ARTICLE IN PRESS
JID: JOES [m5G;September 20, 2021;11:1]

Journal of Ocean Engineering and Science xxx (xxxx) xxx

Contents lists available at ScienceDirect

Journal of Ocean Engineering and Science

journal homepage: www.elsevier.com/locate/joes

Some applications of the least squares-residual power series method for fractional generalized long wave equations

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ARTICLE INFO

Article history:
Received 14 July 2021
Revised 2 September 2021
Accepted 2 September 2021
Available online xxx

Keywords:
LS-RPSM
Caputo fractional derivative (CFD)
Series solution
TFRLWEs,

ABSTRACT

This article examines a new effective method called the least squares-residual power series method (LS-RPSM) and compares this method with the RPSM. The LS-RPSM assembles the least-squares process with the residual power series method. These techniques are applied to investigate the linear and nonlinear time-fractional regularized long wave equations (TFRLWEs). The RLW models define the shallow water waves in oceans and the internal ion-acoustic waves in plasma. Firstly, we apply the well-known RPSM to acquire approximate solutions. In the next step, the Wronskian determinant is searched in fractional order to show that the functions are linearly independent. After these operations, a system of linear equations is obtained. In the last step, the least-squares algorithm is used to find the necessary coefficients. When this article is examined, it can be said that LS-RPSM is more useful because it requires using fewer terms than the required number of terms when applying the RPSM. Additionally, the experiments show that this method converges better than RPSM.

1. Introduction

Many nonlinear events are modeled with the help of fractional order differential and integral equations [1–12]. When modeling physical events, it is more advantageous to calculate with fractional derivatives instead of integer derivatives. It can be said that using the fractional derivatives is common when studying biological systems. In addition, fractional derivatives are used when modeling most of the physical phenomena. One of the most used fractional derivatives is the CFD [3].

Mathematical modelling of tsunamis and tidal oscillations can be used as an example to apply this equation in oceanic engineering. These types of equations can also be applied in engineering, such as diffraction, refractive prediction, flatness, and prediction of harmonic interactions around coastal structures. Nonlinear evolutionary equations are of great interest in various fields such as plasma physics, astrophysics, chemical physics, solid state physics, biology, fiber optics and oceanic engineering.

Fractional differential equations have been used in modeling events in many different fields in recent years. Some of these fields are electromagnetics, signal processing, control theory, communication, physics, mathematics, biology, chemistry, engineering and finance [1–16]. Many methods have been studied for many fractional differential equations. Some of those Lie group analysis and Lie's invariant analysis method [17,18], Laguerre wavelet collocation method [19], fractional reduced differential transform method [20], Double(G'/G,1/G)-expansion method [21,22].

While investigating analytical and semi-analytical solutions of fractional differential equations, the RPSM has been applied in many studies recently [16,23–25]. RPSM gives these solutions via a convergent series. In this method, the serial solution is obtained with an algorithm established by following sequential steps. The RPSM was developed as an effective algorithm for the Fuzzy differential equations [26].

One of the methods applied to obtain approximate solutions of fractional differential equations (linear and non-linear) is the least-

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https://doi.org/10.1016/j.joes.2021.09.001

squares method [27,28]. The LS-RPSM is explained by combining the least-squares method and the RPSM [29]. The advantage of this new combined method is that it reduces the number of operations and increases the reliability of the solutions. The nonlinear wave equation defines wave propagation in dispersive media such as liquid flow including gas bubbles, fluid flow in elastic tubes, rivers, lakes, and the ocean, as well as gravity waves in a corresponding domain and nonlinear wave motion rescaling. Using this type of evolution equation for ocean wave motion and fluid flow research could be substantial.

The present study examines the following time-fractional nonlinear partial differential equations [30–33] :

1) The time-fractional non-linear generalized RLWE,

$$D_\kappa^\beta z + z_\mu + z z_\mu + z_{\mu\mu\kappa} = 0, \quad \kappa > 0, \quad \mu \in \mathbb{R}, \quad 0 < \beta \leq 1, \tag{1.1}$$

with the initial condition

$$z(\mu, 0) = 3a \sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2.$$

2) The time-fractional linear RLWE,

$$D_\kappa^\beta z + z_\mu = 2 z_{\mu\mu\kappa}, \quad \kappa > 0, \quad \mu \in \mathbb{R}, \quad 0 < \beta \leq 1, \tag{1.2}$$

where

$$z(\mu, 0) = e^{-\mu},$$

is defined the initial condition for equation (1.2).

3) The TFRLWE,

$$D_\kappa^\beta z + z_{\mu\mu\mu\mu} = 0, \quad \kappa > 0, \quad \mu \in \mathbb{R}, \quad 0 < \beta \leq 1, \tag{1.3}$$

where the initial condition for this equation is given as

$$z(\mu, 0) = \sin(\mu).$$

Here, the fractional derivative order β refers to the Caputo fractional derivative operator. Expresses the probability density function $z(\mu, \kappa)$ with the spatial variable μ and the temporal variable κ.

Until now, it has been mathematically modeled with RLW equations in order to solve and interpret many physical phenomena in many fields. Examples of some of these areas are dispersal waves in flexible wands and compression waves in liquid gas blister blends, ion-acoustic waves, flow moving in a certain area, magnetohydrodynamic waves in plasma. Also, this equation has been studied when describing some nonlinear wave propagation problems. For example, heat conduction, thermal radiation, mass diffusion, chemical reaction, viscosity, etc[30–33].

The RLW equations, which express the KdV equation, which is discussed in the analysis of the soliton solution, were first revealed by Peregrine. Peregrine introduced this equation to model and investigate solitons from small-amplitude long waves on water [33]. The study of RLW equations is of great importance in particle physics. Because the solutions of the RLW equation have secondary solitary waves. In other words, it can be said that these waves arise from the collision of two solitary waves consisting of sinusoidal solutions. By this property, these solutions are similar to collisions of granules that cause certain granules or radiations in definite domains of physics. This situation emphasizes the importance of the RLW equation once again.

Another considerable characteristic of the fractional RLW equations is that they mathematically type huge ocean waves named tsunamis. In this respect, the RLW models are alternating to the given KdV models for modeling oceanic phenomena [34,35].

The analytical solution of the (1.1) TFGRLWE for $\beta = 1$, is given below [30]

$$z(\mu, \kappa) = 3a \sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}(\mu - (1+a)\kappa)\right)^2. \tag{1.4}$$

The analytical solution of the (1.2) TFRLWE for $\beta = 1$, is given below [30]

$$z(\mu, \kappa) = e^{\kappa - \mu}. \tag{1.5}$$

The analytical solution of the (1.3) TFRLWE for $\beta = 1$, is given below [30]

$$z(\mu, \kappa) = e^{-\kappa} \sin(\mu). \tag{1.6}$$

The aim of this article is to obtain the solutions of the TFRLW and GRLW equations using the LS-RPSM and RPSM algorithms with the help of CFD operators. In the second part, necessary expressions for fractional calculations are given. Chapter 3 explains the LS-RPSM method. In chapter 4, the solutions of the discussed equations were investigated with the help of RPSM and LS-RPSM algorithms. These solutions are compared in Section 5. For this purpose, some physical examinations, comparisons and graphical representations are acquired for the TFRLWE Finally, the results are explained in Section 6.

2. Some expressions for fractional calculus theory

In this section, we give some necessary expressions for fractional computations [1–6].

Definition 2.1. Riemann–Liouville fractional integral operator with β. order is given as [1–6] $(\beta \geq 0)$

$$J^\beta g(\mu) = \frac{1}{\Gamma(\beta)} \int_0^\mu (\mu - \kappa)^{\beta - 1} g(\kappa) d\kappa, \quad \beta > 0, \quad \mu > 0, \tag{2.1}$$

$$J^0 g(\mu) = g(\mu).$$

Definition 2.2. Caputo-FD operator with β.order is defined as [1–6],

$$D^\beta g(\mu) = J^{\lambda - \beta} D^\lambda g(\mu) = \frac{1}{\Gamma(\lambda - \beta)} \int_0^\mu (\mu - \kappa)^{\lambda - \beta - 1} \frac{d^\lambda}{d\kappa^\lambda} g(\kappa) d\kappa, \quad \mu > 0, \quad \lambda - 1 < \beta \leq \lambda, \tag{2.2}$$

where D^λ refers to λ.order classic differential operator.

Definition 2.3. Caputo-FD operator with β.order for $G(\mu, \kappa)$ is given as below [1–6],

$$D_\kappa^\beta G(\mu, \kappa) = \frac{\partial^\beta G(\mu, \kappa)}{\partial \kappa^\beta} = \frac{1}{\Gamma(\lambda - \beta)} \int_0^\kappa (\mu - \kappa)^{\lambda - \beta - 1} \frac{\partial^\lambda G(\mu, \kappa)}{\partial \kappa^\lambda} d\kappa, \quad \lambda - 1 < \beta < \lambda,$$

$$D_\kappa^\lambda G(\mu, \kappa) = \frac{\partial^\lambda G(\mu, \kappa)}{\partial \kappa^\lambda}, \quad \lambda \in N. \tag{2.3}$$

Definition 2.4. The f-PS are defined at $\kappa = \kappa_0$ as below [24],

$$\sum_{\lambda=0}^{\infty} c_\lambda (\kappa - \kappa_0)^{\lambda\beta} = c_0 + c_1 (\kappa - \kappa_0)^\beta + c_2 (\kappa - \kappa_0)^{2\beta} + \ldots, \quad 0 \leq \lambda - 1 < \beta \leq \lambda, \quad \kappa \geq \kappa_0, \tag{2.4}$$

Definition 2.5. The f-PS for function $g(\mu)$ can be defined as below [24],

$$\sum_{\lambda=0}^{\infty} g_\lambda(\mu) (\kappa - \kappa_0)^{\lambda\beta} = g_0(\mu) + g_1(\mu)(\kappa - \kappa_0)^\beta + g_2(\mu)(\kappa - \kappa_0)^{2\beta} + \ldots, \quad 0 \leq \lambda - 1 < \beta \leq \lambda, \ \kappa \geq \kappa_0, \tag{2.5}$$

Theorem 2.1. *f-PS of $G(\mu,\kappa)$ at $\kappa=\kappa_0$ is defined as belows*

$$G(\mu,\kappa)=\sum_{\lambda=0}^{\infty} g_\lambda(\mu)(\kappa-\kappa_0)^{\lambda\beta}, \tag{2.6}$$
$$0\le\lambda-1<\beta\le\lambda,\quad \mu\in I,\quad \kappa_0\le\kappa<\kappa_0+R.$$

If $D_\kappa^{\lambda\beta}G(\mu,\kappa)$ is continuous on $I\times(\kappa_0,\kappa_0+R)$, coefficients $g_\lambda(\mu)$ are expressed as below

$$g_\lambda(\mu)=\frac{D_\kappa^{\lambda\beta}G(\mu,\kappa_0)}{\Gamma(\lambda\beta+1)},\qquad \lambda=\overline{0,\infty}$$

where $D_\kappa^{\lambda\beta}=\frac{\partial^{\lambda\beta}}{\partial\kappa^{\lambda\beta}}=\frac{\partial^{\beta}}{\partial\kappa^{\beta}}.\frac{\partial^{\beta}}{\partial\kappa^{\beta}}\dots\frac{\partial^{\beta}}{\partial\kappa^{\beta}}$ (λ-times) and $R=\min_{c\in I}R_c$. Where R_c is field of convergency for the f-PS $\sum_{\lambda=0}^{\infty}g_\lambda(c)(\kappa-\kappa_0)^{\lambda\beta}$. Where, $g(\mu)$ is analytic on $\mu>0$ [24].

Result 2.1. f-PS for fractional-order of $G(\mu,\kappa)$ extended as below ($\kappa=\kappa_0$),

$$G(\mu,\kappa)=\sum_{\lambda=0}^{\infty}\frac{D_\kappa^{\lambda\beta}G(\mu,\kappa_0)}{\Gamma(\lambda\beta+1)}(\kappa-\kappa_0)^{\lambda\beta}, \tag{2.7}$$
$$0\le\lambda-1<\beta\le\lambda,\ \mu\in I,\ \kappa_0\le\kappa<\kappa_0+R,$$

The above equation is named Generalized Taylor's series expansion. When $\beta=1$ in the above equation, it becomes the known TSE. We can remember the TSE as follows [24],

$$G(\mu,\kappa)=\sum_{\lambda=0}^{\infty}\frac{\partial^\lambda G(\mu,\kappa_0)}{\partial\kappa^\lambda}\frac{(\kappa-\kappa_0)^\lambda}{\lambda!},\quad \mu\in I,\ \kappa_0\le\kappa<\kappa_0+R.$$

Definition 2.6. Suppose $\Upsilon_0,\ \Upsilon_1,\dots,\ \Upsilon_\lambda$ be functions of variables μ and κ. (μ,κ) defined on domain Θ. Fractional partial Wronskian of $\Upsilon_0,\ \Upsilon_1,\dots,\ \Upsilon_\lambda$ can be explained as in the following determinate

$$W^\beta[\Upsilon_0,\Upsilon_1,\dots,\Upsilon_\lambda]=\begin{vmatrix}\Upsilon_0 & \Upsilon_1 & \Upsilon_2 & \dots & \Upsilon_\lambda\\ D^\beta\Upsilon_0 & D^\beta\Upsilon_1 & D^\beta\Upsilon_2 & \dots & D^\beta\Upsilon_\lambda\\ D^{2\beta}\Upsilon_0 & D^{2\beta}\Upsilon_1 & D^{2\beta}\Upsilon_2 & \dots & D^{2\beta}\Upsilon_\lambda\\ & & \vdots & & \\ D^{(\lambda-1)\beta}\Upsilon_0 & D^{(\lambda-1)\beta}\Upsilon_1 & D^{(\lambda-1)\beta}\Upsilon_2 & \dots & D^{(\lambda-1)\beta}\Upsilon_\lambda\end{vmatrix}, \tag{2.8}$$

where $D^\beta\Upsilon_j=(\frac{\partial}{\partial\mu}+\frac{\partial^\beta}{\partial\kappa^\beta})(\Upsilon_j)$ and with $0<\beta\le1$ and $j=0,1,2,3,\dots,\lambda$, $D^{\lambda\beta}=D^\beta.\ D^\beta\dots D^\beta(\lambda-times)$ [27].

3. Disclosure of the LS-RPSM

In this section, the application process of the LS-RPSM algorithm is shown and the solutions of some time-fractional differential equations are obtained with this algorithm.

Let the time-fractional differential equation be defined as:

$$L^\beta[\breve{Z}(\mu,\kappa)]+N[\breve{Z}(\mu,\kappa)]=0,\quad \kappa>0,\ 0<\beta\le1, \tag{3.1}$$

Consider the function $\breve{Res}$ for the (3.1) differential equation as

$$\breve{Res}(\mu,\kappa,\breve{Z})=L^\beta[\breve{Z}(\mu,\kappa)]+N[\breve{Z}(\mu,\kappa)],\quad \mu,\kappa\in\mathbb{R} \tag{3.2}$$

where

$$\breve{Z}(\mu,\kappa)=0,$$

condition is satisfied.

∗ If

$$\lim_{j\to\infty}\breve{Res}(\mu,\kappa,u_j^\beta(\mu,\kappa))=0, \tag{3.3}$$

with $j\in\mathbb{N}$, $u_j^\beta(\mu,\kappa)$ is considered to be the solution of Eq. (3.1).

∗ When

$$\left|\breve{Res}(\mu,\kappa,\ \breve{Z})\right|<\rho,$$

$\breve{Z}$ gives the ρ-numerical RPSM solution for Eq. (3.1) on area Θ.

∗ When

$$\int\int_\Theta \breve{Res}^2(\mu,\kappa,\ \breve{Z})d\mu d\kappa\le\rho.$$

$\breve{Z}$ gives the weak ρ-numerical RPSM solution of Eq. (3.1) on area Θ.

i) To implement the known RPSM, suppose the $z_\lambda(\mu,\kappa)$ solution is written as follows

$$z_\lambda(\mu,\kappa)=\sum_{j=0}^{\lambda}f_j(\mu)\frac{\kappa^{j\alpha}}{\Gamma(1+j\alpha)}, \tag{3.4}$$

and where the λ.th residual function is given by:

$$Res_\lambda(\mu,\kappa)=L^\beta[z_\lambda(\mu,\kappa)]+N[z_\lambda(\mu,\kappa)]$$

Where

$$D_\kappa^{(j-1)\beta}Res_\lambda(\mu,0)=0,\qquad \lambda\in N^+,$$

and we appear for the solution of $f_j(\mu)$.

Then $z_\lambda=\Upsilon_0+\Upsilon_1+\Upsilon_2+\dots+\Upsilon_\lambda$ stands for λ.th-order numerical solutions.

ii) The linearly independent functions expressed by $W^\beta[\Upsilon_0,\Upsilon_1,\dots,\Upsilon_\lambda]\neq0$. With $j=0,1,2,\dots$, let $S_j=\{\Upsilon_0,\ \Upsilon_1,\dots,\ \Upsilon_\lambda\}$, where the factors of S_j are linearly independent in the vector space of continuous functions defined on $\mathbb{R}$.

iii) Let the serial solution of Eq. (3.1) be as follows

$$\breve{Z}_j=\sum_{j=0}^{\lambda}k_j^\lambda\Upsilon_r, \tag{3.5}$$

By using the ns $\breve{Z}_j$in (3.2), we can write as follows

$$\breve{Res}(\mu,\kappa,k_j^\lambda)=\breve{Res}(\mu,\kappa,\breve{Z}_j). \tag{3.6}$$

iv) Then we expressed as follows:

$$\min J=\int\int_\Theta(\breve{Res}^2(\mu,\kappa,k_j^\lambda))d\mu d\kappa, \tag{3.7}$$

Where, we can get the $k_j^\lambda(k_j^{g+1},k_j^{g+2},\dots,k_j^\lambda)$ elements.

We can get the elements of $\breve{k}_j^{g+1},\breve{k}_j^{g+2},\dots,\breve{k}_j^{\lambda}$ as the rates which yield the minimum of (3.7) and the elements of $\breve{k}_j^0,\breve{k}_j^1,\dots,\breve{k}_j^\lambda$ again as functions of $\breve{k}_j^{g+1},\breve{k}_j^{g+2},\dots,\breve{k}_j^\lambda$ with the help of the initial conditions.

Then, we calculate the $s_j^\beta(\mu,\kappa)$ by solving (3.7):

$$s_j^\beta(\mu,\kappa)=\sum_{j=0}^{\lambda}\breve{k}_j^\lambda\Upsilon_r. \tag{3.8}$$

With the help of (3.5)–(3.8), we obtain as below

$$\breve{Res}^2(\mu,\kappa,s_j^\beta(\mu,\kappa))\le\breve{Res}^2(\mu,\kappa,\breve{Z}_j(\mu,\kappa)). \tag{3.9}$$

Theorem 3. Considering (3.8), the expressions $s_j^\beta(\mu,\kappa)$ provide the following characteristic:

$$\lim_{j\to\infty}\int\int_\Theta\breve{Res}^2(\mu,\kappa,s_j^\beta(\mu,\kappa))d\mu d\kappa=0, \tag{3.10}$$

You can look at [29] for proof.

4. Applications on fractional RLW equations

4.1. The solutions obtained using RPSM

4.1.1. Fractional Eq. (1.1) GRLW

Let's think of the fractional GRLW Eq. (1.1) with the help of the initial condition

$$z(\mu,0)=3a\sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2. \quad (4.1)$$

Here we can write the residual function as follows

$$Res(\mu,\kappa)=\frac{\partial^\beta z(\mu,\kappa)}{\partial\kappa^\beta}+\frac{\partial z(\mu,\kappa)}{\partial\mu}+z(\mu,\kappa)\frac{\partial z(\mu,\kappa)}{\partial\mu}+\frac{\partial^3 z(\mu,\kappa)}{\partial\mu^2\partial\kappa}.$$

If we apply the RPSM [16,23–25], we get the RPSM approximate solution of order 2 as,

$$\begin{aligned} z_2(\mu,\kappa) &= 3a\sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2 \\ &+\frac{3}{2}a\sqrt{\frac{a}{1+a}}\left(1+6a+\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right)\right)\sec h(\sqrt{\frac{a}{1+a}}\mu)^4 \\ &\tanh\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)\frac{\kappa^\beta}{\Gamma(1+\beta)} \\ &+\frac{1}{64(1+a)}3a^2(-8-96a-576a^2+(-9-48a+432a^2) \\ &\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right)+48a\cosh(2\sqrt{\frac{a}{1+a}}\mu) \\ &+\cosh(3\sqrt{\frac{a}{1+a}}\mu))\sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^8\frac{\kappa^{2\alpha}}{\Gamma(1+2\alpha)}. \end{aligned} \quad (4.2)$$

4.1.2. Fractional Eq. (1.2) RLW

Let's think of the fractional RLW Eq. (1.2) with the help of the initial condition

$$z(\mu,0)=e^{-\mu}. \quad (4.3)$$

Here we can write the residual function as follows

$$Res(\mu,\kappa)=\frac{\partial^\beta z(\mu,\kappa)}{\partial\kappa^\beta}+\frac{\partial z(\mu,\kappa)}{\partial\mu}-2\frac{\partial^3 z(\mu,\kappa)}{\partial\mu^2\partial\kappa}.$$

If we apply the RPSM, we get the RPSM approximate solution of order 2 below,

$$z_2(\mu,\kappa)=e^{-\mu}+e^{-\mu}\frac{\kappa^\beta}{\Gamma(1+\beta)}+e^{-\mu}\frac{\kappa^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (4.4)$$

4.1.3. Fractional Eq. (1.3) RLW

Let's think of the fractional RLW Eq. (1.3) with the help of the initial condition,

$$z(\mu,0)=\sin(\mu). \quad (4.5)$$

Here we can write the residual function as follows

$$Res(\mu,\kappa)=\frac{\partial^\beta z(\mu,\kappa)}{\partial\kappa^\beta}+\frac{\partial^4 z(\mu,\kappa)}{\partial\mu^4}.$$

If we apply the RPSM, we get the RPSM approximate solution of order 2 as,

$$z_2(\mu,\kappa)=\sin(\mu)-\sin(\mu)\frac{\kappa^\beta}{\Gamma(1+\beta)}+\sin(\mu)\frac{\kappa^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (4.6)$$

4.2. The solutions obtained using LS-RPSM

4.2.1. Fractional Eq. (1.1) GRLW

The second order approximate solution of fractional Eq. (1.1) GRLW is obtained from eq. (4.2) of the previous section as $z_2=\Upsilon_0+\Upsilon_1+\Upsilon_2$.

$$\begin{aligned} \Upsilon_0(\mu,\kappa) &= 3a\sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2, \\ \Upsilon_1(\mu,\kappa) &= \frac{3}{2}a\sqrt{\frac{a}{1+a}}\left(1+6a+\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right)\right) \\ &\quad \sec h\left(\sqrt{\frac{a}{1+a}}\mu\right)^4\tanh\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)\frac{\kappa^\beta}{\Gamma(1+\beta)}, \\ \Upsilon_2(\mu,\kappa) &= \frac{1}{64(1+a)}3a^2\Bigg(-8-96a-576a^2 \\ &\quad +(-9-48a+432a^2)\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right) \\ &\quad +48a\cosh\left(2\sqrt{\frac{a}{1+a}}\mu\right)+\cosh\left(3\sqrt{\frac{a}{1+a}}\mu\right)\Bigg) \\ &\quad \sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^8\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}. \end{aligned} \quad (4.7)$$

When $\beta=1,\mu=1,\kappa=0.5,\ W^1[\Upsilon_0,\ \Upsilon_1,\ \Upsilon_2]\neq 0$. Hence, the functions Υ_0, Υ_1 and Υ_2 are linearly independent.

Using these obtained values, we can express the approximate solution as follows,

$$\begin{aligned} \breve{Z}(\mu,\kappa) &= 3a\sec h\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2 \\ &+k_1\Bigg(\frac{3}{2}a\sqrt{\frac{a}{1+a}}(1+6a+\cosh(\sqrt{\frac{a}{1+a}}\mu)) \\ &\sec h\left(\sqrt{\frac{a}{1+a}}\mu\right)^4\tanh\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)\frac{\kappa^\beta}{\Gamma(1+\beta)}\Bigg) \\ &+k_2\Bigg(\frac{1}{64(1+a)}3a^2\Bigg(-8-96a-576a^2 \\ &+(-9-48a+432a^2)\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right) \\ &+48a\cosh(2\sqrt{\frac{a}{1+a}}\mu) \\ &+\cosh\left(3\sqrt{\frac{a}{1+a}}\mu\right)\Bigg)\sec h(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu)^8\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}\Bigg). \end{aligned} \quad (4.8)$$

From (4.1), we can get the residual function

$$\breve{Res}(\mu,\kappa,\breve{Z})=\frac{\partial^\beta\breve{Z}(\mu,\kappa)}{\partial\kappa^\beta}+\frac{\partial\breve{Z}(\mu,\kappa)}{\partial\mu}+\breve{Z}(\mu,\kappa)\frac{\partial\breve{Z}(\mu,\kappa)}{\partial\mu}+\frac{\partial^3\breve{Z}(\mu,\kappa)}{\partial\mu^2\partial\kappa}. \quad (4.9)$$

Where, the J expression provides the following property

$$J(k_1,k_2)=\int\int_\Theta \breve{Res}^2(\mu,\kappa,k_1,k_2)d\mu d\kappa. \quad (4.10)$$

Using the J function expressed by (4.10), we can form the following system of equations

$$\frac{\partial J}{\partial k_1}=0, \quad (4.11)$$

$$\frac{\partial J}{\partial k_2} = 0,$$

by using $\beta = 1$, the unknown k_1 and k_2 coefficients are calculated from the systems of Eqs. (4.11)

$$k_1 = 1.4635595488831894, \tag{4.12}$$
$$k_2 = 2.5364949762652302.$$

Using these obtained values, we can express the approximate solution as follows,

$$\begin{aligned}\breve{Z}(\mu,\kappa) = {} & 3a\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^2 \\ & + 1.4635595488831894\left(\frac{3}{2}a\sqrt{\frac{a}{1+a}}(1+6a+\cosh(\sqrt{\frac{a}{1+a}}\mu))\right. \\ & \left.\mathrm{sech}\left(\sqrt{\frac{a}{1+a}}\mu\right)^4 \tanh\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)\frac{\kappa^\beta}{\Gamma(1+\beta)}\right) \\ & + 2.5364949762652302\left(\frac{1}{64(1+a)}\right. \\ & 3a^2(-8-96a-576a^2+(-9-48a+432a^2)\cosh\left(\sqrt{\frac{a}{1+a}}\mu\right) \\ & + 48a\cosh\left(2\sqrt{\frac{a}{1+a}}\mu\right) \\ & \left.+\cosh(3\sqrt{\frac{a}{1+a}}\mu))\,\mathrm{sech}\left(\frac{1}{2}\sqrt{\frac{a}{1+a}}\mu\right)^8 \frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}\right).\end{aligned} \tag{4.13}$$

4.2.2. Fractional Eq. (1.2) RLW

The second order approximate solution of fractional Eq. (1.2) RLW is obtained from eq. (4.4) of the previous section as $z_2 = \Upsilon_0 + \Upsilon_1 + \Upsilon_2$.

$$\Upsilon_0(\mu,\kappa) = e^{-\mu},$$
$$\Upsilon_1(\mu,\kappa) = e^{-\mu}\frac{\kappa^\beta}{\Gamma(1+\beta)}, \tag{4.14}$$
$$\Upsilon_2(\mu,\kappa) = e^{-\mu}\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}.$$

When $\beta = 1, \mu = 0.5, \kappa = 0.5$, $W^1[\Upsilon_0,\ \Upsilon_1, \Upsilon_2] \neq 0$. Hence, the functions Υ_0, Υ_1 and Υ_2 are linearly independent.

Using these obtained values, we can express the approximate solution as follows,

$$\breve{Z}(\mu,\kappa) = e^{-\mu} + e^{-\mu}k_1\frac{\kappa^\beta}{\Gamma(1+\beta)} + e^{-\mu}k_2\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}. \tag{4.15}$$

From (4.3), we can get the residual function

$$\breve{Res}(\mu,\kappa,\breve{Z}) = \frac{\partial^\beta \breve{Z}(\mu,\kappa)}{\partial\kappa^\beta} + \frac{\partial \breve{Z}(\mu,\kappa)}{\partial\mu} - 2\frac{\partial^3 \breve{Z}(\mu,\kappa)}{\partial\mu^2\partial\kappa}. \tag{4.16}$$

Where, the J expression provides the following property

$$J(k_1,k_2) = \iint_\Theta \breve{Res}^2(\mu,\kappa,k_1,k_2)\,d\mu\,d\kappa. \tag{4.17}$$

Using the J function expressed by (4.17), we can form the following system of equations

$$\frac{\partial J}{\partial k_1} = 0, \tag{4.18}$$
$$\frac{\partial J}{\partial k_2} = 0,$$

by using $\beta = 1$, the unknown k_1 and k_2 coefficients are calculated from the systems of Eqs. (4.18)

$$k_1 = -0.9427168576104746, \tag{4.19}$$
$$k_2 = 0.6219312602291326.$$

Using these obtained values, we can express the approximate solution as follows,

$$\begin{aligned}\breve{Z}(\mu,\kappa) = {} & e^{-\mu} - 0.9427168576104746e^{-\mu}\frac{\kappa^\beta}{\Gamma(1+\beta)} \\ & +0.6219312602291326e^{-\mu}\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}.\end{aligned} \tag{4.20}$$

4.2.3. Fractional Eq. (1.3) RLW

The second order approximate solution of fractional Eq. (1.3) RLW is obtained from eq. (4.6) of the previous section as $z_2 = \Upsilon_0 + \Upsilon_1 + \Upsilon_2$.

$$\Upsilon_0(\mu,\kappa) = \sin(\mu),$$
$$\Upsilon_1(\mu,\kappa) = -\sin(\mu)\frac{\kappa^\beta}{\Gamma(1+\beta)}, \tag{4.21}$$
$$\Upsilon_2(\mu,\kappa) = \sin(\mu)\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}.$$

When $\beta = 1, \mu = 0.5, \kappa = 0.5$, $W^1[\Upsilon_0,\ \Upsilon_1, \Upsilon_2] \neq 0$. Hence, the functions Υ_0, Υ_1 and Υ_2 are linearly independent.

Using these obtained values, we can express the approximate solution as follows,

$$\breve{Z}(\mu,\kappa) = \sin(\mu) - \sin(\mu)k_1\frac{\kappa^\beta}{\Gamma(1+\beta)} + \sin(\mu)k_2\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}. \tag{4.22}$$

From (4.5), we can get the residual function

$$\breve{Res}(\mu,\kappa,\breve{Z}) = \frac{\partial^\beta \breve{Z}(\mu,\kappa)}{\partial\kappa^\beta} + \frac{\partial^4 \breve{Z}(\mu,\kappa)}{\partial\mu^4}. \tag{4.23}$$

Where, the J expression provides the following property

$$J(k_1,k_2) = \iint_\Theta \breve{Res}^2(\mu,\kappa,k_1,k_2)\,d\mu\,d\kappa. \tag{4.24}$$

Using the J function expressed by (4.24), we can form the following system of equations

$$\frac{\partial J}{\partial k_1} = 0, \tag{4.25}$$
$$\frac{\partial J}{\partial k_2} = 0,$$

by using $\beta = 1$, the unknown k_1 and k_2 coefficients are calculated from the systems of Eqs. (4.25)

$$k_1 = 0.9427168576104746, \tag{4.26}$$
$$k_2 = 0.6219312602291326.$$

Using these obtained values, we can express the approximate solution as follows,

$$\begin{aligned}\breve{Z}(\mu,\kappa) = {} & \sin(\mu) - 0.9427168576104746\sin(\mu)\frac{\kappa^\beta}{\Gamma(1+\beta)} \\ & +0.6219312602291326\sin(\mu)\frac{\kappa^{2\beta}}{\Gamma(1+2\beta)}.\end{aligned} \tag{4.27}$$

5. Physical reviews

In this section, some graphs and tables are drawn to interpret some properties of the numerical solutions obtained in the previous section. In addition, it was expressed with these physical expressions how the change of fractional order affects the solutions.

In Tables 1 and 2, we expressed the absolute error between different values of μ and κ when $\beta = 1, a = 0.02$, $0.1 \leq \kappa \leq 0.5$, $0.1 \leq \mu \leq 0.5$.

Table 1
Absolute error between the exact solution (es) and numerical solution (ns) (RPSM) of Eq. (1.1) for different values of μ and κ. ($\beta = 1, a = 0.02$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	1.51493×10^{-8}	6.01185×10^{-8}	1.34906×10^{-7}	2.39509×10^{-7}	3.73922×10^{-7}
0.2	1.53883×10^{-8}	6.05972×10^{-8}	1.35628×10^{-7}	2.40478×10^{-7}	3.75147×10^{-7}
0.3	1.56267×10^{-8}	6.10734×10^{-8}	1.36344×10^{-7}	2.41438×10^{-7}	3.76358×10^{-7}
0.4	1.58644×10^{-8}	6.15471×10^{-8}	1.37054×10^{-7}	2.42389×10^{-7}	3.77553×10^{-7}
0.5	1.61015×10^{-8}	6.20183×10^{-8}	1.37759×10^{-7}	2.43329×10^{-7}	3.78733×10^{-7}

Table 2
Absolute error between the es and ns (LS-RPSM) of Eq. (1.1) for different values of μ and κ. ($\beta = 1, a = 0.02$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	1.97959×10^{-8}	2.28561×10^{-8}	9.1795×10^{-9}	2.12376×10^{-8}	6.84013×10^{-8}
0.2	4.79596×10^{-8}	7.9187×10^{-8}	9.36834×10^{-8}	9.14475×10^{-8}	7.24758×10^{-8}
0.3	7.61214×10^{-8}	1.35515×10^{-7}	1.78184×10^{-7}	2.04129×10^{-7}	2.1335×10^{-7}
0.4	1.0428×10^{-7}	1.91837×10^{-7}	2.62677×10^{-7}	3.16802×10^{-7}	3.54216×10^{-7}
0.5	1.32435×10^{-7}	2.48152×10^{-7}	3.47159×10^{-7}	4.29463×10^{-7}	4.95068×10^{-7}

Table 3
Comparison of es, LS-RPSM, RPSM and HPSTM [28] solutions at $\beta = 1$, $a = 0.02$ and $\kappa = 0.5$ of Eq. (1.1).

μ	*Exact solution*	*RPSM solution*	*LS − RPSM solution*	*HPSTM solution*
0.1	0.00599952	0.00599954	0.00599965	0.00599954
0.2	0.00599973	0.00599979	0.00599998	0.00599979
0.3	0.00599988	0.00600002	0.00600023	0.00600002
0.4	0.00599997	0.00600021	0.0060004	0.00600021
0.5	0.006	0.00600038	0.0060005	0.00600038
0.6	0.00599997	0.00600051	0.00600051	0.00600051
0.7	0.00599988	0.00600062	0.00600045	0.00600062
0.8	0.00599973	0.00600069	0.00600032	0.00600069
0.9	0.00599952	0.00600074	0.00600011	0.00600074
1	0.00599925	0.00600075	0.00599982	0.00600075

Table 4
Absolute errors by the es and ns (RPSM) of Eq. (1.2) for different values of μ and κ. ($\beta = 1$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	0.000154653	0.00126927	0.00439643	0.0106994	0.0214639
0.2	0.000139936	0.00114848	0.00397806	0.00968124	0.0194213
0.3	0.000126619	0.00103919	0.00359949	0.00875995	0.0175731
0.4	0.00011457	0.000940297	0.00325696	0.00792633	0.0159008
0.5	0.000103667	0.000850816	0.00294702	0.00717204	0.0143877

Table 5
Absolute errors by the es and ns (LS-RPSM) of Eq. (1.2) for different values of μ and κ. ($\beta = 1$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	0.177649	0.35968	0.547143	0.741204	0.943147
0.2	0.160744	0.325452	0.495076	0.670669	0.853394
0.3	0.145447	0.294481	0.447963	0.606846	0.772183
0.4	0.131606	0.266457	0.405334	0.549097	0.6987
0.5	0.119082	0.2411	0.366761	0.496844	0.63221

It can be seen from the obtained solutions that better solutions are acquired as β gets closer to 1.

We compared the es, LS-RPSM, RPSM and the HPSTM solutions in Table 3. Where μ is in the range $0.1 \le \mu \le 1$ when $\beta = 1, a = 0.02$ and $\kappa = 0.5$. Where, values written for HPSTM solution have been acquired by reference [30].

In Tables 4 and 5, we expressed the absolute error between different values of μ and κ when $\beta = 1,\ 0.1 \le \kappa \le 0.5,\ 0.1 \le \mu \le 0.5$.

In Figs. 2, 4 and 6, 2D graphs are drawn for the approximate solution of Eq. (1.2). It can be seen from the obtained solutions that better solutions are acquired as β gets closer to 1.

Table 6
Comparison of the es, LS-RPSM, RPSM and HPSTM [28] solutions at $\beta = 1$ and $\kappa = 0.5$ of Eq. (1.2).

μ	*Exact solution*	*RPSM solution*	*LS − RPSM solution*	*HPSTM solution*
0.1	0.67032	0.670216	0.551238	0.670216
0.2	0.740818	0.739967	0.499718	0.739967
0.3	0.818731	0.815784	0.45197	0.815784
0.4	0.904837	0.897665	0.407994	0.897665
0.5	1	0.985612	0.36779	0.985612
0.6	1.10517	1.07962	0.331358	1.07962
0.7	1.2214	1.1797	0.298699	1.1797
0.8	1.34986	1.28584	0.269812	1.28584
0.9	1.49182	1.39805	0.244697	1.39805
1	1.64872	1.51633	0.223354	1.51633

We compared the es, LS-RPSM, RPSM and the HPSTM solutions in Table 5. We used the values $0.1 \le \mu \le 1$ when $\beta = 1$ and $\kappa = 0.5$. Where, values written for HPSTM solution have been acquired by reference [30].

In Table 6 and 7, we showed the absolute error between different values of μ and κ when $\beta = 1,\ 0.1 \le \kappa \le 0.5,\ 0.1 \le \mu \le 0.5$.

In Figs. 1, 3, and 5, we draw 3-dimensional graphics for the numerical solutions of the (1.3) equation. It can be seen from the obtained solutions that better solutions are obtained as β gets closer to 1.

In Table 8, we compared the es, LS-RPSM, RPSM and the HPSTM solutions. We used the values $0.1 \le \mu \le 1$ when $\beta = 1$ and $\kappa = 0.5$. Where, values written for HPSTM solution have been obtained by reference [30].

6. Final remarks

In this article, approximate solutions of nonlinear TFGRLW and RLWEs was obtained using a new method, LS-RPSM, and known RPSM. Here, it was shown that using LS-RPSM is more advantageous than RPSM at certain intervals. One of these advantages is to reach convergent solutions by spending less processing and time. This method is more useful because it requires using fewer terms than the required number of terms when applying the RPSM. Also, the examples described here show that this method converges better than RPSM. This method is effective and reliable in obtaining solutions for the time-fractional differential equations. The obtained solutions can be used to investigate the hydrodynamics of wide channels or open seas of finite depth, the propagation of shallow-water waves with various dispersion relationships, the travel of shallow-water waves, and the propagation of waves in

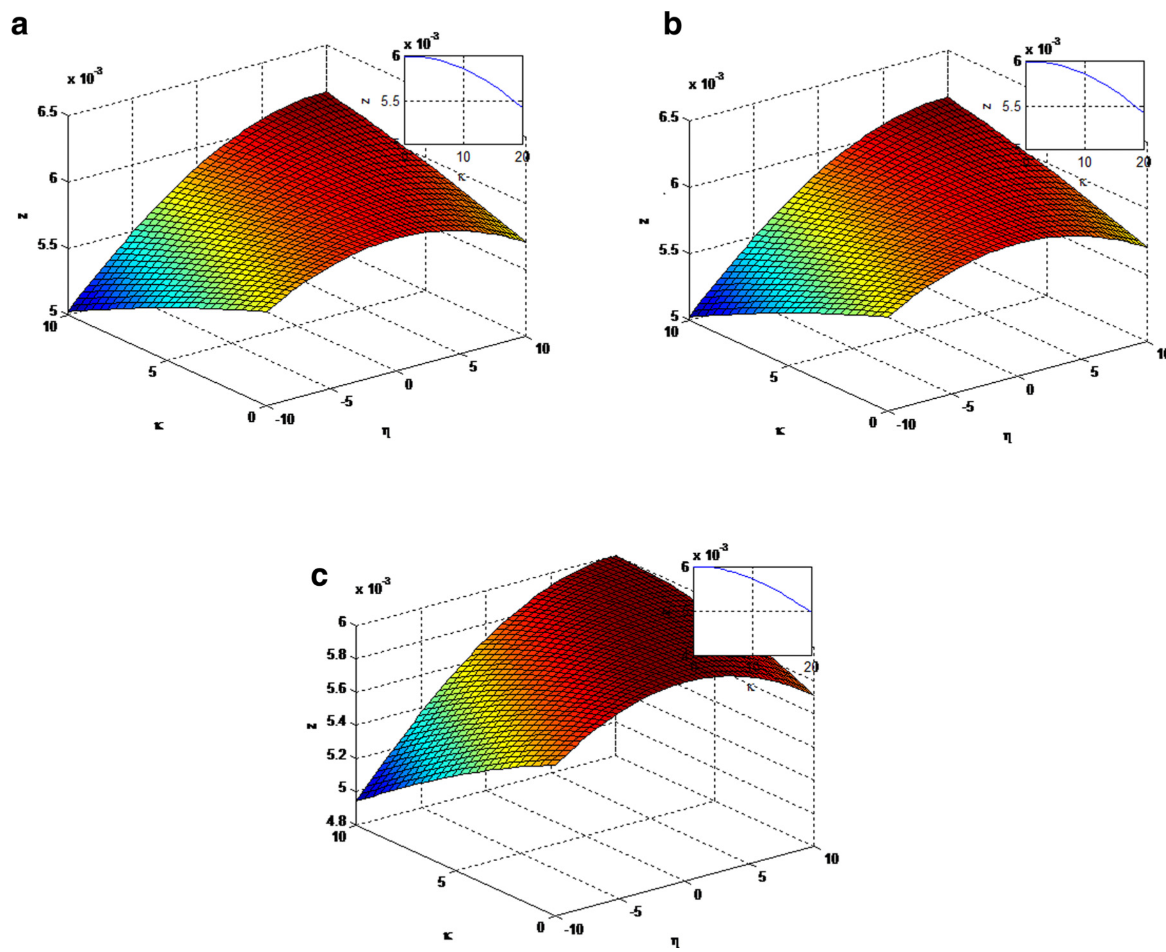

Fig. 1. 3D graphics for the ns with the help of RPSM, LS-RPSM and the es of Eq. (1.1) **a)** (4.2) z numerical solution **B)** (4.13) $\breve{z}$ numerical solution **c)** The exact solution ($\beta = 1, a = 0.02$).

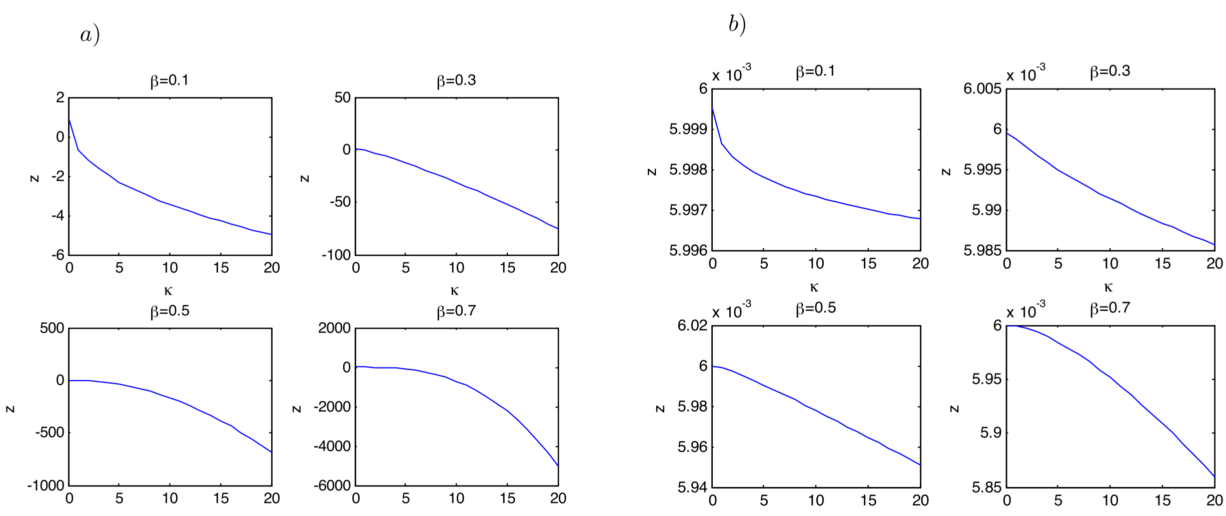

Fig. 2. The 2D graphics of the $\breve{z}$ ns with the help of ns of Eq. (1.1) for different values of β. **a)** RPSM , **b)** LS-RPSM ($\mu = 0.4$).

Table 7
Absolute errors by the es and ns (RPSM) of Eq. (1.3) for different values of μ and κ. ($\beta = 1$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	0.0000162311	0.000126713	0.000417481	0.000966383	0.00184386
0.2	0.0000323	0.00025216	0.000830791	0.00192311	0.00366929
0.3	0.0000480463	0.000375088	0.0012358	0.00286062	0.00545806
0.4	0.0000633124	0.000494268	0.00162846	0.00376955	0.0071923
0.5	0.0000779459	0.000608509	0.00200485	0.00464082	0.00885467

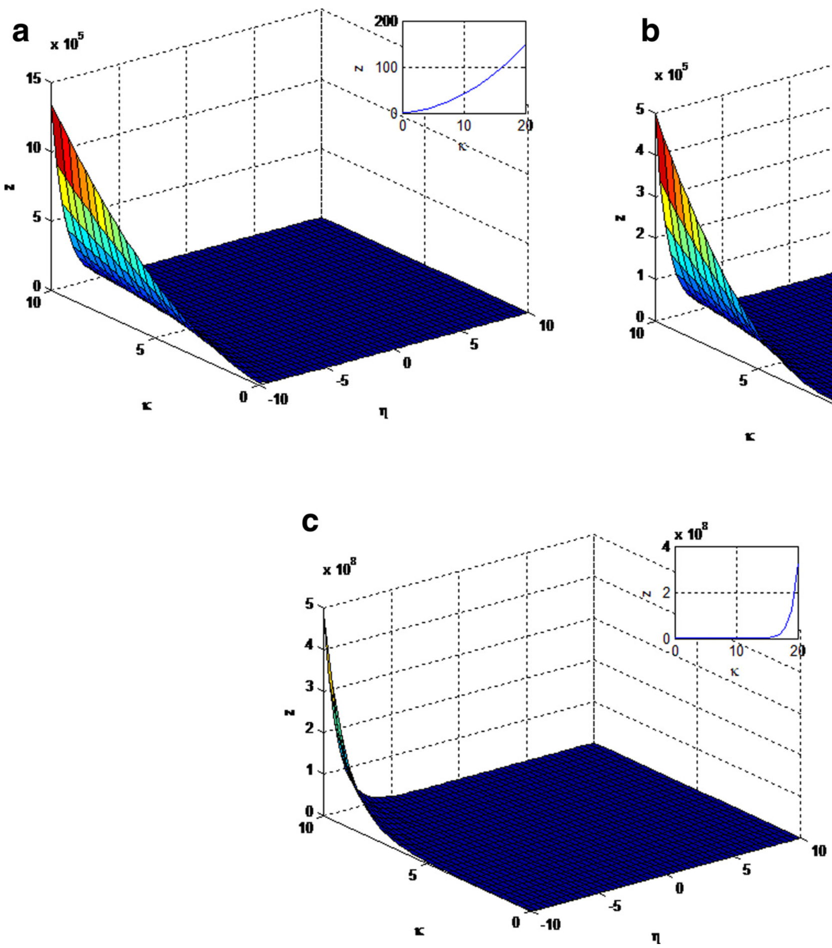

Fig. 3. 3D graphics for the ns with the help of LS-RPSM, RPSM and the es of Eq. (1.2) **a)** (4.4) z numerical solution **b)** (4.20) $\breve{Z}$ numerical solution **c)** The exact solution ($\beta = 1$).

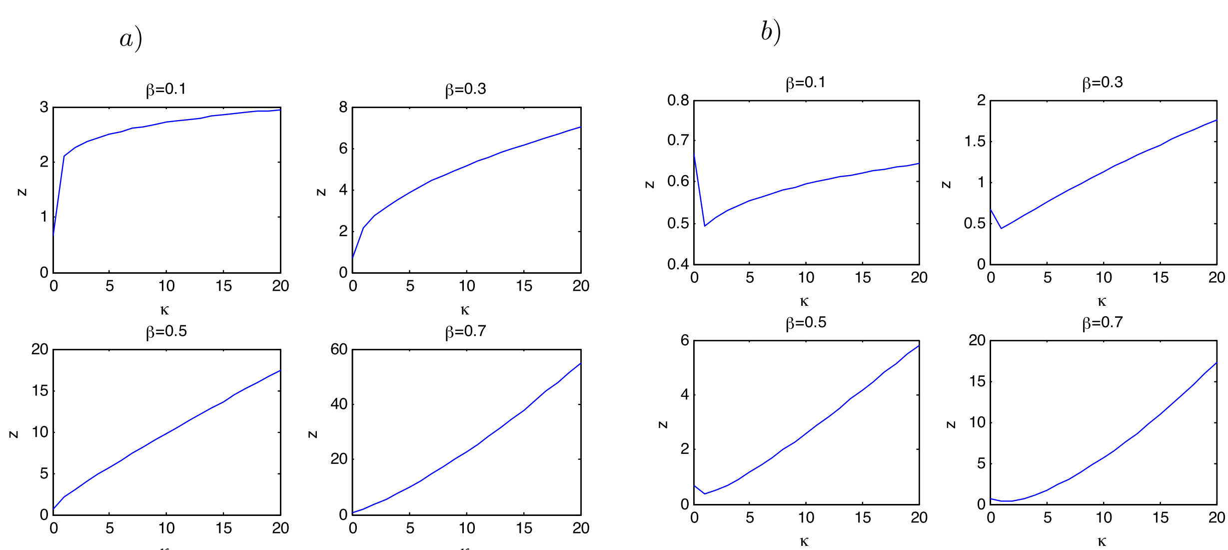

Fig. 4. The 2D graphics of the $\breve{Z}$ numerical solution with the help of RPSM and LS-RPSM for different values of β. **a)** RPSM, **b)** LS-RPSM ($\mu = 0.4$).

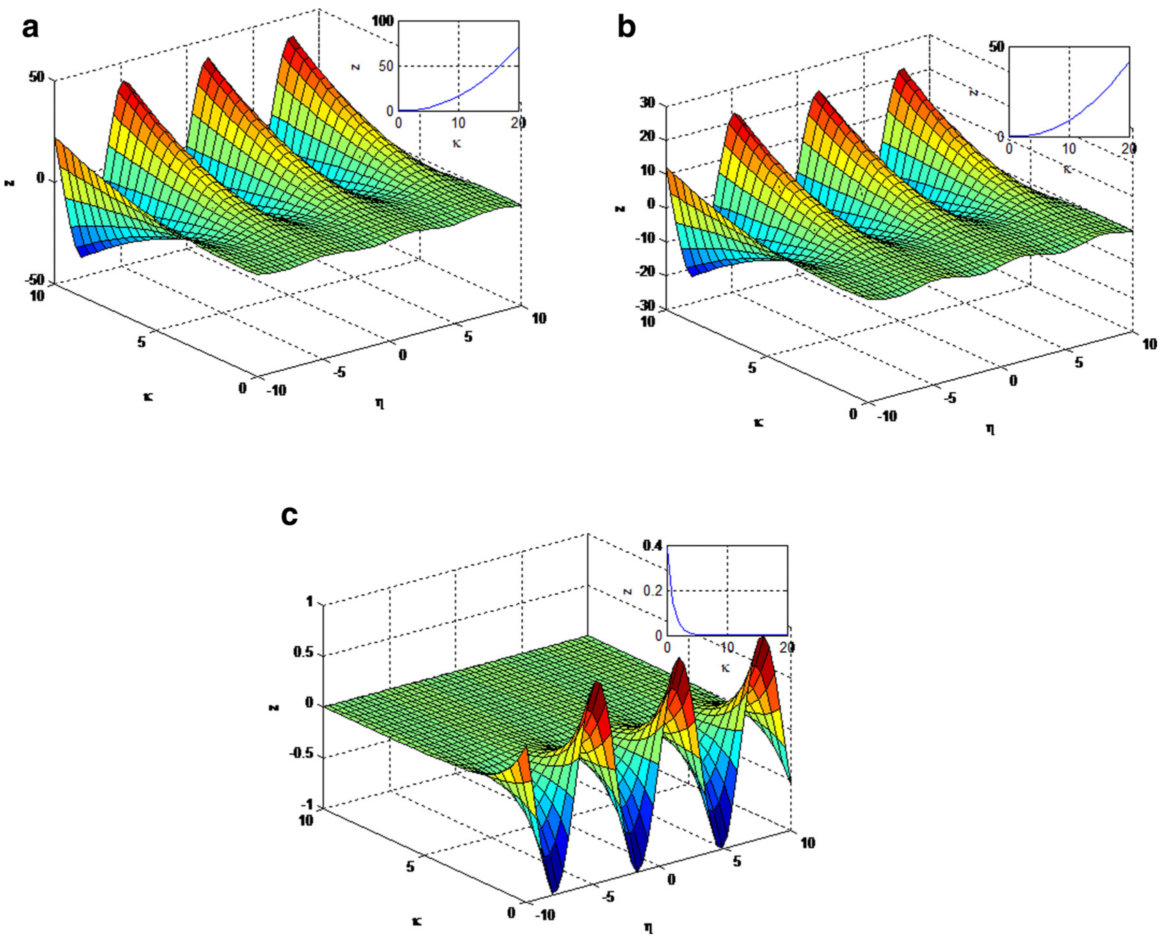

Fig. 5. 3D graphics for the ns with the help of RPSM, LS-RPSM and the exact solution of (1.3) equation **a)** (4.6) z numerical solution **b)** (4.27) $\breve{Z}$ numerical solution **c)** The exact solution ($\beta = 1$).

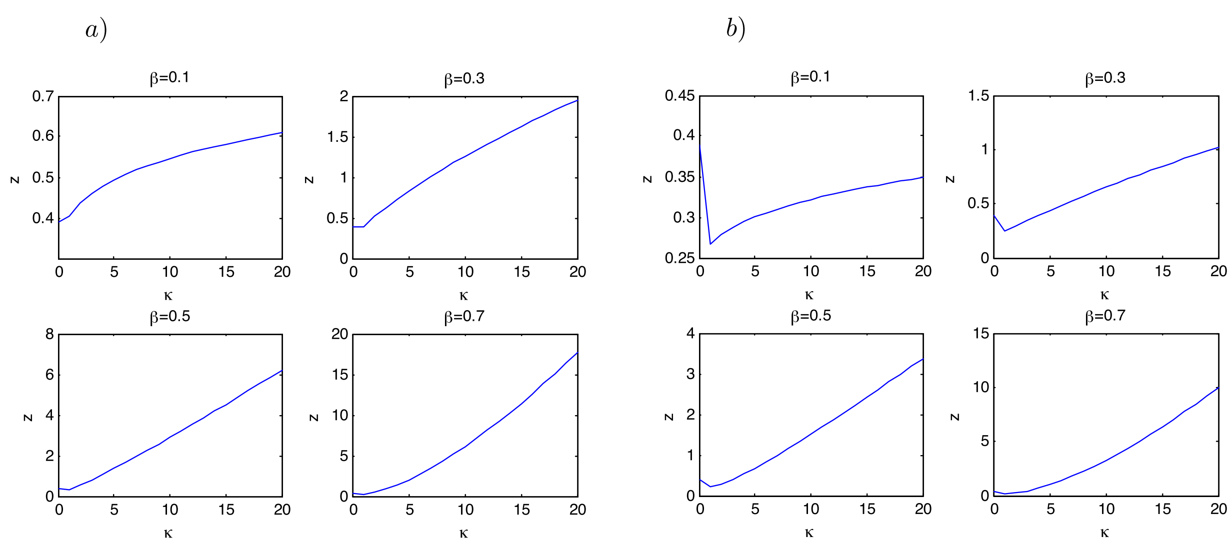

Fig. 6. The 2D graphics of the $\breve{Z}$ ns with the help of RPSM and LS-RPSM for different value of β. **a)** RPSM , **b)** LS-RPSM ($\kappa = 0.4$).

Table 8
Absolute errors by the es and ns (LS-RPSM) of Eq. (1.3) for different values of μ and κ. ($\beta = 1$).

$\mu \setminus \kappa$	0.1	0.2	0.3	0.4	0.5
0.1	0.000399389	0.00051559	0.000434638	0.00023438	0.0000147435
0.2	0.000794787	0.00102603	0.000864933	0.000466418	0.0000293397
0.3	0.00118224	0.00152621	0.00128659	0.000693796	0.0000436427
0.4	0.00155789	0.00201115	0.00169538	0.000914242	0.0000575097
0.5	0.00191797	0.00247599	0.00208724	0.00112555	0.000070802

Table 9
Comparison of the es, LS-RPSM, RPSM and HPSTM [28] solutions at $\beta = 1$ and $\kappa = 0.5$ of Eq. (1.3).

μ	Exact solution	RPSM solution	LS − RPSM solution	HPSTM solution
0.1	0.433802	0.43388	0.43572	0.43388
0.2	0.39252	0.393129	0.394996	0.393129
0.3	0.355167	0.357172	0.357254	0.357172
0.4	0.321369	0.326009	0.322494	0.326009
0.5	0.290786	0.299641	0.290715	0.299641
0.6	0.263114	0.278067	0.261919	0.278067
0.7	0.238076	0.261287	0.236103	0.261287
0.8	0.21542	0.249301	0.21327	0.249301
0.9	0.19492	0.24211	0.193418	0.24211
1	0.176371	0.239713	0.176548	0.239713

dissipative and nonlinear media, as well as beachfront ocean and ocean engineering.

Tables and graphs were drawn to interpret Table 9 the solutions found. It can be said here that as the fractional order β value approaches 1, results close to the real solution are obtained. This study shows that the error rate of the solutions obtained with the new method examined is low. It is important in this aspect and is a good alternative method for finding solutions to nonlinear FDEs.

Declaration of Competing Interest

There are no political, personal, religious, ideological, academic, and intellectual competing interests. The authors declare that they have no competing interests.

Acknowledgement

Bandar Almohsen is supported by Researchers Supporting Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

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