STABILITY AND OPTIMAL CONTROL OF HYBRID MODELS OF FISHERY

NURGÜL GÖKGÖZ KÜÇÜKSAKALLI

SEPTEMBER 2022

ÇANKAYA UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
MASTER'S THESIS IN
ELECTRICAL AND ELECTRONICS ENGINEERING

STABILITY AND OPTIMAL CONTROL OF HYBRID MODELS OF FISHERY

NURGÜL GÖKGÖZ KÜÇÜKSAKALLI

SEPTEMBER 2022

ABSTRACT

STABILITY AND OPTIMAL CONTROL OF HYBRID MODELS OF FISHERY

GÖKGÖZ KÜÇÜKSAKALLI, Nurgül
Master of Science in Electrical and Electronics Engineering

Supervisor: Assist. Prof. Dr. Oğuzhan ÇİFDALÖZ
September 2022, 38 pages

All around the world, there is an increasing interest in the renewable resources which can be mainly listed as fisheries, forests, agricultural lands and freshwater. They are controlled and managed by some foundations and govenments which are searching for the methods to determine how to efficiently manage those resources under the effect of uncertanities caused by social and ecological events such as climate change, adversities in the application procedure of the strategies, and error in the data. Control systems methodologies serve an appropriate tool to overcome the difficulties, uncertanities and errors listed above since the problem can be designed as a mathematical problem. By this way, sustainability of the resource can be investigated with respect to different scenarios in a systematic way. Moreover, since the harvesting of the fishery is applied during only some seasons, the problem has to be modelled by using both discrete and continuous dynamics which are called as the hybrid dynamical systems in the literature.

In this thesis, we define two new hybrid dynamical models of fishery. One of the models is one dimensional and the other one is a two dimensional model and they represent exactly the same sustainable system. We solve the optimal control problem on the one dimensional one and we check the stability of the two dimensional model. By doing so, we determine the optimal effort needed for the sustainability of the system whenever the model is unstable.

Keywords: Stability, Optimal Control, Hybrid Dynamical Systems, Fishery, Bioeconomic Models.

ÖZ

BALIKÇILIK İÇİN HİBRİT MODELLERİN KARARLILIĞI VE OPTİMAL KONTROLÜ

GÖKGÖZ KÜÇÜKSAKALLI, Nurgül
Elektrik-Elektronik Mühendisliği Yüksek Lisans

Danışman: Dr. Öğr. Üyesi Oğuzhan ÇİFDALÖZ
Eylül 2022, 38 sayfa

Tüm dünyada, balıkçılık, ormanlar, tarıma elverişli araziler ve temiz su şeklinde listelenebilecek yenilenebilir kaynaklara artan bir ilgi bulunmaktadır. Bunlar, iklim değişikliği gibi sosyal ve ekolojik olayların sebep olduğu belirsizliklerin, statejilerin uygulanmasındaki prosedürlerin zorluğun ve datadaki hataların etkisi altında nasıl etkili karar vereceğne dair metotlar arayan kurumlar ve hükümetler tarafından kontrol edilmektedir ve yönetilmektedir. Kontrol sistem yöntemleri, problem bir matematiksel problem olarak tasarlanabildiği için, yukarda listelenen zorlukların, belirsizliklerin ve hataların üstesinden gelmek için uygun bir yol sunar. Bu sayede, kaynağın sürdürülebilirliği sistematik bir şekilde farklı senaryolara göre incelenebilir. Dahası, balıkçılık hasatı sadece bazı mevsimlerde yapıldığı için, problem literatürde hibrit dinamik sistemler olarak anılan kesikli ve sürekli dinamikler kullanılarak modellenmelidir.

Bu tezde, balıkçılık için iki yeni hibrit model tanımlıyoruz. Modellerden biri bir boyutlu ve diğeri iki boyutlu ve tamamiyle aynı sistemi gösteriyorlar. Bir boyutlu olan için optimal kontrol problemini çözdük ve iki boyutlu modelin kararlılığını kontrol ettik. Böyle yaparak, model kararsız olduğunda, sistemin sürdürülebilirliği için gereken optimal çabaya karar veriyoruz.

Anahtar Kelimeler: Kararlılık, Optimal Kontrol, Hibrit Dinamik Sistemler, Balıkçılık, Biyo-ekonomik Modeller.

To my family...

ACKNOWLEDGMENTS

I would like to express my sincere thanks to my supervisor Assist. Prof. Dr. Oğuzhan Çifdalöz for his guidance to this new and enjoyable research area and patience in the progress of this thesis. His help and profound knowledge in the area motivated me to prepare this thesis. I would also like to thank my thesis committee members, Prof. Dr. Mehmet Yüceer and Assoc. Prof. Dr. Erdem Akagündüz for their valuable comments.

I would like to thank my lovely family. My beloved husband Ömer, my sweet parents Nurşen and Süleyman, my considerate brother Sinan are shareholders of this success. Their support and unrequited love are the biggest help all the time.

I am also grateful to some friends and collegues. I am deeply indebted to Ezel for never wavering in her support. I would like to extend my sincere thanks to my lovely friends İpek, Aslı. They provided me with encouragement throughout the duration of this thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Motivation and Problem Definition

The world population is increasing day by day which is the main reason of the wars, diseases, famine, climate change etc. The most important question is the sufficiency of the resources and this makes the sustainability of the renewable resources a hot topic for foundations and governments. Sustainability basically deals with the transfer of the renewable resources to the future generations without endangering them. Mostly considered renewable resources are fisheries, forests, water basins and agricultural lands. Even though, a great deal of effort has been made on the sustainability, still we face the endangered resources in some areas of the world. Therefore, a systematic way is needed to investigate this problem. From a scientist's point of view, the questions to be answered are,

- the amount of the resource in question,
- sustainability of the resource,
- economical issues related to the harvesting of the resource,
- the amount and the period of the harvesting.

To make a systematic investigation on this matter, many different researchers from different research areas have developed mathematical and economical models. Every model has its own advantages and disadvantages. Actually, the problem has to be considered by taking into account the three important factors. Economical, environmental and social factors all have to be considered in order to make an optimum decision. Even if an analysis is conducted on the model and the amount of the resource is determined, still, the amount and period of harvesting to be allowed must be determined. At this point, control engineering techniques enter the picture to understand the system dynamics and decide the parameters listed above. The methodologies that a control engineer use already allows one to obtain robust techniques due to the given difficulties and overcome the complexity of the socio-economical problem one may encounter during the implementation of the policy.

In this thesis, sustainability of the fishery is considered among those renewable resources from the perspective of a mathematical point of view. We have equiped the models with the analysis of stability and optimal control in order to determine the difficulties one may face.

1.2 Proposed Models and Analysis of the Models

In the literature of sustainably managed resources, the most attractive models that are examined are the continous and discrete models. Moreover, economic models are also considered. Classical supply-demand analysis and game-theoretic models are used in that case. However, all of those models appear to contain discrete as well as continuous dynamics if investigated in deep. Since the nature of the problem contains continuous dynamics as well as discrete state transitions, we use hybrid dynamical system modeling scheme.

Hybrid dynamical systems has taken attention of vast amount of researchers from various disciplines. Even though the idea of a combination of discrete and continuous dynamics is very natural and sounds familiar, the description in the sense of mathematical modeling has taken a great effort. Impulsive systems, switching systems, hybrid dynamical systems, sliding mode, hybrid automata, differential algebraic equations and many others are all distinct mathematical formulations of this phenomena. Most famous ones are hybrid dynamical systems (combines all of the mentioned models in one model), switching systems (the theory is very-well developed with respect to the others) and impulsive systems (again a very-well known approach). Each one of them has its own pros and cons.

In this work, we use hybrid formulation for the sustainability of fishery. Since the resource is harvested in some seasons and left for production of fish in the non-harvest seasons, the revenue will have a hybrid nature. From this idea, we construct and consider hybrid models. For a depensatory hybrid model, we investigate the Lyapunov stability, global stability and we solve an optimal control problem. By this way, in a dynamic way, we find optimal harvesting rate and optimal harvesting time period and whenever the system is under the effect of an unexpected component we check the stability and we calculate new optimum harvest rate and time for a new harvesting period.

1.3 Contributions

In the literature, there exist some hybrid models that combine continuous and discrete dynamics. For example aquaculture models [31, 32] that consider te shrimp population using an impulsive model, the fishery models with on-off harvesting [5, 6], or the stochastic hybrid system version of the fishery system [17]. This area can be developed in various directions. To the best of our knowledge, the unified framework of hybrid dynamical system approach is not used in this topic. Therefore, we take the first step in this area which can be developed later on both in the sense of theory and application.

1.4 The Outline of the Thesis

A literature survey and the necessary background on the technical definitions, methods and fishery models considered previously in the literature are given in chapter 2. The hybrid system formulations of the problem, optimal control and stability are given in chapter 3. In chapter 4, a summary of the main results, disscussion and future directions are explained.

CHAPTER 2

LITERATURE SURVEY AND BACKGROUND

In this chapter, some of the basic mathematical concepts used throughout this thesis are given. Firstly, we give a collection of related works in the literature. Secondly, we give a summary of hybrid modeling frameworks that are used or mentioned in the rest of the thesis. Then, we give optimal control and stability of hyrid systems, respectively. Finally, we give a review of mathematical models of fishery with different aspects.

2.1 Literature Survey

Sustainability of the renewable resources is getting attention of the foundations and governments since it has the crucial role in the felicity of the public. The mostly considered and important resources in the literature are fishery [16, 46], forests [19, 36], freshwater basins [22, 38] and agricultural lands [41, 42]. Among those resources fish stocks are facing with the danger of a sudden reduction. Approximately one of four fisheries collapsed during the last 50 years [37]. Therefore an immediate action must be taken into account to carry on the well-being of the public and to transfer the resources to future generations. In order to take an effective act to manage those resources, the problem should be considered by taking into account economical, ecological and social aspects. Randomness in any step in the management of the natural resource or disregarding economical apprehensions of the actors in the market leads to the ineligible management of the resources [16]. To overcome the complexities in the implementation process to manage the fishery is the main motivation to use tools and methods from control systems. First of all, a mathematical model which is identical to the real-world problem has to be constructed in the simplest way possible. Secondly, the control system techniques are used to get over the difficulties mentioned above. Mathematical models of fishery are dating back to population dynamics [12, 14], however they are reconsidered in the last decades due to the developments in the modeling techniques [5, 17, 18, 39].

Control of the sustainable resources and their social and political management issues are rather newly considered aspects of the problem in the literature [13]. Robust control of a fishery [2, 13], feedback control of a prey-predator system where only the harvesting of the prey is considered [20], dynamic deterrence problem considering an optimal control problem [21] and social-economical viewpoints with respect to public policies [3, 45] are some basic but powerful steps taken in this direction.

As the improvements in modeling has been mentioned, among them, hybrid dynamical systems are a challenging one which has taken attention of many researchers from various disciplines. Even though the idea of combinining discrete and continuous dynamics may seem easy and reasonable to implement to a real-world problem, mathematical formulation of the problem was rather challenging and differed among the researchers choice/need. This type of modeling, despite its rather complex contruction, is giving us a chance to obtain new directions in the sense of stability [8, 9, 23, 29].

2.2 Hybrid Systems

Hybrid dynamical systems are a combination of continuous and discrete dynamics. Since real world problems mostly contain and combine discrete and continuous dynamics, they arise naturally in modeling problems in computer science, control, robotics, etc. Even though a mixture of continuous and discrete dynamics is a natural way to see a real world problem, the mathematical formulation of the hybrid systems has taken an effort and this led to a huge amount of development in various research areas. Since the beginning of the hybrid dynamical systems research, many different modeling approaches have been used. Hybrid automata [1, 25, 34], sliding mode control [48], switching systems [29], set valued analysis [23] frameworks are some of them. Every model has advantages and disadvantages. Among them, set valued analysis version captures and unifies all other formulations in one representation [23]. By this way, all the results from different modeling frameworks can be unified.

Since some of them are used in modeling fishery, they are explained in the following subsections.

2.2.1 Hybrid Dynamical Systems

Hybrid dynamical systems are described by the following equations,

$$\begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^{+} \in G(x) \end{cases} \tag{2.1}$$

where C and D are called the *flow set* and the *jump set*, respectively, and F and G are called as the *flow map* and the *jump map*, respectively.

2.2.2 Hybrid Automata

A finite state machine is a mathematical model with finite set of continuous functions that are described by a set of ordinary differential equations. Formally those systems are described by [10, 35]

- $Q = 1, ..., q_{max}$ is a set of modes,
- $Domain(q)$ stands for the domain of the continuous variable x for each q,
- f is a flow map,
- $Edges$ stands for a set of edges,
- $Guard(q, q^{'})$ indicates the guard conditions for each edge
- $Reset$ is the reset map.

2.2.3 Switching Systems

Consider an index set P which indicates a family of regular functions $f_p : p \in P$ and $f_p : \mathbb{R}^n \to \mathbb{R}^n$. The differential equation stated by this family of functions is [29]

$$\dot{x} = f_p(x). \tag{2.2}$$

If p is taken as $\sigma : [0, \infty] \to P$ which is a piecewise constant function then we call σ a *switching signal* and the system is called a *switching system*.

2.3 Optimal Control and Maximum Principle for Switching Sytems

Consider the following time-invariant control systems [30],

$$\dot{x} = f_q(x, u), \quad q \in Q \tag{2.3}$$

where Q is an index set and $U \subset \mathbb{R}^m$ is a control set. Moreover, assume that switching surfaces (i.e. guards) are $S_{q,q'} \subset \mathbb{R}^{2n}$, for every $(q, q^{'}) \subset Q \times Q$. Assume that there is a function $x : [t_0, t_f] \to \mathbb{R}^n$ which is a trajectory the given switching system (i.e. hybrid system) with a control $u : [t_o, t_f] \to U$ prrovided that for the following time instants

$$t_0 < t_1 < ... < t_k < t_{k+1} := t_f$$

and states $q_0, q_1, ..., q_k \in Q$ such that $x(\cdot)$ captures

$$\dot{x} = f_{q_i}(x(t), u(t)), \;\; \forall t \in (t_i, t_{i+1}), \; i = 0, 1, ..., k \tag{2.4}$$

and

$$\begin{pmatrix} x(t_i^-) \\ x(t_i^+) \end{pmatrix} \in S_{q_{i-1}, q_i}, \;\; i = 1, ..., k. \tag{2.5}$$

where $x(t_i^-)$ and $x(t_i^+)$ stands for the values of x just before and just after t_i, respectively, and the value $x(t_i)$ may be equal to right limit or left limit. At every t_i, there exists a discontinuity which is called a switching event or a discrete transition. Here, q is the discrete state of the hybrid system and $q : [t_0, t_f] \longrightarrow Q$ which is given by by $q(t) := q_i$ for $t \in [t_i, t_{i+1})$ portrays q on the trajectory. The cost functionals for this case are [30]

$$J(u, t_i, q_i) := \Sigma_{i=0}^{k} \int_{t_i}^{t_{i+1}} L_{q_i}(x(t), u(t))dt + \Sigma_{i=0}^{k} \Phi_{q_{i-1},q_i}(x(t_i^-), x(t_i^+)), \tag{2.6}$$

where $L_q : \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ is the running cost and $\Phi_{q,q'} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is the switching cost for $()q, q') \in Q$. The constraint for this optimal control problem is the endpoints which is given by

$$\begin{pmatrix} x(t_0) \\ x(t_f) \end{pmatrix} \in E_{q_0,q_k}. \tag{2.7}$$

So, the switching system (or hybrid system) optimal control problem is to find a control that minimizes the cost 2.6 subject to the constraint 2.7. Moreover, in this case the following Hamiltonian is considered.

$$H_q(x, u, p, p_0) = \langle p, f_q(x, u) \rangle + p_0 L_q(x, u), \quad q \in Q. \tag{2.8}$$

There is a vast amount of work in the literature related to the optimal control of switching systems. They find place in the application of the real world problems. For instance, using the time-scale transformation the optimal control problem of switching systems is solved in [31, 49]. Then, to decide a feedback optimal control strategy, a neighboring extremal procedure is used in [28]. A different variation of this optimal control problem where x is constructed as a factor of a variable that are piecewise constant functions is considered in [7]. For other applications one may see the following and the references therein [32, 40] and other theoretical approaches please see [4, 9, 24, 44] and cited papers therein.

2.4 Stability of Hybrid Dynamical Systems and Switching Systems

A basic tool to investigate the local stability of a dynamical system is to use Lyapunov theorems. For a hybrid dynamical system given with the representation described in equations 2.1, Lyapunov stability is also investigated. The most important results and the related references can be found in [23]. Similar results also exist for the switching system literature (see, for example, [29, 11]) In this section, we mention some of them that are used in the next chapter. Firstly we give two results due to Liberzon

[29]. Then, we give the result due to Goebel, et al [23].

Theorem 1 ([29]). *If all subsystems in the following family*

$$\dot{x} = f_p \quad p \in \mathcal{P}$$

share a radially unbounded common Lyapunov Function, then the switching system with a signal, σ,

$$\dot{x} = f_\sigma(x)$$

is globally uniformly asymptotically stable.

Note: We call V a *a common Lyapunov function* for the switching systems if there exists a positive definite continuous function W s.t

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x), \ \ \forall x, \ \ \forall p \in \mathcal{P}. \tag{2.9}$$

Theorem 2 (A converse Lyapunov theorem,[29]). *Suppose that the switching system*

$$\dot{x} = f_\sigma(x)$$

is globally uniformly asymptotically stable, the set of family of functions $\{f_p\}$ is bounded and the function f_p is locally Lipschitz. Then all systems in the switched system share a common Lyapunov function.

Combining those two theorems gives us a quick result which is given by the corollary in [29] and we obtain the following inequality,

$$\alpha \frac{\partial V}{\partial x} f_p(x) + (1-\alpha) \frac{\partial V}{\partial x} f_q(x) \leq -W(x), \ \ \forall x. \tag{2.10}$$

Theorem 3 (Sufficient Lyapunov conditions,[23]). *Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous $\rho \in \mathcal{PD}$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x), \quad \forall x \in \quad C \cup D \cup G(D) \tag{2.11}$$

$$\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in \quad C, f \in F(x) \tag{2.12}$$

$$V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in \quad D, g \in G(x) \tag{2.13}$$

then $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$.

Basically, for a continous-time model we have a potential-like function with weaker conditions. However, for a hybrid dynamical system (or a switching system) even though we again have a potential-like function, this time those weaker conditions of the continuous model may not hold. To see the illustration of the potential-like functions of the continuous and hybrid cases, please see figure (2.1).

Figure 2.1: Lyapunov function illustration for the continuous and the hybrid cases. The left one stands for the continuous case, and the left one is for the hybrid case.

2.5 Mathematical Models on Fishery

In the literature, there are different kinds of mathematical models. Just to mention some of them here: generalized logistic models, economic models, supply and demand models,models that consider growth and aging, multispecies models are the mostly used ones among them. Mathematically, each one of them may be continuous, discrete, stochastic. In this thesis, continuous time general logistic models are used which is the topic of the next subsection.

2.5.1 Continuous Time Models

Consider the biological model [14]

$$\dot{x} = F(x) - h(t), \tag{2.14}$$

where $x(t)$ is the fish stock or biomass, $h(t)$ is the harvest rate, and $F(x)$ stands for the is the net natural growth rate. Assume that we define a function $G(x)$ such that $G(x) = \dfrac{F(x)}{x}$. This is the common way to analyze population dynamics models since by this way we obtain $G(x)$ which is a density function or in other words $G(x)$ defines the *per capita growth rate*. One of the mostly considered models is, obviously, the logistic model where $G(x) = r(1 - x/k)$. Other important models that has been considered in the literature are,

- Gompertz's model in 1825 where $G(x) = r \log \frac{K}{x}$
- F. Smith's model in 1963 where $G(x) = r\frac{r(K-x)}{K+ax}$
- Ayala, et al.'s model in 1973 where $G(x) = r(1 - (\frac{x}{K})^{\theta}$
- Nisbet and Gurney's model in 1982 where $G(x) = re^{1-x/K} - d$.

Next, we give a classification of the logistic models depending on different expressions of $h(t)$. As a first case, assume $h(t) = 0$ which means no harvesting and our equation is $\dot{x} = xG(x)$ and assume that there is a stable equilibrium at $x = k$. In the literature, the curves of this equation is called as the *depensation curve*.
Case I : $h = 0$ (For detailed description, see [12].)

- If the density function $G(x)$ is greater than 0 and decreasing for $0 \leq x \leq k$ then it is called as the *compensation model*.
- If $G(x)$ is increasing for small x, the model is called as the *depensation model*.
- If $G(x)$ is smaller than 0 for small x, then the model is called as the *critical depensation model*.

The main difference between a compensation model and a depensation model is that if fishing stops, the fish population will catch up with its old values no matter how low the population becomes in a compensation model [14, 27]. Those models can be seen in figures 2.2 and 2.3. In figures 2.2 and 2.3, the value k_c is also called as *minimum viable population level*. If the population level tends toward below this value then the extinction of the population cannot be reversed. This is called as *Allee effect* in the literature. The density function, $G(x)$, versus x graph can be seen on figure 2.3.
Case II : $h \neq 0$.

- If $h(x,t) = constant$, then the model is called as *constant-yield harvesting*.
- If $h(x,t) = Ex(t)$, then the model is called as *constant-effort harvesting*. In this expression, E stands for the effort spent in fishing $x(t)$ amount of fish.
- If $h(x,t) = qEx(t)$, then the model assumes that *catch-per-unit-effort* is proportional to the fish stock. Here q stands for the *catchability coefficient*.Then the

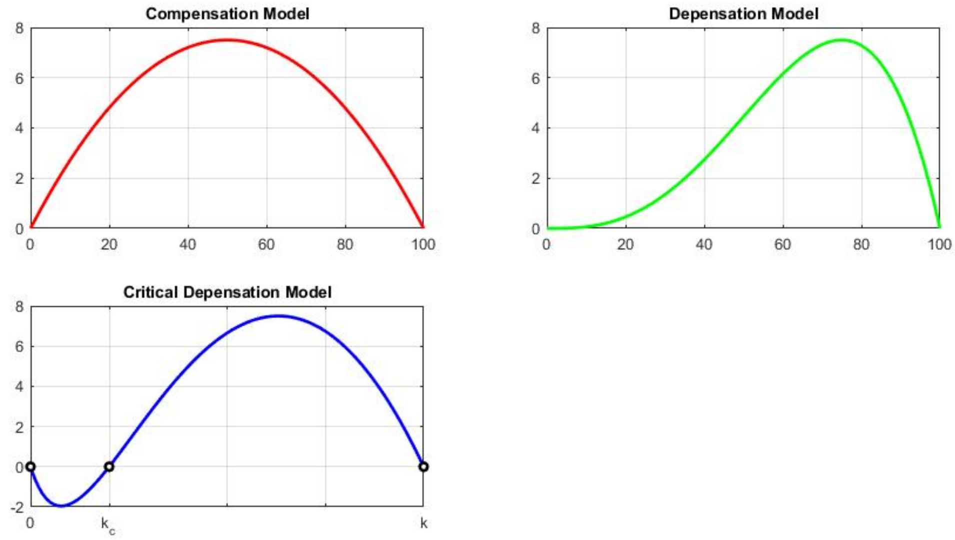

Figure 2.2: Graphs of compensation, depensation and critical depensation models. The vertical axes correspond to $\dot{x} = F(x)$ and the horizantal axes correspond to x.

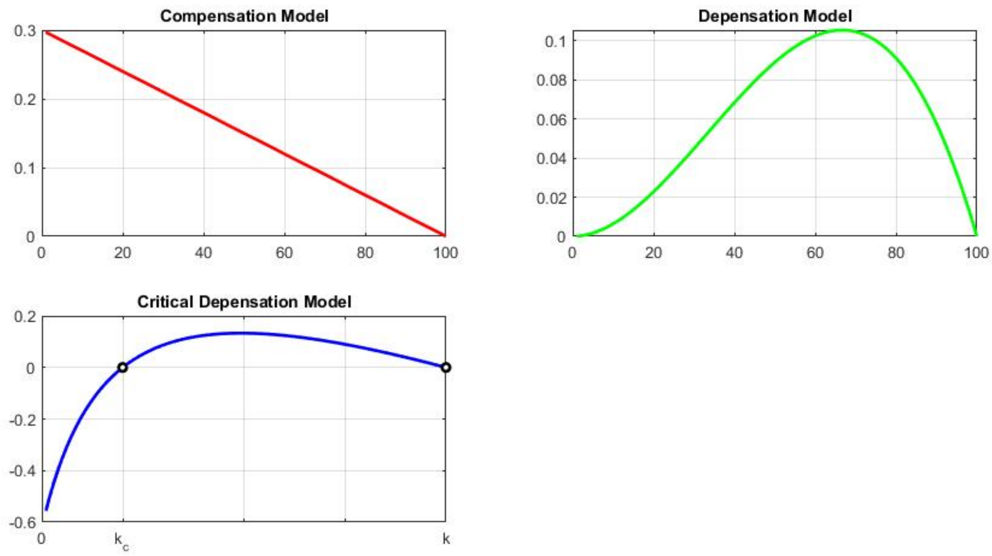

Figure 2.3: Graphs of compensation, depensation and critical depensation models. The vertical axes correspond to $G(x)$ and the horizantal axes correspond to x.

model takes the form,

$$\dot{x} = xG(x) - qEx$$

and if $G(x)$ is chosen as $G(x) = r(1 - x/K)$, it is called as the *Schafer model.*

Consider the following model used in [13],

$$\dot{x} = rx^2(1 - \frac{x}{k}) - qux. \quad (2.15)$$

In that case, effort given by E is replaced by a control variable u and an optimal control problem is constructed and solved using the bioeconomic model[15],

$$R = pqux - cu \quad (2.16)$$

where R is the revenue, p is the price of the resource, and c is the opportunity cost of harvesting. To avoid the confusion of parameters of sustainable fishery models we give a table of parameters in table 2.1.

Since the sustainability of the fish stock is also very related to economic parameters,

Table 2.1: Model parameters and their units.

Parameter	Explanation	Unit
x	biomass	KT (kilotons)
p	price	Million dollars/KT
E	effort	-
q	technology	1/year
c	cost of fishing	Million dollars/year
r	intrinsic growth rate	1/year
k_c	minimum viable population level	KT

in the literature there exist bioeconomic (or bionomic) models that combine the just mentioned models with economic values. Smith's economic model [47]

$$\frac{dE}{dt} = k(pqxE - cE) \quad (2.17)$$

combined with [14]

$$\frac{dx}{dt} = xG(x) - qEx \quad (2.18)$$

give a two dimensional dynamical system. Investigating the stability of this system, the bionomic equilibrium is unstable since eigenvalues of the system gives $Re\ \lambda_i > 0$ [14].

2.5.2 Optimal Control of a Fishery Model

Consider the equation

$$\frac{dx}{dt} = F(x) - h(t) \tag{2.19}$$

with initial condition $x(0) = x_0$ and terminal condition $x(T) = x_T$. Assume that we have calculated revenue and obtained a smooth, convex and nonnegative function, R. The objective functional is defined as [14]

$$J(h) = \int_0^\infty e^{-\delta t} R(h) dt \tag{2.20}$$

where $h \geq 0$. The Hamiltonian is defined for this case as [14]

$$\mathcal{H} = e^{-\delta t} R(h) + \lambda\{F(x) - h\}. \tag{2.21}$$

Using the maximum principle and the adjoint equation, we obtain

$$\frac{dh}{dt} = \frac{R^{'}(h)}{R''(h)}[\delta - F^{'}(x)]. \tag{2.22}$$

Considering this equation together with equation 2.19 we get an autonomous system of differential equations. By considering the isoclines $\dot{x} = 0$ and $\dot{h} = 0$, we get the equilibrium point (x^*, h^*).

2.5.3 Dynamical Systems Approach for Fishery Models

Now, turn back to the equations 2.17 and 2.18

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}) - qEx \tag{2.23}$$

$$\frac{dE}{dt} = kE(pqx - c) \tag{2.24}$$

where instead of equation 2.18 the Schafer model is used. The equilibrium and the eigenvalues of this system respectively given as [14],

$$x_\infty = \frac{c}{pq}, \quad E_\infty = \frac{r}{q}(1 - \frac{x_\infty}{K}) \tag{2.25}$$

$$\lambda_i = -\frac{rx_\infty}{2K} \pm \sqrt{\frac{r^2 x_\infty^2}{4K^2} - 4kpq^2 x_\infty E_\infty}\}. \tag{2.26}$$

Then $Re\ \lambda_i < 0$ which means the equilibrium point is stable. However, if instead of Schafer model a depensation model is used in equation 2.18 then the eigenvalues of the system are calculated at (x_{infty}, E_∞) as [14]

$$\lambda_i = \frac{1}{2}\{[F(x_\infty) - qE_\infty] \pm \sqrt{[F(x_\infty) - qE_\infty]^2 - 4kpq^2x_\infty E_\infty}\} \tag{2.27}$$

which indicates that the equilibrium is unstable. If it is the case of a noncritical depensation then a trajectory near $(K, 0)$ must exhibit a limit cycle. In other words, the fish stock x and effort E must oscillate.

In the literature, there are stability analysis that consider threshold harvesting [43]. This system produces a hybrid system with different continuous dynamics. Moreover, to consider the dynamics of fisherey discrete-time models are also considered. They exhibit similar methodologies with respect to their continuous counterparts. However, to check the stability of the system, the iteration of the model is taken into account. These methodologies requires some different procedures other than the very weell-known Lyapunov methods. In those models, again threshold harvesting is considered and investigated. For example, one may see [26, 33]. Obviously, in those cases, threshold is also a parameter to be determined as well as the harvesting quota.

CHAPTER 3

PROPOSED MATHEMATICAL MODELS AND ANALYSIS

In this section, we give two different hybrid models. As a first case, a one dimensional hybrid system is constructed using the equaion (3.1). Then a hybrid optimal control problem is solved. Secondly, a two dimensional model for the same model is constructed and an analysis of the model from the perspective of dynamical systems is given.

3.1 Depensation Model of Fishery with Control

Consider again the fishery model in[13],

$$\dot{x} = rx^2(1 - \frac{x}{k}) - qux, \;\; x(0) = x_0. \tag{3.1}$$

Moreover, an optimal control law is obtained by solving an optimal control problem as [13]

$$u(t) = \begin{cases} 0 & G(x,t) < 0, \\ u_{max} & G(x,t) > 0. \end{cases} \tag{3.2}$$

where $G(x,t) = e^{-\delta t}(pqx - c) - \lambda qx$. It is obtained by rearranging and solving the Hamiltonian. This problem can be considered as a hybrid dynamical system which is the topic of the next section.

3.2 One Dimensional Hybrid Model and Optimal Control

For the one dimensional case, we consider hybrid automata representation. Consider the states $Q = 1,2$ and the flows of those states are $f(1,x) = F(x)$ and $f(2,x) = F(x) - qu_{max}x$. Edges are $Edges = (1,2),(2,1)$. Guard conditions are $Guard(1,2) = (c,\infty)$ and $Guard(2,1) = (-\infty,c]$. Hybrid automata of this system can be seen in figure 3.1. An optimal control is applied to this hybrid model. By applying a hybrid formulation to the system, we only find the trajectories that switch along the way. And find the optimum among them.

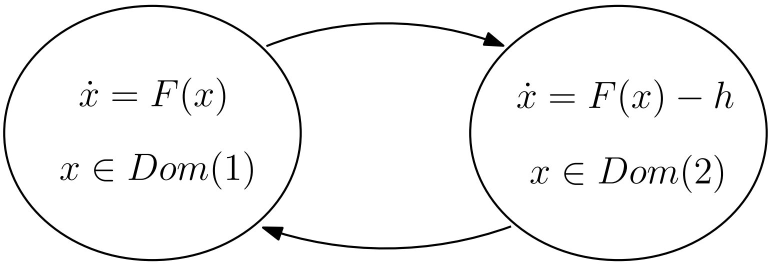

Figure 3.1: Fishery hybrid automata representation.

3.2.1 Problem Statement for the Optimal Control

Assume that ψ_i denote the fraction of fish stock subtracted from the ecosystem at ith harvest. Actually, this corresponds to the qu expression in the equation 3.1. Therefore, the control variable u can be restated as $u_i = \frac{\psi_i}{q}$. Let τ_i stand for the ith harvest time. Then the decision variables $\tau_i,\ \ i = 1, ..., m$ and $\psi_i,\ \ i = 1, ..., m$ captivate the conditions

$$0 \leq \tau_1 \leq \tau_2 \leq ... \leq \tau_m \tag{3.3}$$

and

$$0 \leq \psi_i \leq 1, \quad i = 1, ..., m. \tag{3.4}$$

Then the total revenue obtained within the time period $[0, \tau_m]$ is given by

$$R = \sum_{i=1}^{m} \{p\psi_i x(\tau_i^-) - H\} \tag{3.5}$$

where p is the price and H (or cu) is the fixed cost of a single harvest. The problem is to select the optimal harvest times $\tau_i\ i = 1, ..., m$ and the optimal harvest rate $\psi_i,\ i = 1, ...m$ to maximize the total revenue subject to the costraints 3.4 and 3.5 the dynamic system

$$\dot{x} = F(x) - qux$$
$$x(0) = x_0$$

where x_0 is a given constant and

$$x(\tau_i^+) = x(\tau_i^-) - \psi_i x(\tau_i^-), \quad i = 1, ..., p. \tag{3.6}$$

This problem is equivalent to the following equivalent optimal control problem [31].
Problem: Find a pair $(\theta^*, \xi^*) \in \Theta \times W$ such that

$$\bar{J} = \underset{(\theta,\xi)\in\Theta\times W}{inf} \bar{J}(\theta, \xi) \tag{3.7}$$

For the rest of the problem, optimal values of harvesting time and harvesting rate by using the procedure given in [31]. In this case, the Hamiltonian is

$$H = \langle p, F(x) - qu_i x \rangle + p_0. \tag{3.8}$$

For this fishery problem, optimal value of the fish stock is taken as $x_0 = 72.549$. This value is calculated due to the results of [13] by simplifying the functional with inflation rate equal to 0, ignoring measurement noise, etc. Then, by taking $c = 45.23$ and initial fish stock $x_0 = 55$ where $x_0 < x_{opt}$. One may see the results listed in Table (3.1). We

Table 3.1: Periodicities and the corresponding total revenue for $x_0 = 55$.

x_0	**Harvesting Periodicity**	**Total Revenue**
	1 day	1628.691
	3 days	1627.924
55	12 days	1628.926
	1 month	1630.164
	2 months	1647.470
	6 months	1665.567

calculate the revenue with respect to the periodicities: 1 day, 3 days, 12 days, 1 month, 2 months and 6 months. With the same manner we choose $x_0 = 90$, i.e. $x_0 > x_{opt}$ and we calculate the revenue with respect to the same periodicities. The results are given in the Table (3.2). Those values are just taken to represent the result that optimal periodicity

Table 3.2: Periodicities and the corresponding total revenue for $x_0 = 90$.

x_0	**Harvesting Periodicity**	**Total Revenue**
	1 day	1911.800
	3 days	1913.436
90	12 days	1911.623
	1 month	1912.917
	2 months	1902.134
	6 months	1866.425

for the case $x_0 > x_{opt}$ and $x_0 < x_{opt}$ totally differs from each other. For the given values of x_0 and x_opt we observe that, if $x_0 < x_{opt}$ then then 6 months gives the best choice for periodicity of the harvesting. On the other hand, if $x_0 > x_{opt}$ then 3 days show the best harvesting periodicity. This is a result of the fact that if $x_0 < x_{opt}$ the population has to increase first of all and then the harvesting starts. For the case $x_0 < x_{opt}$, there is no need to wait for the population to increase and the harvesting should start immediately. As an illustration of those two cases that gives the most profitable choices, one may see the figures.

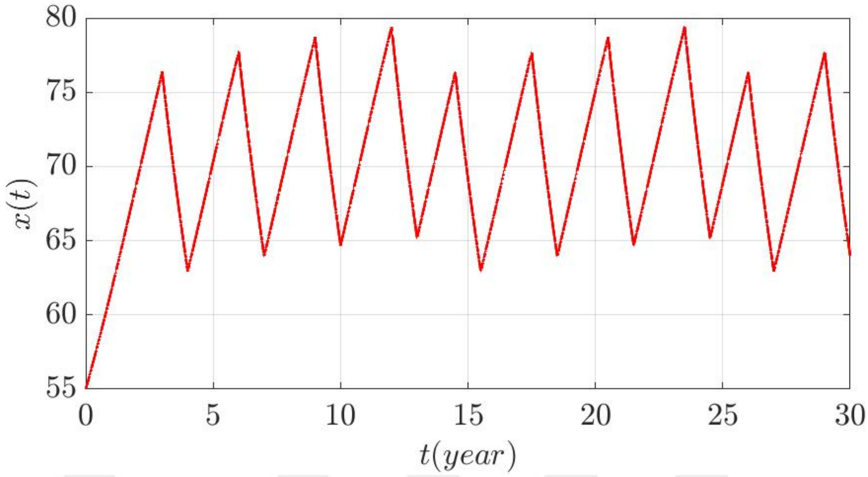

Figure 3.2: Graph of fish stock with $x_0 = 55$ and 6 months of harvesting periodicity.

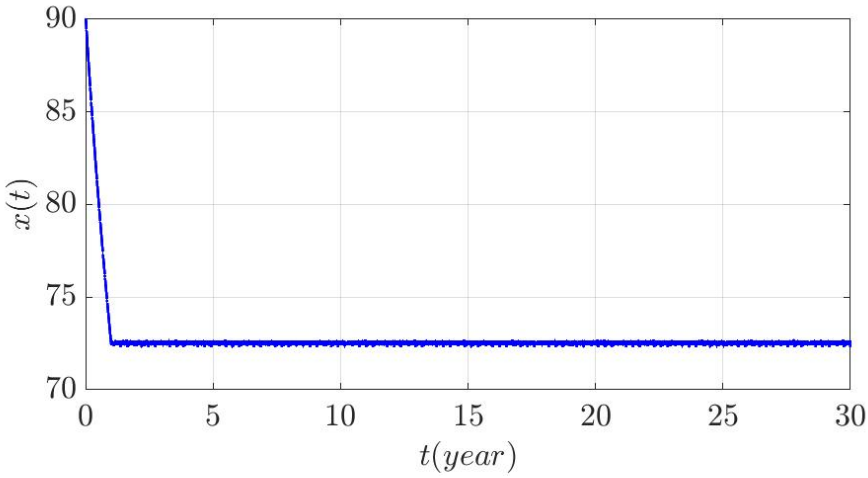

Figure 3.3: Graph of fish stock with $x_0 = 90$ and 3 days of harvesting periodicity.

3.3 Stability Analysis of One Dimensional Hybrid Model

For the hybrid model described by the hybrid automata in figure 3.1, we have two different flows described by

$$\frac{dx}{dt} = F(x) - qu_i x \tag{3.9}$$

where $i \in \{0,1\}$ and moreover assume that $u_0 = 0$ and $u_1 = u_{max}$. This indicates that the system switches between two flows,

$$\frac{dx}{dt} = F(x) - qu_{max}x \quad \text{or} \tag{3.10}$$

$$\frac{dx}{dt} = F(x). \tag{3.11}$$

Using the theorem from section 2, we choose Lyapunov function as $V(x) = \dfrac{x^2}{2}$. Then

$$\frac{\partial V}{\partial x} f_i(x) = x(F(x) - qu_i x) = xF(x) - qu_i x^2.$$

For $F(x) = rx(1 - \dfrac{x}{K})$, we obtain,

$$\begin{aligned} \frac{\partial V}{\partial x} f_i(x) &= xrx(1 - \frac{x}{K}) - qu_i x^2 \\ &= rx^2 - \frac{rx^3}{K} - qu_i x^2 \\ &= x^2(r - \frac{rx}{K} - qu_i). \end{aligned}$$

Then it is stable, if

$$r - \frac{rx}{K} - qu_i \leq -W(x), \tag{3.12}$$

where $W(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a positive definite continuous function. This condition gives us some conditions on the parameter values.

3.4 Two Dimensional Hybrid Model and Stability Analysis

Consider the following model

$$\frac{dx}{dt} = F(x) - qu_i x \tag{3.13}$$

$$\frac{du}{dt} = u_i(pqx - c) \tag{3.14}$$

which is the same system of equations given by the equations 2.17 and 2.18 except u in equation 3.13 is replaced by u_i. This effect creates a switching system that will be examined in terms of the global stability analysis. Choose $F(x) = rx(\frac{x}{k_c} - 1)(1 - \frac{x}{k})$ and calculate the binomic equilibrium ($\dot{x} = \dot{u_i} = 0$). A straightforward calculation gives the equilibrium points

$$x^* = \frac{c}{pq} \tag{3.15}$$

$$u_i^* = \frac{r(xk - kk_c - x^2 + xk_c)}{qkk_c}. \tag{3.16}$$

which is a unique equilibrium point. Since $F(x)$ is a depensation model, the equilibrium point is unstable as described in the previous chapter.

CHAPTER 4

CONCLUSION AND FUTURE DIRECTIONS

In this thesis, hybrid dynamical system formulation for the fishery problem is considered. Firstly, an optimal control of the one dimensional case is solved. Then, two dimensional version of the same model is obtained and the stability analysis is investigated. Due to the results, we are able to obtain an optimal contol and obtain the region of stability due to this optimal control. When the system is unstable, a new optimal control is obtained and the same stability analysis is done. This area of research is a newly developed one and therefore open to new applications. The system is very open to randomness. Therefore, a meaningful first step will be to construct stochastic counterparts of those models, investigate stability analysis and obtaining optimal control solutions.

Delay can be added to the system since the whole population is not able to reproduce. This leads us to age-structured models and the simplest way to represent this is to use delay hybrid system models. Like in the stochastic case, again stability and optimal control problems will be solved. Moreover, more complex models such as multispecies models can be considered. Real world measurements will allow us to calculate the system parameters in a reasonable way.

At this point classical methods like data fitting can be used or machine learning techniques can be considered which are highly in fashion during the last decades.

REFERENCES

[1] ALUR Rajeev, COURCOUBETIS Costas, HALBWACHS Nicolas, HENZINGER Thomas A., HO P-H, NICOLLIN Xavier, OLIVERO Alfredo, SIFAKIS Joseph and YOVINE Sergio (1995), "The algorithmic analysis of hybrid systems", *Theoretical Computer Science*, 138(1):3-34.

[2] ANDERIES John M., RODRIGUEZ Armando A., JANSSEN Marco A. and CIFDALOZ Oguzhan (2007), “Panaceas, uncertainty, and the robust control framework in sustainability science”, *Proceedings of the National Academy of Sciences*, 104(39): 15194-15199.

[3] ANDERIES John M. and JANSSEN Marco A. (2013), “Robustness of social-ecological systems: implications for public policy”, *Policy Studies Journal*, 41(3): 513-536.

[4] BENGEA Sorin C. and DECARLO Raymond A. (2005), "Optimal control of switching systems", *Automatica*, 41:11 – 27.

[5] BISCHI Gian Italo , LAMANTIA Fabio and RADI Davide (2013), "A prey–predator fishery model with endogenous switching of harvesting strategy", *Applied Mathematics and Computation*, Volume 219, Issue 20, Pages 10123-10142.

[6] BISCHI Gian Italo , LAMANTIA Fabio and TRAMONTANA Fabio (2014), "Sliding and oscillations in fisheries with on–off harvesting and different switching times", *Commun Nonlinear Sci Numer Simulat*, 19:216–229.

[7] BLANCHARD Eunice A, LOXTON Ryan and REHBOCK Volker (2013), "A computational algorithm for a class of non-smooth optimal control problems arising in aquaculture operations", *Appl. Math. Comput.* 219, 8738–8746.

[8] BRANICKY Michael S. (1994), "Stability of switched and hybrid systems," *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, pp. 3498-3503 vol.4.

[9] BRANICKY Michael S. and MITTER Sanjoy K. (1995), "Algorithms for Optimal Hybrid Control", *Proceedings of the 34th Conference on Decision & Control*, pp. 2661-2666 vol.3.

[10] BRANICKY Michael S., BORKAR Vivek S. and MITTER Sanjoy K. (1998), "A unified framework for hybrid control: Model and optimal control", *IEEE*

Transactions on Automatic Control, 43(1):31-45.

[11] BRANICKY Michael S. (1998), "Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems", *IEEE Transactions on Automatic Control*, 43(4):475-481.

[12] BRAUER Fred and CASTILLO-CHAVEZ Carlos (2012), *Mathematical Models in Population Biology and Epidemiology*. Springer.

[13] CIFDALOZ Oguzhan (2022), "Sustainable Management of a Renewable Fishery Resource with Depensation Dynamics from a Control Systems Perspective", *Gazi University Journal of Science*, 35(3): 936-955.

[14] CLARK Colin W. (1990), *Mathematical Bioeconomics*. John Wiley & Sons, Inc., New York.

[15] CLARK Colin W. (2006), *The worldwide crisis in fisheries: economic models and human behavior*. Cambridge University Press, New York.

[16] CLARK, Colin W. (2006), “Fisheries bioeconomics: why is it so widely misunderstood?”, *Population Ecology*, 48(2): 95-98.

[17] COUNCILL Elizabeth L. (2016), "Incorporating stochasticity in the study of exploited fish population dynamics: Implications for the study of post-recruitment harvest strategies". *Mathematical Biosciences*, Volume 273, Pages 16-22.

[18] CREPIN Anne-Sophie, NORBERG Jon and MALER Karl-Goran (2011), “Coupled economic-ecological systems with slow and fast dynamics - modelling and analysis method”, *Ecological Economics*, 70: 1448-1458.

[19] CUNNIFFE Nik J., COBB Richard C., MEENTEMEYER Ross K., RIZZO David M. and GILLIGAN Christopher A. (2016), “Modeling when, where, and how to manage a forest epidemic, motivated by sudden oak death in California”, *Proceedings of the National Academy of Sciences*, 113(20): 5640-5645.

[20] DAS Debabrata and KAR Tapan Kumar (2019), “Feedback control and its impact on generalist predator-prey system with prey harvesting”, *Nonlinear Analysis-Modelling and Control*, 24(5): 718-732.

[21] DE GIOVANNI Domenico and LAMANTIA, Fabio (2018), “Dynamic harvesting under imperfect catch control”, *Journal of Optimization Theory and Applications*, 176(1): 252-267.

[22] GELICK Peter H. (2018), "Transitions to freshwater sustainability", *Proceedings of the National Academy of Sciences*, 115 (36) 8863-8871.

[23] GOEBEL Rafal , SANFELICE Richardo G. and TEEL A. R. (2012), *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton University Press, Princeton.

[24] HEDLUND Sven and RANTZER Anders (1999), "Optimal Control of Hybrid Systems", *Proceedings of the 38th IEEE Conference on Decision and Control*, vol.4 pp. 3972-3977.

[25] HENZINGER Thomas A. (1996), "The theory of hybrid automata", In *Proc. 11th Annual Symp. on Logic in Comp. Science*, pages 278-292. IEEE CS Press.

[26] HILKER Frank M. and LIZ Eduardo (2020), "Threshold harvesting as a conservation or exploitation strategy in population management", *Theoretical Ecology*, 13:519–536.

[27] IBRAHIM Mahmud (2020), "Optimal Control of a Fishery Utilizing Compensation and Critical Depensation Models", *Applied Mathematics&Information Sciences* 14, No. 3, 467-479.

[28] JIANG Canghua, TEO Kok Lay, LOXTON Ryan and DUAN Guang-Ren (2012), "A neighboring extremal solution for an optimal switched impulsive control problem", *J. Ind. Manag. Optim.*, 8, 591–609.

[29] LIBERZON Daniel (2003), *Switching in Systems and Control*, Systems and Control: Foundations and Applications Series. Birkhauser.

[30] LIBERZON Daniel (2012), *Calculus of Variations and Optimal Control Theory*, Princeton University Press.

[31] LIN Qun, LOXTON Ryan, TEO Kok Lay and WU Yong Hong (2011), "A new computational method for optimizing nonlinear impulsive systems", *Dyn. Contin. Discrete Impuls. Syst. Ser. B, Appl. Algorithms* 18, 59–76.

[32] LIN Qun, LOXTON Ryan and TEO Kok Lay (2013), "Optimal Control of Nonlinear Switched Systems: Computational Methods and Applications", *Journal of the Operations Research Society of China* 1:275–311.

[33] LIZ Eduardo and LOIS-PRADOS Cristina (2020), "Dynamics and bifurcations of a family of piecewise smooth maps arising in population models with threshold harvesting", *Chaos*, 30, 073108, 2020.

[34] LYGEROS John (1996), *Hierarchical, hybrid control of large scale systems*, PhD Thesis, University of California, Berkeley.

[35] LYGEROS John, TOMLIN Claire and SASTRY Shankar (1999), "Controllers for reachability specifications for hybrid systems", *Automatica*, 35(3): 349-370.

[36] MORAN Emilio F. and OSTROM Elinor (2005), *Seeing the forest and the trees: human-environment interactions in forest ecosystems*, The MIT Press, Massachusetts.

[37] MULLON Christian, FREON Pierre and CURY Philippe (2005), "The dynamics of collapse in world fisheries", *Fish and Fisheries*, Volume 6, Issue 2, Pages 111-120.

[38] PAULOO Richard A., ESCRIVA-BOU Alvar, DAHLKE Helen, FENCL Amanda, GUILLON Herve and FOGG Graham E. (2020), "Domestic well vulnerability to drought duration and unsustainable groundwater management in California's central valley", *Environmental Research Letters*, 15(4): 044010.

[39] PEI Yongzhen, CHEN Miaomiao, LIANG Xiyin and LI Changguo (2019), "Model-based on fishery management systems with selective harvest policies", *Mathematics and Computers in Simulation*, 156: 377-395.

[40] PEPYNE David L. and CASSANDRAS Christos G. (2000), "Optimal Control of Hybrid Systems in Manufacturing", *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1108-1123.

[41] POWLSON David S., GREGORY Peter J., WHALLEY W. Richard, QUINTON John N., HOPKINS David W., WHITMORE Andrew Peter, HIRSCH Penny R. and GOULDING Keith W.T. (2011), "Soil management in relation to sustainable agriculture and ecosystem services", *Food Policy*, 36: 72–87.

[42] PRETTY Jules (2008), "Agricultural sustainability: concepts, principles and evidence", *Phil. Trans. R. Soc.*, B363447–465.

[43] REBAZA Jorge (2012), "Dynamics of prey threshold harvesting and refuge", *Journal of Computational and Applied Mathematics*, 236, 1743-1752.

[44] RIEDINGER Pierre, IUNG Claude and KRATZ Frederic (2003), "An optimal Control Approach for Hybrid Systems", *European Journal of Control*, 9:449-458.

[45] SCHOON Michael L. and COX Michael E. (2012), "Understanding disturbances and responses in social-ecological systems", *Society and Natural Resources*, 25(2): 141-155.

[46] SETHI Gautam, COSTELLO Christopher, FISHER Anthony, HANEMANN Michael and KARP Larry (2005), "Fishery management under multiple uncertainty", *Journal of Environmental Economics and Management*, 50(2): 300-318.

[47] SMITH Vernon L. (1969), "On models of commercial fishing", *Journal of Political Economy*, 77, 181-198.

[48] UTKIN Vadim I. (1992), *Sliding Modes in Control and Optimization*. Communica-

tions and Control Engineering Series. Springer-Verlag Berlin, Heidelberg.

[49] YU Run and LEUNG Pingsun (2006), "Optimal partial harvesting schedule for aquaculture operations", *Mar. Resour. Econ.* 21, 301–315.