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MASTER'S THESIS

(ALPHA-PSI) TYPE CONTRACTIVE MAPPINGS AND RELATED FIXED POINT THEOREMS

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ABSTRACT

(ALPHA-PSI) TYPE CONTRACTIVE MAPPINGS AND RELATED FIXED POINT THEOREMS

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This thesis consists of three sections. The first section is reserved for introduction, some basic definitions and declaration of literature about alpha-psi type contractive mappings and related fixed point theorems have been presented. The second section deals with the main results. We have applied definition of alpha-admissible for two different functions of f and T. Furthermore, we have investigated their results. Finally, the third section includes discussion and conclusion.

Keywords: Fixed Point, Contractive Mappings, (α-ψ)-Contractive Condition, α-admissible maps, Multi-Valued Maps, Coincidence Points

ÖZ

ALPHA-PSİ TİPİNDEN BÜZÜLME DÖNÜŞÜMLERİ VE İLGİLİ SABİT NOKTA TEOREMLERİ

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Bu çalışma üç bölümden oluşmuştur. Birinci bölüm giriş,temel tanımlar,literatür bilgileri alpha-psi tipinden büzülme dönüşümleri ve ilgili sabit nokta teoremlerine ayrılmıştır. İkinci bölümde ana fikri verdik. Alpha-admissible tanımını iki farklı fonksiyon olan f ve T için uyguladık ve sonuçlarını inceledik. Üçüncü bölümde sonuç ve tartışmalara yer verdik.

Anahtar Kelimeler: Sabit Nokta, Büzülme Dönüşümleri, alpha-psi tipinden büzülme koşulu, α-admissible dönüşümler, Çoğul değerli dönüşümler, Çakışma noktaları

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CHAPTER I

Introduction

1.1 SOME BASIC DEFINITIONS AND THEOREMS

In 1922, Banach published his fixed point theorem also known as Banach's Contraction Principle using the concept of Lipschitz mappings.

Definition 1.1.1. *Let (M, d) be a metric space. The map $T : M \to M$ is said to be lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d(Tx, Ty) \leq kd(x, y)$$

A lipschitzian mapping with a lipschitz constant k less than 1, i.e. $k < 1$, is called contraction.

Theorem 1.1.1 (Banach's Contraction Principle)**.** *Let (M, d) be a complete metric space and let $T : M \to M$ be a contraction mapping. Then, T has a unique fixed point x_0; and for each $x \in M$, we have*

$$lim_{n\to\infty} T^n x = x_0$$

Moreover, for each $x \in M$, we have

$$d(T^n x, x_0) \leq \frac{k^n}{1-k} d(Tx, x).$$

Definition 1.1.2 (Fixed Point)**.** *Let f be a function which maps a set of X into itself; i.e. $f : X \to X$. A fixed point of the mapping f is an element x belonging to X such that $f(x) = x$.*

Definition 1.1.3 (Contraction Mapping)**.** *Let (X, d) be a metric space. Then, a map $T : X \to X$ is called as a contraction mapping on X if there exists $q \in [0, 1)$ such that $d(Tx, Ty) \leq q\ d(x, y)$ for all x, y in X.*

Definition 1.1.4 (Coincidence Point)**.** *The Coincidence point (or simply coincidence) of two mappings in their domain has the same image point under both mappings.*
Formally, given two mappings $f, g \colon X \to Y$, it can be said that a point x in X is a coincidence point of f and g if $f(x) = g(x)$.
Coincidence points is, in most settings, a generalization of fixed point theory, the study of points x with $f(x) = x$. Fixed point theory is the special case obtained from the above by letting $X = Y$ and taking g to be the identity mapping. Just as fixed point theory has its fixed-point theorems, there are theorems that also guarantee the existence of coincidence points for pairs of mappings.

1.2 SOME KNOWN RESEARCH

In this part, We have investigated what has happened about (α-ψ) Type Contractive Mappings and Related Fixed Point Theorems, and who the founders of this theory are and how they have been classified by these researchers. We have gone through some articles about all of them.

In the last decades, metric fixed point theory has had many applications in functional analysis. The contractive conditions on underlying functions play an important role to find solutions for metric fixed point problems. The Banach contraction principle is a remarkable result in metric fixed point theory. Metric fixed point theory has been appreciated by a number of authors who have improved the celebrated Banach fixed point theorem for various contractive mapping in the context of different abstract spaces. The authors introduced the notions of (α-ψ)-contractive mappings, and investigated the existence and uniqueness of a fixed point for such mappings. Further, they showed that several well-known fixed point theorems can be derived from the fixed point theorem of (α-ψ)-contractive mappings.

In 2012, Samet [1] introduced the concepts of (α-ψ)-contractive and α-admissible mappings, and established various fixed point theorems for such mappings in complete metric spaces.

Afterwards, Karapınar and Samet [2] generalized the notion (α-ψ)-contractive mappings, and obtained a fixed point for this generalized version.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

In the literature, these functions are known as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Recently, Samet et al.[1] introduced the following concepts.

Definition 1.2.1. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We can say that T is an (α-ψ)-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \Psi(d(x,y)), \quad for \quad all \quad x,y \in X.$$

Clearly, any contractive mapping, which could be, a mapping satisfying Banach contraction is an (α-ψ)-contractive mapping with $\alpha(x,y) = 1$ for all $x,y \in X$ and $\Psi(t) = kt$, $k \in (0,1)$.

Definition 1.2.2. *Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is α-admissible if for all $x,y \in X$, and we have*

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1$$

We have shown some examples.

Example 1.2.1. *Let $X = (0,\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Tx = ln(x+1)$ for all $x \in X$ and*

$$\alpha(x,y) = \begin{cases} e & : if \quad x \geq y \\ 0 & : if \quad x < y \end{cases}$$

Then, T is α-admissible.

Example 1.2.2. *Let $X = [1,\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Tx = x^2$ for all $x \in X$ and*

$$\alpha(x,y) = \begin{cases} x+y & : if \quad x \geq y \\ 0 & : if \quad x < y \end{cases}$$

Then, T is α-admissible.

Example 1.2.3. *Let $X = (0,+\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Tx = lnx$ for all $x \in X$ and*

$$\alpha(x,y) = \begin{cases} 2 & : if \quad x \geq y \\ 0 & : if \quad x < y \end{cases}$$

Then, T is α-admissible.

Example 1.2.4. *Let $X = [0, +\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} e^{x-y} & : if \quad x \geq y \\ 0 & : if \quad x < y \end{cases}$$

Then, T is α-admissible.

Remark 1.2.1. *In the examples (1.2.3) and (1.2.4), every nondecreasing self-mapping T is α-admissible.*

Theorem 1.2.1. *(Samet [1]) Let (X, d) be a complete metric space, and $T : X \times X$ be an $(\alpha$-$\psi)$-contractive mapping. Suppose that*

1. *T is α admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *T is continuous.*
   Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.2.2. *Let (X, d) be a complete metric space and $T : X \times X$ be an $(\alpha$-$\psi)$-contractive mapping. Suppose that*

1. *T is α admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*
   Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.2.3. *In addition to the hypotheses of Theorem* 1.2.1 *(resp., Theorem* 1.2.2*) the condition, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ $\alpha(y, z) \geq 1$, and one obtains uniqueness of the fixed point.*
In this study, we have introduced the concept of generalized $(\alpha$-$\psi)$-contractive type mappings, and we have studied the existence and uniqueness of fixed points for such mappings. The theorems presented in this study extend and generalize the above results derived by Samet et al. in [1].Moreover, from the fixed point theorems, we have had the possibility deduce various fixed point results on metric spaces endowed with a partial order and fixed point results for cyclic contractive mappings.

On the other hand, Asl [3] characterized the notions of (α-ψ)-contractive mapping and α-admissible mappings with the notions of (α-ψ)-contractive and α -admissible mappings to investigate the existence of a fixed point for a multivalued function.

Furthermore, Rezapour and Shahzad [4] generalized the notion of (α-ψ)-Ciric in fixed point results for multivalued mappings.
Denoted by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{+} \infty\ \psi^n(t) < +\infty$ for all $t > 0$. It is well-known that $\psi(t) < t$ for all $t > 0$. Let (X, d) be a metric space, $\beta : 2^x \times 2^x \to [0, \infty)$ be a mapping and $\psi \in \Psi$. A multivalued operator $T : X \to 2^x$ is said to be (β-ψ) contractive whenever $\beta(Tx, Ty)H(Tx, Ty) \leqslant \psi(d(x, y))$ for all $x, y \in X$, where H is the Hausdorff metric. Alikhani, Rezapour and Shahza proved the fixed point results for (β-ψ) contractive multifunctions. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping and $\psi \in \Psi$. It can be stated $T : X \to 2^x$ is an (α-ψ)-Ciric generalized multifunction if

$$\alpha(x, y)d(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$. Also, it can be said that the self-map F on X is α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Fx, Fy) \geq 1$ [1]. In this study, we it can be said fixed point results for (α-ψ)-Ciric generalized multifunctions.
In 2012, Haghi, Rezapour and Shahzad proved that some fixed point generalizations are not the real ones. Here, by presenting a result and an example, we are going to show that obtained results in this new field are the real generalizations in respect to the previous ones in the literature.

Lemma 1.2.1. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ be a function, $\psi \in \Psi$ and T be a self-map on X such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi \left(max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)] \right\} \right)$$

for all $x, y \in X$. Suppose that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, T_{x0}) \geq 1$. Assume that if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all n. Then T has a fixed point.

Next, MU Ali,Kamran and Karapınar [5] proved fixed point theorems for non-self multivalued (α-ψ)-contractive type mappings using a new condition. Sallimi's and Hussain's [6] work was to modify the notions of (α-ψ)-contractive and α-admissible mappings further, and establish fixed point theorems for such mappings in complete metric spaces. After that, Karapınar, Salimi and Vetro [7] introduced the notion of a G-(α-ψ)-Meir-Keeler contractive mapping and proved some fixed points theorems for this class of G-metric spaces. Following them, Gordji, Karapınar and Sintunavarat [8] introduced a new type of generalized (α-ψ)-Meir-Keeler contractive maapping and established some interesting theorems on the existence of fixed points for such mappings via admissible mappings.

In 1969, Meir and Keeler established a fixed point theorem in a metric space (X, d) for mappings satisfying the condition that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \geq d(x, y) < \epsilon + \delta(\epsilon) \quad implies \quad d(Tx, Ty) < \epsilon$$

for all $x, y \in X$. This condition is called as the Meir-Keeler contractive type condition. We have introduced a new type of contractive mapping based on Meir-Keeler type contractive condition. For such mappings, we have studied and established fixed point theorems via admissible mappings. Moreover, we have presented some applications of our new results.
In this regard, let $\mathbb{N}$ denote the set of positive integers. Let ψ stands for the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Remark 1.2.2. *For every function $\psi : [0, \infty) \to [0, \infty)$ the following holds:*

$$\lim_{n\to\infty} \psi(t) = 0 \longrightarrow \psi(t) < t \longrightarrow \psi(0) = 0$$

Therefore, if $\psi \in \Psi$, then for each $t > 0$, $\psi(t) < t$ and $\psi(0) = 0$.

Example 1.2.5. *Let $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ be defined in the following way:*

$$\psi_1 = \frac{1}{2}t \ and \ \psi_2 = \begin{cases} \frac{t}{3} & : if \quad 0 \leq t < 1 \\ \frac{t}{5} & : if \quad t \geq 1 \end{cases}$$

It is clear that $\psi_1, \psi_2 \in \Psi$. Notice that ψ_1, ψ_2 are examples of continuous and discontinuous functions in Ψ.

Remark 1.2.3. *If $T : X \to X$ satisfies the Banach contraction principle in a metric space (X, d), then T is an $(\alpha\text{-}\psi)$-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t > 0$, where $k \in [0, 1)$.*

Afterwards, Karapınar [9] investigated the existence and uniqueness of fixed points of $(\alpha\text{-}\psi)$-contractive mappings in complete generalized metric spaces, introduced by Branciari.

In addition, Karapınar [10] considered a generalization of $(\alpha\text{-}\psi)$-Geraghty contractions and investigated the existence and uniqueness of fixed point for the mapping satisfying this condition. It is important to recall Geraghty's theorem. For this purpose, it will be significant first to remind the class of F all functions $\beta : [0, \infty) \to [0, 1)$which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0$$

Theorem 1.2.4. *(Geraghty) Let (X, d) be a complete metric space and $T : X \to X$ be an operator. If t satisfies the following inequality:*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

for any $x, y \in X$, where $\beta \in F$ the T has a unique fixed point.

Definition 1.2.3. *Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbf{R}$ be a function. A mapping $T : X \to X$ is said to be generalized $(\alpha\text{-}\psi)$-Geraghty contraction if there exists $\beta \in F$ such that*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$$

for any $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

and $\psi \in \Psi$.
Notice that if we take $\psi(t) = t$ in Definition[1.2.3],then T is called generalized α-Geraghty contraction mapping.

After, MU Ali, Kamran and Karapınar [11] considered the characterization of the notions of $(\alpha\text{-}\psi)$-contractive and α-admissible mappings in the context uniform spaces.

Lastly, Karapınar, Shahi and Tas [12] introduced two classes of generalized (α-ψ)-contractive type mappings of integral type in order to analyze the existence of fixed points for these mappings in complete metric spaces.
We have touched upon some necessary definitions and basic results from the literature. Throughout thesis, let $\mathbb{N}$ denote the set of all nonnegative integers. Berzig and Rus [13] introduced the following definition.

Definition 1.2.4. *(see [13]) Let $N \in \mathbb{N}$. We can say that α is N-transitive (on X) if*

$$x_0, x_1, ..., x_{N+1} \in X : \quad \alpha(x_i, x_{i+1}) \geq 1$$

for all $i \in \{0, 1, ..., N\} \Rightarrow \alpha(x_0, x_{N+1}) \geq 1$
In particular, we can say that α is transitive if it is 1-transitive, i.e.,

$$x, y, z \in X : \alpha(x, y) \geq 1 \quad and \quad \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$$

As consequences of Definition [1.2.4], we obtain the following remarks.

Remark 1.2.4. *(see [13])*

1. *Any function $\alpha : X \times X \to [0, +\infty)$ is 0-transitive.*
2. *If α is N transitive, then it is kN-transitive for all $k \in \mathbb{N}$.*
3. *If α is transitive, then it is N-transitive for all $N \in \mathbb{N}$.*
4. *If α is N-transitive, then it is not necessarily transitive for all $N \in \mathbb{N}$.*

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(Ψ_1) *ψ is nondecreasing;*
(Ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for all $t > 0$, where ψ^n is the nth iterate of ψ.*
In the literature, such mappings are called in two different ways: (c)-comparison functions in some sources(see, e.g., [11]), and Bianchini-Grandolfi gauge functions in some others (see, e.g., [12-14]).
It can be easily verifed that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.
Define $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}\}$ be such that φ is nonnegative,Lebesgue integrable and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad for \quad each \quad \epsilon > 0$$

Shahi et al. in [14] introduced the following new concept of (α-ψ)-contractive type mappings of integral type.

Definition 1.2.5. *Let (X, d) be a metric space and $T : X \times X$ be a given mapping. We say that T is an (α-ψ)-contractive mapping of integral type if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$,*

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi \left(\int_0^{d(x,y)} \varphi(t)dt \right).$$

where $\varphi \in \Phi$. In what follows, we recollect the main results of Shahi et al. [14].

Theorem 1.2.5. *Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping. Suppose that $T : X \times X$ is an (α-ψ)-contractive mapping of integral type and satisfies the following conditions:*

1. *T is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *T is continuous.*

Then, T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Theorem 1.2.6 (14)**.** *Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping. Suppose that $T : X \times X$ is an (α-ψ)-contractive mapping of integral type and satisfies the following conditions:*

1. *T is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then, T has a fixed point. In this point, there exists $z \in X$ such that $Tz = z$.
It is important to pay attention that in the theorems above, the authors proved only the existence of a fixed point. To guarantee the uniqueness of the fixed point, the following condition is needed. (U): For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

The last one is [15] to employ multivalued maps for (φ-ψ)-contraction condition in a complete metric space. Murthy and Tiwari have obtained a common fixed point theorem for (φ-ψ)-contraction under compatible maps of $type(A)$.
After the Banach fixed point theorem a lot of researches have been carried out to extend and generalize the Banach fixed point theorem in different spaces. We have divided these researchers into two groups:

1. One group of researchers tried to obtain fixed points by using different contraction conditions such as Edelstein [21], Kannan [23], Browder [18], Ciric [20], etc.

2. Another group of researchers tried to weaken the contraction condition by introducing a control functions in place of the contraction constant $\alpha \in (0, 1)$. In particular, Rakotch [25] and Boyd and Wong [19] obtained fixed points of a self map which were employed on the contraction condition in a complete metric space which follows:

Let $T : X \to X$ be such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad for \quad all \quad x, y \in X$$

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad for \quad all \quad x, y \in X$$

In the earlier results, the control functions $\varphi, \alpha : [0, \infty] \to [0, \infty]$ are continuous and monotonically decreasing. Using these control functions, we have a healthy literature in the context of fixed point theory dealt with it and applications.
Later, it was an open problem before the researchers working in the area of fixed point theory and applications existing in any contraction condition which is weaker than of Banach [17].
It was Rhoades [26] who responded the question and established a theorem in a complete metric space by implementing the result of Alber et. al [16] in Hilbert spaces to a complete metric space.
The contraction condition used by Rhoades ([26]) is in the following.
A mapping $T : X \to X$ satisfies the following condition

$$d(Tx, Ty) \leq d(x, y)\varphi(d(x, y)), \quad for \quad all \quad x, y \in X.$$

After Rhoades [26], a good number of results appeared in the literature of fixed point theory and applications.

Definition 1.2.6 (22)**.** *Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be compatible of type(A) if*

$$\lim_{n\to\infty} d(TS(x_n)), SS(x_n)) = 0 \quad \text{and} \quad d(ST(x_n), TT(x_n)) = 0$$

whenever x_n is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some $t \in X$.

Theorem 1.2.7. *Let (X, d) be a complete metric space and let A, B, S and T : $X \to X$ be a mapping satisfying*

$$\psi(d^2(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y))$$

for all $x, y \in X$ with $x \neq y$ and

$$M(x, y) = \max\{d^2(Sx, Ty), (Sx, Ax).d(Ty, By), d(Sx, Ax).d(Ty, Ax), \\ \frac{1}{2}(d(Sx, By).d(Ty, By)), (d(Sx, By).d(Ty, Ax))\}$$

and

$$N(x, y) = \min\{d^2(Sx, Ty), (Sx, Ax).d(Ty, By), d(Sx, Ax).d(Ty, Ax), \\ \frac{1}{2}(d(Sx, By).d(Ty, By)), (d(Sx, By).d(Ty, Ax))\}$$

$A(X) \subset T(X)$ and $B(X) \subset S(X)$;
One of A, B, S, and T is continuous; A, S and B, T are compatible pair of type(A);

$\varphi : [0, \infty) \to [0, \infty)$ is a mapping such that $\varphi(t) > 0$; which is lower semi-continuous for all $t > 0$ and φ is discontinuous at $t = 0$ with $\varphi(0) = 0$ and $\psi : [0, \infty) \to [0, \infty)$ is an alternating function.
Then A, B, S and T have a unique common fixed point in X.

In the recent years, Chandok, Taş and Ansari [27] have investigated some fixed point results for TAC-type contractive mappings. They have proved some fixed point results for new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces. Their results have extended, generalized and improved some of the well-known results from literature.

Let X be a nonempty set and $T : X \to X$ be an arbitrary mapping. It can be said that $x \in X$ is a fixed point for T , if $x = Tx$. We denote Fix(T) the set of all fixed points of T .

Definition 1.2.7 (28). *Let $T : X \to X$ be a mapping and $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. We say that T is a cyclic (α, β)-admissible mapping if*

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(Tx) \geq 1$,

(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(Tx) \geq 1$.

Example 1.2.6 (28). *Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(-x) = -T(x)$. Suppose that $\alpha, \beta : \mathbb{R} \to \mathbb{R}^+$ are given by $\beta(x) = 5^x$ for all $x \in \mathbb{R}$ and $\alpha(y) = 5^{-y}$ for all $y \in \mathbb{R}$. Then T is a cyclic (α, β) -admissible mapping.*

Let Ψ denote the set of all monotone increasing continuous functions
$\psi : [0, \infty) \to [0, \infty)$, with $\psi^{-1}(\{0\}) = 0$.

Let Φ denote the set of all continuous functions
$\phi : [0, \infty) \to [0, \infty)$, with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

Lemma 1 (19). *Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon$ and*
(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$;
(iii)$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$.

Remark 2. *in a same way to the proof of Lemma 1, we get*

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon$$

.

In 2014, the concept of C-class functions(see Definition 1.2.8) was introduced by H. Ansari in [29] and is important to see the number of examples (1),(2) from Example 1.2.7.

Definition 1.2.8. *[29] It can be pointed out that $f : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

(1) $f(s, t) \leq s$;

(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$;

for all $s, t \in [0, \infty)$,.

Note that $f(0, 0) = 0$.

We denote C-class functions as $\mathcal{C}$.

Example 1.2.7. *[29] The following functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$.*

(1) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;

(2) $f(s, t) = ks$,$k \in (0, 1)$, $f(s, t) = s \Rightarrow t = 0$;

(3) $f(s, t) = \frac{s}{(1+t)}$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $f(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $f(s, t) = \ln(1 + a^s)/2$, $a > e$, $f(s, t) = s \Rightarrow s = 0$;

(6) $f(s, t) = (s + l)^{(1/(1+t))} - l$, $l > 1$, $f(s, t) = s \Rightarrow t = 0$.

Definition 1.2.9. *Let (X, d) be a metric space and $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. It can be claimed that $T : X \to X$ is a TAC-contractive mapping if*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))) \tag{1.2.1}$$

for $x, y \in X$, where $f \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$.

Now, let us prove our first theorem.

Theorem 1.2.8. *Let (X,d) be a complete metric space and $T : X \to X$ be a cyclic (α,β)-admissible mapping. Assume that T is a TAC-contractive mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and either of the following conditions hold:*

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.

Then T has a fixed point.

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic (α,β)-admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. Again, since T is a cyclic (α,β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. From (1.2.1), we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq f(\psi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n))) \\ &\leq \psi(d(x_{n-1}, x_n)). \end{aligned} \tag{1.2.2}$$

Using monotonicity of ψ, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Hence the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. So for the nonnegative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{1.2.3}$$

Assume that $r > 0$. On letting $n \to \infty$ in (1.2.2), using the continuity of ψ and f and inequality (1.2.3), we obtain

$$\psi(r) \leq f(\psi(r), \phi(r)) \leq \psi(r), \tag{1.2.4}$$

thus $f(\psi(r), \phi(r)) = \psi(r)$. Now, by using Definition 1.2.8, we get that either $\psi(r) = 0$ or $\phi(r) = 0$, in both cases it follows that $r = 0$, which implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.2.5}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then by lemma 1 there exists an $\delta > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with, $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \delta. \tag{1.2.6}$$

Now, by setting $x = x_{m_k}$ and $y = x_{n_k}$ in (1.2.1), and using $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$, we obtain

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq f(\psi(d(x_{m(k)}, x_{n(k)})), \phi(d(x_{m(k)}, x_{n(k)})))$$

On letting $k \to \infty$, using (1.2.6) , we obtain

$$\psi(\delta) \leq f(\psi(\delta), \phi(\delta)) \leq \psi(\delta), \tag{1.2.7}$$

$\psi(\delta) = 0$, or $\phi(\delta) = 0$, that is, $\delta = 0$ which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, we firstly suppose that T is continuous. Hence,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So z is a fixed point of T

In the second part, we suppose that condition (b) holds, that is, $\alpha(x_n)\beta(z) \geq 1$. So, we have

$$\psi(d(x_{n+1}, Tz)) \leq f(\psi(d(x_n, z)), \phi(d(x_n, z))) \leq \psi(d(x_n, z)).$$

By taking the limit $n \to \infty$ and using the properties of ψ, we obtain $d(z, Tz) = 0$. Hence z is a fixed point of T.

To prove the uniqueness of fixed point, let us suppose that z_1 and z_2 are two fixed points of T. Since $\alpha(z_1)\beta(z_2) \geq 1$, from (1.2.1), we have

$$\psi(d(z_1,z_2)) = \psi(d(Tz_1,Tz_2)) \le f(\psi(d(z_1,z_2)),\phi(d(z_1,z_2))) \le \psi(d(z_1,z_2)).$$

Hence by using the properties of ψ, we have $z_1 = z_2$. □

Example 1.2.8. *Let $X = \mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $T : X \to X$ be defined by*

$$T(x) = \begin{cases} -\frac{x}{4}, & x \in [-2,1] \\ 3x, & \mathbb{R} \setminus [-2,1] \end{cases}$$

and $\alpha, \beta : X \to \mathbb{R}^+$ be given by

$$\alpha(x) = \begin{cases} 2 & x \in [-2,0] \\ 0 & \mathbb{R} \setminus [-2,0] \end{cases} \text{ and } \beta(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \mathbb{R} \setminus [0,1] \end{cases}$$

Also define $\psi \in \Psi$ as $\psi(t) = t$, $\phi \in \Phi$ as $\phi(t) = \frac{1}{3}$ and $F \in \mathcal{C}$ as $F(s,t) = \frac{s}{1+t}$.

Now, we firstly prove that T is a cyclic (α,β)-admissible mapping.

If $\alpha(x) \ge 1$. Then $x \in [-2,0]$ and $Tx \in [0,1]$. Therefore, $\beta(Tx) \ge 1$. Similarly, if $\beta(x) \ge 1$, then $\alpha(Tx) \ge 1$. Then T is a cyclic (α,β)-admissible mapping.

Now, let us check the hypotheses (b) of Theorem 1.2.8.

Let $\{x_n\} \subseteq X$ such that $\beta(x_n) \ge 1$ and $x_n \to x$. Therefore, $x_n \in [0,1]$. Hence $x \in [0,1]$,

Let $\alpha(x)\beta(y) \ge 1$. Then $x \in [-2,0]$ and $y \in [0,1]$ and so we have
$\psi(d(Tx,Ty)) = |Tx - Ty| = \frac{1}{4}|x-y| \le \frac{3}{4}|x-y| = \frac{|x-y|}{1+\frac{1}{3}} = \frac{\psi(d(x,y))}{1+\phi(d(x,y))}$.
Hence inequality (1.2.1) is satisfied. Therefore by Theorem 1.2.8, T has a fixed point.

Corollary 1.2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a cyclic $(\alpha,\beta)-$admissible mapping. Assume that T is a $(\alpha,\beta)-$contractive mapping, that is, for all $x,y \in X$,*

$$\alpha(x)\beta(y)\psi(d(Tx,Ty)) \le f(\psi(d(x,y)),\phi(d(x,y))). \tag{1.2.8}$$

Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and either of the following conditions hold:

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.

Then T has a fixed point.

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for $x, y \in X$. Hence by using (1.2.8), we have T is a TAC-contractive mapping. Therefore, by applying Theorem 1.2.8, we have reached the result. □

Definition 1.2.10. *Let (X, d) be a metric space and and $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. be a cyclic (α, β)-admissible mapping. A mapping $T : X \to X$ is called a weak $TAC-$ rational contraction if $\alpha(x)\beta(y) \geq 1$ for some $x, y \in X$ implies*

$$d(Tx, Ty) \leq f(M(x, y), \phi(M(x, y))), \tag{1.2.9}$$

where $f \in \mathcal{C}$, $\phi \in \Phi$ and $M(x, y) = \max\left\{d(x, y), \frac{[1+d(x,Tx)]d(y,Ty)}{d(x,y)+1}\right\}$.

Theorem 1.2.9. *Let (X, d) be a complete metric space and $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a weak $TAC-$ rational contraction. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and one of the following assertions hold:*

(a) T is continuous, or

(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.

Then T has a fixed point.

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in$ Fix(T), then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$. Since T is a cyclic (α, β)-admissible mapping and $\alpha(x_0) \geq 1$ then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get$\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. Again, since T is a cyclic (α, β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. That is,$\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Therefore by (1.2.9), we have

$$d(x_n, x_{n+1}) \leq f(M(x_{n-1}, x_n), \phi(M(x_{n-1}, x_n))), \tag{1.2.10}$$

where $M(x_{n-1}, x_n) = \{d(x_{n-1}, x_n, d(x_n, x_{n+1}))\}$.

Now, suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) > d(x_{n_0-1}, x_{n_0})$. Therefore, $M(x_{n_0-1}, x_{n_0}) = d(x_{n_0}, x_{n_0+1})$ and so from (1.2.10), we get

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\leq f(d(x_{n_0}, x_{n_0+1}), \phi(d(x_{n_0}, x_{n_0+1}))) \quad (1.2.11)\\ &\leq d(x_{n_0}, x_{n_0+1}). \quad (1.2.12) \end{aligned}$$

This implies that $d(x_{n_0}, x_{n_0+1}) = 0$, or $\phi(d(x_{n_0}, x_{n_0+1})) = 0$, that is $d(x_{n_0}, x_{n_0+1}) = 0$, which is a contradiction. Hence, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. As a result, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Thus, for the nonnegative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{1.2.13}$$

Assume that $r > 0$. On letting $n \to \infty$ in (1.2.11), using the continuity of ψ and f and (1.2.13), we obtain

$$r \leq f(r, \phi(r)) \leq r, \tag{1.2.14}$$

which implies that either $r = 0$, or $\phi(r) = 0$, that is in both cases it follows that $r = 0$, which implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.2.15}$$

Now, let us prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then by lemma 1 there exists an $\delta > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with, $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \delta. \tag{1.2.16}$$

Now, by setting $x = x_{n_k+1}$ and $y = y_{m_k+1}$ in (1.2.9), and using $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$, we obtain

$$d(x_{m_k+1}, x_{n_k+1}) \leq f(M(x_{n_k}, x_{m_k}), \phi(M(x_{n_k}, x_{m_k}))), \tag{1.2.17}$$

where $M(x_{n_k}, x_{m_k}) = \max\left\{d(x_{n_k}, x_{m_k}), \frac{[1+d(x_{n_k}, x_{n_k+1})]d(x_{m_k}, x_{m_k+1})}{d(x_{n_k}, x_{m_k})+1}\right\}$.

On letting $k \to \infty$, using (1.2.16) and (1.2.17), we obtain

$$\delta \leq f(\delta, \phi(\delta)), \tag{1.2.18}$$

$\psi(\delta) = 0$, or $\phi(\delta) = 0$, that is, $\delta = 0$ which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

First, we consider that T is continuous. Hence,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

Therefore, z is a fixed point of T.

As a second, we suppose that condition (b) holds, that is, $\alpha(x_n)\beta(z) \geq 1$. So, we have

$$d(x_{n+1}, Tz) \leq f(M(x_n, z), \phi(M(x_n, z))) \leq M(x_n, z)$$

,

where $M(x_n, z) = \max\left\{d(x_n, z), \frac{[1+d(x_n, x_{n+1})]d(z, Tz)}{d(x_n, z)+1}\right\}$. By taking the limit $n \to \infty$ and using the properties of ψ, we can obtain $d(z, Tz) = 0$. Hence z is a fixed point of T.

To prove the uniqueness of fixed point, suppose that z_1 and z_2 are two fixed points of T. Since $\alpha(z_1)\beta(z_2) \geq 1$, from (1.2.9), we have

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq f(M(z_1, z_2), \phi(M(z_1, z_2))) \leq M(z_1, z_2),$$

where $M(z_1, z_2) = \max\left\{d(z_1, z_2), \frac{[1+d(z_1, Tz_1)]d(z_2, Tz_2)}{d(z_1, z_2)+1}\right\}$.
This implies that $d(z_1, z_2) = 0$ or $\phi(d(z_1, z_2)) = 0$ and hence $z_1 = z_2$. $\square$

Example 1.2.9. *Let $X = [0, +\infty)$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $T : X \to X$ be defined by*

$$T(x) = \begin{cases} -\frac{x}{8}, & x \in [0, 1] \\ \frac{1}{2}, & x \in (1, +\infty) \end{cases}$$

and $\alpha, \beta : X \to \mathbb{R}^+$ be given by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & otherwise \end{cases}.$$

Also, define $\phi \in \Phi$ as $\phi(t) = \frac{t}{2}$ and $F \in \mathcal{C}$ as $F(s, t) = s - t$.

It is not so difficult to verify that T is a cyclic (α, β)-admissible mapping.

Now, we check the hypotheses (b) of Theorem 1.2.9.

Let $\{x_n\} \subseteq X$ such that $\beta(x_n) \geq 1$ and $x_n \to x$. Therefore, $x_n \in [0, 1]$. Hence $x \in [0, 1]$,

Let $\alpha(x)\beta(y) \geq 1$. Then $x \in [0, 1]$ and $y \in [0, 1]$, so we have $d(Tx, Ty) = |Tx - Ty| = \frac{1}{8}|x - y| \leq \max\left\{d(x, y), \frac{[1+d(x,Tx)]d(y,Ty)}{d(x,y)+1}\right\}$. In this way, inequality (1.2.9) is satisfied. Therefore by Theorem 1.2.9, T has a fixed point, that is, 0 is a fixed point of T.

Now, we apply some cyclic contraction via cyclic (α, β)-admissible mapping in a natural way. The Theorem (1.2.8) to prove a fixed point theorem involving a cyclic mapping.

Theorem 1.2.10. *Let A and B be two closed subsets of complete metric space (X, d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \to A \cup B$ be a mapping such that $TA \subset B$ and $TB \subset A$. Assume that*

$$\psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))) \tag{1.2.19}$$

for all $x \in A$ and $y \in B$ where $f \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique fixed point in $A \cap B$.

Proof. Define $\alpha, \beta : X \to \mathbb{R}^+$ by

$$\alpha(x) = \begin{cases} 1, & x \in A \\ 0, & otherwise \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & x \in B \\ 0, & otherwise \end{cases}.$$

Let $\alpha(x)\beta(y) \geq 1$. Then $x \in A$ and $y \in B$. Hence, by (1.2.19) we have

$$\psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

for all $x, y \in A \cup B$.

Let $\alpha(x) \geq 1$ for some $x \in X$, then $x \in A$. Hence, $Tx \in B$ and so $\beta(Tx) \geq 1$. Now, let $\beta(x) \geq 1$ for some $x \in X$, so $x \in B$. Hence, $Tx \in A$ and then $\alpha(Tx) \geq 1$. Therefore T is a cyclic (α, β)-admissible mapping. Since $A \cap B$ is nonempty, there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $x_n \in B$ for all $n \in \mathbb{N}$. Therefore $x \in B$. This implies that $\beta(x) \geq 1$. So the condition (b) of Theorem 3.2 hold. Therefore, T has a fixed point in $A \cup B$, say z. Since $z \in A$, then $z = Tz \in B$ and since $z \in B$, then $z = Tz \in A$. Therefore $z \in A \cap B$. The uniqueness of the fixed point follows easily from (1.2.19). □

Example 1.2.10. *Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $T : A \cup B \to A \cup B$ be defined by $Tx = -\frac{x}{3}$ where $A = [-1, 0]$ and $B = [0, 1]$. Also define ψ, $\phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\phi(t) = \frac{2}{3}t$. Indeed, for all $x \in A$ and all $y \in B$, we have*
$\psi(d(Tx, Ty)) = |Tx - Ty| = \frac{1}{3}|x - y| = \psi(d(x, y)) - \phi(d(x, y)) = f(\psi(d(x, y)), \phi(d(x, y)))$. Therefore, the conditions of Theorem 1.2.10 hold and T has a unique fixed point, that is, 0 is a fixed point of T.

Corollary 1.2.2. *Let A and B be two closed subsets of complete metric space (X, d) such that $A \cap B \neq \emptyset$, and $T : A \cup B \to A \cup B$ be a mapping such that $TA \subset B$ and $TB \subset A$. Assume that*

$$d(Tx, Ty) \leq f(d(x, y), \phi(d(x, y))), \tag{1.2.20}$$

for all $x \in A$ and $y \in B$ where $f \in \mathcal{C}$, and $\phi \in \Phi$. Then T has a unique fixed point in $A \cap B$.

CHAPTER II

Main Results

Now, we are going to apply α-admissible in Definition (1.2.2) for two different functions of f and T.
In addition to this, we are going to explain some definitions. To begin with, α-admissibility for a pair of mappings will be defined.

Definition 2.0.1. *Let T and f be self-mappings on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be another mapping. It can be that T and f are α-admissible if the following condition holds:*
$x, y \in X, \quad \alpha(fx, fy) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$

Definition 2.0.2. *Let (X, d) be a metric space and $T : X \to X$ and $f : X \to X$ is called f-weak compatible $\iff$ $(fT)(x) \subseteq X \quad \forall x \in X$ and*

- $\lim d(fTx_n, Tfx_n) \leq \lim d(Tfx_n, Tx_n)$
- $\lim d(fTx_n, fx_n) \leq \lim d(Tfx_n, Tx_n)$

whenever $x_n \in X$ such that $Tx_n \to t$, $fx_n \to t$ for some $t \in X$.

Now, the result for single-valued f-weak compatible mappings is to proved.

Theorem 2.0.1. *Let (X, d) be a complete metric space and $f : X \to X$ and $T : X \to X$ be the f-weak compatible pair such that $TX \subseteq fX$.Suppose that the following conditions hold:*

1. *T and f are α-admissible mappings;*
2. $\alpha(fx, fy) \geq 1 \Rightarrow \xi d(Tx, Ty) \leq \psi\xi(M(x, y))$ (2.0.1)
   where $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}$
   and ξ and ψ are as defined earlier.

3. *There exists* $fx_0 \in X$ *and* $fx_1 \in Tx_0$ *such that*

$$\alpha(fx_0, fx_1) \geq 1;$$

If one of the mappings T *and* f *is continuous,then there exists a point* $t \in X$ *such that* $ft = Tt = t$.

Proof. It is seen that the sequence $\{Tx_n\}$,where $Tx_n = fx_{n+1}$ for each n,is a Cauchy sequence. Hence it converges to some point $z \in X$. Suppose that T is continuous.Then $T^2x_n \to Tz$ and $Tfx_n \to Tz$. By f-weak compatibility of f and T, we have

1. $\lim\limits_{n\to\infty} d(fTx_n, Tfx_n) \leq \lim\limits_{n\to\infty} d(Tfx_n, Tx_n)$,

   and

2. $\lim\limits_{n\to\infty} d(fTx_n, fx_n) \leq \lim\limits_{n\to\infty} d(Tfx_n, Tx_n)$ (2.0.2).

Now,using (2.0.1),(2.0.2) and the continuity of T, we get

$$\begin{aligned}\xi d(T^2x_n, Tx_n) &\leq \psi\xi(M(x,y))\\ &\leq \psi\xi(\max\{d(fTx_n, fx_n), d(fTx_n, T^2x_n), d(fx_n, Tx_n), d(fTx_n, Tx_n),\\ &d(fx_n, T^2x_n))\}\\ &\leq \psi\xi(\max\{d(fTx_n, fx_n), d(fTx_n, Tfx_n), d(Tfx_n, T^2x_n), d(fTx_n, Tx_n),\\ &d(fx_n, Tx_n), d(fTx_n, Tx_n), d(fx_n, T^2x_n))\}\end{aligned}$$

that is,

$$\xi d(Tz, z) \leq \psi\xi(\max\{d(Tz, z), d(Tz, z), 0, d(Tz, z), d(z, Tz))\} \quad as \quad n \to \infty,$$

that is, $Tz = z$. Since $Tx \subseteq fX$,there exists a point z',such that $z = Tz = fz'$ and using (2.0.1) again,

$$\xi d(T^2x_n, Tx_n) \leq \psi\xi(\max\{d(fTx_n, z), d(fTx_n, T^2x_n), d(z, Tz'), d(fTx_n, Tz'), d(z, T^2x_n))\}$$

As $n \to \infty$ we deduce that $\xi d(z, Tz') \leq \psi\xi d(z, Tz')$;that is, $z = Tz' = fz'$ and we get

$fz = fTz' = Tfz' = Tz = z$.

Now, suppose that f is continuous. Then, $f^2x_n \to fz$ and $fTx_n \to fz$.By f-weak compatibility of f and T and continuity of f, we have

1. $\lim_{n\to\infty} d(fz, Tfx_n) \leq \lim_{n\to\infty} d(Tfx_n, z)$
   and

2. $\lim_{n\to\infty} d(fz, z) \leq \lim_{n\to\infty} d(Tfx_n, z)$ (2.0.3).

Now, using (2.0.1),(2.0.3) and continuity of f, we get

$$\xi d(Tfx_n, Tx_n) \leq \psi\xi(M(x,y))$$

$\leq \psi\xi(\max\{d(f^2x_n, fx_n), d(f^2x_n, Tfx_n), d(fx_n, Tx_n), d(f^2x_n, Tx_n), d(fx_n, Tfx_n))\}$

that is,

$\xi d(fz, z) \leq \xi d(Tfx_n, z) \leq \psi\xi(\max\{d(fz, z), d(fz, Tfx_n), 0, d(fz, z), d(z, Tfx_n)\}$as $n \to \infty$

$\xi d(fz, z) \leq \xi d(Tfx_n, z) \leq \psi\xi(\max\{d(fz, z), d(Tfx_n, z), 0, d(Tfx_n, z), d(z, Tfx_n)\}$as $n \to \infty$

that is,$Tfx_n \to z$ as $n \to \infty$ and $fz = z$. Again using (2.0.1) and (2.0.3), we have

$\xi d(Tz, Tfx_n) \leq \psi\xi(\max\{d(fz, f^2x_n), d(fz, Tz), d(f^2x_n, Tfx_n), d(fz, Tfx_n), d(f^2x_n, Tz)\}$

that is,

$$\xi d(Tz, z) \leq \psi\xi(\max\{d(0, d(z, Tz), 0, 0, d(z, Tz)\}\text{as } n \to \infty,$$

a contradiction. Therefore, z is a common fixed point of f and T.
Finally, example to discuss the validity of Theorem (2.0.1) is provided. □

Example 2.0.1. *Let $X = [0, \infty)$ be endowed with the Euclidean metric d. Let $fx = \frac{1}{2}(x^2 + x)$ and $Tx = \frac{1}{3}(x^2 + 2)$ for each $x \geq 0$.T and f are clearly continuous and $T(X) = f(X) = X$. Since $fx = Tx$ iff $x_n \to 1$. Also, it can be presented that f and T are f-weak compatible.*
let $\alpha : X \times X \to [0, \infty)$ by $\alpha(x,y) = \begin{cases} 1 & : \textit{when} \quad x, y \geq 0 \\ 0 & : \textit{otherwise} \end{cases}$
Take $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \sqrt{t}$ for each $t \geq 0$.
Then, T and f satisfy condition (2.0.1).
Moreover, T and f are α-admissible mappings. Thus, all the conditions of Theorem (2.0.1) are satisfied. Therefore, T and f have the coincidence point such as 1 is coincidence point of T and f.

Example 2.0.2. *Let $X = [0, \infty)$, and let $d(x, y) = |x - y|$. Let $fx = \frac{1}{4}(x^2 + 3)$ and $Tx = \frac{1}{3}(x^2 + 2x)$ for each $x \geq 0$. T and f are clearly continuous and $T(X) = f(X) = X$. Since $fx = Tx$ iff $x_n \to 1$. Also we can show that f and T are f-weak compatible.*

let $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \begin{cases} 1 & : when \quad x, y \geq 0 \\ 0 & : otherwise \end{cases}$

Take $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \sqrt{t}$ for each $t \geq 0$. Then, T and f satisfy condition (2.0.1).

Moreover, T and f are α-admissible mappings. Thus, all the conditions of Theorem (2.0.1) are satisfied. Therefore, T and f have the coincidence point such as 1 is coincidence point of T and f.

Remark 2.0.1. *Coupled fixed point theorems can be constructed for multi-valued as well as single-valued mappings by taking T defined as $T : X \times X \to CL(X)$ and $T : X \times X \to X$ in the theorem proved above. In order to deduce the results for coupled fixed point, it is crucial to take α defined as $\alpha : X^2 \times X^2 \to [0, \infty)$.*

CHAPTER III

Conclusion

In this thesis, We have worked on the topic of (α-ψ)-Type Contractive Mappings and Related Fixed Point Theorems. We have examined the (α-ψ)-Type Contractive Mappings And Related Fixed Point Theorems under the various headings and included some other researches from the literature.

In the last decades, metric fixed point theory has been appreciated by a number of authors who have extended the celebrated Banach fixed point theorem for various contractive mapping in the context of different absract spaces.Recently, Samet, Vetro and Vetro have introduced the notion of (α-ψ)-Type Mappings. After those researchers, Karapınar and Samet have generalized the notion of (α-ψ)-Contractive Mappings and obtained a fixed point for this generalized version. Furthermore, Asl has characterized the notions of (α-ψ)-contractive mapping and α-admissible mappings with the notions of (α-ψ)-contractive and α -admissible mappings to investigate the existence of a fixed point for a multivalued function.

Denote with Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$.

Let (X, d) be a metric space, T be a self-map on X, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is called an (α-ψ)-contraction mapping whenever $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Also, it can be stated that T is α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

These kind of researches are to be continued.

In this study, definition of α-admissibility has been applied for one function; however, the definition of α-admissibility has been expanded for two mappings T and f. Firstly, we have defined α-admissibility for a pair of mappings.

Let T and f be self-mappings on a nonempty set X and
$\alpha : X \times X \to [0, \infty)$ be another mapping. It can be pointed out that T and f are α-admissible if the following condition holds:
$x, y \in X, \quad \alpha(fx, fy) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$

After all, we have concluded that these functions are f-weak compatible, and they have a common fixed point. Consequently, T and f have a coincidence point such as 1. The equation of $fx = Tx$, x is applied to 1 as it is a condition of coincidence point. If we take the position that f is an identity, we will find α-admissible.
In the end, it could be said that the main part that we have approached in this study is not in the literature yet. Therefore there is a lot of corollary for it to be expanded and included in the results. This is an open-ended question.

REFERENCES

[1] *B. Samet, C. Vetro, P. Vetro: Fixed point theorems for (α-ψ)-contractive type mappings. Nonlinear Anal. 75, 2154-2165 (2012)*

[2] *E. Karapınar, B. Samet: Generalized (α-ψ)-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. 2012, Article ID 793486 (2012)*

[3] *JH. Asl, S. Rezapour, N. Shahzad: On fixed points of (α-ψ)-contractive multifunctions. Fixed Point Theory Appl. 2012, Article ID 212 (2012)*

[4] *B. Mohammadi, S. Rezapour, N. Shahzad: Some results on fixed points of (α-ψ)-Ciric generalized multifunctions. Fixed Point Theory Appl. 2013, Article ID 24 (2013)*

[5] *MU. Ali, T. Kamran, E. Karapınar:A new approach to (α-ψ)-contractive nonself multivalued mappings. J.Inequal.Appl. 2014:71 , (2012)*

[6] *P. Salimi, A. Latif, N. Hussain: Modified (α-ψ)-contractive mappings with applications. Fixed Point Theory Appl. 2013, Article ID 2013:151 (2013)*

[7] *N. Hussain, E. Karapınar, P. Salimi, P. Vetro : On fixed points of (α-ψ)-contractive multifunctions. Fixed Point Theory Appl. 2013, Article ID (2013:34)*

[8] *A. Latif, MA. Gordji, E. Karapınar, W. Sintunavarat: Fixed points results for generalized (α-ψ)-Meir-Keeler contractive mappings and applications. J.Inequal.Appl 2014, Article ID 68 (2014)*

[9] *E. Karapınar: Discussion on (α-ψ)-contractions on generalized metric spaces.Hindawi Appl. Article ID (962784) , (2014)*

[10] *E. Karapınar: (α-ψ)-Geraghty-contractive type mappings and some related fixed point results. Pmf. 2014, Article ID 37-48 (2014)*

[11] *MU. Ali, E. Karapınar, T. Kamran:Fixed points of (α-ψ)-contractive type mappings in uniform spaces. Fixed Point Theory Appl. 2014, article ID (2014:150)*

[12] *E. Karapınar, P. Shahi, K. Taş: Generalized (α-ψ)-contractive type mappings of integral type and related fixed point theorems. J.Inequal.Appl 2014, Article ID 160 (2014)*

[13] *M. Berzig, M. Rus: Fixed point theorems for α-contractive mappings of Meir-Keeler type and applications.arXiv:1303.5798[math.GN].*

[14] *P. Shahi, J. Kaur, S. Bhatia: Fixed point theorems for (α-ψ)-contractive type mappings of integral type with applications. J.Nonlinear Convex Anal., 2015.*

[15] *P.P. Murthy, S. Tiwai: (α-ψ)-contraction condition for various types of maps in a complete metric space. Mathematics Subject Classification. 2015, Article ID (47H10,54H25)*

[16] *Ya. I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces: I. Gohberg, Yu. Lyubich(Eds.), New Results in Operator Theory, Advances and Appl. 98, Birkhuser, Basel(1997), 7-22.*

[17] *S. Banach, Sur les Operations dans les Ensembles Abstraits et leur Applications aux Equations Integrales, Fund. Math. 3(1922), 133-181.*

[18] *F. E. Browder, The Topological Fixed Point Theory and Its Application to Functional Equations, Princeton Univ. princeton, 1948.*

[19] *D.W. Boyd, T.S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc., 20(1969), 458-464.*

[20] *Lj. Ciric, J. S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory and Applications, 2008.*

[21] *M. Edelstein, An extention of Banach's contraction principle, Proc. AMS, 12, 1961, 7-10.*

[22] *G. Jungck, P. P. Murthy and Y. J. Cho, Compatible Mappings of type (A) and Common Fixed Points, Math. Japonica 38, No. 2 (1993), 381-390.*

[23] *R. Kannan,Some Results on Fixed Points, Bull.Cal.Soc. 6(1968), 71-78..*

[24] *M. S. Khan, M. Swalesh, S. Sessa, Fixed Points Theorems by Altering Distances Between the Points, Bull. Austral. Math. Soc., 30(1984), 1-9..*

[25] *A. Rakotch, A note on Contraction Mappings, Proc.Aneer.Math.Soc. 13(1962), 459-462..*

[26] *B.E. Rhoades, Some theorem on weakly contraction maps, Nonlinear Anal. 47(2001), 2683-2693.*

[27] *S. Chandok, K. Taş, A.H. Ansari, Some Fixed Point Results For TAC Type Contractive Mappings, Journal of Function Spaces, 2016*

[28] *S. Alizadeh, F. Moradloub, P. Salimi, Some fixed point results for (α ,β)-(ψ ,ϕ)-contractive mappings, Filomat 28:3 (2014), 635–647.*

[29] *A. H.Ansari,Note on" $(\varphi$-$\psi)$-contractive type mappings and related fixed point",The 2nd Regional Conference onMathematics And Applications,Payame Noor University,Tonekabon, 2014,pages377-380*

CHAPTER IV

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