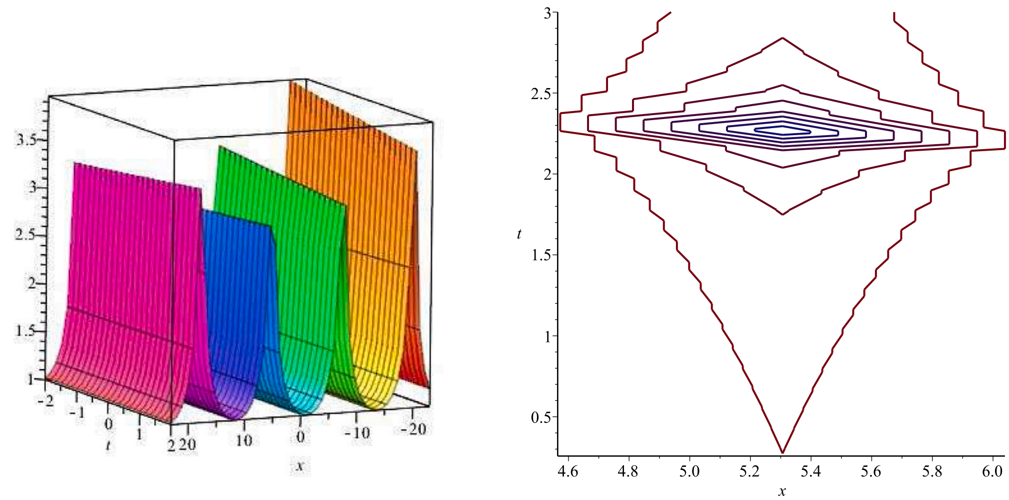

Fig. 7. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 1, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 0 = .01, \lambda = 1, A_0 = 2 - I, A_1 = 1, B_1 = 0, n = 0.4 - 0.53I, \sigma = -4.53, l = 4.2667, C = 1, \theta = 0.2$.

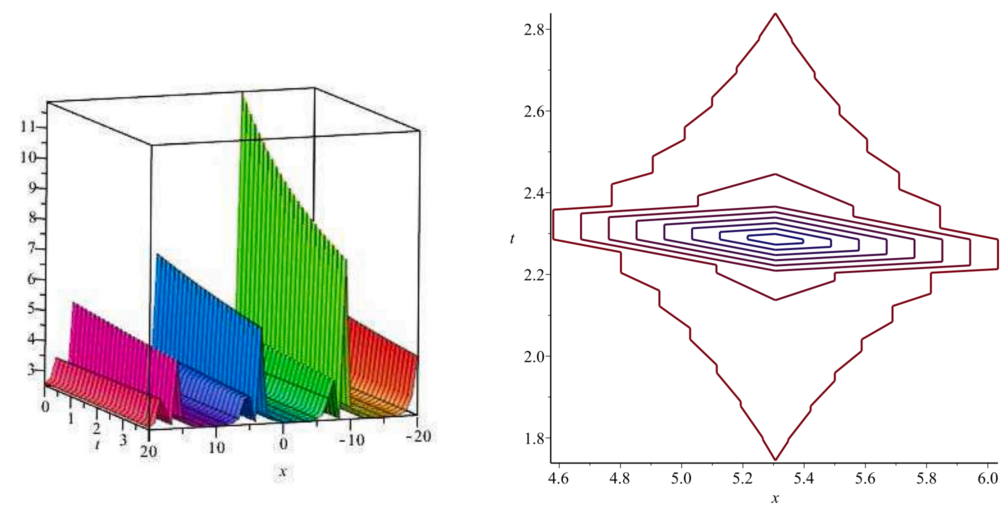

Fig. 8. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 1, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 0.01, \lambda = 1, A_0 = 4.4721, A_1 = 1, B_1 = 5, n = -3.0518, \sigma = 10.622, l = 4.2667, C = 1, \theta = 0.2$.

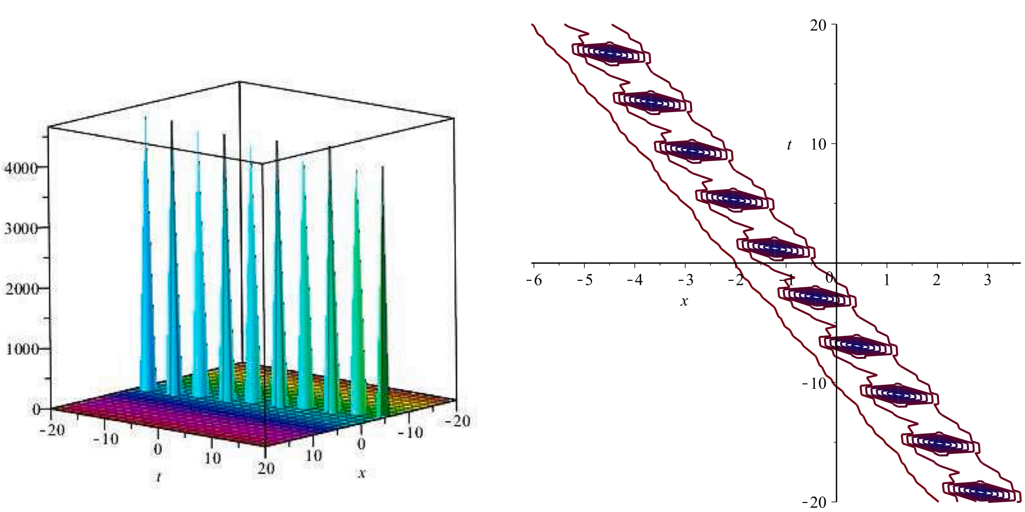

Fig. 9. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 1, a_2 = 1, \eta = 0.2, \zeta = 0.1, \mu = -0.2, \lambda = 1, A_0 = 1, A_1 = 0, B_1 = 1, n = 0.667, \sigma = 1, l = 0.667, C = 1, \theta = 0.2, m = 1, k = 1$.

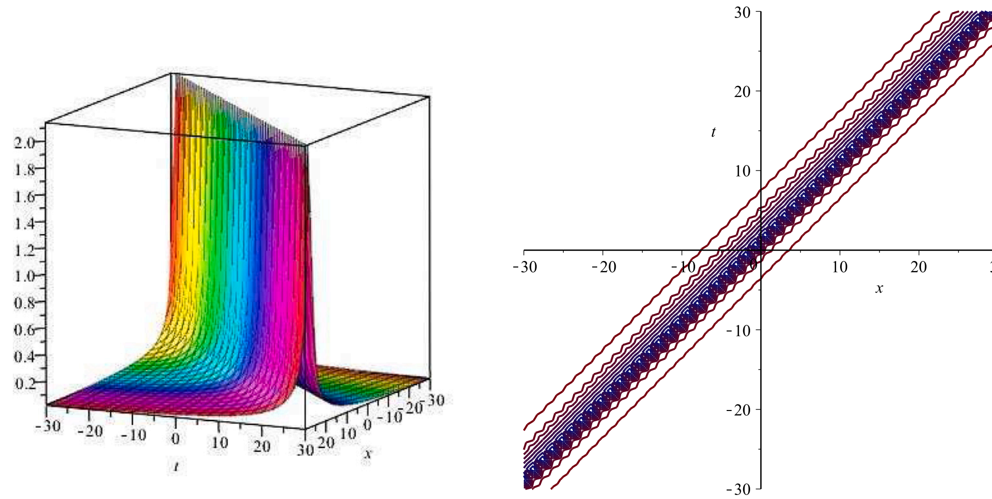

Fig. 10. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 0, a_2 = 1, \eta = 0.2, \zeta = 0.1, \mu = 1, \lambda = 1, A_0 = 1, A_1 = 0.5, B_1 = 0.5, n = 0.667, \sigma = -0.2667, l = 0.2667, C = 1, \theta = 0.2, m = 0, k = 1$.

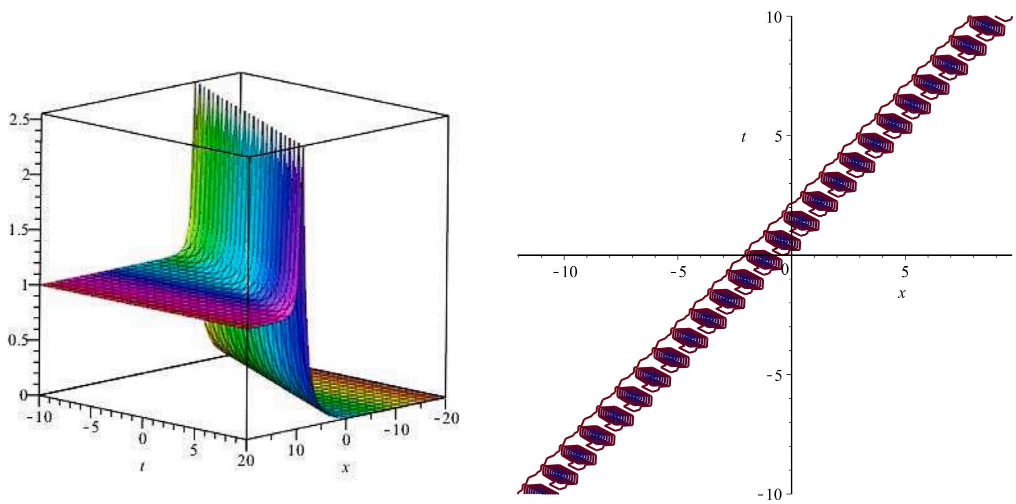

Fig. 11. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1, a_1 = 1, a_2 = 1, \eta = 0.2, \zeta = 0.1, \mu = 1, \lambda = 1, A_0 = 1, A_1 = 0, B_1 = 1, n = 0.667, \sigma = -0.2, l = 0.2667, C = 1, \theta = 0.2, m = 0, k = 1$.

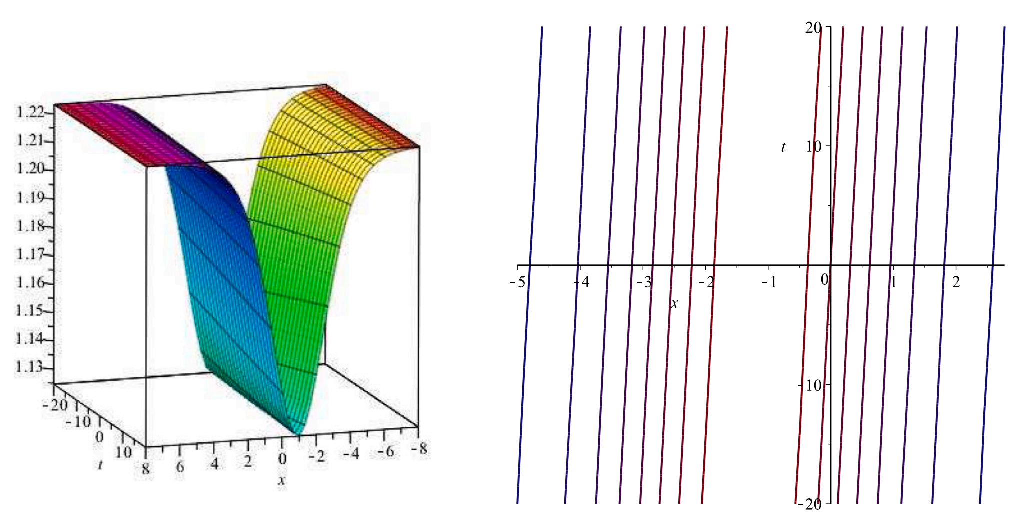

Fig. 12. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = -1.5, a_1 = -1, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 0.01, \lambda = 1, A_0 = 0.8944, A_1 = 1, B_1 = 0.2, n = 0.2429, \sigma = -0.2373, l = 0.1706, C = 1, \theta = 0.2, k = 1$.

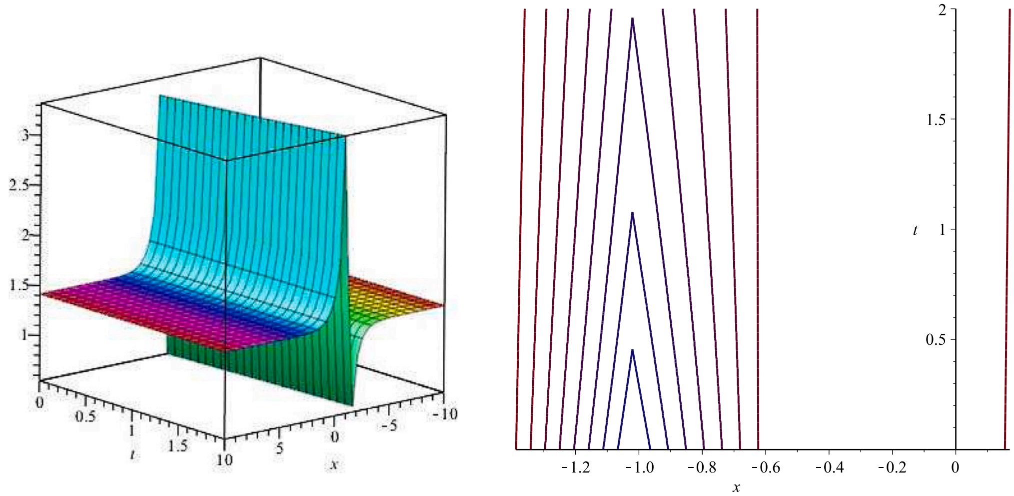

Fig. 13. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 0, a_1 = 1.5, a_2 = 0, \eta = 0.2, \zeta = 0.1, \mu = 1, \lambda = 1, A_0 = 1, A_1 = 0.5, B_1 = 0.5, n = 0.0667, \sigma = -0.1185, l = 0.1185, C = 1, \theta = 0.2, m = 0, k = 1$.

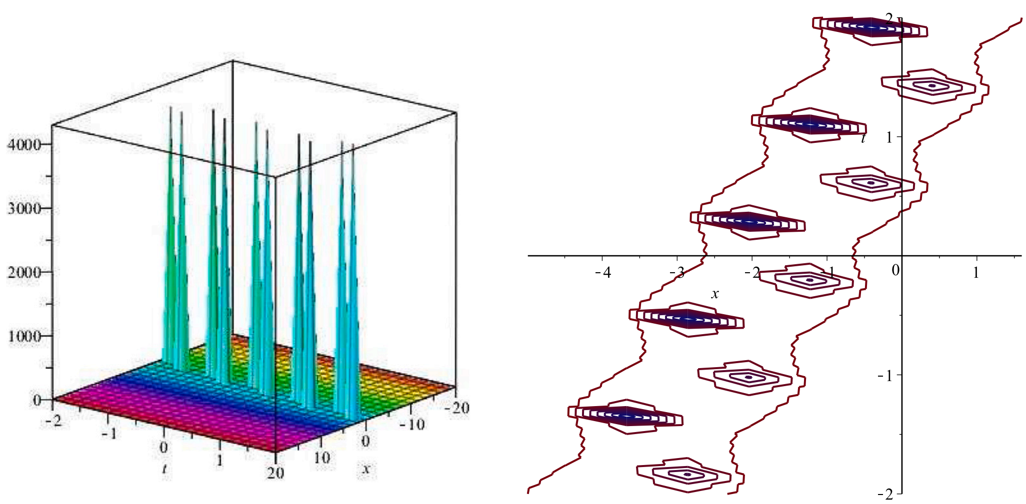

Fig. 14. The left graph shows the 3D plot and the right graph is the corresponding contour for Kerr law of non-linearity by assuming the values $a_0 = 1.5, a_1 = 0, a_2 = 1.5, \eta = 0.2, \zeta = 0.1, \mu = 0.01, \lambda = 1, A_0 = 2I, A_1 = 1, B_1 = -1, n = 0.4 + 1.0667I, \sigma = -1.807 + 0.1I, l = 0.4740, C = 1, \theta = 0.2$.

$$p_{65}(\eta) = \left[\sqrt{-\frac{4a_1+4a_2}{a_1-a_2}}A_1 + A_1\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right) - \frac{(a_1+a_2)A_1}{a_1-a_2}\left(\frac{e^{a_1(\eta+C)}+a_1-a_0}{e^{a_1(\eta+C)}-a_1-a_0}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (92)$$

$$p_{66}(\eta) = \left[\sqrt{-\frac{4a_1+4a_2}{a_1-a_2}}A_1 + A_1\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right) - \frac{(a_1+a_2)A_1}{a_1-a_2}\left(-\frac{a_0e^{a_0(\eta+C)}}{a_2e^{a_0(\eta+C)}-1}\right)^{-1}\right]^{\frac{1}{2}} (\exp(i(\lambda x+\sigma t+\theta))), \quad (93)$$

$$p_{67}(\eta) = \left[\sqrt{-\frac{4a_1+4a_2}{a_1-a_2}}A_1 + A_1\left(-\frac{a_2(\eta+C)+2}{a_2(\eta+C)}\right) - \frac{(a_1+a_2)A_1}{a_1-a_2}\left(\left(-\frac{a_2(\eta+C)+2}{a_2(\eta+C)}\right)\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))), \quad (94)$$

$$p_{68}(\eta) = \left[\sqrt{-\frac{4a_1+4a_2}{a_1-a_2}}A_1 + A_1\left(-\frac{1}{a_2(\eta+C)}\right) - \frac{(a_1+a_2)A_1}{a_1-a_2}\left(-\frac{1}{a_2(\eta+C)}\right)^{-1}\right]^{\frac{1}{2}}(\exp(i(\lambda x+\sigma t+\theta))). \quad (95)$$

Graphical illustration

The graphical illustration of some of the constructed solutions is presented in this section. Traveling wave solutions of four types i.e. rational, trigonometric, exponential and hyperbolic function solutions have been retrieved. A variety of these solutions have been plotted to properly understand the mechanism of the original equation (1). The retrieved solutions can be used to investigate the propagation of solitons in the related physical problems. Abundant soliton solutions have been observed by taking suitable choice of free parameters. The obtained solutions include kink, periodic, dark, bright and singular solitons. Particularly in Fig. 1, a kink soliton is obtained by plotting Family 13 of Set 1. In Fig. 2, 3D plot for Family 4 of Set 1 is presented. Fig. 3 represents the bright soliton by plotting Set 5 for Family 2. In Fig. 4, Set 2 for Family 2 is presented by the graph. Fig. 5 is indicating graph of Set 5 for Family 5. Plot of Set 5 for Family 1 is given in Fig. 6. In Fig. 7, Set 3 for Family 1 is represented. Likewise, in Fig. 8, Set 6 for Family 1 is given. 3D plot and the contour of solution Set 2 for Family 9 is illustrated in Fig. 9. Plot of Set 7 for Family 8 is introduced in Fig. 10. The plot of Set 7 for Family 9 is introduced in Fig. 11. Figs. 12 and 13 are denoting the graphs of Set 6 for Family 13 and Family 6 respectively.

A dark soliton is indicated in Fig. 12. Last Figure is the graph of Set 5 for Family 15 (see Fig. 14).

Comparison with other methods

It is remarkable that improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method is more rewarding and valuable due to the wide variety of soliton solutions obtained. The extracted soliton solutions depict that this technique is more effective than many other methods like semi-inverse variational principle [5], Jacobi's elliptic function scheme, $\exp(\phi(\eta))$ expansion method [6] and modified simple equation method [23], as [5] reported bright solitons, [6] derived dark and singular solitons and [23] conferred dark and singular soliton solutions only.

Results and discussion

In this work, Lakshmanan–Porsezian–Daniel equation has been studied with Kerr law non-linearity. The model was solved by a well-known integration technique *i.e.* improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method to construct a variety of optical solitons for arbitrary initial condition. It has been graphically demonstrated that by applying this method dark, bright, singular, kink and periodic soliton solutions of LPD equation are retrieved. For a better insight into the physical structure of waves, the corresponding contour plots are also illustrated. The computations were performed by using Maple software.

It is worth-mentioning that the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method is applied on this model for the first time to the best of our knowledge. The solutions of LPD equation that are listed in this paper are novel and not reported elsewhere in literature. Thus, observations validate that the improved $\tan\left(\frac{\psi(\eta)}{2}\right)$ method is a worthwhile technique for the analytic study of many other non-linear PDEs in mathematical physics, optical fibers and engineering fields.

Conclusion

This research work extracted a series of traveling wave solutions such as kink, singular, periodic, bright and dark soliton solutions of the LPD equation with Kerr law of non-linearity by employing the $\tan\left(\frac{\psi(\eta)}{2}\right)$-expansion method. Comparison of the results obtained with already published results reveal that many new solutions are retrieved in this paper. The proposed method can be applied to many other non-linear PDEs in mathematical biology, engineering and physics. The novel solutions reported in this study may help to understand the dynamical framework of many optical problems and other physical phenomena.

CRediT authorship contribution statement

Ghazala Akram: Conceptualization, Data curation, Investigation, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing - review & editing. **Maasoomah Sadaf:** Conceptualization, Formal analysis, Investigation, Resources, Software, Validation, Writing – original draft, Writing - review & editing. **Mirfa Dawood:** Data curation, Formal analysis, Methodology, Software, Writing – original draft, Writing - review & editing. **Dumitru Baleanu:** Project administration, Supervision, Visualization, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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