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Research paper

A numerical method based on the piecewise Jacobi functions for distributed-order fractional Schrödinger equation

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ABSTRACT

In this work, the distributed-order time fractional version of the Schrödinger problem is defined by replacing the first order derivative in the classical problem with this kind of fractional derivative. The Caputo fractional derivative is employed in defining the used distributed fractional derivative. The orthonormal piecewise Jacobi functions as a novel family of basis functions are defined. A new formulation for the Caputo fractional derivative of these functions is derived. A numerical method based upon these piecewise functions together with the classical Jacobi polynomials and the Gauss–Legendre quadrature rule is constructed to solve the introduced problem. This method converts the mentioned problem into an algebraic problem that can easily be solved. The accuracy of the method is examined numerically by solving some examples.

1. Introduction

The precedent of fractional calculus dates back to more than 300 years ago [1]. This topic is an extension of the traditional one that studies derivative and integral operators whose order may be non-integer. It has been shown that fractional operators have diverse applications in science and engineering [1]. Indeed, the lucrative attributes of these operators, such as a greater degree of freedom than the ordinary ones and their memory attributes have made it suitable for these extensive applications [2–4]. In addition, a lot of attention has been paid to the numerical solution of fractional equations in recent years. For instance, see [5–9].

As a generalization of the classical fractional derivatives, the distributed-order fractional derivatives are defined by the integration of the classical ones over the order of the derivative within a specific domain [10,11]. These types of derivatives play a middle role between the ordinary and fractional derivatives [12]. Differential equations defined by these fractional derivatives can be seen as generalizations of single- and multi-order fractional differential equations [11,13]. During the last years, such differential equations have been successfully applied to more accurately model diverse problems in the fields of electrochemistry [14], signal processing [15], control [16], viscoelastic [17], diffusion [18] and etc.

The Schrödinger equation is an illustrious differential equation that mostly becomes manifest in physical problems [19] It describes the quantum treatment alteration of disparate physical systems relative to time [19]. More precisely, this equation explains the transmutation of a wave package of low rate alteration in amplitude in weak systems with very

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dispersive mediums [20]. The fractional models of this equation play a vital role in quantum mechanics [21]. Recently, much attention has been paid to the numerical solution of fractional forms of this equation. In [22], the Legendre spectral method is used to solve a fractional form of the one- and two-dimensional Schrödinger equations. A numerical method based upon the reproducing kernel theory and collocation approach is developed in [23] for the time fractional form of this problem. In [24], the linearized compact ADI algorithms are adopted for the time fractional version of this equation. A hybrid approach based on the clique functions is proposed in [25] to solve space–time fractional form of this problem.

Due to the importance of the fractional form of this problem, in this paper, with the help of distributed-order fractional derivatives, we introduce another fractional version of this equation and present a numerical method to solve it. So, we concentrate on the distributed-order time fractional problem

$$i\int_0^1 \rho(\alpha)\,{}_0^C D_t^{\alpha}\Psi(z,t)d\alpha+\sigma\Psi_{zz}(z,t)+\eta|\Psi(z,t)|^2\Psi(z,t)+\tilde{w}(z)\Psi(z,t)=G(z,t)\,,\quad (z,t)\in[0,a]\times[0,b]\,, \tag{1.1}$$

under the conditions

$$\Psi(z,0)=\hat{\Psi}(z), \tag{1.2}$$

and

$$\Psi(0,t)=\tilde{\Psi}_0(t),\qquad \Psi(a,t)=\tilde{\Psi}_1(t), \tag{1.3}$$

where $i=\sqrt{-1}$ is the imaginary unit, Ψ is the undetermined solution of the problem, $\tilde{w}$ is a real function (expresses the trapping potential), G, $\hat{\Psi}$, $\tilde{\Psi}_0$ and $\tilde{\Psi}_1$ are known functions, and σ and η are real numbers. Also, ${}_0^C D_t^{\alpha}\Psi(z,t)$ is the Caputo fractional derivative of order α relative to t of Ψ. Moreover, the distribution function $\rho:[0,1]\longrightarrow\mathbb{R}^+$ satisfies the following conditions [26]:

$$\forall\ \alpha\in[0,1],\quad \rho(\alpha)>0,\quad \text{and}\quad \int_0^1\rho(\alpha)d\alpha=c_0>0.$$

Piecewise basis functions manufactured by the orthogonal polynomials possess many lucrative properties, such as exponential accuracy, orthogonality, and locality [27]. During the last years, these functions have been efficiently employed to construct appropriate methods for solving diverse problems. The orthonormal piecewise functions constructed using the Legendre cardinal polynomials are applied in [28] for the fractional version of the Riccati equation. The orthonormal piecewise functions generated by the Chebyshev cardinal polynomials have been used in [29] to solve fractional optimization problems. The orthonormal piecewise functions generated based on the Legendre polynomials are utilized in [30] for partial differential equations, in [31] for integro-differential equations and in [32] for fractional differential equations. The orthonormal piecewise functions defined using the Taylor polynomials are employed in [33] to construct a numerical method for fractional delay optimization problems.

To solve the above expressed equation, we first define another class of the piecewise functions called the orthonormal piecewise Jacobi functions (JFs) and then obtain some useful properties about them. After that, we proposed a hybrid method based on the orthonormal Jacobi polynomials (JPs) and orthonormal piecewise Jacobi functions. In fact, we expand the solution of the problem in terms of the orthonormal JPs (in the spatial domain) and orthonormal piecewise Jacobi functions (in the temporal domain) simultaneously. By computing fractional derivative of the orthonormal piecewise JFs, employing the Gauss–Legendre quadrature formula as well as applying the derivative operational matrix of the orthonormal Jacobi polynomials and the collocation approach, we transform solving this problem into solving an algebraic system. The credibility of the method is examined in some examples.

This article is arranged as follows: Required preliminaries about fractional calculus are given in Section 2. The orthonormal JPs and piecewise Jacobi functions are defined in Section 3. Some matrix relationships regarding the orthonormal JPs and orthonormal piecewise JFs are obtained in Section 4. The numerical technique is explained in Section 5. Some examples are provided in Section 6. The conclusion of this study is provided in Section 7.

2. Definitions and preliminaries

Here, we provide some preparations about fractional calculus that are needed in this study.

Definition 2.1 (*[1]*)**.** The Mittag-Leffler function is defined as

$$\mathbf{E}_{\mu}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(j\mu+1)},\qquad \mu\in\mathbb{R}^+,\ t\in\mathbb{C}. \tag{2.1}$$

Definition 2.2 (*[1]*)**.** Let g is a differential function over $[c_1,c_2]$ and $0<\alpha\leq 1$ is a real number. The Caputo fractional derivative of order α of g is given by

$${}_{c_1}^{C}D_t^{\alpha}g(t)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_{c_1}^{t}(t-s)^{-\alpha}g'(s)ds, & 0<\alpha<1,\\ g'(t), & \alpha=1.\end{cases} \tag{2.2}$$

Note that for $\alpha = 0$, we have ${}_{c_1}^{C}D_t^0 g(t) = g(t)$.

Property 2.3 ([1]). *For $0 < \alpha < 1$ and $k \in \mathbb{N} \cup \{0\}$, we have*

$$ {}_{c_1}^{C}D_t^{\alpha}\,(t-c_1)^k = \begin{cases} 0, & k = 0,\\ \dfrac{k!\,(t-c_1)^{k-\alpha}}{\Gamma(k-\alpha+1)}, & k = 1,2,\ldots. \end{cases} \tag{2.3} $$

3. Orthonormal Jacobi polynomials and piecewise Jacobi functions

Here, we introduce the orthonormal JPs and employ them to define piecewise JFs.

3.1. Orthonormal Jacobi polynomials

The JPs are defined on $[0, b]$ as follows [34]:

$$ \begin{aligned} &\hat{J}_{b,0}^{(\beta,\gamma)}(t) = 1,\\ &\hat{J}_{b,1}^{(\beta,\gamma)}(t) = \frac{1}{2}\left(\beta+\gamma+2\right)\left(\frac{2}{b}t-1\right) + \frac{1}{2}\left(\beta-\gamma\right),\\ &\hat{J}_{b,j+1}^{(\beta,\gamma)}(t) = \left(a_j^{(\beta,\gamma)}\left(\frac{2}{b}t-1\right)\gamma - b_j^{(\beta,\gamma)}\right)\hat{J}_{b,j}^{(\beta,\gamma)}(t) - c_j^{(\beta,\gamma)}\hat{J}_{b,j-1}^{(\beta,\gamma)}(t), \qquad j \geq 1, \end{aligned} $$

where $\beta > -1$ and $\gamma > -1$ are given constants, and

$$ \begin{aligned} a_j^{(\beta,\gamma)} &= \frac{(2j+\beta+\gamma+1)(2j+\beta+\gamma+2)}{2(j+1)(j+\beta+\gamma+1)},\\ b_j^{(\beta,\gamma)} &= \frac{\left(\gamma^2-\beta^2\right)(2j+\beta+\gamma+1)}{2(j+1)(j+\beta+\gamma+1)(2j+\beta+\gamma)},\\ c_j^{(\beta,\gamma)} &= \frac{(j+\beta)(j+\gamma)(2j+\beta+\gamma+2)}{(j+1)(j+\beta+\gamma+1)(2j+\beta+\gamma)}. \end{aligned} $$

The above polynomials can also be generated as follows [34]:

$$ \hat{J}_{b,j}^{(\beta,\gamma)}(t) = \sum_{k=0}^{j} \hat{h}_{b,jk}^{(\beta,\gamma)} t^k, \qquad j \geq 0, \tag{3.1} $$

where

$$ \hat{h}_{b,jk}^{(\beta,\gamma)} = \begin{cases} 1, & j = 0,\\ (-1)^{j+k}\dfrac{\Gamma(j+\gamma+1)\Gamma(j+k+\beta+\gamma+1)}{b^k k!(j-k)!\Gamma(k+\gamma+1)\Gamma(j+\beta+\gamma+1)}, & j > 0. \end{cases} $$

The set $\{\hat{J}_{b,j}^{(\beta,\gamma)}(t)\}_{j=0}^{\infty}$ produces an orthogonal set over $[0, b]$ with respect to the weigh function

$$ w_b^{(\beta,\gamma)}(t) = \left(\frac{2}{b}\right)^{\beta+\gamma}(b-t)^{\beta}\,t^{\gamma}. $$

In addition, the orthogonal property of these polynomials is as follows:

$$ \int_0^b \hat{J}_{b,i}^{(\beta,\gamma)}(t)\hat{J}_{b,j}^{(\beta,\gamma)}(t) w_b^{(\beta,\gamma)}(t)dt = \delta_{ij}\sigma_{b,i}^{(\beta,\gamma)}, \tag{3.2} $$

where δ_{ij} is the Kronecker delta, and

$$ \sigma_{b,i}^{(\beta,\gamma)} = \frac{b}{2}\begin{cases} \dfrac{2^{\beta+\gamma+1}\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}, & i = 0,\\ \dfrac{2^{\beta+\gamma+1}\Gamma(i+\beta+1)\Gamma(i+\gamma+1)}{(2i+\beta+\gamma+1)i!\Gamma(i+\beta+\gamma+1)}, & i > 0. \end{cases} $$

Based on the JPs, we define the orthonormal JPs over $[0, b]$ by the following formula:

$$ J_{b,j}^{(\beta,\gamma)}(t) = \sum_{k=0}^{j} h_{b,jk}^{(\beta,\gamma)} t^k, \qquad j \geq 0, \tag{3.3} $$

where

$$h_{b,jk}^{(\beta,\gamma)} = \frac{\hat{h}_{b,jk}^{(\beta,\gamma)}}{\sqrt{\sigma_{b,j}^{(\beta,\gamma)}}}.$$

Any function $f \in L^2_{w_b}([0,b])$ can be expanded via the orthonormal JPs as follows:

$$f(t) \simeq \sum_{j=0}^{\hat{M}-1} f_{b,j}^{(\beta,\gamma)} J_{b,j}^{(\beta,\gamma)}(t) \triangleq \left(\mathbf{F}_{\hat{M}}^{(\beta,\gamma)}\right)^T \mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t), \tag{3.4}$$

where $\hat{M} \in \mathbb{Z}^+$,

$$\mathbf{F}_{\hat{M}}^{(\beta,\gamma)} = \left[f_{b,0}^{(\beta,\gamma)}\ f_{b,1}^{(\beta,\gamma)}\ \dots\ f_{b,\hat{M}-1}^{(\beta,\gamma)}\right]^T,$$

with

$$f_{b,j}^{(\beta,\gamma)} = \int_0^b w^{(\beta,\gamma)}(t) f(t) J_{b,j}^{(\beta,\gamma)}(t)dt,$$

and

$$\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t) = \left[J_{b,0}^{(\beta,\gamma)}(t)\ J_{b,1}^{(\beta,\gamma)}(t)\ \dots\ J_{b,\hat{M}-1}^{(\beta,\gamma)}(t)\right]^T. \tag{3.5}$$

3.2. Orthonormal piecewise Jacobi functions

The orthonormal piecewise JFs can be defined over $[0, b]$ as

$$\varphi_{b,nm}^{(\beta,\gamma)}(t) = \begin{cases} \sqrt{N} J_{b,m}^{(\beta,\gamma)}(Nt - nb), & t \in \left[\frac{nb}{N}, \frac{(n+1)b}{N}\right], \\ 0, & otherwise, \end{cases} \tag{3.6}$$

where $N, M \in \mathbb{Z}^+$, $n = 0, 1, \dots, N-1$ and $m = 0, 1, \dots, M-1$. They are orthonormal with respect to the weigh function

$$w_{b,n}^{(\beta,\gamma)}(t) = \begin{cases} \left(\frac{2}{b}\right)^{\beta+\gamma} ((n+1)b - Nt)^\beta (Nt - nb)^\gamma, & t \in \left[\frac{nb}{N}, \frac{(n+1)b}{N}\right], \\ 0, & otherwise, \end{cases} \tag{3.7}$$

where $n = 0, 1, \dots, N-1$. The orthonormal piecewise JFs can be employed to express a function $g(t) \in L^2_{w_{b,n}}[0, b]$ as follows:

$$g(t) \simeq \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g_{nm}^{(\beta,\gamma)} \varphi_{b,nm}^{(\beta,\gamma)}(t) \triangleq \left(\mathbf{G}_{NM}^{(\beta,\gamma)}\right)^T \Phi_{b,NM}^{(\beta,\gamma)}(t), \tag{3.8}$$

where

$$\mathbf{G}_{NM}^{(\beta,\gamma)} = \left[g_{00}^{(\beta,\gamma)}\ g_{01}^{(\beta,\gamma)}\ \dots\ g_{0(M-1)}^{(\beta,\gamma)} | g_{10}^{(\beta,\gamma)}\ g_{11}^{(\beta,\gamma)}\ \dots\ g_{1(M-1)}^{(\beta,\gamma)} |\ \dots\ | g_{(N-1)\,0}^{(\beta,\gamma)}\ g_{(N-1)\,1}^{(\beta,\gamma)}\ \dots\ g_{(N-1)(M-1)}^{(\beta,\gamma)}\right]^T,$$

with

$$g_{nm}^{(\beta,\gamma)} = \int_0^b w_{b,n}^{(\beta,\gamma)}(t) \varphi_{b,nm}^{(\beta,\gamma)}(t) g(t)dt,$$

and

$$\begin{aligned} \Phi_{b,NM}(t) = \Big[&\varphi_{b,00}^{(\beta,\gamma)}(t)\ \varphi_{b,01}^{(\beta,\gamma)}(t)\ \dots \varphi_{b,0(M-1)}^{(\beta,\gamma)}(t) | \varphi_{b,10}^{(\beta,\gamma)}(t)\ \varphi_{b,11}^{(\beta,\gamma)}(t)\ \dots\ \varphi_{b,1(M-1)}^{(\beta,\gamma)}(t) |\ \dots \\ &| \varphi_{b,(N-1)\,0}^{(\beta,\gamma)}(t)\ \varphi_{b,(N-1)\,1}^{(\beta,\gamma)}(t)\ \dots \varphi_{b,(N-1)(M-1)}^{(\beta,\gamma)}(t)\Big]^T. \end{aligned} \tag{3.9}$$

3.3. Hybrid approximation

A function $u(z, t)$ defined on $[0, a] \times [0, b]$ may be approximated via the above two classes of basis functions as follows:

$$u(z,t) \simeq \sum_{i=0}^{\hat{M}-1} \sum_{j=1}^{NM} u_{ij}^{(\beta,\gamma)} J_{a,i}^{(\beta,\gamma)}(z) \hat{\varphi}_{b,j}^{(\beta,\gamma)}(t) \triangleq \left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T \mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)} \Phi_{b,NM}^{(\beta,\gamma)}(t), \tag{3.10}$$

where $\hat{\varphi}_{b,j}^{(\beta,\gamma)}(t)=\varphi_{b,nm}^{(\beta,\gamma)}(t)$ with $j=nM+m+1$ for $n=0,1,\ldots,N-1$ and $m=0,1,\ldots,M-1$, and $\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}=\left[u_{ij}^{(\beta,\gamma)}\right]$ is an $\hat{M}\times NM$ matrix with entries

$$u_{ij}^{(\beta,\gamma)}=\int_0^b\int_0^a w_a^{(\beta,\gamma)}(z)w_{b,n}^{(\beta,\gamma)}(t)J_{a,i}^{(\beta,\gamma)}(z)\hat{\varphi}_{b,j}^{(\beta,\gamma)}(t)u(z,t)dzdt,\qquad 0\le i\le\hat{M}-1,\ 1\le j\le NM.$$

4. Operational matrices

In this section, some matrix relationships regarding the ordinary and fractional derivatives of the basis functions introduced in the previous section are expressed.

Theorem 4.1. *The differentiation of the vector $\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)$ in (3.5) can be represented as*

$$\frac{d\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)}{dt}=\mathbf{D}_{b,\hat{M}}^{(1,\beta,\gamma)}\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t),\tag{4.1}$$

where $\mathbf{D}_{b,\hat{M}}^{(1,\beta,\gamma)}$ is an $\hat{M}\times\hat{M}$ matrix as

$$\mathbf{D}_{b,\hat{M}}^{(1,\beta,\gamma)}=\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\hat{\mathbf{D}}_{\hat{M}}^{(1)}\left(\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\right)^{-1},$$

with

$$\left[\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\right]_{ij}=\begin{cases}\sqrt{\dfrac{2\Gamma(\beta+\gamma+2)}{b2^{\beta+\gamma+1}\Gamma(\beta+1)\Gamma(\gamma+1)}}, & i=j=1,\\ \mathbf{B}_{b,ij}^{(\beta,\gamma)}, & 2\le i\le\hat{M},\ 1\le j\le i,\\ 0, & otherwise,\end{cases}\tag{4.2}$$

in which

$$\mathbf{B}_{b,ij}^{(\beta,\gamma)}=(-1)^{i+j}\frac{\Gamma(i+\gamma)\Gamma(i+j+\beta+\gamma-1)}{b^{j-1}(j-1)!(i-j)!\Gamma(j+\gamma)\Gamma(i+\beta+\gamma)}\sqrt{\frac{2(2i+\beta+\gamma-1)(i-1)!\Gamma(i+\beta+\gamma)}{b2^{\beta+\gamma+1}\Gamma(i+\beta)\Gamma(i+\gamma)}},$$

and

$$\left[\hat{\mathbf{D}}_{\hat{M}}^{(1)}\right]_{ij}=\begin{cases}i-1, & 2\le i\le\hat{M},\ 1\le j\le i-1,\ i=j+1,\\ 0, & otherwise.\end{cases}\tag{4.3}$$

Proof. From (3.3) and (3.5), it is obvious that $\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)$ can be rewritten as

$$\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)=\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\begin{pmatrix}1\\ t\\ t^2\\ \vdots\\ t^{\hat{M}-1}\end{pmatrix},$$

where the entires of the matrix $\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}$ are calculated using (4.2). Thus, we have

$$\frac{d\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)}{dt}=\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\frac{d}{dt}\begin{pmatrix}1\\ t\\ t^2\\ \vdots\\ t^{\hat{M}-1}\end{pmatrix}=\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\begin{pmatrix}0\\ 1\\ 2t\\ \vdots\\ (\hat{M}-1)t^{\hat{M}-2}\end{pmatrix}=\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\hat{\mathbf{D}}_{\hat{M}}^{(1)}\begin{pmatrix}1\\ t\\ t^2\\ \vdots\\ t^{\hat{M}-1}\end{pmatrix},$$

where the entires of the matrix $\hat{\mathbf{D}}_{\hat{M}}^{(1)}$ are computed using (4.3). Thus, from the above two relations, we get

$$\frac{d\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)}{dt}=\left(\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\hat{\mathbf{D}}_{\hat{M}}^{(1)}\left(\mathbf{A}_{b,\hat{M}}^{(\beta,\gamma)}\right)^{-1}\right)\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t)\triangleq\mathbf{D}_{b,\hat{M}}^{(1,\beta,\gamma)}\mathbf{J}_{b,\hat{M}}^{(\beta,\gamma)}(t),$$

which completes the proof. □

Corollary 4.2. *From the above Theorem, we have*

$$\frac{d^2\mathbf{J}^{(\beta,\gamma)}_{b,\hat{M}}(t)}{dt^2} = \mathbf{D}^{(1,\beta,\gamma)}_{b,\hat{M}} \times \mathbf{D}^{(1,\beta,\gamma)}_{b,\hat{M}}\mathbf{J}^{(\beta,\gamma)}_{b,\hat{M}}(t) \triangleq \mathbf{D}^{(2,\beta,\gamma)}_{b,\hat{M}}\mathbf{J}^{(\beta,\gamma)}_{b,\hat{M}}(t). \tag{4.4}$$

As a numerical example, for $\hat{M} = 5$, we have

$$\mathbf{D}^{(1,1,1)}_{b,5} = \frac{1}{b}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{70} & 0 & 0 & 0 \\ \frac{3}{5}\sqrt{30} & 0 & \frac{6}{5}\sqrt{105} & 0 & 0 \\ 0 & 2\sqrt{11} & 0 & 2\sqrt{66} & 0 \end{pmatrix}, \quad \mathbf{D}^{(2,1,1)}_{b,5} = \frac{1}{b^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 10\sqrt{14} & 0 & 0 & 0 & 0 \\ 0 & 42\sqrt{6} & 0 & 0 & 0 \\ \frac{56\sqrt{55}}{5} & 0 & \frac{36\sqrt{770}}{5} & 0 & 0 \end{pmatrix}.$$

Theorem 4.3. *Assume $\varphi^{(\beta,\gamma)}_{b,nm}(t)$ are the functions expressed in (3.6) and $0 \le \alpha \le 1$ is a real number. Then, we have*

$${}^{C}_{0}D^{\alpha}_{t}\varphi^{(\beta,\gamma)}_{b,nm}(t) \triangleq \tilde{\varphi}^{(\beta,\gamma)}_{b,nm}(t,\alpha) = \begin{cases} \varphi^{(\beta,\gamma)}_{b,nm}(t), & \alpha = 0, \\ \dfrac{d\varphi^{(\beta,\gamma)}_{b,nm}(t)}{dt}, & \alpha = 1, \\ \varphi^{(\alpha,\beta,\gamma)}_{b,nm}(t), & 0 < \alpha < 1, \end{cases} \tag{4.5}$$

where

$$\frac{d\varphi^{(\beta,\gamma)}_{b,nm}(t)}{dt} = \begin{cases} 0, & m = 0, \\ \begin{cases} N^{\frac{3}{2}}\displaystyle\sum_{k=1}^{m} k\,h^{(\beta,\gamma)}_{b,mk}\,(Nt-nb)^{k-1}, & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ 0, & otherwise, \end{cases} & m = 1,2,\ldots,M-1, \end{cases} \tag{4.6}$$

and

$$\varphi^{(\alpha,\beta,\gamma)}_{b,nm}(t) = \begin{cases} 0, & m = 0, \\ \begin{cases} \theta^{(\alpha,\beta,\gamma)}_{b,nm}(t), & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ \vartheta^{(\alpha,\beta,\gamma)}_{b,nm}(t), & t \in \left[\dfrac{(n+1)b}{N}, b\right], \\ 0, & otherwise, \end{cases} & m = 1,2,\ldots,M-1, \end{cases} \tag{4.7}$$

with

$$\theta^{(\alpha,\beta,\gamma)}_{b,nm}(t) = N^{\alpha+\frac{1}{2}}\sum_{k=1}^{m}\frac{k!h^{(\beta,\gamma)}_{b,mk}}{\Gamma(k-\alpha+1)}(Nt-nb)^{k-\alpha}, \tag{4.8}$$

and

$$\begin{aligned}\vartheta^{(\alpha,\beta,\gamma)}_{b,nm}(t) =& \frac{N^{\alpha+\frac{1}{2}}}{\Gamma(1-\alpha)}\sum_{k=1}^{m}kh^{(\beta,\gamma)}_{b,mk}\left\{\prod_{l=1}^{k}\frac{1}{l-\alpha}\prod_{l=1}^{k-1}(k-l)\,(Nt-nb)^{k-\alpha}\right. \\ &\left.-\sum_{r=1}^{k}\prod_{l=1}^{r}\frac{1}{l-\alpha}\prod_{l=1}^{r-1}(k-l)\left[N^{k-r}\left(\frac{b}{N}\right)^{k-r}(Nt-(n+1)b)^{r-\alpha}\right]\right\}.\end{aligned} \tag{4.9}$$

Proof. For $\alpha = 0$, the proof is obvious. For $\alpha = 1$, from (3.3) and (3.6), we get $\frac{d\varphi^{(\beta,\gamma)}_{b,n0}(t)}{dt} = 0$. Meanwhile, for $m = 1,2,\ldots,M-1$, we have

$$\frac{d\varphi^{(\beta,\gamma)}_{b,nm}(t)}{dt} = \begin{cases} N^{\frac{3}{2}}\displaystyle\sum_{k=1}^{m} k\,h^{(\beta,\gamma)}_{b,mk}\,(Nt-nb)^{k-1}, & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ 0, & otherwise. \end{cases}$$

In the case of $0 < \alpha < 1$, from Property 2.3 and relations (3.3) and (3.6), we get

$${}^{C}_{0}D^{\alpha}_{t}\varphi^{(\beta,\gamma)}_{b,n0}(t) = 0,$$

and for $m = 1, 2, \ldots, M-1$, we obtain

$$
\begin{aligned}
{}_0^C D_t^{\alpha} \varphi_{b,nm}^{(\beta,\gamma)}(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d\varphi_{b,nm}^{(\beta,\gamma)}(s)}{ds} ds \\
&= \begin{cases} \dfrac{\sqrt{N}}{\Gamma(1-\alpha)} \displaystyle\int_{\frac{nb}{N}}^{t} (t-s)^{-\alpha} \frac{d}{ds}\left(J_{b,m}^{(\beta,\gamma)}(Ns-nb)\right) ds, & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ \dfrac{\sqrt{N}}{\Gamma(1-\alpha)} \displaystyle\int_{\frac{nb}{N}}^{\frac{(n+1)b}{N}} (t-s)^{-\alpha} \frac{d}{ds}\left(J_{b,m}^{(\beta,\gamma)}(Ns-nb)\right) ds, & t \in \left[\dfrac{(n+1)b}{N}, b\right], \\ 0, & otherwise. \end{cases}
\end{aligned}
\tag{4.10}
$$

Using (3.3) and (4.10), we obtain

$$
{}_0^C D_t^{\alpha} \varphi_{b,nm}^{(\beta,\gamma)}(t) = \begin{cases} \dfrac{N^{\frac{1}{2}}}{\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{m} k h_{b,mk}^{(\beta,\gamma)} N^k \int_{\frac{nb}{N}}^{t} (t-s)^{-\alpha} \left(s-\frac{nb}{N}\right)^{k-1} ds, & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ \dfrac{N^{\frac{1}{2}}}{\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{m} k h_{b,mk}^{(\beta,\gamma)} N^k \int_{\frac{nb}{N}}^{\frac{(n+1)b}{N}} (t-s)^{-\alpha} \left(s-\frac{nb}{N}\right)^{k-1} ds, & t \in \left[\dfrac{(n+1)b}{N}, b\right], \\ 0, & otherwise. \end{cases}
\tag{4.11}
$$

In addition, Property 2.3 yields

$$
\int_{\frac{nb}{N}}^{t} (t-s)^{-\alpha} \left(s-\frac{nb}{N}\right)^{k-1} ds = \frac{(k-1)!\,\Gamma(1-\alpha)}{\Gamma(k-\alpha+1)} \left(t-\frac{nb}{N}\right)^{k-\alpha},
\tag{4.12}
$$

and integration by parts gives

$$
\begin{aligned}
\int_{\frac{nb}{N}}^{\frac{(n+1)b}{N}} (t-s)^{-\alpha} \left(s-\frac{nb}{N}\right)^{k-1} ds &= \prod_{l=1}^{k} \frac{1}{l-\alpha} \prod_{l=1}^{k-1}(k-l) \left(t-\frac{nb}{N}\right)^{k-\alpha} \\
&\quad - \sum_{r=1}^{k} \prod_{l=1}^{r} \frac{1}{l-\alpha} \prod_{l=1}^{r-1}(k-l) \left[\left(\frac{b}{N}\right)^{k-r} \left(t-\frac{(n+1)b}{N}\right)^{r-\alpha}\right].
\end{aligned}
\tag{4.13}
$$

Substituting (4.12) and (4.13) into (4.11) gives

$$
{}_0^C D_t^{\alpha} \varphi_{b,nm}^{(\beta,\gamma)}(t) = \begin{cases} \theta_{b,nm}^{(\alpha,\beta,\gamma)}(t), & t \in \left[\dfrac{nb}{N}, \dfrac{(n+1)b}{N}\right], \\ \vartheta_{b,nm}^{(\alpha,\beta,\gamma)}(t), & t \in \left[\dfrac{(n+1)b}{N}, b\right], \\ 0, & otherwise, \end{cases}
$$

where $\theta_{b,nm}^{(\alpha,\beta,\gamma)}(t)$ and $\vartheta_{b,nm}^{(\alpha,\beta,\gamma)}(t)$ are introduced in (4.8) and (4.9), respectively. Thus, the proof is completed. □

5. The proposed method

For gaining a numerical solution for the distributed-order time fractional problem introduced in (1.1)–(1.3), we let

$$
\begin{aligned}
&\Psi(z,t) = \Psi_1(z,t) + i\,\Psi_2(z,t), \quad G(z,t) = G_1(z,t) + i\,G_2(z,t), \quad \hat{\Psi}(z) = \hat{\Psi}_1(z) + i\,\hat{\Psi}_2(z), \\
&\tilde{\Psi}_0(t) = \tilde{\Psi}_{01}(t) + i\,\tilde{\Psi}_{02}(t), \qquad \tilde{\Psi}_1(t) = \tilde{\Psi}_{11}(t) + i\,\tilde{\Psi}_{12}(t),
\end{aligned}
\tag{5.1}
$$

where Ψ_1, Ψ_2, G_1, G_2, $\hat{\Psi}_1$, $\hat{\Psi}_2$, $\tilde{\Psi}_{01}$, $\tilde{\Psi}_{02}$, $\tilde{\Psi}_{11}$ and $\tilde{\Psi}_{12}$ are real functions. This leads to the equivalent problem

$$
\begin{aligned}
&i\left(\int_0^1 \rho(\alpha)\, {}_0^C D_t^{\alpha} \Psi_1(z,t) d\alpha + \sigma \Psi_{2zz}(z,t) + \eta\left(\Psi_1^2(z,t)+\Psi_2^2(z,t)\right)\Psi_2(z,t) + \tilde{w}(z)\Psi_2(z,t) - G_2(z,t)\right) \\
&-\left(\int_0^1 \rho(\alpha)\, {}_0^C D_t^{\alpha} \Psi_2(z,t) d\alpha - \sigma \Psi_{1zz}(z,t) - \eta\left(\Psi_1^2(z,t)+\Psi_2^2(z,t)\right)\Psi_1(z,t) - \tilde{w}(z)\Psi_1(z,t) + G_1(z,t)\right) = 0,
\end{aligned}
\tag{5.2}
$$

or equivalently

$$
\begin{aligned}
&\int_0^1 \rho(\alpha)\, {}_0^C D_t^{\alpha} \Psi_1(z,t) d\alpha + \sigma \Psi_{2zz}(z,t) + \eta\left(\Psi_1^2(z,t)+\Psi_2^2(z,t)\right)\Psi_2(z,t) + \tilde{w}(z)\Psi_2(z,t) - G_2(z,t) = 0, \\
&\int_0^1 \rho(\alpha)\, {}_0^C D_t^{\alpha} \Psi_2(z,t) d\alpha - \sigma \Psi_{1zz}(z,t) - \eta\left(\Psi_1^2(z,t)+\Psi_2^2(z,t)\right)\Psi_1(z,t) - \tilde{w}(z)\Psi_1(z,t) + G_1(z,t) = 0.
\end{aligned}
\tag{5.3}
$$

under the conditions

$$\Psi_1(z,0)=\hat{\Psi}_1(z), \qquad \Psi_2(z,0)=\hat{\Psi}_2(z), \tag{5.4}$$

and

$$\begin{aligned}\Psi_1(0,t)&=\tilde{\Psi}_{01}(t), \qquad \Psi_2(0,t)=\tilde{\Psi}_{02}(t),\\ \Psi_1(a,t)&=\tilde{\Psi}_{11}(t), \qquad \Psi_2(a,t)=\tilde{\Psi}_{12}(t).\end{aligned} \tag{5.5}$$

To solve the system extracted in (5.3)–(5.5), we assume

$$\begin{aligned}\Psi_1(z,t)&\simeq\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}u_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\hat{\varphi}_{b,j}^{(\beta,\gamma)}(t)\triangleq\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t),\\ \Psi_2(z,t)&\simeq\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}v_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\hat{\varphi}_{b,j}^{(\beta,\gamma)}(t)\triangleq\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t),\end{aligned} \tag{5.6}$$

where $\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}=\left[u_{ij}^{(\beta,\gamma)}\right]$ and $\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}=\left[v_{ij}^{(\beta,\gamma)}\right]$ are undetermined $\hat{M}\times NM$ matrices. Using (4.5) and (5.6), we have

$$\begin{aligned}\int_0^1\rho(\alpha)\,{}_0^CD_t^\alpha\Psi_1(z,t)d\alpha&\simeq\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}u_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\int_0^1\rho(\alpha)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}(t,\alpha)d\alpha,\\ \int_0^1\rho(\alpha)\,{}_0^CD_t^\alpha\Psi_2(z,t)d\alpha&\simeq\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}v_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\int_0^1\rho(\alpha)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}(t,\alpha)d\alpha,\end{aligned} \tag{5.7}$$

where $\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}(t,\alpha)=\tilde{\varphi}_{b,nm}^{(\beta,\gamma)}(t,\alpha)$ with $j=nM+m+1$ for $n=0,1,\ldots,N-1$ and $m=0,1,\ldots,M-1$. The integrals in (5.7) can be calculated by an $\hat{N}$-point Gauss–Legendre integration formula as

$$\int_0^1\rho(\alpha)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}(t,\alpha)d\alpha\simeq\frac{1}{2}\sum_{r=1}^{\hat{N}}\hat{w}_r\rho\left(\frac{1}{2}\left(\hat{t}_r+1\right)\right)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}\left(t,\frac{1}{2}\left(\hat{t}_r+1\right)\right), \tag{5.8}$$

where

$$\hat{w}_r=\frac{2}{\left(1-\hat{t}_r^2\right)\left(L'_{\hat{N}}\left(\hat{t}_r\right)\right)^2},$$

and $\left\{\hat{t}_r\right\}_{r=1}^{\hat{N}}$ are the Gauss–Legendre integration nodes in $[-1,1]$. For more details, see [35]. Substituting (5.8) into (5.7), results in

$$\begin{aligned}\int_0^1\rho(\alpha)\,{}_0^CD_t^\alpha\Psi_1(z,t)d\alpha&\simeq U^{(\beta,\gamma)}(z,t),\\ \int_0^1\rho(\alpha)\,{}_0^CD_t^\alpha\Psi_2(z,t)d\alpha&\simeq V^{(\beta,\gamma)}(z,t),\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}U^{(\beta,\gamma)}(z,t)&=\frac{1}{2}\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}u_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\sum_{r=1}^{\hat{N}}\hat{w}_r\rho\left(\frac{1}{2}\left(\hat{t}_r+1\right)\right)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}\left(t,\frac{1}{2}\left(\hat{t}_r+1\right)\right),\\ V^{(\beta,\gamma)}(z,t)&=\frac{1}{2}\sum_{i=0}^{\hat{M}-1}\sum_{j=1}^{NM}v_{ij}^{(\beta,\gamma)}J_{a,i}^{(\beta,\gamma)}(z)\sum_{r=1}^{\hat{N}}\hat{w}_r\rho\left(\frac{1}{2}\left(\hat{t}_r+1\right)\right)\tilde{\tilde{\varphi}}_{b,j}^{(\beta,\gamma)}\left(t,\frac{1}{2}\left(\hat{t}_r+1\right)\right).\end{aligned}$$

From (4.4) and (5.6), we get

$$\begin{aligned}\Psi_{1zz}(z,t)&\simeq\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\left(\mathbf{D}_{a,\hat{M}}^{(2,\beta,\gamma)}\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t),\\ \Psi_{2zz}(z,t)&\simeq\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\left(\mathbf{D}_{a,\hat{M}}^{(2,\beta,\gamma)}\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t).\end{aligned} \tag{5.10}$$

Substituting (5.6), (5.9) and (5.10) into (5.3) yields

$$U^{(\beta,\gamma)}(z,t)+\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\left[\sigma\left(\mathbf{D}_{a,\hat{M}}^{(2,\beta,\gamma)}\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}+\tilde{w}(z)\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\right]\Phi_{b,NM}^{(\beta,\gamma)}(t)$$

$$\begin{aligned}&+\eta\left\{\left(\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2+\left(\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2\right\}\\&\times\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)-G_2(z,t)\triangleq\mathbf{\Lambda}_1^{(\beta,\gamma)}(z,t)\simeq 0,\end{aligned}\quad(5.11)$$

$$\begin{aligned}V^{(\beta,\gamma)}(z,t)&-\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\left[\sigma\left(\mathbf{D}_{a,\hat{M}}^{(2,\beta,\gamma)}\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}+\tilde{w}(z)\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\right]\Phi_{b,NM}^{(\beta,\gamma)}(t)\\&-\eta\left\{\left(\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2+\left(\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2\right\}\\&\times\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)+G_1(z,t)\triangleq\mathbf{\Lambda}_2^{(\beta,\gamma)}(z,t)\simeq 0\end{aligned}\quad(5.12)$$

From (5.4)–(5.6), we get

$$\begin{aligned}&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(0)-\hat{\Psi}_1(z)\triangleq\mathbf{R}_1^{(\beta,\gamma)}(z)\simeq 0,\\&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(0)-\hat{\Psi}_2(z)\triangleq\mathbf{R}_2^{(\beta,\gamma)}(z)\simeq 0,\end{aligned}\quad(5.13)$$

and

$$\begin{aligned}&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(0)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)-\tilde{\Psi}_{01}(t)\triangleq\mathbf{S}_1^{(\beta,\gamma)}(t)\simeq 0,\\&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(0)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)-\tilde{\Psi}_{02}(t)\triangleq\mathbf{S}_2^{(\beta,\gamma)}(t)\simeq 0,\\&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(a)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)-\tilde{\Psi}_{11}(t)\triangleq\mathbf{S}_3^{(\beta,\gamma)}(t)\simeq 0,\\&\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(a)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)-\tilde{\Psi}_{12}(t)\triangleq\mathbf{S}_4^{(\beta,\gamma)}(t)\simeq 0.\end{aligned}\quad(5.14)$$

From (5.11)–(5.14), we extract the following $2(\hat{M}\times NM)$ system:

$$\begin{cases}\mathbf{\Lambda}_l^{(\beta,\gamma)}\left(z_i,t_j\right)=0, & l=1,2, \quad 2\le i\le\hat{M}-1,\ 2\le j\le NM,\\ \mathbf{R}_l^{(\beta,\gamma)}(z_i)=0, & l=1,2, \quad 2\le i\le\hat{M}-1,\\ \mathbf{S}_l^{(\beta,\gamma)}(t_j)=0, & l=1,2,3,4, \quad 1\le j\le NM,\end{cases}\quad(5.15)$$

where

$$\begin{aligned}z_i&=\frac{a}{2}\left(1-\cos\left(\frac{(2i-1)\pi}{2\hat{M}}\right)\right),\\ t_j&=\frac{b}{2}\left(1-\cos\left(\frac{(2j-1)\pi}{2NM}\right)\right).\end{aligned}$$

After solving system (5.15) and obtaining matrices $\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}$ and $\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}$, a numerical solution for the problem is found via (5.6). The "fsolve" command in Maple 18 (with 25 decimal digits) is used to solve the system (5.15).

6. Numerical examples

In this section, we have applied the method stated in the previous section for some examples. The accuracy of the derived results is evaluated by the formulae

$$\begin{aligned}e_{\Psi_1}&=\left(\int_0^b\int_0^a\left(\Psi_1(z,t)-\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{U}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2dzdt\right)^{1/2},\\ e_{\Psi_2}&=\left(\int_0^b\int_0^a\left(\Psi_2(z,t)-\left(\mathbf{J}_{a,\hat{M}}^{(\beta,\gamma)}(z)\right)^T\mathbf{V}_{\hat{M}\times NM}^{(\beta,\gamma)}\Phi_{b,NM}^{(\beta,\gamma)}(t)\right)^2dzdt\right)^{1/2},\end{aligned}$$

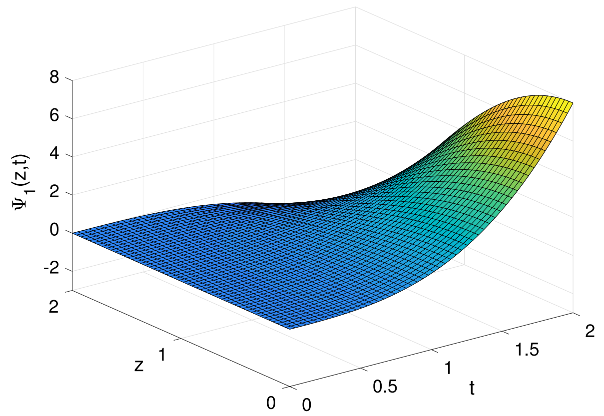
where Ψ_1 and Ψ_2 are respectively the real and imaginary parts of the exact solution. For numerical integration, we put $\hat{N}=15$.

Example 1. Consider the problem

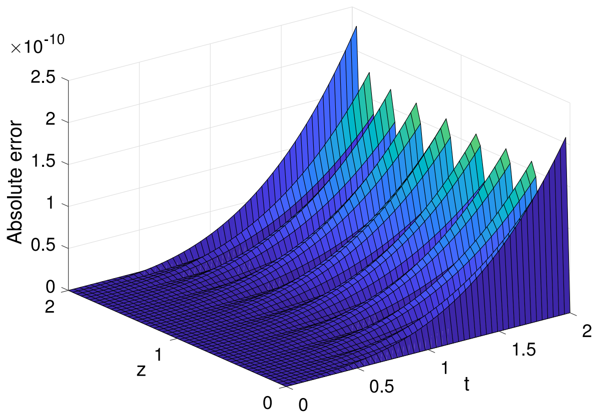
$$i\int_0^1\Gamma(4-\alpha)\,{}_0^CD_t^\alpha\Psi(z,t)d\alpha+\Psi_{zz}(z,t)+|\Psi(z,t)|^2\Psi(z,t)+\cos(z)\Psi(z,t)=G(z,t),\quad(z,t)\in[0,2]\times[0,2],$$

Table 1
The results obtained for errors via ($M = 4, N = 2$) and several $\hat{M}$ in Example 1.

$\hat{M}$	M	N	$\beta = \gamma = 0.0$		$\beta = \gamma = -\frac{1}{2}$		$\beta = \gamma = \frac{1}{2}$		$\beta = -\frac{1}{2}, \gamma = \frac{1}{2}$	
			$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$
5	4	2	2.2657E−03	1.4715E−03	2.2657E−03	1.4715E−03	2.2657E−03	1.4715E−03	2.2657E−03	1.4715E−03
7			1.2471E−05	8.0589E−06	1.2471E−05	8.0589E−06	1.2471E−05	8.0589E−06	1.2471E−05	8.0589E−06
9			4.1630E−08	2.6836E−08	4.1630E−08	2.6836E−08	4.1630E−08	2.6836E−08	4.1630E−08	2.6836E−08
11			9.2514E−11	5.9561E−11	9.2514E−11	5.9561E−11	9.2514E−11	5.9561E−11	9.2514E−11	5.9561E−11



(a)



(b)

Fig. 1. Approximate solution $\Psi_1(z,t)$ (1(a)) and corresponding absolute error function (1(b)) with ($\hat{M} = 11, M = 4, N = 2$) and $\beta = \gamma = 0.0$ in Example 1.

where

$$G(z,t) = \left\{ \frac{6i\,t^2(t-1)}{\ln(t)} + \left(t^9 - t^3\right) + t^3\cos(z) \right\} e^{iz},$$

with

$$\Psi(z,0) = 0,$$

and

$$\Psi(0,t) = t^3, \qquad \Psi(2,t) = t^3 e^{2i}.$$

The exact solution is

$$\Psi(z,t) = t^3 e^{iz}.$$

The results derived from applying the proposed method with some values of β and γ are shown in Table 1 and Figs. 1 and 2. It can be seen from Table 1 that by increasing the basis functions, the accuracy of the results improves. Moreover, for all selected values of β and γ, the same results are obtained.

Example 2. Consider the problem

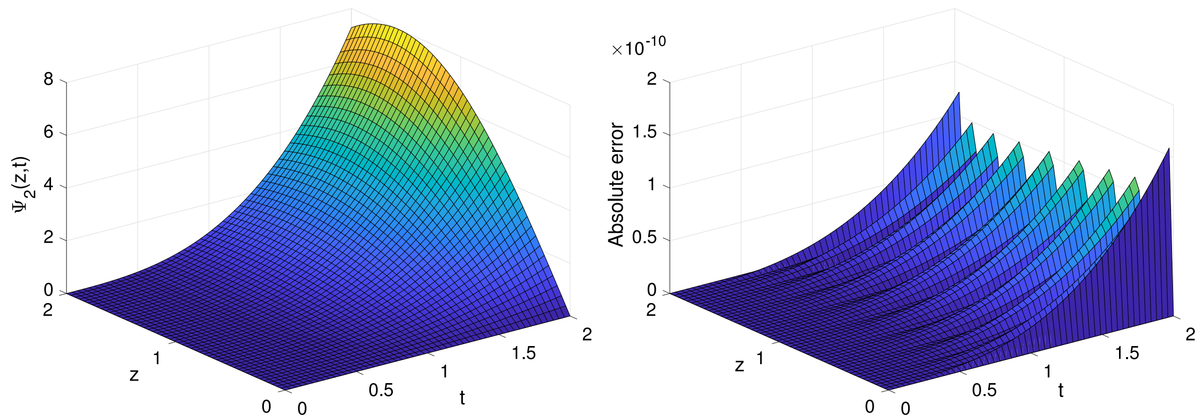
$$i\int_0^1 \rho(\alpha)\, {}_0^C D_t^{\alpha} \Psi(z,t) d\alpha + 2\Psi_{zz}(z,t) + |\Psi(z,t)|^2 \Psi(z,t) + e^z \Psi(z,t) = G(z,t)\,, \quad (z,t) \in [0,1] \times [0,1]\,,$$

where

$$G(z,t) = \left\{ i\int_0^1 \rho(\alpha) t^{1-\alpha} \mathbf{E}_{2,2-\alpha}\left(-t^2\right) d\alpha + \left(\sin^2(t) + e^z - 2\right)\sin(t) \right\} e^{-iz},$$

with

$$\Psi(z,0) = 0,$$



(a)   (b)

Fig. 2. Approximate solution $\Psi_2(z,t)$ (2(a)) and corresponding absolute error function (2(b)) with $(\hat{M}=11, M=4, N=2)$ and $\beta=\gamma=0.0$ in Example 1.

Table 2
The results obtained for errors via $N=1$ and several $\hat{M}$ and M whenever $\beta=\gamma=-\frac{1}{2}$ in Example 2.

$\hat{M}$	M	N	$\rho(\alpha)=\Gamma(2-\alpha)$		$\rho(\alpha)=\Gamma(3-\alpha)$		$\rho(\alpha)=\Gamma(4-\alpha)$		$\rho(\alpha)=\Gamma(5-\alpha)$	
			$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$
4	4	1	3.3017E−04	1.7287E−04	3.0228E−04	1.3646E−04	2.8389E−04	1.3203E−04	2.6342E−04	1.4181E−04
5	5		3.3703E−05	1.8539E−05	2.9360E−05	1.4819E−05	2.6690E−05	1.3883E−05	2.3928E−05	1.4524E−05
6	6		8.5375E−07	4.5519E−07	7.3668E−07	3.5183E−07	6.6749E−07	3.2166E−07	6.0390E−07	3.3258E−07
7	7		6.1384E−08	3.3727E−08	5.1297E−08	2.5878E−08	4.5428E−08	2.3557E−08	4.0207E−08	2.4045E−08
8	8		1.1263E−09	6.0369E−10	9.3815E−10	4.5546E−10	8.3125E−10	4.0873E−10	7.4143E−10	4.1401E−10

and

$$\Psi(0,t)=\sin(t), \qquad \Psi(1,t)=\sin(t)e^{-i}.$$

The exact solution is

$$\Psi(z,t)=\sin(t)e^{-iz}.$$

The results derived for this system using different choices of $\rho(\alpha)$ whenever $\left(\beta=\gamma=-\frac{1}{2}\right)$ are provided in Table 2. These results show the high accuracy of the method. Figs. 3 and 4 are shown the obtained results with $(\hat{M}=M=8, N=1)$ and $\rho(\alpha)=\Gamma(5-\alpha)$. Note that we have utilized the first 25 terms of the Mittag-Leffler function in the computations. Moreover, the appeared integral is computed using a 15-point Gauss–Legendre integration formula. This work is also made in the below example.

Example 3. Consider the problem

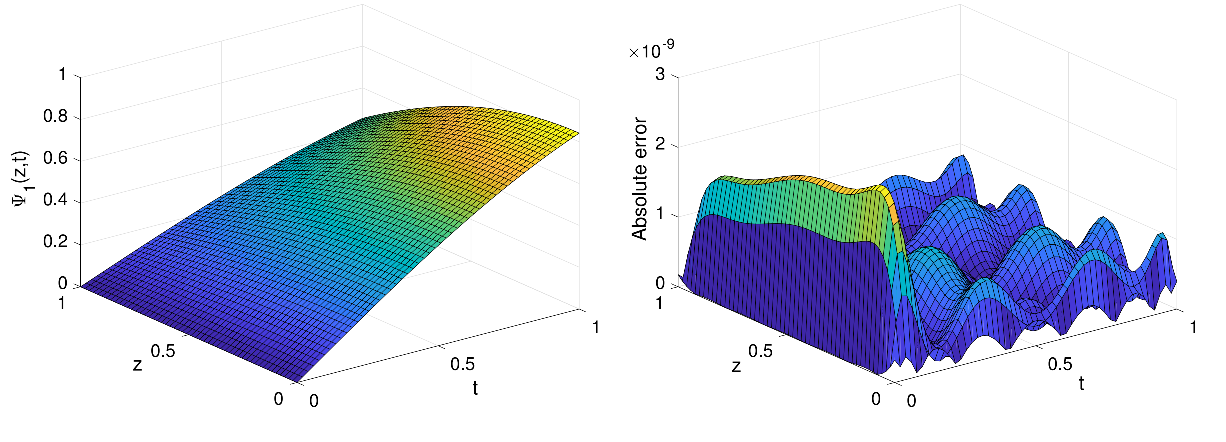
$$i\int_0^1 \rho(\alpha)\,{}_0^C D_t^{\alpha}\Psi(z,t)d\alpha+\frac{1}{3}\Psi_{zz}(z,t)+\frac{1}{2}|\Psi(z,t)|^2\Psi(z,t)+\sin(z)\Psi(z,t)=G(z,t), \quad (z,t)\in[0,1]\times[0,3],$$

where

$$G(z,t)=\left(i\int_0^1 F(t,\alpha)d\alpha+\left[\frac{1}{3}+\frac{1}{2}\left(e^{-2z}+e^{2z}\right)\begin{cases}t^6, & 0\le t<\frac{3}{2},\\ t^4, & \frac{3}{2}\le t\le 3,\end{cases}\right]\begin{cases}t^3, & 0\le t<\frac{3}{2},\\ t^2, & \frac{3}{2}\le t\le 3,\end{cases}\right)\left(e^{-z}+ie^{z}\right),$$

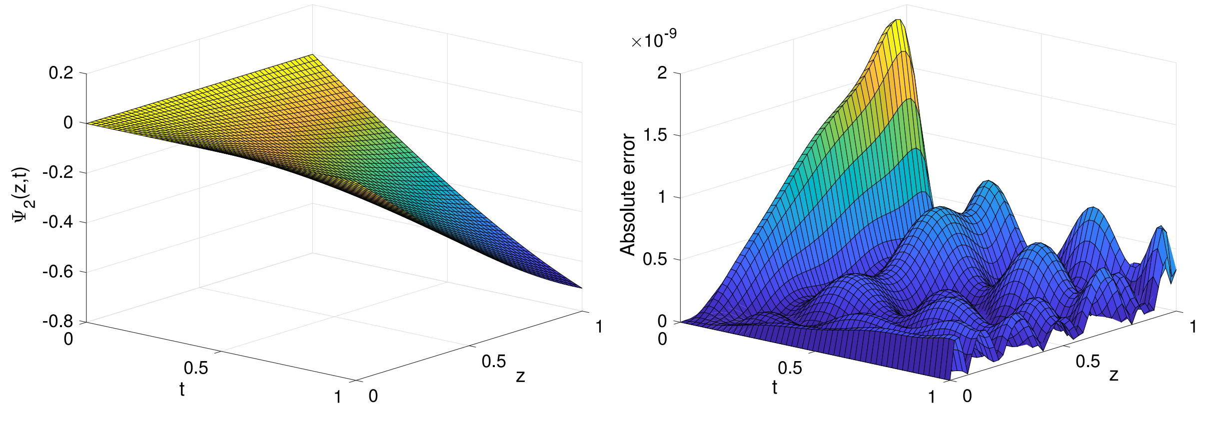
in which

$$F(t,\alpha)=\rho(\alpha)\begin{cases}\dfrac{6t^{3-\alpha}}{\Gamma(4-\alpha)}, & 0\le t<\dfrac{3}{2},\\ \hat{g}(t,\alpha), & \dfrac{3}{2}\le t\le 3,\end{cases}$$



(a) (b)

Fig. 3. Approximate solution $\Psi_1(z,t)$ (Fig. 3(a)) and corresponding absolute error function (Fig. 3(b)) with ($\hat{M} = M = 8, N = 1$) and $\rho(\alpha) = \Gamma(5-\alpha)$ in Example 2.



(a) (b)

Fig. 4. Approximate solution $\Psi_2(z,t)$ (4(a)) and corresponding absolute error function (Fig. 4(b)) with ($\hat{M} = M = 8, N = 1$) and $\rho(\alpha) = \Gamma(5-\alpha)$ in Example 2.

and

$$\begin{aligned}\hat{g}(t,\alpha) = &\frac{3}{\Gamma(1-\alpha)}\left(\prod_{l=1}^{3}\frac{1}{l-\alpha}\prod_{l=1}^{2}(3-l)t^{3-\alpha} - \sum_{r=1}^{3}\prod_{l=1}^{r}\frac{1}{l-\alpha}\prod_{l=1}^{r-1}(3-l)\left[\left(\frac{3}{2}\right)^{3-r}\left(t-\frac{3}{2}\right)^{r-\alpha}\right]\right)\\ &+\frac{2}{\Gamma(2-\alpha)}\left(\frac{3}{2}\left(t-\frac{3}{2}\right)^{1-\alpha} + \frac{1}{2-\alpha}\left(t-\frac{3}{2}\right)^{2-\alpha}\right),\end{aligned}$$

under the conditions

$$\Psi(z,0) = 0,$$

and

$$\Psi(0,t) = (1+i)\begin{cases} t^3, & 0 \le t < \frac{3}{2},\\ t^2, & \frac{3}{2} \le t \le 3,\end{cases} \qquad \Psi(1,t) = \left(e^{-1} + ie\right)\begin{cases} t^3, & 0 \le t < \frac{3}{2},\\ t^2, & \frac{3}{2} \le t \le 3,\end{cases}$$

Table 3
The results obtained for errors via ($M = 4, N = 2$) and several $\hat{M}$ whenever $\beta = \gamma = 1$ in Example 3.

$\hat{M}$	M	N	$\rho(\alpha) = \Gamma(2-\alpha)$		$\rho(\alpha) = \Gamma(3-\alpha)$		$\rho(\alpha) = \Gamma(4-\alpha)$		$\rho(\alpha) = \Gamma(5-\alpha)$	
			$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$	$e_{\Psi 1}$	$e_{\Psi 2}$
5	4	2	6.2522E−05	1.6995E−04	6.2522E−05	1.6995E−04	6.2522E−05	1.6995E−04	6.2522E−05	1.6995E−04
6			2.4672E−06	6.7066E−06	2.4672E−06	6.7066E−06	2.4672E−06	6.7066E−06	2.4672E−06	6.7066E−06
7			8.4871E−08	2.3070E−07	8.4871E−08	2.3070E−07	8.4871E−08	2.3070E−07	8.4871E−08	2.3070E−07
8			2.5815E−09	7.0173E−09	2.5815E−09	7.0173E−09	2.5815E−09	7.0173E−09	2.5815E−09	7.0173E−09
9			7.0281E−11	1.9104E−10	7.0281E−11	1.9104E−10	7.0281E−11	1.9104E−10	7.0281E−11	1.9104E−10

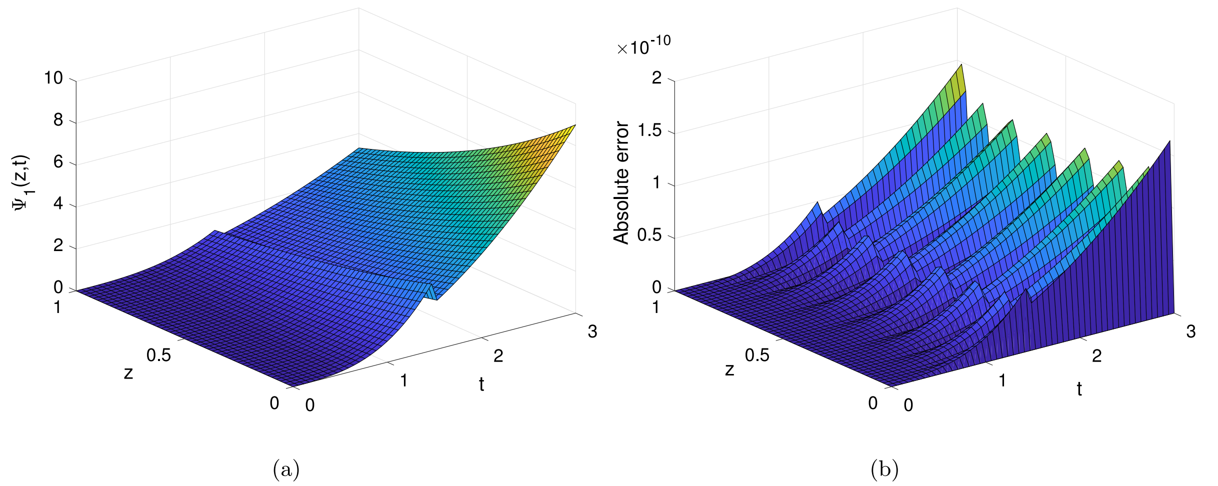


Fig. 5. Approximate solution $\Psi_1(z,t)$ (5(a)) and corresponding absolute error function (5(b)) with ($\hat{M} = 5, M = 4, N = 2$) and $\rho(\alpha) = \Gamma(2-\alpha)$ in Example 3.

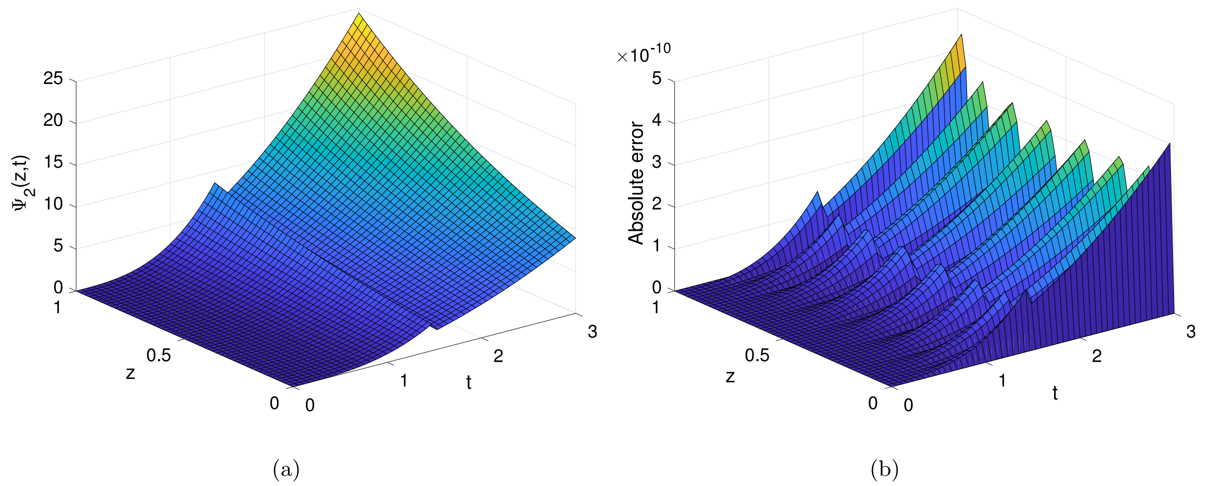


Fig. 6. Approximate solution $\Psi_2(z,t)$ (6(a)) and corresponding absolute error function (6(b)) with ($\hat{M} = 5, M = 4, N = 2$) and $\rho(\alpha) = \Gamma(2-\alpha)$ in Example 3.

The exact solution is

$$\Psi(z,t) = \left(e^{-z} + ie^{z}\right)\begin{cases} t^3, & 0 \le t < \dfrac{3}{2}, \\ t^2, & \dfrac{3}{2} \le t \le 3. \end{cases}$$

Table 3 and Figs. 5 and 6 are used to show the accuracy of the method whenever $\beta = \gamma = 1$ in solving this example.

7. Conclusion

In this work, the orthonormal piecewise Jacobi functions were defined. A formulation for the Caputo fractional derivative of these functions was provided. These basis functions together with the classical Jacobi polynomials and the Gauss–Legendre quadrature rule were applied to construct a hybrid method for solving the distributed-order time fractional version of the nonlinear Schrödinger equation. The established method converts solving the expressed problem into solving an algebraic system of equations. Some examples were studied to show the accuracy of the method. The derived results confirmed the high accuracy of the method.

CRediT authorship contribution statement

M.H. Heydari: Conceptualization, Methodology, Software, Validation, Writing – original draft, Visualization, Supervision. **M. Razzaghi:** Conceptualization, Methodology, Software, Validation, Review and editing. **D. Baleanu:** Conceptualization, Methodology, Software, Validation, Review and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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